Inventory Control with Unobservable Lost Sales and Bayesian Updates

Xiangwen Lu • Jing-Sheng Song • Kaijie Zhu

Cisco Systems, 210 West Tasman Drive,San Jose, CA 95134

Fuqua School of Business, Duke University, Durham, NC 27708

Department of Industrial Engineering and Logistics Management, Hong Kong University of Science and Technology, Clear Water Bay, Kowloon, Hong Kong

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We study a finite-horizon lost-sales inventory model. The demand distribution is unknown and is dynamically updated based on the previous sales data in a Bayesian fashion. We derive a sample-path representation of the first order optimality condition, which characterizes the key tradeoff of the problem. The expression allows us to see why the computation of the optimal policy is difficult and why the myopic solution is not a bound on the optimal solution. It enables us to develop simpler solution bounds and approximations. It also helps us to develop cost bounds as well as cost error bounds of the approximations. Numerical examples indicate that our approximations are most effective for products with short life-cycle. Otherwise, the myopic policy may be a reasonable choice.

1 Introduction

Stochastic inventory models mostly assume that the demand distribution is known. While this is often not the case in reality, this assumption makes the models much more tractable to yield simple solutions. These simple solutions can then be implemented as approximate solutions to real systems with unknown demand distributions. The drawback, however, is that it is difficult to assess the effectiveness of these solutions when the demand distribution is indeed unknown.

Many researchers have taken a different approach – They develop models in which demand distributions have unknown parameters that can be updated as sales information becomes available. These problems are considerably more difficult to solve, because the state space grows rapidly as time evolves due to a longer demand history. The problem becomes even harder when demand data is not completely observable, such as unobservable lost sales (termed censored data). The thrusts of this stream of research are therefore 1) identifying conditions under which simple solutions can be devised and 2) developing insights and effective approximate solutions.

This paper falls into this second category of research. Our focus is on inventory-control problems of *nonperishable products* with *unobservable lost sales*. As explained below, the combination of these two features in the model makes the problem the most complicated to analyze thus far. Our results represent one of the earliest efforts to tackle the difficulty.

We consider a periodic-review inventory system over a finite planning horizon. Demand in each period is stochastic and follows a distribution in a parametric family. However, the parameter of the demand distribution is unknown and is subject to a prior belief, which can be updated periodically with the observed sales data using the Bayesian approach. At the beginning of each period, based on the current status of inventory and the latest knowledge on demand distribution, we make the replenishment decision and place an order accordingly. The order arrives immediately. (We can imagine orders are placed at the end of each day and the shipments arrive the next morning. This is valid in many retail settings, especially with the advanced information technology and cross-docking logistics.) During the period, demand occurs and is fulfilled from the inventory if feasible. At the end of the period, if there is any inventory left, it will be carried over to the next period and an inventory-holding cost is incurred. If the demand exceeds the inventory level, the unsatisfied portion is lost, resulting in a shortage-penalty cost. The objective is to minimize the total expected cost over the planning horizon.

Earlier works on dynamic Bayesian inventory models with unknown demand distributions assume demand data is fully observable. To overcome the high dimensionality of the dynamic program, two approaches have been used. The first approach is to identify a sufficient statistic of the sales data, which reduces the state space to a fixed dimension. The second approach is state-space reduction technique (see Scarf 1959, 1960, Karlin 1960, and Azoury 1985), which reduces the fixed-dimensional problem to a single-dimension problem. This technique is applicable to certain types of demand distributions, including Uniform, Weibull, and Gamma distributions. Lovejoy (1990) further develops conditions under which the single-dimension Bayesian dynamic program can be reduced to a static optimization problem (the myopic optimization problem).

When demand data is not fully observable due to unobservable lost sales, the information dynamics become more complex. On the one hand, the current inventory decision surely depends on the knowledge of the demand distribution. On the other hand, the demand-distribution updating relies on all the sales data obtained previously, which, in turn, depends on inventory decisions made previously. This complication greatly affects the applicability of dimension-reduction approaches mentioned above. So far, we also know that a sufficient statistic exists for the newsvendor distributions (Braden and Freimer 1991) and that the state-space reduction technique is applicable to the Weibull distribution, a specific newsvendor distribution (Lariviere and Porteus 1999).

Most of the work on inventory problems with censored demand data focus on the so-called the censored newsvendor problem, in which inventory is perishable and therefore must be salvaged at the end of each period. Nahmias (1994) and Agrawal and Smith (1996) study how to use censored data to obtain reliable demand estimates. Harpaz, Lee, and Winkler (1982), Lariviere and Porteus (1999), Ding, Puterman, and Bisi (2002), Lu, Song, and Zhu (2005, 2008), Bensoussan, Cakanyildirim, and Sethi (2006, 2007), and Bisi and Dada (2007) analyze the key tradeoffs in various dynamic settings. One major finding of this literature is that the myopic solution, which minimizes the expected inventory cost in the current period but ignores the impact of learning from observed sales data on future periods, is a lower bound on the optimal inventory level. That is, to obtain the informational benefit, we should stock more than the myopic solution.

When the product is not perishable so inventory leftovers can be carried over to the next period, as the case considered in this paper, the above lower bound result no longer holds, as shown by Chen and Plambeck (2008) using a discrete-demand model. We also remark that practical methods of inventory management haven't been well studied in the literature, except the adaptive policies developed by Huh and Rusmevichientong (2006) using a non-parametric approach.

Our work complements Chen and Plambeck (2008) and Huh and Rusmevichientong (2006) in several ways. First, using a continuous-demand setting, which are consistent with the majority of the work in the censored newsvendor problem, our result allows us to see more clearly the connection between the perishable and non-perishable product problems. In particular, we derive an explicit sample-path expression for the first derivative of the optimality equation. The expression reveals that the optimal decision needs to tradeoff three parts – the inventory cost in the current period, the inventory cost in the future, and the informational benefit due to higher inventory. This result generalizes that of the censored newsvendor model in which the tradeoff is between the inventory cost in the current period and the future informational benefit (see Lu, Song, and Zhu 2008). It explains why, unlike in the censored newsvendor model, the myopic solution can be either higher or lower than the optimal inventory level. The expression also allows us to compare the present problem with two well-studied problems mentioned earlier – one with the demand distribution known and the other with unknown demand distribution but observable lost-sales. This, in turn, yields bounds on the optimal cost and demonstrates the value of information.

Second, to facilitate the understanding of various tradeoffs, we perform an *incremental analysis* of the inventory decision in each period. That is, we study the effect of increasing the inventory level by a small amount in a period, i.e., how such an increment of the inventory level brings better demand information in that period and is carried over to future periods. This approach is expected to be applicable to many inventory-control problems with information updating, pricing, and inventory carryover.

Third, the first derivative of the optimality equation provides an alternative and convenient approach for constructing tractable bounds on the optimal inventory level. We use the weighted averages of these bounds as approximations of the optimal policy. We further derive an upper bound on the relative cost errors of any heuristic policy. We show that the cost error bound for a modified myopic policy is asymptotically robust. We compare the weighted solution bounds and the modified myopic policy through a numerical study. We observe that the former performs much better than the latter when the product-life-cycles are relatively short.

Note that the computational intractability is true even for the simpler perishable-product case. In fact, the previous perishable-product literature mentioned above focuses on the qualitative properties of the problem and leaves the issue of computations of the optimal policy open (with the exception of, as mentioned, Huh and Rusmevichientong 2006). Our results, therefore, are also useful for the perishable-product systems.

Several authors have developed solution bounds and/or the relative cost errors for inventory models with demand forecasting updates using non-censored data; see, e.g., Morton (1978), Lovejoy (1990, 1992), Morton and Pentico (1995), and Lu, Song, and Regan (2006). Unfortunately, those *results* and *analyses* are not applicable here. For one thing, when demand data is non-censored (i.e., is always observable regardless of inventory decisions), increasing inventory does not bring any informational benefit. This implies that, for a given sample path, the demand information obtained from using any heuristic policy is exactly the same as that obtained from the optimal policy. This is no longer true here, however.

The remainder of the paper is organized as follows: In Section 2, we introduce the basic notation and discuss preliminaries. In Section 3, we derive the first derivative function for the optimality equation. In Section 4, we develop bounds on the optimal cost and optimal order-up-to level, from which we construct approximations for the optimal inventory level and present an analytical approach to evaluating the performance of the approximations. Then, we illustrate the main result in Sections 3 and 4 by an example, followed by an extensive numerical study. Finally, we make concluding remarks. All the proofs are presented in the appendix.

2 Notation and Preliminaries

We consider a finite-horizon periodic-review inventory control problem. We use t as the time index, $t = 1, ..., T$. We denote by Z_t the demand in period t and assume $Z_1, \cdots, Z_T$ to be identically distributed random variables. The demand distribution has a continuous probability density function (pdf) $f(\cdot|\theta)$ and a cumulative distribution function (cdf) $F(\cdot|\theta)$, where $\theta \in \Theta$ represents a parameter (or a vector of parameters) for which we have only some prior knowledge. Assuming that the prior distribution of θ is π_t at the beginning of period t, we obtain

$$m_t(z_t|\pi_t) = \int_\Theta f(z_t|\theta)\,\pi_t(\theta)\,d\theta, \tag{1}$$

$$M_t(z_t|\pi_t) = \int_\Theta F(z_t|\theta)\,\pi_t(\theta)\,d\theta \tag{2}$$

as the pdf and cdf for Z_t, respectively.

For any integer $j \geq 1$, define $Z[t, t+j) = Z_t + ... + Z_{t+j-1}$. Let $\mathbf{u}_{t,t+j}$ denote the vector $(u_t, ..., u_{t+j})$. For example, $\mathbf{Z}_{t,t+j} = \mathbf{z}_{t,t+j}$ means $Z_t = z_t, ..., Z_{t+j} = z_{t+j}$. We use $\mathbf{1}(A)$ to denote the indicator function of event A.

Let h and p be the unit inventory-holding and shortage-penalty costs, respectively. To simplicity, we assume that the inventory leftover at the end of the planning horizon can be salvaged at the unit ordering cost, so the problem can be transformed to one with the zero ordering cost (see Veinott 1963 and Morton and Pentico 1995). Given that y_t is the inventory level in period t after ordering and z_t is the realized demand, we obtain the inventory-holding or shortage-penalty cost as

$$r(y_t, z_t) = h(y_t - z_t)^+ + p(z_t - y_t)^+ .$$

Then, the expected inventory-holding and shortage-penalty cost incurred in period t is

$$C_t(y_t, \pi_t) = \int_0^{+\infty} r(y_t, z_t)\, m_t(z_t|\pi_t)\, dz_t.$$

To facilitate the understanding of the notation and the later analysis, in the rest of this section, we relate our problem to two simpler models that are well studied in the literature.

2.1 Lost Sales with Known Demand Distribution

Consider the standard lost-sales inventory model in which the demand distribution is known, i.e., the value of θ is given. Therefore we can suppress θ and π_t in the notation and denote by $f(\cdot)$ and $F(\cdot)$ the pdf and cdf of Z_t.

Let $V_t(x_t)$ be the optimal expected cost from period t to T given that the inventory level before ordering in period t is x_t. We have the following optimality equations:

$$\begin{aligned} V_{T+1}(x_t) &= 0, \quad x_t \geq 0, \\ V_t(x_t) &= \min_{y_t \geq x_t} \{G_t(y_t)\}, \quad x_t \geq 0, \quad 1 \leq t \leq T, \end{aligned}$$

where

$$G_t(y_t) = C_t(y_t) + E\left[V_{t+1}\left((y_t - Z_t)^+\right)\right].$$

It is well known that $G_t(y_t)$ is convex. Let s_t^* be its minimizer. Then, the optimal policy is an order-up-to policy with the optimal order-up-to level s_t^* for period t. (See, e.g., Karlin 1958 and Morton and Pentico 1995.)

For any given period t, if we minimize only the cost in the current period by solving

$$C_t'(y_t) = (p+h)F(y_t) - p = 0,$$

then we obtain the myopic order-up-to level

$$s^m = F^{-1}\left(\frac{p}{p+h}\right),$$

which is independent of t.

However, to determine the optimal inventory level s_t^*, we need to tradeoff the marginal inventory cost in the current period, $C_t'(y_t)$ and the marginal inventory cost in the future periods, $\psi_t(y_t) \triangleq \frac{d}{dy_t}E\left[V_{t+1}\left((y_t - Z_t)^+\right)\right]$. Below we take a sample-path approach to show how to do this by backward iterations. This approach is instrumental to understand the analysis for more complex systems later. (Note that a similar result has been shown in the literature. See, e.g., Morton 1978.)

Note that, because $V_{T+1} = 0$, $s_T^* = s^m$. Suppose we have obtained s_{t+1}^*, ..., s_T^*. We shall show that $G_t'(y_t)$ can be computed through C_t' and s_{t+1}^*, ..., s_T^*. Thus, $G_t'(y_t) = 0$ is readily solvable to yield s_t^*. Suppose we are in period t and the inventory level after ordering is y_t. Then, no replenishment order will be placed until the stopping time

$$\tau(t, y_t) = \min\{t + j \mid (y_t - Z[t, t+j))^+ < s_{t+j}^*,\ j \geq 1\}.$$

(Throughout the paper, $\min\{\emptyset\} = \infty$.) If $\tau(t, y_t) > t + i$, then no lost sales would happen in periods t through $t + i - 1$. On the other hand, if $\tau(t, y_t) = t + 1$, then lost sales may happen in period t, in which case the initial inventory level in period $t+1$ is zero. We have:

Theorem 1 *In the lost-sales inventory-control problem with a known demand distribution, the first derivative function of $G_t(y_t)$ is*

$$G_t'(y_t) = C_t'(y_t) + \psi_t(y_t),$$

where

$$\psi_t(y_t) = \frac{d}{dy_t}E\left[V_{t+1}\left((y_t - Z_t)^+\right)\right] = \sum_{i=1}^{T-t} E[\mathbf{1}(\tau(t, y_t) > t + i)C_t'(y_t - Z[t, t+i))] \geq 0. \qquad (3)$$

Because $\psi_t(y_t)$ is always nonnegative, we have $s_t^* \leq s^m$. On the other hand, recall that $s_T^* = s^m$, we have $\tau(T-1, s^m) = T$. This implies $\psi_{T-1}(s^m) = 0$ and hence $C_{T-1}'(s^m) = 0$. So, $s_{T-1}^* = s^m$. Continuing in the same fashion, we can show that $s_t^* = s^m$ for all t. That is, the myopic policy is optimal.

2.2 Observable Lost Sales with Unknown Distribution

Next, suppose θ is unknown but lost sales are observable, as in the context of mail order and e-tailing. At the end of period t, after observing the demand realization z_t, we update π_t to π_{t+1}, the posterior distribution of θ, based on the Bayes formula:

$$\pi_{t+1}(\theta|\pi_t, z_t) = \frac{f(z_t|\theta)\pi_t(\theta)}{\int_\Theta f(z_t|\theta')\pi_t(\theta')d\theta'}. \tag{4}$$

In this setting, the state of the inventory system comprises both the current inventory level and the prior distribution of the demand. Let $V_t^{FI}(x_t, \pi_t)$ (here, FI stands for "full information") be the optimal expected cost from period t to T, given that the inventory level before ordering in period t is x_t and the prior distribution of θ is π_t. Then, we have the following optimality equations:

$$\begin{aligned} V_{T+1}^{FI}(x_{T+1}, \pi_{T+1}) &= 0, \quad x_{T+1} \geq 0, \\ V_t^{FI}(x_t, \pi_t) &= \min_{y_t \geq x_t} \left\{ G_t^{FI}(y_t, \pi_t) \right\}, \quad x_t \geq 0, \quad 1 \leq t \leq T, \end{aligned} \tag{5}$$

where

$$\begin{aligned} G_t^{FI}(y_t, \pi_t) &= C_t(y_t, \pi_t) + E\left[V_{t+1}^{FI}\left((y_t - Z_t)^+, \pi_{t+1}\right)\right] \\ &= C_t(y_t, \pi_t) + \int_0^{+\infty} V_{t+1}^{FI}\left((y_t - z_t)^+, \pi_{t+1}(\cdot|\pi_t, z_t)\right) m_t(z_t|\pi_t) dz_t. \end{aligned} \tag{6}$$

It can be verified that $G_t^{FI}(y_t, \pi_t)$ is a convex function in y_t (see, for instance, Scarf 1959, 1960 and Azoury 1985). Let $s_t^{FI}(\pi_t)$ be its minimizer (which is state-dependent, i.e., dependent on the prior π_t). Then, the optimal policy is an order-up-to policy with state-dependent order-up-to level $s_t^{FI}(\pi_t)$.

Similar to the case of a known demand distribution, (6) indicates that the tradeoff is between the marginal inventory cost in the current period, $C_t{}'(y_t, \pi_t)$, and the marginal inventory cost in the future periods, $\psi_t^{FI}(y_t, \pi_t) \triangleq \frac{d}{dy_t} E\left[V_{t+1}^{FI}\left((y_t - Z_t)^+, \pi_{t+1}\right)\right]$. Similar to Morton (1978), Morton and Pentico (1995) and Lu, Song, and Regan (2006), we can show the following theorem:

Theorem 2 *In the inventory-control problem with an unknown demand distribution and observable lost sales, the first derivative function of $G_t^{FI}(y_t, \pi_t)$ is*

$$\frac{d}{dy_t} G_t^{FI}(y_t, \pi_t) = C_t{}'(y_t, \pi_t) + \psi_t^{FI}(y_t, \pi_t), \tag{7}$$

where

$$\psi_t^{FI}(y_t, \pi_t) = \sum_{i=1}^{T-t} E[\mathbf{1}(\tau^{FI}(t, y_t) > t+i) C'_{t+i}(y_t - Z[t, t+i), \pi_{t+i})] \geq 0, \tag{8}$$

$$\tau^{FI}(t, y_t) = \min\{t + j | (y_t - Z[t, t+j))^+ < s_{t+j}^{FI}(\pi_{t+j}), j \geq 1\}.$$

The myopic order-up-to level equals $M_t^{-1}\left(\frac{p}{h+p}\middle|\pi_t\right)$, which minimizes $C_t(y_t,\pi_t)$ and hence solves $C_t{}'(y_t,\pi_t)=(h+p)M_t(y_t|\pi_t)-p=0$. The optimal inventory level $s_t^{FI}(\pi_t)$, on the other hand, needs to solve $\frac{d}{dy_t}G_t^{FI}(y_t,\pi_t)=0$, i.e., $C_t{}'(y_t,\pi_t)+\psi_t^{FI}(y_t,\pi_t)=(h+p)M_t(y_t|\pi_t)-p+\psi_t^{FI}(y_t,\pi_t)=0$. Because $\psi_t^{FI}\geq 0$ (see (8)), we have

$$s_t^{FI}(\pi_t)=M_t^{-1}\left(\frac{p-\psi_t^{FI}(s_t^{FI}(\pi_t),\pi_t)}{h+p}\middle|\pi_t\right)\leq M_t^{-1}\left(\frac{p}{h+p}\middle|\pi_t\right).$$

Thus, unlike in the case with known demand distribution discussed in subsection 2.1, here, the myopic order-up-to level is not optimal but an upper bound on the optimal order-up-to level.

In general, the computation of $s_t^{FI}(\pi_t)$ is complicated. This is because π_t is updated from π_1 successively based on the history of the observed demands, $z_1,...,z_{t-1}$. As a result, the state in period t consists of x_t and $(z_1,...,z_{t-1})$, whose dimension is increasing in t.

Fortunately, for a wide range of demand distributions, the history of $(z_1,...,z_{t-1})$ can be represented by a sufficient statistic. This implies that the state in period t can be reduced to consisting of only the starting inventory position and the sufficient statistic, whose dimension is fixed for all t. So the computation of the optimal policy becomes much easier. Scarf (1959, 1960) and Azoury (1985) show that the state space can be further reduced to a single dimension if $f(\cdot|\theta)$ satisfies certain additional conditions. The distributions that satisfy these conditions include Uniform, Weibull, and Gamma.

3 Unobservable Lost Sales with Unknown Distribution

We now turn to the main focus of this paper – The parameter θ in the demand distribution is *unknown* and lost sales are *unobservable*.

3.1 Bayesian Updates

At the beginning of period t, our knowledge about the demand in that period is a prior distribution of θ, denoted by π_t. Using this prior, we can derive the pdf m_t and cdf M_t in the same fashion as in (1) and (2). However, because the lost sales are unobservable, the sales quantity recorded in each period may or may not equal the demand realization of that period. When there is inventory leftover at the end of the period, i.e., when the stocked amount is larger than the sales quantity, the sales quantity equals the demand realization and hence represents an *exact* observation of the demand. Otherwise, the sales quantity represents a *censored* observation of the demand, in which case we can only tell that the demand should be no less than the stocked inventory. Thus, the demand-distribution updating at the end of t is much more complicated than (4).

To facilitate the later sample-path analysis in which we investigate the impact of inventory decision on demand observations, we denote the observed sales information in period t by two parts:

$$o_t = (o_{t,1}, o_{t,2}),$$

where $o_{t,1}$ is the sales quantity and can take any nonnegative value, while $o_{t,2}$ is the observation status taking values of e or c, representing that the observation of the demand is exact or censored, respectively.

For an illustration, we suppose the stocked inventory is y_t and analyze the different demand realizations of z_t. If $z_t < y_t$, then the sales quantity is z_t and the observation of the demand is exact. Otherwise, with $z_t \geq y_t$, the sales quantity is y_t and the observation is censored. That is, o_t is determined by both y_t and z_t, which we express as

$$o_t = y_t \otimes z_t \triangleq \begin{cases} (z_t, e), & \text{if } z_t < y_t, \\ (y_t, c), & \text{if } z_t \geq y_t. \end{cases}$$

For expositional simplicity, in the remainder of the paper, we use the short-hand notation z_t^e and y_t^c to represent (z_t, e) and (y_t, c), respectively.

This notation will enable us to communicate the incremental analysis of inventory decisions in Subsection 3.3: If the sales quantity (observed demand) equals the stocked amount, i.e., $z_t = y_t$, then the situation can be represented by $o_t = y_t^c$. If, however, the inventory level were $y_t + dy_t$, where $dy_t > 0$, and the demand realization $z_t = y_t$ was the same, then the situation can be represented by $o_t = y_t^e$.

At the end of period t, based on the observed sales information o_t, we update π_t to π_{t+1}, the posterior distribution of θ. We define

$$l(o_t|\theta) = \begin{cases} f(o_{t,1}|\theta), & \text{if } o_{t,2} = e, \\ 1 - F(o_{t,1}|\theta), & \text{if } o_{t,2} = c, \end{cases} \tag{9}$$

as the likelihood function of the sales information o_t for given θ. Then, based on the observed sales information o_t in period t, we use the Bayes formula to update π_t to π_{t+1}, the posterior distribution of θ, as follows:

$$\pi_{t+1}(\theta|\pi_t, o_t) = \frac{l(o_t|\theta)\,\pi_t(\theta)}{\int_\Theta l(o_t|\theta')\,\pi_t(\theta')\,d\theta'}. \tag{10}$$

As is clear, the above updating is more complicated and general than that in the case of observable lost sales, i.e., (4). Specifically, when $o_t = z_t^e$, (10) reduces to be (4).

3.2 The Dynamic Program and Its Complications

Let $V_t(x_t, \pi_t)$ be the optimal expected cost from period t to T given that the inventory level before ordering in period t is x_t and the prior distribution of θ is π_t. The optimality equations for this problem are

$$
\begin{aligned}
V_{T+1}\left(x_{T+1},\pi_{T+1}\right) &= 0, \quad x_{T+1}\geq 0, && (11)\\
V_t\left(x_t,\pi_t\right) &= \min_{y_t\geq x_t}\left\{G_t\left(y_t,\pi_t\right)\right\}, \quad x_t\geq 0, \quad 1\leq t\leq T, && (12)
\end{aligned}
$$

where

$$
\begin{aligned}
G_t\left(y_t,\pi_t\right) &= C_t\left(y_t,\pi_t\right)+E\left[V_{t+1}\left((y_t-Z_t)^+,\pi_{t+1}\right)\right] && (13)\\
&= C_t\left(y_t,\pi_t\right)+\int_0^{y_t}V_{t+1}\left(y_t-z_t,\pi_{t+1}(\cdot|\pi_t,z_t^e)\right)m_t(z_t|\pi_t)dz_t\\
&\quad +V_{t+1}\left(0,\pi_{t+1}(\cdot|\pi_t,y_t^c)\right)\left[1-M_t(y_t|\pi_t)\right].
\end{aligned}
$$

Similar to the model with observable lost sales studied in Subsection 2.2, the sufficient statistic and the state-space reduction technique can reduce the dimensionality of the dynamic program. However, the applicability of these methods is extremely limited. In fact, the literature shows that a sufficient statistic exists only for newsvendor distributions (Braden and Freimer 1991) and that the state-space reduction technique is applicable only to the Weibull distribution, a specific newsvendor distribution (Lariviere and Porteus 1999).

For a general demand distribution, unobservable lost sales make it difficult to identify a sufficient statistic with a fixed dimension or to apply the state-space reduction technique. Therefore, the computation of the optimal policy is very difficult or even intractable. This is well recognized by the literature. For example, Braden and Freimer (1991) claim that "if a modeler feels that no member of the families we characterize is a reasonable approximation, then he will almost surely encounter serious analytic and computational problems if his data includes censored observations."

A second complication in this model is about the convexity of expected costs. In the two settings studied in Section 2, inventory decisions have no impact on demand realizations, and thus the convexity are shown to be true via induction. But in the setting of censored demand, an inventory decision can affect demand realizations in future periods, which may make the preservation of convexity collapse (see Chen and Plambeck 2008; in a setting of profit maximization, they find that the optimal expected profit, in the presence of censored demand data, is not necessarily concave in the inventory level). Therefore, the optimal policy is not necessarily an order-up-to policy.

Observe that $\lim_{y_t\to\pm\infty}C_t(y_t,\pi_t)=+\infty$. By induction, it can be shown $\lim_{y_t\to\pm\infty}G_t(y_t,\pi_t)=+\infty$. Thus, for any given x_t, we have a finite

$$
s_t^*\left(x_t,\pi_t\right)=\arg\min_{y_t\geq x_t}G_t\left(y_t,\pi_t\right).
$$

(If there are multiple minimizers, we take the smallest one.) Clearly, $s_t^*\left(x_t,\pi_t\right)$ is the optimal inventory level in period t, given that the inventory level before ordering is x_t and the prior distribution of θ is π_t. Because $C_t(y_t,\pi_t)$ is differentiable in y_t, by induction we can conclude that $G_t(y_t,\pi_t)$ is also differentiable. Therefore, to solve $s_t^*\left(x_t,\pi_t\right)$, it is important to understand ${G_t}'(y_t,\pi_t)$.

3.3 The First Order Condition

In this subsection, similar to Theorems 1 and 2, to gain a deeper understanding of the key tradeoff of the problem, we derive a sample-path expression for $G_t'(y_t, \pi_t)$.

For this purpose, we set forth a few compact notation. Consider a sample path from period t to $t+i-1$, on which observations $\mathbf{o}_{t,t+i-1} = (o_t, \cdots, o_{t+i-1})$ are obtained. Because we can compute the posterior distribution $\pi_{t+i}(\theta|\pi_t, \mathbf{o}_{t,t+i-1})$ (see Lemma 18 in the appendix), we replace π_{t+i} by $(\pi_t, \mathbf{o}_{t,t+i-1})$ to highlight the impact of inventory decisions in-between on the posterior distribution (through their effect on the observations $\mathbf{o}_{t,t+i-1}$). For example, we equate $G_{t+i}(y_{t+i}, \pi_{t+i})$ with $G_{t+i}(y_{t+i}, \pi_t, \mathbf{o}_{t,t+i-1})$, with the understanding that π_{t+i} is computed from π_t and $\mathbf{o}_{t,t+i-1}$. For ease of exposition, we also define $H_{t,t+i} = (\pi_t, \mathbf{o}_{t,t+i-1})$. In other words, we shall use π_{t+i}, $(\pi_t, \mathbf{o}_{t,t+i-1})$, and $H_{t,t+i}$ interchangeably in the remainder of the paper.

We now present two cost terms that will be useful in the following analysis. First, suppose that the sales information obtained in period t is $o_t = y_t^e$, which implies that the demand realization is y_t and is faithfully observed due to stocked amount being strictly higher than y_t. In this case, if we follow the optimal policy from period $t+1$ on, we can obtain an expected cost

$$G_{t+1}\left(s_{t+1}^*\left(x_{t+1}, \pi_t, y_t^e\right), \pi_t, y_t^e\right) = \sum_{i=1}^{T-t} EC_{t+i}\left(s_{t+i}^*(x_{t+i}, \pi_t, y_t^e, \mathbf{o}_{t+1,t+i-1}), \pi_t, y_t^e, \mathbf{o}_{t+1,t+i-1}\right),$$

where $o_{t+i} = s_{t+i}^*(x_{t+i}, \pi_t, y_t^e, \mathbf{o}_{t+1,t+i-1}) \otimes z_{t+i}$ and $x_{t+i+1} = [s_{t+i}^*(x_{t+i}, \pi_t, y_t^e, \mathbf{o}_{t+1,t+i-1}) - z_{t+i}]^+$.

Second, suppose the sales information is still $o_t = y_t^e$. But instead of the optimal inventory policy, we follow a sub-optimal one. More specifically, from period $t+1$ on, the inventory decisions are made as if the obtained observation in period t were y_t^c. The resulting expected cost of following this policy from period $t+1$ to T is

$$J_{t+1}\left(s_{t+1}^*\left(x_{t+1}, \pi_t, y_t^c\right), \pi_t, y_t^e\right) = \sum_{i=1}^{T-t} EC_{t+i}\left(s_{t+i}^*(x_{t+i}, \pi_t, y_t^c, \mathbf{o}_{t+1,t+i-1}), \pi_t, y_t^e, \mathbf{o}_{t+1,t+i-1}\right), \tag{14}$$

where $o_{t+i} = s_{t+i}^*(x_{t+i}, \pi_t, y_t^c, \mathbf{o}_{t+1,t+i-1}) \otimes z_{t+i}$ and $x_{t+i+1} = [s_{t+i}^*(x_{t+i}, \pi_t, y_t^c, \mathbf{o}_{t+1,t+i-1}) - z_{t+i}]^+$. Because this policy cannot perform better than the optimal one, we have

$$\begin{aligned} V_{t+1}\left(x_{t+1}, \pi_t, y_t^e\right) &= G_{t+1}\left(s_{t+1}^*(x_{t+1}, \pi_t, y_t^e), \pi_t, y_t^e\right) \\ &\leq J_{t+1}\left(s_{t+1}^*(x_{t+1}, \pi_t, y_t^c), \pi_t, y_t^e\right). \end{aligned} \tag{15}$$

In addition, let

$$\tau(t, y_t) = \min\{t+j \mid (y_t - Z[t, t+j))^+ < s_{t+j}^*((y_t - Z[t, t+j))^+, \pi_{t+j}), j \geq 1\}.$$

Then, the next order placement will not occur until period $\tau(t, y_t)$. Note that for any sample path such that $\tau(t, y_t) > t+i$, we have $y_t - z[t, t+i) \geq s_{t+i}^*((y_t - z[t, t+i))^+, \pi_{t+i}) > 0$, which implies

z_{t+k} is less than the initial inventory level in period $t+k$, $k=0,1,...,i-1$. So, prior to $t+i$, all demand observations are exact. This means that, conditional on $\tau(t,y_t) > t+i$, we can equate π_{t+i} with $(\pi_t, \mathbf{z}^e_{t,t+i-1})$, where $\mathbf{z}^e_{t,t+i-1}$ is the exact demand realizations from period t to $t+i-1$. In addition, let $\mathbf{Z}^e_{t,t+i-1} = \left(Z^e_t, ..., Z^e_{t+i-1}\right)$, where Z^e_{t+k} is defined recursively as the random event that $o_{t+k,1} = Z_{t+k}$ and $o_{t+k,2} = e$ (e.g., due to $Z_{t+k} < s^*_{t+k}(y_t - Z[t,t+k), \pi_t, \mathbf{Z}^e_{t,t+k-1})$).

Using the above notation, we can now present one of the main results of the paper:

Theorem 3 *In the inventory-control problem with an unknown demand distribution and unobservable lost sales, the first derivative of $G_t(y_t,\pi_t)$ is*

$$G_t'(y_t,\pi_t) = C_t'(y_t,\pi_t) + \psi_t(y_t,\pi_t) + \varphi_t(y_t,\pi_t), \qquad (16)$$

where

$$\psi_t(y_t,\pi_t) = \sum_{i=1}^{T-t} \psi_{t,t+i}(y_t,\pi_t) \geq 0, \qquad \varphi_t(y_t,\pi_t) = \sum_{i=1}^{T-t} \varphi_{t,t+i-1}(y_t,\pi_t) \leq 0 \qquad (17)$$

$$\begin{aligned}
\psi_{t,t+i}(y_t,\pi_t) &= E[\mathbf{1}(\tau(t,y_t) > t+i) C'_{t+i}(y_t - Z[t,t+i), \pi_t, \mathbf{Z}^e_{t,t+i-1})], \qquad i \geq 1, \\
\varphi_{t,t}(y_t,\pi_t) &= \left[G_{t+1}\left(s^*_{t+1}\left(0, H^e_{t,t+1}\right), H^e_{t,t+1}\right) - J_{t+1}\left(s^*_{t+1}\left(0, H^c_{t,t+1}\right), H^e_{t,t+1}\right)\right] m_t\left(y_t|\pi_t\right), \\
\varphi_{t,t+i-1}(y_t,\pi_t) &= E[\left(G_{t+i}(s^*_{t+i}(0,H^e_{t,t+i}), H^e_{t,t+i}) - J_{t+i}(s^*_{t+i}(0,H^c_{t,t+i}), H^e_{t,t+i})\right) \\
&\quad \times \mathbf{1}(\tau(t,y_t) > t+i-1) m_{t+i-1}(y_t - Z[t,t+i-1)|\pi_t, \mathbf{Z}^e_{t,t+i-2})], \qquad i \geq 2,
\end{aligned}$$

and $H^e_{t,t+1} = (\pi_t, y^e_t)$, $H^c_{t,t+1} = (\pi_t, y^c_t)$, $H^e_{t,t+i} = (\pi_t, \mathbf{Z}^e_{t,t+i-2}, (y_t - Z[t,t+i-1))^e)$, and $H^c_{t,t+i} = (\pi_t, \mathbf{Z}^e_{t,t+i-2}, (y_t - Z[t,t+i-1))^c)$ for $i \geq 2$.

The decomposition of $G_t'(y_t,\pi_t)$ in (16) indicates that to make the right inventory decision, in addition to trade off the marginal inventory cost in the current period $C_t'(y_t,\pi_t)$ and the marginal inventory cost in the future periods $\psi_t(y_t,\pi_t)$ as in the case of known demand distribution or observable lost sales, we also need to take into account the marginal informational benefit in the future resulting from increasing the inventory level in period t (since $\varphi_t(y_t,\pi_t) \leq 0$, its absolute value is the marginal benefit).

Note that $G_t'(y_t,\pi_t)$ measures the marginal cost of having additional inventory dy_t (where $dy_t > 0$) at the beginning of period t. In fact, in deriving this function, we actually compared two scenarios – the inventory level after ordering in period t is y_t (scenario 1) or $y_t + dy_t$ (scenario 2). The two scenarios have a cost difference in period t as follows:

$$C_t(y_t + dy_t, \pi_t) - C_t(y_t, \pi_t) = C_t'(y_t,\pi_t)\, dy_t + o(dy_t),$$

which gives the first term on the right-hand side of (16), the marginal cost in the current period. To gain insights from the other terms, we discuss three cases of demand realization z_t: $y_t < z_t < y_t + dy_t$, $z_t > y_t + dy_t$, and $z_t < y_t$.

Case 1: $y_t < z_t < y_t + dy_t$. This case will happen with probability $m_t(y_t|\pi_t)\,dy_t$. In this case, under scenario 1, we obtain a censored observation of y_t^c and make the inventory decision in *any* future period based on this observation, which results in a cost of $J_{t+1}\left(s_{t+1}^*(0, \pi_t, y_t^c), \pi_t, y_t^c\right)$. Under scenario 2, we obtain an exact observation of y_t^e and make future inventory decisions accordingly, which results in a cost of $G_{t+1}\left(s_{t+1}^*(0, \pi_t, y_t^e), \pi_t, y_t^e\right)$. In short, the additional inventory dy_t in scenario 2 allows us to obtain a more accurate demand observation in period t. This leads to a cost reduction (i.e., an informational value) of $\varphi_{t,t}(y_t, \pi_t)dy_t$, resulting in the first term in $\varphi_t(y_t, \pi_t)$.

Case 2: $z_t > y_t + dy_t$. In this case, the demand observations are censored under both scenarios 1 and 2; they are y_t^c and $(y_t + dy_t)^c$, respectively. Thus, having additional inventory dy_t in scenario 2 brings a negligible informational value (i.e., a value of only $o(dy_t)$). We will see an illustration of this case in Section 4.

Case 3: $z_t < y_t$. Here, the inventory levels before ordering in period $t+1$ are $y_t - z_t$ and $y_t + dy_t - z_t$ under scenarios 1 and 2, respectively. But but the demand observations in both scenarios are the same z_t^e. So the prior knowledge in period $t+1$ becomes $\pi_{t+1} = (\pi_t, z_t^e)$.

Case 3a. If an order is placed in period $t+1$ in scenario 1, i.e., $\tau(t, y_t) = t+1$, then an order will be placed in scenario 2 as well, which will bring the inventory to the same level as that in scenario 1. Therefore, the additional inventory dy_t can be offset by reducing the ordering quantity in scenario 2 by dy_t. Because both the updated distribution and the inventory level (*after ordering*) in period $t+1$ are the same, the expected costs incurred from period $t+1$ to T will be the same in the two scenarios.

Case 3b. If no order is placed in period $t+1$ in scenario 1, i.e., $\tau(t, y_t) > t+1$, then there is no need to place any order in scenario 2 either. The inventory level (*after ordering*) in period $t+1$ will then be $y_t - z_t$ and $y_t + dy_t - z_t$ for scenarios 1 and 2, respectively. Thus, the respective inventory costs incurred in period $t+1$ are $C_{t+1}(y_t - z_t, \pi_t, z_t^e)$ and $C_{t+1}(y_t + dy_t - z_t, \pi_t, z_t^e)$. This leads to a cost difference of $C_{t+1}(y_t + dy_t - z_t, \pi_t, z_t^e) - C_{t+1}(y_t - z_t, \pi_t, z_t^e) = C_{t+1}'(y_t - z_t, \pi_t, z_t^e)dy_t + o(dy_t)$, from which we obtain $\psi_{t,t+1}(y_t, \pi_t)$, the first term in $\psi_t(y_t, \pi_t)$. Meanwhile, the additional inventory dy_t carried to period $t+1$ can allow a more accurate demand observation in $t+1$ (if $y_t - z_t < z_{t+1} < y_t + dy_t - z_t$

happens) and thus can bring an informational value, $\varphi_{t,t+1}(y_t, \pi_t)dy_t$, yielding the second term in $\varphi_t(y_t, \pi_t)$.

Similarly, if $\tau(t, y_t) > t + i$ (which implies $z_t < y_t$, $z_t + z_{t+1} < y_t$, till $\mathbf{z}_{[t,t+i)} < y_t$), the additional inventory dy_t carried from period t to period $t + i$ brings both the marginal inventory cost incurred in that period, $\psi_{t,t+i}(y_t, \pi_t)$, and the informational value due to a more accurate observation obtained in that period, $\varphi_{t,t+i}(y_t, \pi_t)$. Finally, we have $\varphi_{t,T}(y_t, \pi_t) = 0$, because a more accurate observation in the last period has no value at all.

We note that (16) generalizes the first derivative of the optimal equation for the censored newsvendor model. In that model, without inventory carryover, we place an order in every period. Thus, $\tau(t, y_t) \equiv t + 1$ and $\mathbf{1}(\tau(t, y_t) > t + i) = 0$ for any $i = 1, \cdots, T - t$ and y_t. So, the terms $\psi_{t,t+1}(y_t, \pi_t), \cdots, \psi_{t,T}(y_t, \pi_t)$ and $\varphi_{t,t+1}(y_t, \pi_t), \cdots, \varphi_{t,T-1}(y_t, \pi_t)$ vanish from (16), yielding

$$G_t'(y_t, \pi_t) = C_t'(y_t, \pi_t) + \varphi_{t,t}(y_t, \pi_t), \tag{18}$$

an expression shown in Lu, Song, and Zhu (2008) for the censored newsvendor model.

We next present some properties of $\psi_t(y_t, \pi_t)$ and $\varphi_t(y_t, \pi_t)$. These will be useful later in developing the solution bounds.

Theorem 4 *The terms of $\psi_t(y_t, \pi_t)$ and $\varphi_t(y_t, \pi_t)$ have the following properties:*

$$\varphi_{t,t+i-1}(y_t, \pi_t) \leq 0, \tag{19}$$

$$\sum_{j=i}^{T-t} \psi_{t,t+j}(y_t, \pi_t) + \sum_{j=i+1}^{T-t} \varphi_{t,t+j-1}(y_t, \pi_t) \geq 0, \tag{20}$$

$$\sum_{j=i}^{T-t} \psi_{t,t+j}(y_t, \pi_t) \geq 0, \tag{21}$$

where $1 \leq i \leq T - t$.

More specifically, (19) means that the higher the inventory level y_t in period t, the larger the informational benefit obtained in any future periods. (20) indicates that when there is no order placed between period $t + 1$ to $t + i$, increasing the inventory level y_t in period t increases the total expected cost from period $t+i$ to T. Because this cost consists of both the marginal inventory cost and the marginal informational benefit, (20) essentially implies that the magnitude of the former outweighs that of the latter, i.e., $\sum_{j=i}^{T-t} \psi_{t,t+j}(y_t, \pi_t) \geq -\sum_{j=i+1}^{T-t} \varphi_{t,t+j-1}(y_t, \pi_t)$. For example, with $i = 1$, $\psi_{t,t+1}(y_t, \pi_t) + \cdots + \psi_{t,T}(y_t, \pi_t) \geq -\varphi_{t,t+1}(y_t, \pi_t) - \cdots - \varphi_{t,T-1}(y_t, \pi_t)$ means that the total marginal inventory cost incurred in periods $t+1, \cdots, T$ due to stocking dy_t in period t is no less than the total marginal informational benefit from exact demand observations in periods $t+1, \cdots, T-1$ brought by dy_t. As a result, (21) shows that the aggregate marginal inventory cost incurred from period $t+i$ to T is nonnegative (but the marginal inventory cost in a particular period, $\psi_{t,t+i}(y_t, \pi_t)$, is not necessarily nonnegative).

3.4 The Myopic Solution

Based on the first derivative function, we can compare the myopic and optimal solutions. The myopic solution of this model is

$$s_t^m(x_t, \pi_t) = \arg\min_{y_t \geq x_t} C_t\left(y_t, \pi_t\right).$$

Note that

$$s_t^m(0, \pi_t) = M_t^{-1}\left(\frac{p}{h+p}\middle|\, \pi_t\right), \qquad s_t^m(x_t, \pi_t) = \max\{x_t, s_t^m(0, \pi_t)\}. \tag{22}$$

Recall that in the censored newsvendor model, the myopic solution is a lower bound on the optimal inventory level. Because that model is a special case of our model, we can recover this result from (18)

$$s_t^m(0, \pi_t) = M_t^{-1}\left(\frac{p}{h+p}\middle|\, \pi_t\right) \leq s_t^*(0, \pi_t) = M_t^{-1}\left(\frac{p - \varphi_{t,t}(s_t^*(0, \pi_t), \pi_t)}{h+p}\middle|\, \pi_t\right)$$

due to $\varphi_{t,t}(y_t, \pi_t) \leq 0$.

We now show that this result may not hold in the general case when inventory can be carried over to future periods. (Chen and Plambeck 2008 make a similar observation in a discrete-demand setting, using a different approach.) Observe that from (16)

$$s_t^*(0, \pi_t) = M_t^{-1}\left(\frac{p - \psi_t(s_t^*(0, \pi_t), \pi_t) - \varphi_t(s_t^*(0, \pi_t), \pi_t)}{h+p}\middle|\, \pi_t\right),$$

where $\psi_t(s_t^*(0, \pi_t), \pi_t) \geq 0$ and $\varphi_t(s_t^*(0, \pi_t), \pi_t) \leq 0$ (recall $\psi_t(s_t^*(0, \pi_t), \pi_t)$ and $\varphi_t(s_t^*(0, \pi_t), \pi_t)$ represent the marginal inventory cost in the future periods and the marginal informational benefit, respectively). Therefore, depending on the magnitudes of $\psi_t(s_t^*(0, \pi_t), \pi_t)$ and $\varphi_t(s_t^*(0, \pi_t), \pi_t)$, the myopic solution $s_t^m(0, \pi_t)$ may not be lower than $s_t^*(0, \pi_t)$.

Below we present a sufficient condition for the myopic solution to be a lower bound.

Theorem 5 *When the updated myopic inventory level is reachable for any demand realization in period t, i.e., $s_t^m(0, \pi_t) - z_t \leq s_{t+1}^m(0, \pi_{t+1})$ for all $\{\pi_t : t = 1, 2, \ldots, T\}$, then the myopic inventory level is a lower bound on the optimal inventory level, i.e., $s_t^m(x_t, \pi_t) \leq s_t^*(x_t, \pi_t)$.*

It remains an open question how to derive a condition for the myopic solution to be an upper bound.

4 Approximation and Performance Evaluation

In this section, we develop bounds on the optimal cost and inventory levels. We then use the solution bounds to construct heuristic policies. We also discuss how to assess the effectiveness of these heuristic policies.

4.1 Cost Bounds

Consider a case in which the demand distribution is unknown but no observed demand information is ever used to update the prior distribution π_1. Thus, the knowledge on θ will remain to be π_1 throughout the planning horizon. (This can represent a situation where an inventory manager does not know how to update the distribution, so he uses only the inventory control model of lost sales with a known demand distribution, as discussed in Subsection 2.1.) In this case, the decision maker wants to solve the following optimality equations:

$$\begin{aligned} V_{T+1}^{NI}\left(x_{T+1}, \pi_{T+1}\right) &= 0, \quad x_{T+1} \geq 0, \\ V_t^{NI}\left(x_t, \pi_t\right) &= \min_{y_t \geq x_t} \left\{G_t^{NI}\left(y_t, \pi_t\right)\right\}, \quad x_t \geq 0, \quad 1 \leq t \leq T, \end{aligned} \tag{23}$$

where

$$\begin{aligned} G_t^{NI}\left(y_t, \pi_t\right) &= C_t\left(y_t, \pi_t\right) + E\left[V_{t+1}^{NI}\left(\left(y_t - Z_t\right)^+, \pi_{t+1}\right)\right], \\ \pi_{t+1}\left(\theta\right) &= \pi_t\left(\theta\right), \text{ for any } \theta \in \Theta. \end{aligned}$$

Note that these equations are parallel to those for the lost sales model with a known demand distribution discussed in Subsection 2.1. As a result, the solution approach is similar. We differentiate the notation by using a superscript NI, indicating "no information updating for unknown demand distribution."

The analytical details of Theorem 3 allow us to compare the optimal cost in the presence of censored data, as shown in (12), with that of no information updating and that of observable lost sales, as shown in (23) and (5), respectively. We demonstrate the comparison in the theorem below.

Theorem 6 *For any given π_1, we have*

$$V_1^{FI}\left(0, \pi_1\right) \leq V_1\left(0, \pi_1\right) \leq V_1^{NI}\left(0, \pi_1\right).$$

Theorem 6 provides a linkage between all the lost sales models discussed in the paper and in essence verifies the value of information. Specifically, compared to the model of censored demand data (in which the demand information is only partially observable), improving the information observability (such that the information becomes fully observable) reduces the total cost, i.e., $V_1^{FI}\left(0, \pi_1\right) \leq V_1\left(0, \pi_1\right)$. On the other hand, even if the information is partially observable, being ignorant in information updating causes a higher cost, i.e., $V_1\left(0, \pi_1\right) \leq V_1^{NI}\left(0, \pi_1\right)$.

4.2 Solution Bounds Based on First Derivative Function

Because ${G_t}'(y_t, \pi_t)$ as shown in Theorem 3 is computationally intractable, we seek approximate solutions. Our approach is to first approximate ${G_t}'(y_t, \pi_t)$ by eliminating some elements of $\varphi_t(y_t, \pi_t)$ and $\psi_t(y_t, \pi_t)$. Then, we bound the remaining elements. Finally, we use those bounds on ${G_t}'$ to develop lower and upper bounds on $s_t^*(x_t, \pi_t)$, denoted by $s_t^\ell(x_t, \pi_t)$ and $s_t^u(x_t, \pi_t)$, respectively.

Step I. Approximating ${G_t}'(y_t, \pi_t)$

Applying Theorem 4 to (16), we can approximate ${G_t}'(y_t, \pi_t)$ as follows:

Lemma 7 ${C_t}'(y_t, \pi_t) + \varphi_{t,t}(y_t, \pi_t) \le {G_t}'(y_t, \pi_t) \le {C_t}'(y_t, \pi_t) + \psi_t(y_t, \pi_t)$.

Note that $\varphi_{t,t}(y_t, \pi_t)$ is the marginal informational benefit obtained from an exact demand observation brought by dy_t, while $\psi_t(y_t, \pi_t)$ is the marginal inventory cost incurred in periods $t+1, \cdots, T$ due to stocking dy_t additionally. Our next step is to derive a lower bound on $\varphi_{t,t}(y_t, \pi_t)$ and an upper bound on $\psi_t(y_t, \pi_t)$, as will be demonstrated below.

Remark We can obtain closer approximations of ${G_t}'(y_t, \pi_t)$ than the above. For example, using Theorem 4 slightly differently, we can show ${C_t}'(y_t, \pi_t) + \varphi_{t,t}(y_t, \pi_t) + \varphi_{t,t+1}(y_t, \pi_t) + \psi_{t,t+1}(y_t, \pi_t) \le {G_t}'(y_t, \pi_t) \le {C_t}'(y_t, \pi_t) + \psi_t(y_t, \pi_t) + \varphi_{t,t}(y_t, \pi_t)$. The techniques presented below continue to be applicable, although the exposition would become more complex.

Step II. Deriving a lower bound on $\varphi_{t,t}(y_t, \pi_t)$

From the expression of $\varphi_{t,t}(y_t, \pi_t)$ in Theorem 3, we need to approximate $G_{t+1}\left(s_{t+1}^*\left(0, H_{t,t+1}^e\right), H_{t,t+1}^e\right)$ and $J_{t+1}\left(s_{t+1}^*\left(0, H_{t,t+1}^c\right), H_{t,t+1}^e\right)$. We first analyze $J_{t+1}\left(s_{t+1}^*\left(0, H_{t,t+1}^c\right), H_{t,t+1}^e\right)$. As discussed in Case 1 following Theorem 3, this term represents a cost from making the inventory decisions in future periods based on a censored observation of y_t^c in period t, while the actual demand realization is exactly y_t. We now compare this cost with $G_{t+1}\left(s_{t+1}^*\left(0, H_{t,t+1}^c\right), H_{t,t+1}^c\right)$, as shown in Lemma 8 below, and then bound the latter. Denote

$$\xi(y_t|\pi_t) = \frac{[1 - M_t(y_t|\pi_t)]}{m_t(y_t|\pi_t)} \max_{\theta \in \Theta} \left\{ \frac{f(y_t|\theta)}{1 - F(y_t|\theta)} \right\}. \tag{24}$$

Lemma 8 $J_{t+1}\left(s_{t+1}^*\left(0, H_{t,t+1}^c\right), H_{t,t+1}^e\right) \le G_{t+1}\left(s_{t+1}^*\left(0, H_{t,t+1}^c\right), H_{t,t+1}^c\right) \xi(y_t|\pi_t)$.

Note that $J_{t+1}\left(s_{t+1}^*\left(0, H_{t,t+1}^c\right), H_{t,t+1}^e\right)$ and $G_{t+1}\left(s_{t+1}^*\left(0, H_{t,t+1}^c\right), H_{t,t+1}^c\right)$ differ in the observations used to update the prior (i.e., $H_{t,t+1}^e = (\pi_t, y_t^e)$ vs. $H_{t,t+1}^c = (\pi_t, y_t^c)$). Besides, the likelihood functions of y_t^e and y_t^c are $f(y_t|\theta)$ and $1 - F(y_t|\theta)$, respectively. So the above lemma essentially indicates that the difference between these two likelihood functions determines the difference between $J_{t+1}\left(s_{t+1}^*\left(0, H_{t,t+1}^c\right), H_{t,t+1}^e\right)$ and $G_{t+1}\left(s_{t+1}^*\left(0, H_{t,t+1}^c\right), H_{t,t+1}^c\right)$.

Note $V_{t+1}(0, H_{t,t+1}) = G_{t+1}\left(s^*_{t+1}(0, H_{t,t+1}), H_{t,t+1}\right)$. The following result comes immediately from Theorem 6.

Lemma 9 $V^{FI}_{t+1}(0, H_{t,t+1}) \leq G_{t+1}\left(s^*_{t+1}(0, H_{t,t+1}), H_{t,t+1}\right) \leq V^{NI}_{t+1}(0, H_{t,t+1})$.

Applying Lemmas 8 and 9 yields a lower bound on $\varphi_{t,t}(y_t, \pi_t)$ as follows:

Lemma 10 $\varphi_{t,t}(y_t, \pi_t) \geq \left[V^{FI}_{t+1}\left(0, H^e_{t,t+1}\right) - V^{NI}_{t+1}\left(0, H^c_{t,t+1}\right)\xi(y_t|\pi_t)\right] m_t(y_t|\pi_t)$.

Step III. Deriving an upper bound on $\psi_t(y_t, \pi_t)$

We next present a general approach to computing upper bounds on $\psi_t(y_t, \pi_t)$.

Lemma 11 *Given $s^\ell_{t+i}((y_t - z[t, t+i])^+, \pi_{t+i})$ for $i = 1, \cdots, T-t$, we have the following upper bound on $\psi_t(y_t, \pi_t)$:*

$$\psi_t(y_t, \pi_t) \leq \sum_{i=1}^{T-t} E[\mathbf{1}(\tau^\ell(t, y_t) > t+i-1)\mathbf{1}(A^m_{t+i})C_{t+i}'(y_t - Z[t, t+i), \pi_t, \mathbf{Z}^e_{t,t+i-1})],$$

where

$$\begin{aligned} \tau^\ell(t, y_t) &= \min\{t+j|(y_t - Z[t, t+j))^+ < s^\ell_{t+j}((y_t - Z[t, t+j))^+, \pi_{t+j}), j \geq 1\}, \\ A^m_{t+i} &= \{y_t - Z[t, t+i) \geq s^m_{t+i}(0, \pi_{t+i})\}. \end{aligned}$$

Applying Lemmas 7, 10, and 11, we can derive bounding functions on $G_t'(y_t, \pi_t)$ as follows.

Theorem 12 $g^\ell_t(y_t, \pi_t) \leq G_t'(y_t, \pi_t) \leq g^u_t(y_t, \pi_t)$, *where*

$$\begin{aligned} g^\ell_t(y_t, \pi_t) &= C_t'(y_t, \pi_t) + \left[V^{FI}_{t+1}\left(0, H^e_{t,t+1}\right) - V^{NI}_{t+1}\left(0, H^c_{t,t+1}\right)\xi(y_t|\pi_t)\right] m_t(y_t|\pi_t), \\ g^u_t(y_t, \pi_t) &= C_t'(y_t, \pi_t) + \sum_{i=1}^{T-t} E[\mathbf{1}(\tau^\ell(t, y_t) > t+i-1)\mathbf{1}(A^m_{t+i})C_{t+i}'(y_t - Z[t, t+i), \pi_t, \mathbf{Z}^e_{t,t+i-1})]. \end{aligned}$$

Solving $g^u_t(y_t, \pi_t) = 0$ and $g^\ell_t(y_t, \pi_t) = 0$ yields a lower and an upper bound on the optimal inventory level, respectively. More specifically, we have:

Corollary 13 *Let $s^u_t(0, \pi_t)$ be the largest solution to $g^\ell_t(y_t, \pi_t) = 0$ and $s^\ell_t(0, \pi_t)$ the smallest solution to $g^u_t(y_t, \pi_t) = 0$ over $y_t \geq 0$. Then, $s^\ell_t(x_t, \pi_t) \leq s^*_t(x_t, \pi_t) \leq s^u_t(x_t, \pi_t)$, where $s^\ell_t(x_t, \pi_t) = s^\ell_t(0, \pi_t)$ and $s^u_t(x_t, \pi_t) = s^u_t(0, \pi_t)$ for $x_t < s^u_t(0, \pi_t)$ and $s^\ell_t(x_t, \pi_t) = s^u_t(x_t, \pi_t) = x_t$ otherwise.*

The above methodology of deriving solution bounds based on the first derivative function is applicable to the censored newsvendor model in which the inventory is perishable. In that case, as shown in (18), ${G_t}'(y_t, \pi_t)$ is simply ${G_t}'(y_t, \pi_t) = {C_t}'(y_t, \pi_t) + \varphi_{t,t}(y_t, \pi_t)$. Since $\varphi_{t,t}(y_t, \pi_t) \leq 0$, we obtain immediately that an upper bound on ${G_t}'(y_t, \pi_t)$ is ${C_t}'(y_t, \pi_t)$. We can also use Lemma 10 to derive a lower bound on $\varphi_{t,t}(y_t, \pi_t)$ and hence a lower bound on ${G_t}'(y_t, \pi_t)$.

The following is corollary of Lemma 11, which compares the first derivative functions of the unobservable lost sales model and the observable lost sales model (stated in Theorems 3 and 2, respectively), and their respective optimal solutions. Note that $\psi_t^{FI}(y_t, \pi_t)$ is defined in (8), i.e., the marginal inventory cost in the model of observable lost sales.

Corollary 14 *(a)* $\psi_t(y_t, \pi_t) \leq \psi_t^{FI}(y_t, \pi_t)$ *and* ${G_t}'(y_t, \pi_t) \leq \frac{dG_t^{FI}(y_t,\pi_t)}{dy_t}$ *hold for any* $y_t \geq 0$ *and* π_t. *(b)* $s_t^{FI}(x_t, \pi_t) \leq s_t^*(x_t, \pi_t)$ *holds for any* $x_t \geq 0$ *and* π_t.

That is, given the same prior π_t and initial inventory x_t, the optimal inventory level in the observable lost sales model is no more than that in the unobservable lost sales model. This result, together with Theorem 6, implies that making the lost sales information transparent eliminates the need of stocking high inventory levels to observe such information *and* reduces the total expected cost. We note that Chen and Plambeck (2008) have proved a similar result of part (b) of the above corollary under the discrete demand assumption.

We now discuss the computation complexity of the solution bounds presented in Corollary 13. First, the computation of these bounds involves the computation of V_t^{FI} and V_t^{NI}. Let n_x be the state-space size of the demand realization. The complexity of computing V_t^{NI} is $O(n_x)$. This is because in the case of no information updating, a stationary myopic policy is optimal, and the computation of the value function comes from calculating the integration of the inventory-holding and shortage-penalty cost. As for the complexity of computing V_t^{FI}, recall that there are T periods. Assume the the state-space reduction technique as discussed in Subsection 2.2 is applicable. In each period, the state space is of only one dimension, which resembles the starting inventory level and brings a complexity of $O(n_x)$. For each state, we need to calculate the one-period cost and the expected future cost, in which the integration of the expectation brings another $O(n_x)$. So the total complexity is $O\left(Tn_x^2\right)$. If the state-space reduction technique is not applicable but a sufficient statistic exists, the sufficient statistic needs to be included in the state space, bringing another $O(n_x)$, which leads to a total complexity of $O\left(Tn_x^3\right)$.

Second, assume that all the values of V_t^{FI} and V_t^{NI} over the planning horizon $t = 1, \ldots, T$ have been calculated and become available for use. For an upper solution bound derived from $g_t^{\ell}(y_t, \pi_t)$, the computation complexity is $O(n_x n_\theta)$, for which searching $g_t^{\ell}(y_t, \pi_t)$ brings $O(n_x)$ and updating

π_t to $H^e_{t,t+1}$ and $H^c_{t,t+1}$ and computing $\max_{\theta\in\Theta}\left\{\frac{f(y_t|\theta)}{1-F(y_t|\theta)}\right\}$ require $O(n_\theta)$. For the lower bound, the one given by Corollary 14 is immediate, that is, $s^{FI}_t(x_t,\pi_t)$ calculated in solving V^{FI}_t is already a lower bound. The general computation of the lower bound based on $g^u_t(y_t,\pi_t)$ is complicated. The complexity mainly comes from $\tau^\ell(t,y_t)$, which in turn depends on the information of s^ℓ_{t+j} for $j\geq 1$ in future (see Lemma 11). To make the computation manageable, the implementation can follow the so-called k-step-ahead approximation suggested in the literature of inventory models with nonstationary demand (see, for instance, Lu, Song, and Regan 2006). That is, we use the derived solutions (such as $s^{FI}_{t+1},...,s^{FI}_{t+k}$) for the lower bounds of the next k periods, but for periods further beyond, we simply use 0 as the lower bounds. With this approach, we can show that the computation of the lower bound based on $g^u_t(y_t,\pi_t)$ is $O(n_x^{k+1}n_\theta)$.

4.3 Solution Bounds Based on Cost-to-Go Function

Note that when $\max_{\theta\in\Theta}\left\{\frac{f(y_t|\theta)}{1-F(y_t|\theta)}\right\}=\infty$, we cannot derive a valid upper bound on the optimal inventory level from Theorem 12. To circumvent this deficiency, we develop another technique to derive the upper solution bound, as shown in the theorem below.

Theorem 15 *(a)* $G^{FI}_t(y_t,\pi_t)\leq G_t(y_t,\pi_t)\leq C_t(y_t,\pi_t)+(T-t)\,hE(y_t-Z_t)^+ + V^{NI}_{t+1}(0,\pi_t)$.

(b) For given x_t *and* π_t*, define* $Q\triangleq\min_{y'_t\geq x_t}\left\{C_t(y'_t,\pi_t)+(T-t)\,hE(y'_t-Z_t)^+ + V^{NI}_{t+1}(0,\pi_t)\right\}$. *Then, the solution to*

$$G^{FI}_t(y_t,\pi_t)=Q,$$

where $y_t>s^{FI}_t(0,\pi_t)$*, exists and is an upper bound on* $s^*_t(x_t,\pi_t)$.

Now we briefly explain Theorem 15. Part (a) presents two bounding functions on the cost-to-go function, $G_t(y_t,\pi_t)$. It is straightforward that both bounding functions are convex. For part (b), we first identify the minimum of the upper bounding function, $C_t(y'_t,\pi_t)+(T-t)\,hE(y'_t-Z_t)^+ + V^{NI}_{t+1}(0,\pi_t)$, and assign the objective value to Q (the minimum is unique due to the convexity). Then, solving $G^{FI}_t(y_t,\pi_t)=Q$ yields a solution that generates the same cost Q on the lower bounding function $G^{FI}_t(y_t,\pi_t)$ (due to the convexity, there can be two solutions to the equation, but with $y_t>s^{FI}_t(0,\pi_t)$, we select the larger one). We claim the obtained solution is an upper solution bound on the optima of $G_t(y_t,\pi_t)$ because any point beyond the solution will give a cost strictly higher than the optima.

The lower solution bound can be derived in a similar way.

It is straightforward that the complexity of computing the above solution bound is dominated by the computation of $G^{FI}_t(y_t,\pi_t)$. But the values of G^{FI}_t have been calculated when computing V^{FI}_t, the complexity is $O\left(Tn_x^3\right)$.

4.4 Heuristics and Performance Evaluation

We can obtain a general class of heuristics by averaging the lower and upper bounds on the optimal inventory level with weights, i.e.,

$$s_t^H(x_t, \pi_t) = \gamma s_t^u(x_t, \pi_t) + (1-\gamma) s_t^\ell(x_t, \pi_t), \text{ where } \gamma \in [0,1]. \tag{25}$$

Generating heuristics by weighting bounds has been widely used in the literature, see, e.g., Morton and Pentico (1995), Shang and Song (2003), Lu, Song, and Regan (2006). An appropriate value for γ can come from trial and error, based on the cost error bounds derived below.

For a given heuristic, we denote by $V_t^H(x_t, \pi_t)$ the resulting expected cost from period t to period T, given that the inventory level before ordering is x_t in period t and the prior distribution of θ is π_t. The relative cost error of the heuristic to the optimal policy is

$$\frac{V_1^H(0,\pi_1) - V_1(0,\pi_1)}{V_1(0,\pi_1)} \times 100\%.$$

Due to the curse of dimensionality of the dynamic program, it is very difficult or even intractable to compute the optimal cost, i.e., $V_1(0,\pi_1)$. Instead of computing $\frac{V_1^H(0,\pi_1) - V_1(0,\pi_1)}{V_1(0,\pi_1)}$ exactly, we will develop an upper bound on this ratio, which is called *the relative cost error bound*, and use it to measure the performance of the heuristic. Recall that $V_1^{FI}(0,\pi_1)$ is a lower bound on $V_1(0,\pi_1)$ (Theorem 6). What remains is to derive an upper bound on $V_1^H(0,\pi_1) - V_1(0,\pi_1)$. We have:

Theorem 16 $V_1^H(0,\pi_1) - V_1(0,\pi_1)$ *has the following upper bound:*

$$\min\left\{\max\{B_1(1,\pi_1), B_2(1,\pi_1)\} + E\left[\sum_{t=2}^{T}\{\max\{B_1(t,\pi_t), B_2(t,\pi_t)\}\}\right], B_3(\pi_1)\right\}$$

where, for $1 \le t \le T$,

$$\begin{aligned} B_1(t,\pi_t) &= \max_{s_t^\ell(0,\pi_t) \le y_t \le s_t^H(0,\pi_t)}\{g_t^u(y_t,\pi_t)\}\left(s_t^H(0,\pi_t) - s_t^\ell(0,\pi_t)\right), \\ B_2(t,\pi_t) &= -\min_{s_t^H(0,\pi_t) \le y_t \le s_t^u(0,\pi_t)}\left\{g_t^\ell(y_t,\pi_t)\right\}\left(s_t^u(0,\pi_t) - s_t^H(0,\pi_t)\right), \\ B_3(\pi_1) &= V_1^H(0,\pi_1) - V_1^{FI}(0,\pi_1). \end{aligned}$$

Here, the posterior distributions (π_t for $t \ge 2$) are computed based on the observations obtained from applying the heuristic.

In this theorem, we have presented two approaches to developing an upper bound $V_1^H(0,\pi_1) - V_1(0,\pi_1)$. The first one is to use the bounding functions on the first derivative function to measure the possible cost difference caused by each heuristic inventory decision, compared with the corresponding optimal decision. The second approach comes immediately from the fact that $V_1^{FI}(0,\pi_1)$ is a lower bound on $V_1(0,\pi_1)$.

4.4.1 Capacitated Myopic Policy and Robustness of the Cost Error Bound

Consider the following heuristic policy, termed *capacitated myopic policy*:

$$\min \{s_t^m(x_t, \pi_t), s_1^m(0, \pi_1)\}.$$

Because $s_1^m(0, \pi_1)$ is the solution to the model without information updating, this construction ensures that the resulting marginal inventory costs in the future (due to overstock) is no more than that in the model without information updating. On the other hand, the capacitated myopic policy still enjoys informational benefits associated with inventory decisions (due to the updating of the demand distribution), while this is not the case for the model without information updating. As a result, using the latter model as a benchmark, we can quantify the relative cost error bound of this heuristic, as shown below.

Theorem 17 *The relative cost error of the capacitated myopic policy is bounded by*

$$\frac{V_1^{NI}(0, \pi_1) - V_1^{FI}(0, \pi_1)}{V_1^{FI}(0, \pi_1)} \times 100\%. \tag{26}$$

This cost error bound does not depend on sample-path information and solution bounds between periods. In addition, this error bound is asymptotically robust, because $V^{NI}(0, \pi_1)$ is linear in planning horizon T, and $V^{FI}(0, \pi_1)$ is bounded below by knowing θ exactly, the cost of which case is linear in planning horizon T.

Combining Theorems 16 and 17 allows us to ensure the asymptotic robustness of the relative cost error. The strategy is the following:

- Step 1: Develop a heuristic policy by weighting the solution bounds, as indicated by (25), and evaluate its relative cost error bound by Theorem 16.
- Step 2: If the calculated cost error bound in Step 1 is greater than the bound stated in (26), we employ the capacitated myopic policy.

This way, we can ensure that the resulting cost error bound is no more than (26).

5 Example - Newsvendor Distribution

5.1 Example Setup

In this section, we provide an example to illustrate the results in Theorem 3. We assume the demand follows a newsvendor distribution, as in Lariviere and Porteus (1999). That is, $F(z|\theta) = 1 - exp(-\theta w(z))$, where $w(z)$ is positive, differentiable, and increasing. (When $w(z) = z^k$, this is a Weibull distribution with parameters k and θ.) As shown in Braden and Freimer (1991), the gamma is a conjugate for all newsvendor distributions. Let the prior on θ be a gamma distribution with shape parameter a and scale parameter S. These parameters also constitute the sufficient statistic. At the beginning of period t, a_t and S_t denote the updated parameters, which can be used to represent π_t. In particular,

$$\begin{aligned} f(z_t|\theta) &= \theta w'(z_t)e^{-\theta w(z_t)}, \\ \pi_t(\theta|a_t, S_t) &= \frac{S^{a_t}\theta^{a_t-1}e^{-S_t\theta}}{\Gamma(a_t)}, \\ m(z_t|a_t, S_t) &= \frac{a_t S_t^{a_t} w'(z_t)}{(S_t + w(z_t))^{a_t+1}}, \\ M(z_t|a_t, S_t) &= 1 - \left(\frac{S_t}{S_t + w(z_t)}\right)^{a_t}. \end{aligned}$$

Updating π_t to π_{t+1} with an observation $o_t = (o_{t,1}, o_{t,2})$ yields

$$\begin{aligned} a_{t+1} &= a_t + \mathbf{1}(o_{t,2} = e), \\ S_{t+1} &= S_t + w(o_{t,1}). \end{aligned}$$

To simplify exposition, we assume $w(z_t) = z_t$ and consider a two-period problem. The expected one-period cost is

$$C_t(y_t, a_t, S_t) = hy_t + \frac{(h+p)S_t^{a_t}}{(a_t - 1)(S_t + y_t)^{(a_t-1)}} - \frac{hS_t}{a_t - 1}$$

and

$$C_t'(y_t, a_t, S_t) = h - (h+p)\left(\frac{S_t}{S_t + y_t}\right)^{a_t}. \tag{27}$$

Because period two is the last period, we have

$$G_2(y_2, a_2, S_2) = C_2(y_2, a_2, S_2).$$

The optimal solution is the myopic solution

$$s_2^*(x_2, a_2, S_2) = \max\left\{x_2, \left[\left(\frac{h+p}{h}\right)^{\frac{1}{a_2}} - 1\right]S_2\right\}. \tag{28}$$

This leads to

$$V_2(x_2, a_2, S_2) = \begin{cases} \left[\left(\frac{h+p}{h}\right)^{\frac{1}{a_2}} - 1\right]\frac{ha_2S_2}{(a_2-1)}, & \text{if } x_2 < \left[\left(\frac{h+p}{h}\right)^{\frac{1}{a_2}} - 1\right]S_2, \\ G_2(x_2, a_2, S_2), & \text{otherwise.} \end{cases}$$

For period one, the cost-to-go is

$$\begin{aligned} G_1(y_1, a_1, S_1) &= C_1(y_1, a_1, S_1) + \int_0^{y_1} V_2(y_1 - z_1, a_1 + 1, S_1 + z_1)\, m_1(z_1|a_1, S_1)\, dz_1 \\ &\quad + V_2(0, a_1, S_1 + y_1)\left[1 - M_1(y_1|a_1, S_1)\right]. \end{aligned}$$

5.2 Illustration of Theorem 3

Theorem 3 states that the derivative of $G_1(y_1, a_1, S_1)$ with respect to y_1 constitutes the following three terms:

$$G_1'(y_1, a_1, S_1) = C_1'(y_1, a_1, S_1) + \varphi_{1,1}(y_1, a_1, S_1) + \psi_{1,2}(y_1, a_1, S_1).$$

The first term, $C_1'(y_1, a_1, S_1)$, given by (27), is the marginal inventory cost in the current period.

The second term, $\varphi_{1,1}(y_1, a_1, S_1)$, is the marginal informational benefit. In particular,

$$\begin{aligned}\varphi_{1,1}(y_1, a_1, S_1) &= [G_2(s_2^*(0, a_1+1, S_1+y_1), a_1+1, S_1+y_1) \\ &\quad - J_2(s_2^*(0, a_1, S_1+y_1), a_1+1, S_1+y_1)]m_1(y_1|a_1, S_1),\end{aligned}$$

with

$$\begin{aligned}G_2(s_2^*(0, a_1+1, S_1+y_1), a_1+1, S_1+y_1) &= C_2(s_2^*(0, a_1+1, S_1+y_1), a_1+1, S_1+y_1) \quad (29)\\ &= \left[\left(\frac{h+p}{h}\right)^{\frac{1}{a_1+1}} - 1\right]\frac{h(a_1+1)(S_1+y_1)}{a_1}\end{aligned}$$

and

$$\begin{aligned}J_2(s_2^*(0, a_1, S_1+y_1), a_1+1, S_1+y_1) &= C_2(s_2^*(0, a_1, S_1+y_1), a_1+1, S_1+y_1) \quad (30)\\ &= \left[\left(\frac{h+p}{h}\right)^{\frac{1}{a_1}} - 1\right] h(S_1+y_1).\end{aligned}$$

Here, given that the demand update is $a_2 = a_1 + 1$ (i.e., an exact demand observation) and $S_2 = S_1 + y_1$, (29) is the cost-to-go following the true optimal decision $s_2^*(0, a_1+1, S_1+y_1)$. In contrast, (30) is the cost-to-go following a suboptimal decision based on $a_2 = a_1$ (i.e., a censored demand observation). (Note that the shape parameter a represents the precision of the system's information as the coefficient of variation for the gamma prior is $\sqrt{1/a}$.) A detailed derivation yields

$$\varphi_{1,1}(y_1, a_1, S_1) = h\left[(a_1+1)\left(\frac{h+p}{h}\right)^{\frac{1}{a_1+1}} - a_1\left(\frac{h+p}{h}\right)^{\frac{1}{a_1}} - 1\right]\frac{S_1^{a_1}}{(S_1+y_1)^{a_1}},$$

from which we can see that when y_1 increases, the marginal informational benefit shrinks (i.e., the magnitude of $\varphi_{1,1}$ reduces; recall $\varphi_{1,1} \leq 0$). That is, when the inventory level y_1 is stocked at a low level, the demand information is very likely to be censored. Thus, increasing the inventory will bring a nontrivial informational benefit. On the other hand, if y_1 is already high enough, the demand information is observable in most cases, and hence an increase in the inventory level will bring little value.

The third term, $\psi_{1,2}$, is the marginal inventory cost incurred in period two:

$$\psi_{1,2}(y_1, a_1, S_1) = E[\mathbf{1}(\tau(1, y_1) > 2)C_2'(y_1 - z_1, a_1 + 1, S_1 + z_1)].$$

Here, $\tau(t, y_1) > 2$ indicates that no order is placed in period two. In fact, we have two cases.
Case 1 $y_1 \leq \left[\left(\frac{h+p}{h}\right)^{\frac{1}{a_1+1}} - 1\right] S_1$. In this case, $\mathbf{1}(\tau(1, y_1) > 2) \equiv 0$ in any sample path. The argument comes from the fact that the optimal newsvendor solution, as implied in (28), is $s_2^*(0, a_2, S_2) = \left[\left(\frac{h+p}{h}\right)^{\frac{1}{a_2}} - 1\right] S_2$. With $a_2 = a_1 + \mathbf{1}(o_{t,2} = e) \leq a_1 + 1$ and $S_2 = S_1 + o_{t,1} \geq S_1$, we conclude

$$s_2^*(0, a_2, S_2) = \left[\left(\frac{h+p}{h}\right)^{\frac{1}{a_2}} - 1\right] S_2 \geq \left[\left(\frac{h+p}{h}\right)^{\frac{1}{a_1+1}} - 1\right] S_1 \geq y_1.$$

That is, when y_1 is sufficiently low, the second-period optimal solution is always reachable, and hence overstock never occurs, leading to zero marginal inventory cost, i.e.,

$$\psi_{1,2}(y_1, a_1, S_1) = 0.$$

In this case, Theorem 5 indicates that the myopic inventory level is a lower bound on the optimal inventory level in period one.
Case 2 $y_1 > \left[\left(\frac{h+p}{h}\right)^{\frac{1}{a_1+1}} - 1\right] S_1$. We can show $\tau(1, y_1) > 2$, i.e., that there is no order placement in period two, is equivalent to

$$z_1 < \frac{y_1 - \left[\left(\frac{h+p}{h}\right)^{\frac{1}{a_1+1}} - 1\right] S_1}{\left(\frac{h+p}{h}\right)^{\frac{1}{a_1+1}}}.$$

Algebraic manipulation gives

$$\psi_{1,2}(y_1, a_1, S_1) = h - (a_1 + 1)\, h^{\frac{1}{a_1+1}} (h + p)^{\frac{a_1}{a_1+1}} \left(\frac{S_1}{S_1 + y_1}\right)^{a_1} + a_1 (h + p) \left(\frac{S_1}{S_1 + y_1}\right)^{a_1+1},$$

which is an increasing function in y_1.

In short, we see that the marginal information benefit (the absolute value of $\varphi_{1,1}(y_1, a_1, S_1)$) is increasing in the inventory level y_1 while the marginal inventory cost $\psi_{1,2}(y_1, a_1, S_1)$ is nonincreasing in y_1 (keeps at 0 for $y_1 \leq \left[\left(\frac{h+p}{h}\right)^{\frac{1}{a_1+1}} - 1\right] S_1$ and increases afterwards). According to our discussion in Subsection 3.4, if the optimal inventory level is relatively low, the value of $\varphi_{1,1}$ dominates $\psi_{1,2}$, and hence the myopic inventory level is a lower bound on the optimal inventory level. But when the optimal inventory level is high, $\psi_{1,2}$ will dominate $\varphi_{1,1}$, indicating that the myopic inventory level is increased further and becomes an upper bound. Therefore, compared to the myopic level, the optimal inventory level tends to be more robust in some sense.

Moreover, as discussed in Case 2 following Theorem 3,

$$\begin{aligned} & C_2\left(s_2^*\left(0, a_1, S_1 + y_1 + dy_1\right), a_1, S_1 + y_1 + dy_1\right) - C_2\left(s_2^*\left(0, a_1, S_1 + y_1\right), a_1, S_1 + y_1 + dy_1\right) \\ = \quad & o(dy_1). \end{aligned}$$

Thus, given that the demand observation is $(y_1 + dy_1)^c$, the optimal decision based on the corresponding demand update, $a_2 = a_1$ and $S_2 = S_1 + y_1 + dy_1$, makes a negligible impact on cost compared to that based on y_1^c (which implies $a_2 = a_1$ and $S_2 = S_1 + y_1$).

5.3 Illustration of Approximation

Now we illustrate our approximation. First, we present the result obtained from the method described in Subsection 4.2.

Step I. Approximating $G_1'(y_1, a_1, S_1)$

In this step, we note Lemma 7 yields

$$C_1'(y_1, a_1, S_1) + \varphi_{1,1}(y_1, a_1, S_1) \leq G_1'(y_1, a_1, S_1) \leq C_1'(y_1, a_1, S_1) + \psi_{1,2}(y_1, a_1, S_1).$$

Step II. Deriving a lower bound on $\varphi_{1,1}(y_1, a_1, S_1)$

We can obtain from Lemma 8

$$J_2\left(s_2^*\left(0, a_1, S_1 + y_1\right), a_1 + 1, S_1 + y_1\right) \leq G_2\left(s_2^*\left(0, a_1, S_1 + y_1\right), a_1, S_1 + y_1\right) \xi\left(y_1 | a_1, S_1 + y_1\right).$$

Then, applying Lemma 9 and 10 yields

$$\varphi_{1,1}(y_1, a_1, S_1) \geq \left[V_2^{FI}\left(0, a_1 + 1, S_1 + y_1\right) - V_2^{NI}\left(0, a_1, S_1 + y_1\right) \xi(y_1 | a_1, S_1)\right] m_1\left(y_1 | a_1, S_1\right).$$

Here we make two comments. Firstly, as we consider a two-period example, both $V_2^{FI}(0, a_1 + 1, S_1 + y_1)$ and $V_2^{NI}(0, a_1, S_1 + y_1)$ can be computed as the newsvendor-type costs. Second, the value of $\xi(y_1 | a_1, S_1)$ for the newsvendor distribution is ∞. Hence, we need to reply on the approach derived in Subsection 4.3 to derive approximations (on the other hand, deriving approximations for this newsvendor distribution is unnecessary because sufficient statistic and dimension reduction are applicable; see Braden and Freimer 1991 and Lariviere and Porteus 1999).

Step III. Deriving an upper bound on $\psi_{1,2}(y_1, a_1, S_1)$

Lemma 11 yields the following upper bound on $\psi_{1,2}(y_1, a_1, S_1)$ (note the term of $\tau^{\ell}(t, y_t) > t + i - 1$ becomes vacuous in this two-period model):

$$\psi_1(y_1, a_1, S_1) \leq E[\mathbf{1}(A_2^m) C_2'(y_1 - z_1, a_1 + 1, S_1 + z_1)],$$

where $A_2^m = \{y_1 - z_1 \geq s_2^m(0, a_1+1, S_1+z_1)$.

The above steps thus allows us to construct bounding functions, $g_1^\ell(y_1, a_1, S_1)$ and $g_1^u(y_1, a_1, S_1)$, on $G_1'(y_1, a_1, S_1)$, as indicated in Theorem 12.

On the other hand, if we apply Subsection 4.3 to this example, Theorem 15 indicates that we rely on the following bounding functions:

$$G_1^{FI}(y_1, a_1, S_1) \leq G_1(y_1, a_1, S_1) \leq C_1(y_1, a_1, S_1) + hE(y_1 - Z_1)^+ + V_2^{NI}(0, a_1, S_1).$$

In fact, we can even compute $Q = \frac{3ha_2S_2}{(a_2-1)}\left[\left(\frac{3h+2p}{3h}\right)^{\frac{1}{a_2}} - 1\right]$, which in turn allow us to calculate solution bounds according to Theorem 15.

6 Numerical Study

We now present a numerical study to illustrate our approach of developing and evaluating heuristics. We also compare the performance of the capacitated myopic policy (termed myopic policy in short) with that of the developed heuristics.

We assume $f(\cdot|\theta)$ is a normal distribution with unknown mean θ and known variance σ^2, where θ takes its value from the set $\Theta = \{\theta_1, \theta_2, \theta_3\} = \{100, 200, 300\}$. In other words, there are three possible values for the mean of the normal distribution. Besides, we let $\sigma = 100$, $h = 1$, and $p = 10$. Although we treat θ as a continuous variable in the previous analysis, by replacing the integration $\int_\Theta$ with the summation $\sum_\Theta$, we can show that the analysis holds for the case where θ is a discrete variable. We note that assigning a discrete set of values to the unknown parameter is often used in practical applications of the Bayesian approach (see, e.g., Eppen and Iyer 1997 for an application to fashion products).

In the first set of numerical experiments (summarized Table 1), we assume a planning horizon of $T = 10$ and parameterize π_1. To parameterize π_1 over a single parameter, we assume that the prior probability of θ_1 is η and the prior probability of θ_2 equals that of θ_3 (i.e., $\pi_1 = \left(\eta, \frac{1-\eta}{2}, \frac{1-\eta}{2}\right)$). We can compute the variance of θ as $Var(\theta) = 2500\left(1 + 8\eta - 9\eta^2\right)$, which is increasing in η for $\eta \in \left[0, \frac{4}{9}\right]$ and decreasing for $\eta \in \left[\frac{4}{9}, 1\right]$.

We report in Table 1 solution bounds obtained based on the first derivative function $G_t'(y_t, \pi_t)$ (see Subsection 5.2) and those obtained on the cost-to-go $G_t(y_t, \pi_t)$ (see Subsection 5.3). We report the solution bounds for period 1 because those for $t \geq 2$ depend on the sample path of observations and decisions. We choose the tighter bounds to construct the heuristic policy, in which process we uniformly choose $\gamma = 0.65$. Then, Theorem 16 is applied to derive the relative cost error bound for the heuristic policy in each scenario. We find that for the scenarios studied, the relative cost error bounds under this value of γ are relatively small. Of course, for a particular scenario, searching for the best value of γ may lead to a further reduction of the relative cost error bound.

We also report in Table 1 the myopic policy in period 1 (i.e., $s_1^m(0, \pi_1)$) for the same values of π_1. We can modify Theorem 16 slightly to compute the relative cost error bound for the myopic policy. So we report this bound in the table as well. In addition, we also report the robust bound as presented in Theorem 17.

Prior π_1	Heuristic of weighted bounds		
	Solution bounds based on first derivative G_t' $[s_1^\ell(0,\pi_1), s_1^u(0,\pi_1)]$	Solution bounds based on cost-to-go G_t $[s_1^\ell(0,\pi_1), s_1^u(0,\pi_1)]$	Relative cost error bound (Thm 16)
$(0, \frac{1}{2}, \frac{1}{2})$	$[395, 471]$	$[275, 656]$	5.74%
$(\frac{1}{9}, \frac{4}{9}, \frac{4}{9})$	$[389, 466]$	$[245, 726]$	5.68%
$(\frac{2}{9}, \frac{7}{18}, \frac{7}{18})$	$[379, 459]$	$[213, 784]$	5.53%
$(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$	$[369, 451]$	$[183, 830]$	5.36%
$(\frac{4}{9}, \frac{5}{18}, \frac{5}{18})$	$[355, 442]$	$[152, 853]$	5.25%
$(\frac{5}{9}, \frac{2}{9}, \frac{2}{9})$	$[339, 430]$	$[123, 853]$	4.82%
$(\frac{2}{3}, \frac{1}{6}, \frac{1}{6})$	$[318, 414]$	$[100, 824]$	4.78%
$(\frac{7}{9}, \frac{1}{9}, \frac{1}{9})$	$[292, 391]$	$[87, 755]$	4.51%
$(\frac{8}{9}, \frac{1}{18}, \frac{1}{18})$	$[260, 352]$	$[89, 622]$	3.43%
$(1, 0, 0)$	$[229, 244]$	$[127, 418]$	0.10%

Table 1a: Heuristic of weighted solution bounds

Prior π_1	Myopic policy		
	$s_1^m(0,\pi_1)$	Relative cost error bound (similar to Thm 16)	Relative cost error bound (Thm 17)
$(0, \frac{1}{2}, \frac{1}{2})$	400	7.42%	10.46%
$(\frac{1}{9}, \frac{4}{9}, \frac{4}{9})$	393	9.78%	14.65%
$(\frac{2}{9}, \frac{7}{18}, \frac{7}{18})$	384	9.73%	18.89%
$(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$	375	10.45%	23.03%
$(\frac{4}{9}, \frac{5}{18}, \frac{5}{18})$	362	13.24%	26.55%
$(\frac{5}{9}, \frac{2}{9}, \frac{2}{9})$	346	18.51%	29.33%
$(\frac{2}{3}, \frac{1}{6}, \frac{1}{6})$	326	18.89%	30.64%
$(\frac{7}{9}, \frac{1}{9}, \frac{1}{9})$	300	19.34%	29.18%
$(\frac{8}{9}, \frac{1}{18}, \frac{1}{18})$	268	17.25%	22.96%
$(1, 0, 0)$	234	0.00%	0.00%

Table 1b: Myopic policy

We see from Table 1 that the solution bounds obtained from approximating the first derivative function tend to be tighter than those from approximating the cost-to-go. It also appears that the heuristic policy generated from weighting the solution bounds performs better than the myopic policy. To study whether this is a prevailing phenomenon, we conduct additional numerical experiments by extending the length of the planning horizon substantially. The results are summarized in Table 2, in which we choose a prior of $\pi_1 = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$.

From Table 2, we can see that the weighted-average heuristic continues to outperform the myopic solution. However, the marginal gain shrinks as the time length of the planning horizon increases. When the planning horizon is relatively short, the reduction of cost error gained from weighted-average heuristic can be nontrivial. For example, when $T = 12$, the cost error bound is reduced nearly by half, from 12.02% to 6.33%. But when the planning horizon is long, we can see that the marginal gain over the myopic policy becoming small, given that the magnitudes of cost error bounds are increasing.

Planning horizon T	Heuristic of weighted bounds	Myopic policy	
	Relative cost error bound (Thm 16)	Relative cost error bound (similar to Thm 16)	Relative cost error bound (Thm 17)
4	1.18%	5.23%	11.06%
8	3.96%	8.84%	19.69%
12	6.33%	12.02%	26.04%
16	8.62%	15.03%	31.47%
20	10.78%	17.90%	36.40%
24	12.95%	20.66%	41.00%
28	15.19%	23.34%	45.40%
32	17.54%	25.96%	49.65%
36	20.06%	28.54%	53.79%
40	22.55%	31.09%	57.85%
44	25.14%	33.62%	61.84%
48	27.90%	36.14%	65.78%
52	30.65%	38.65%	69.67%
56	33.48%	41.16%	73.52%
60	36.36%	43.66%	77.34%
64	39.44%	46.16%	81.13%
68	42.48%	48.66%	84.88%
72	45.60%	51.16%	88.60%
76	48.79%	53.66%	92.30%
80	52.02%	56.17%	95.96%
84	55.25%	58.67%	99.60%
88	58.74%	61.18%	103.21%
92	62.07%	63.70%	106.79%
96	65.48%	66.21%	110.35%
100	68.96%	68.73%	113.87%

Table 2: Performance Comparison over Various Planning Horizons

Table 2 also indicates that $\frac{V_1^{NI}(0,\pi_1)-V_1^{FI}(0,\pi_1)}{V_1^{FI}(0,\pi_1)} \times 100\%$, the asymptotically robust cost error bound presented in Theorem 17, is also increasing in T. This bound turns out to be larger than the others presented in the same table.

Table 2 implies an important message for retail inventory management in the presence of censored data. Observe from Table 2 that the length of the planning horizon, which is equivalent to the number of inventory replenishment opportunities, is critical in determining the performance of sophisticated inventory-control methods (such as our weighted solution bounds) over simpler ones (such as the myopic policy). In particular, a shorter planning horizon favors a more sophisticated method. Thus, for products with short life-cycles, such as seasonal products, because the replenishment opportunities are limited, managers should value each of these opportunities to learn demand information and make inventory decisions carefully. Our weighted solution bounds can serve as a viable approach in this case. In contrast, many fast-moving consumer goods (FMCGs) have much longer life-cycles and require frequent inventory replenishments. In this case, the modified myopic policy can be a reasonable choice.

7 Concluding Remarks

We analyzed a finite-horizon periodic-review inventory model with an unknown demand distribution and unobservable lost sales. In each period, an inventory decision needs to be made based on the prior belief of the demand distribution, and at the end of the period, the prior distribution of the next period will be updated based on this period's censored demand information. Apart from the majority of the inventory literature on censored demand data, we assumed inventory leftovers at each period can be carried over to future periods. This particular feature complicates the analysis tremendously.

We derived an explicit sample-path representation of the first-order condition for determining the optimal inventory level in each period. The result depicts a clear picture of various tradeoffs in the decision making. It allows us to see that, unlike in the models without inventory carryover, the myopic inventory level in this model can be either smaller or larger than the optimal inventory level. The first-order condition also helped verify the value of information in the censored data setting: We showed that the total cost of the problem considered here is lower than that of a problem in which the demand distribution was assumed to be known and hence no information updating is conducted over time, but is higher than that of a problem in which the lost sales are observable (i.e., the traditional Bayesian inventory problem).

We then used these cost bounds, together with the first derivative function, to develop lower and upper solution bounds on the optimal inventory level. We also developed an analytical approach to evaluating the performance of the approximations and discussed how to ensure the asymptotic robustness of the resulting relative cost error. From extensive numerical experiments, we found that our approach is valuable for products with short life-cycles (such as seasonal products) while a myopic policy may perform reasonably well for products with long life-cycles (such as FMCGs).

By comparing three inventory models – one with unobservable lost sales, one with observable lost sales, and one with known demand distribution, we demonstrated more clearly how better information acquisition can facilitate inventory management. In particular, if the demand distribution is difficult to estimate and the lost sales are neither observed nor recorded, then managers will face an extremely challenging task in inventory control. It is our hope that the present work can provide some guidance and analytical approach to meet the challenge. On the other hand, if advanced information technologies and/or new management mechanisms are developed to track the lost sales, then the task becomes much easier. For a variety of distribution families, the computation of the optimal inventory policy can be reduced to a single-dimension problem, which is quite manageable. In an extreme case in which a company has learned the demand information sufficiently and knows with confidence how customer demand is distributed, so that a known demand distribution can be truthfully employed, then the inventory policy is of the simplest form - the static myopic policy.

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Online Supplement to "Inventory Control with Unobservable Lost Sales and Bayesian Updates"

Proof of Theorems 1 and 2 The proofs are similar to those in Morton and Pentico (1995) or Lu, Song, and Regan (2006). We omit the details for conciseness. □

We present in Lemma 18 and 19 some technical properties used in the proof of Theorem 3. For conciseness, in the following proofs, we let $x_{t+i} = s^*_{t+i-1}(x_{t+i-1}, H^c_{t,t+i-1}) - z_{t+i-1}$, $H^e_{t,t+i} = (\pi_t, y^e_t, \mathbf{o}_{t+1,t+i-1})$ and $H^c_{t,t+i} = (\pi_t, y^c_t, \mathbf{o}_{t+1,t+i-1})$.

Lemma 18 *(a)* $\pi_{t+i}(\theta|\pi_t, \mathbf{o}_{t,t+i-1}) = \frac{\Pi_{j=0}^{i-1} l(o_{t+j}|\theta)\pi_t(\theta)}{\int_\Theta \Pi_{j=0}^{i-1} l(o_{t+j}|\theta')\pi_t(\theta')d\theta'}$.

(b) $m_{t+i}(z_{t+i}|\pi_t, \mathbf{o}_{t,t+i-1}) = \frac{\int_\Theta f(z_{t+i}|\theta)\Pi_{j=0}^{i-1} l(o_{t+j}|\theta)\pi_t(\theta)d\theta}{\int_\Theta \Pi_{j=0}^{i-1} l(o_{t+j}|\theta)\pi_t(\theta)d\theta}$.

Proof of Lemma 18 We prove by induction. It is straightforward that the result holds for period $t+1$. Assume that the result holds for period $t+i$. For period $t+i+1$, we have

$$\begin{aligned}
\pi_{t+i+1}(\theta|\pi_t, \mathbf{o}_{t,t+i}) &= \frac{l(o_{t+i}|\theta)\,\pi_{t+i}(\theta|\pi_t, \mathbf{o}_{t,t+i-1})}{\int_\Theta l(o_{t+i}|\theta')\,\pi_{t+i}(\theta'|\pi_t, \mathbf{o}_{t,t+i-1})\,d\theta'} \\
&= \frac{l(o_{t+i}|\theta)\,l(o_{t+i-1}|\theta)\cdots l(o_t|\theta)\,\pi_t(\theta)}{\int_\Theta l(o_{t+i}|\theta')\,l(o_{t+i-1}|\theta')\cdots l(o_t|\theta')\,\pi_t(\theta')\,d\theta'}, \text{ and} \\
m_{t+i+1}(z_{t+i+1}|\pi_t, \mathbf{o}_{t,t+i}) &= \int_\Theta f(z_{t+i+1}|\theta)\,\pi_{t+i+1}(\theta|\pi_t, \mathbf{o}_{t,t+i})\,d\theta \\
&= \frac{\int_\Theta f(z_{t+i+1}|\theta)\,l(o_{t+i}|\theta)\,l(o_{t+i-1}|\theta)\cdots l(o_t|\theta)\,\pi_t(\theta)\,d\theta}{\int_\Theta l(o_{t+i}|\theta)\,l(o_{t+i-1}|\theta)\cdots l(o_t|\theta)\,\pi_t(\theta)\,d\theta}.
\end{aligned}$$

□

Lemma 19 *We have the following:*

(a) $G_{t+1}\left(s^*_{t+1}\left(0, H^c_{t,t+1}\right), H^c_{t,t+1}\right) = \left[1 - M_t(y_t|\pi_t)\right]^{-1}$
$\times \sum_{i=1}^{T-t} \int_0^{+\infty} \cdots \int_0^{+\infty} r\left(s^*_{t+i}(x_{t+i}, H^c_{t,t+i}), z_{t+i}\right) \int_\Theta \Pi_{j=1}^i f(z_{t+j}|\theta)\left[1 - F(y_t|\theta)\right]\pi_t(\theta)\,d\theta dz_{t+i}\cdots dz_{t+1}$.

(b) $J_{t+1}\left(s^*_{t+1}\left(0, H^c_{t,t+1}\right), H^e_{t,t+1}\right) = \left[m_t(y_t|\pi_t)\right]^{-1}$
$\times \sum_{i=1}^{T-t} \int_0^{+\infty} \cdots \int_0^{+\infty} r\left(s^*_{t+i}(x_{t+i}, H^c_{t,t+i}), z_{t+i}\right) \int_\Theta \Pi_{j=1}^i f(z_{t+j}|\theta)\,f(y_t|\theta)\,\pi_t(\theta)\,d\theta dz_{t+i}\cdots dz_{t+1}$.

Proof of Lemma 19 The proofs of parts (a) and (b) are similar. Below we demonstrate the proof of part (b) in detail. We let $o_t = y_t^e$ and $l(o_t|\theta) = f(y_t|\theta)$. We claim for given $1 \leq k < i$,

$$\begin{aligned}
&\int_0^{+\infty} \cdots \int_0^{+\infty} r\left(s_{t+i}^*(x_{t+i}, H_{t,t+i}^c), z_{t+i}\right) m_{t+k}\left(z_{t+k}|H_{t,t+k}^e\right) \\
&\times \frac{\int_\Theta \Pi_{j=k+1}^i f\left(z_{t+j}|\theta\right) \Pi_{j=0}^k l\left(o_{t+j}|\theta\right) \pi_t(\theta)\, d\theta}{\int_\Theta \Pi_{j=0}^k l\left(o_{t+j}|\theta\right) \pi_t(\theta)\, d\theta} dz_{t+i} \cdots dz_{t+k} \\
= \quad &\int_0^{+\infty} \cdots \int_0^{+\infty} r\left(s_{t+i}^*(x_{t+i}, H_{t,t+i}^c), z_{t+i}\right) \frac{\int_\Theta \Pi_{j=k}^i f\left(z_{t+j}|\theta\right) \Pi_{j=0}^{k-1} l\left(o_{t+j}|\theta\right) \pi_t(\theta)\, d\theta}{\int_\Theta \Pi_{j=0}^{k-1} l\left(o_{t+j}|\theta\right) \pi_t(\theta)\, d\theta} dz_{t+i} \cdots dz_{t+k}. \quad (31)
\end{aligned}$$

Applying Lemma 18(b) to $m_{t+k}\left(z_{t+k}|H_{t,t+k}^e\right)$, the left handside of (31) equals

$$\begin{aligned}
&\int_0^{+\infty} \cdots \int_0^{+\infty} r\left(s_{t+i}^*(x_{t+i}, H_{t,t+i}^c), z_{t+i}\right) \frac{\int_\Theta f\left(z_{t+k}|\theta\right) \Pi_{j=0}^{k-1} l\left(o_{t+j}|\theta\right) \pi_t(\theta)\, d\theta}{\int_\Theta \Pi_{j=0}^{k-1} l\left(o_{t+j}|\theta\right) \pi_t(\theta)\, d\theta} \\
&\times \frac{\int_\Theta \Pi_{j=k+1}^i f\left(z_{t+j}|\theta\right) l\left(o_{t+k}|\theta\right) \Pi_{j=0}^{k-1} l\left(o_{t+j}|\theta\right) \pi_t(\theta)\, d\theta}{\int_\Theta l\left(o_{t+k}|\theta\right) \Pi_{j=0}^{k-1} l\left(o_{t+j}|\theta\right) \pi_t(\theta)\, d\theta} dz_{t+i} \cdots dz_{t+k} \\
= \quad &\int_0^{s_{t+k}^*(x_{t+k}, H_{t,t+k}^c)} \int_0^{+\infty} \cdots \int_0^{+\infty} r\left(s_{t+i}^*(x_{t+i}, H_{t,t+i}^c), z_{t+i}\right) \frac{\int_\Theta f\left(z_{t+k}|\theta\right) \Pi_{j=0}^{k-1} l\left(o_{t+j}|\theta\right) \pi_t(\theta)\, d\theta}{\int_\Theta \Pi_{j=0}^{k-1} l\left(o_{t+j}|\theta\right) \pi_t(\theta)\, d\theta} \\
&\times \frac{\int_\Theta \Pi_{j=k+1}^i f\left(z_{t+j}|\theta\right) l\left(o_{t+k}|\theta\right) \Pi_{j=0}^{k-1} l\left(o_{t+j}|\theta\right) \pi_t(\theta)\, d\theta}{\int_\Theta l\left(o_{t+k}|\theta\right) \Pi_{j=0}^{k-1} l\left(o_{t+j}|\theta\right) \pi_t(\theta)\, d\theta} dz_{t+i} \cdots dz_{t+k+1} dz_{t+k} \\
&+ \int_{s_{t+k}^*(x_{t+k}, H_{t,t+k}^c)}^{+\infty} \int_0^{+\infty} \cdots \int_0^{+\infty} r\left(s_{t+i}^*(x_{t+i}, H_{t,t+i}^c), z_{t+i}\right) \frac{\int_\Theta f\left(z_{t+k}|\theta\right) \Pi_{j=0}^{k-1} l\left(o_{t+j}|\theta\right) \pi_t(\theta)\, d\theta}{\int_\Theta \Pi_{j=0}^{k-1} l\left(o_{t+j}|\theta\right) \pi_t(\theta)\, d\theta} \\
&\times \frac{\int_\Theta \Pi_{j=k+1}^i f\left(z_{t+j}|\theta\right) l\left(o_{t+k}|\theta\right) \Pi_{j=0}^{k-1} l\left(o_{t+j}|\theta\right) \pi_t(\theta)\, d\theta}{\int_\Theta l\left(o_{t+k}|\theta\right) \Pi_{j=0}^{k-1} l\left(o_{t+j}|\theta\right) \pi_t(\theta)\, d\theta} dz_{t+i} \cdots dz_{t+k+1} dz_{t+k}. \quad (32)
\end{aligned}$$

The first term on the right hand of (32) equals (note $l(o_{t+k}|\theta) = f(z_{t+k}|\theta)$ for $z_{t+k} < s_{t+k}^*(x_{t+k}, H_{t,t+k}^c)$)

$$\begin{aligned}
&\int_0^{s_{t+k}^*(x_{t+k}, H_{t,t+k}^c)} \int_0^{+\infty} \cdots \int_0^{+\infty} r\left(s_{t+i}^*(x_{t+i}, H_{t,t+i}^c), z_{t+i}\right) \frac{\int_\Theta f\left(z_{t+k}|\theta\right) \Pi_{j=0}^{k-1} l\left(o_{t+j}|\theta\right) \pi_t(\theta)\, d\theta}{\int_\Theta \Pi_{j=0}^{k-1} l\left(o_{t+j}|\theta\right) \pi_t(\theta)\, d\theta} \\
&\times \frac{\int_\Theta \Pi_{j=k+1}^i f\left(z_{t+j}|\theta\right) f\left(z_{t+k}|\theta\right) \Pi_{j=0}^{k-1} l\left(o_{t+j}|\theta\right) \pi_t(\theta)\, d\theta}{\int_\Theta f\left(z_{t+k}|\theta\right) \Pi_{j=0}^{k-1} l\left(o_{t+j}|\theta\right) \pi_t(\theta)\, d\theta} dz_{t+i} \cdots dz_{t+k+1} dz_{t+k} \\
= \quad &\int_0^{s_{t+k}^*(x_{t+k}, H_{t,t+k}^c)} \int_0^{+\infty} \cdots \int_0^{+\infty} \frac{\int_\Theta \Pi_{j=k+1}^i f\left(z_{t+j}|\theta\right) f\left(z_{t+k}|\theta\right) \Pi_{j=0}^{k-1} l\left(o_{t+j}|\theta\right) \pi_t(\theta)\, d\theta}{\int_\Theta \Pi_{j=0}^{k-1} l\left(o_{t+j}|\theta\right) \pi_t(\theta)\, d\theta} \\
&\times r\left(s_{t+i}^*(x_{t+i}, H_{t,t+i}^c), z_{t+i}\right) dz_{t+i} \cdots dz_{t+k+1} dz_{t+k}, \quad (33)
\end{aligned}$$

while the second term equals (note $l\left(o_{t+k}|\theta\right) = 1-F\left(s^*_{t+k}(x_{t+k}, H^c_{t,t+k})|\theta\right)$ for $z_{t+k} \geq s^*_{t+k}(x_{t+k}, H^c_{t,t+k})$)

$$\begin{aligned}
&\int_{s^*_{t+k}(x_{t+k},H^c_{t,t+k})}^{+\infty}\int_0^{+\infty}\cdots\int_0^{+\infty} r\left(s^*_{t+i}(x_{t+i}, H^c_{t,t+i}), z_{t+i}\right)\frac{\int_\Theta f\left(z_{t+k}|\theta\right)\Pi_{j=0}^{k-1} l\left(o_{t+j}|\theta\right)\pi_t\left(\theta\right)d\theta}{\int_\Theta \Pi_{j=0}^{k-1} l\left(o_{t+j}|\theta\right)\pi_t\left(\theta\right)d\theta} \\
&\times\frac{\int_\Theta \Pi_{j=k+1}^{i} f\left(z_{t+j}|\theta\right)\left[1-F\left(s^*_{t+k}(x_{t+k}, H^c_{t,t+k})|\theta\right)\right]\Pi_{j=0}^{k-1} l\left(o_{t+j}|\theta\right)\pi_t\left(\theta\right)d\theta}{\int_\Theta \left[1-F\left(s^*_{t+k}(x_{t+k}, H^c_{t,t+k})|\theta\right)\right]\Pi_{j=0}^{k-1} l\left(o_{t+j}|\theta\right)\pi_t\left(\theta\right)d\theta} dz_{t+i}\cdots dz_{t+k+1}dz_{t+k} \\
=\ &\int_0^{+\infty}\cdots\int_0^{+\infty} r\left(s^*_{t+i}(x_{t+i}, H^c_{t,t+i}), z_{t+i}\right)\frac{\int_\Theta \left[1-F\left(s^*_{t+k}(x_{t+k}, H^c_{t,t+k})|\theta\right)\right]\Pi_{j=0}^{k-1} l\left(o_{t+j}|\theta\right)\pi_t\left(\theta\right)d\theta}{\int_\Theta \Pi_{j=0}^{k-1} l\left(o_{t+j}|\theta\right)\pi_t\left(\theta\right)d\theta} \\
&\times\frac{\int_\Theta \Pi_{j=k+1}^{i} f\left(z_{t+j}|\theta\right)\left[1-F\left(s^*_{t+k}(x_{t+k}, H^c_{t,t+k})|\theta\right)\right]\Pi_{j=0}^{k-1} l\left(o_{t+j}|\theta\right)\pi_t\left(\theta\right)d\theta}{\int_\Theta \left[1-F\left(s^*_{t+k}(x_{t+k}, H^c_{t,t+k})|\theta\right)\right]\Pi_{j=0}^{k-1} l\left(o_{t+j}|\theta\right)\pi_t\left(\theta\right)d\theta} dz_{t+i}\cdots dz_{t+k+1} \\
=\ &\int_0^{+\infty}\cdots\int_0^{+\infty}\frac{\int_\Theta \Pi_{j=k+1}^{i} f\left(z_{t+j}|\theta\right)\left[1-F\left(s^*_{t+k}(x_{t+k}, H^c_{t,t+k})|\theta\right)\right]\Pi_{j=0}^{k-1} l\left(o_{t+j}|\theta\right)\pi_t\left(\theta\right)d\theta}{\int_\Theta \Pi_{j=0}^{k-1} l\left(o_{t+j}|\theta\right)\pi_t\left(\theta\right)d\theta} \\
&\times r\left(s^*_{t+i}(x_{t+i}, H^c_{t,t+i}), z_{t+i}\right)dz_{t+i}\cdots dz_{t+k+1} \\
=\ &\int_{s^*_{t+k}(x_{t+k},H^c_{t,t+k})}^{+\infty}\int_0^{+\infty}\cdots\int_0^{+\infty}\frac{\int_\Theta \Pi_{j=k+1}^{i} f\left(z_{t+j}|\theta\right)f\left(z_{t+k}|\theta\right)\Pi_{j=0}^{k-1} l\left(o_{t+j}|\theta\right)\pi_t\left(\theta\right)d\theta}{\int_\Theta \Pi_{j=0}^{k-1} l\left(o_{t+j}|\theta\right)\pi_t\left(\theta\right)d\theta} \\
&\times r\left(s^*_{t+i}(x_{t+i}, H^c_{t,t+i}), z_{t+i}\right)dz_{t+i}\cdots dz_{t+k+1}dz_{t+k}. \qquad (34)
\end{aligned}$$

Combining (33) and (34) leads to the claim stated in (31).

Note the definition of J_{t+1} in (14), that is,

$$\begin{aligned}
&J_{t+1}\left(s^*_{t+1}\left(x_{t+1},\pi_t, y^c_t\right),\pi_t, y^e_t\right) = \sum_{i=1}^{T-t} EC_{t+i}\left(s^*_{t+i}(x_{t+i},\pi_t, y^c_t, \mathbf{o}_{t+1,t+i-1}),\pi_t, y^e_t, \mathbf{o}_{t+1,t+i-1}\right) \\
=\ &\sum_{i=1}^{T-t}\int_0^{+\infty}\cdots\int_0^{+\infty} r\left(s^*_{t+i}(x_{t+i}, H^c_{t,t+i}), z_{t+i}\right)\Pi_{j=1}^{i} m_{t+j}\left(z_{t+j}|H^e_{t,t+j}\right)dz_{t+i}\cdots dz_{t+1}. \qquad (35)
\end{aligned}$$

For each term on the right handside of (35), we obtain

$$\begin{aligned}
&\int_0^{+\infty}\cdots\int_0^{+\infty} r\left(s^*_{t+i}(x_{t+i}, H^c_{t,t+i}), z_{t+i}\right)\Pi_{j=1}^{i} m_{t+j}\left(z_{t+j}|H^e_{t,t+j}\right)dz_{t+i}\cdots dz_{t+1} \\
=\ &\int_0^{+\infty}\cdots\int_0^{+\infty} r\left(s^*_{t+i}(x_{t+i}, H^c_{t,t+i}), z_{t+i}\right)\frac{\int_\Theta f\left(z_{t+i}|\theta\right)\Pi_{j=0}^{i-1} l\left(o_{t+j}|\theta\right)\pi_t\left(\theta\right)d\theta}{\int_\Theta \Pi_{j=0}^{i-1} l\left(o_{t+j}|\theta\right)\pi_t\left(\theta\right)d\theta} \\
&\times\Pi_{j=1}^{i-1} m_{t+j}\left(z_{t+j}|H^e_{t,t+j}\right)dz_{t+i}\cdots dz_{t+1} \\
=\ &\int_0^{+\infty}\cdots\int_0^{+\infty} r\left(s^*_{t+i}(x_{t+i}, H^c_{t,t+i}), z_{t+i}\right)\frac{\int_\Theta \Pi_{j=1}^{i} f\left(z_{t+j}|\theta\right)l\left(o_t|\theta\right)\pi_t\left(\theta\right)d\theta}{\int_\Theta l\left(o_t|\theta\right)\pi_t\left(\theta\right)d\theta}dz_{t+i}\cdots dz_{t+1},
\end{aligned}$$

where the first equality comes from applying Lemma 18(b) to $m_{t+i}\left(z_{t+i}|H^e_{t,t+i}\right)$ and the second one comes from applying the above claim recursively from $k = i-1$ to 1. So we obtain the result stated in part (b). We can prove part (a) by following the same logic as above but replacing $m_{t+i}\left(z_{t+i}|H^e_{t,t+i}\right)$ by $m_{t+i}\left(z_{t+i}|H^c_{t,t+i}\right)$. □

Proof of Theorem 3 In short, the proof of this theorem consists of the following two parts: First, we analyze the derivative of $G_t(y_t,\pi_t)$, which yields $\varphi_{t,t}(y_t,\pi_t)$, the marginal informational benefit obtained from an exact demand observation in period t brought by dy_t. Then, we analyze inventory carryover between periods, which yields $\varphi_{t,t+1}(y_t,\pi_t),\cdots,\varphi_{t,T-1}(y_t,\pi_t)$ and $\psi_{t,t+1}(y_t,\pi_t),\cdots,\psi_{t,T}(y_t,\pi_t)$, the other marginal informational benefits and the marginal inventory costs.

First, to analyze $G_t'(y_t,\pi_t)$, we note

$$\begin{aligned} G_t(y_t,\pi_t) &= C_t(y_t,\pi_t)+EV_{t+1}\left((y_t-Z_t)^+,\pi_t,y_t\otimes Z_t\right) \\ &= C_t(y_t,\pi_t)+EG_{t+1}\left(s^*_{t+1}\left((y_t-Z_t)^+,\pi_t,y_t\otimes Z_t\right),\pi_t,y_t\otimes Z_t\right) \\ &= C_t(y_t,\pi_t)+\int_0^{y_t}G_{t+1}\left(s^*_{t+1}\left(y_t-z_t,\pi_t,z^e_t\right),\pi_t,z^e_t\right)m_t\left(z_t|\pi_t\right)dz_t \\ &\quad +G_{t+1}\left(s^*_{t+1}\left(0,\pi_t,y^c_t\right),\pi_t,y^c_t\right)\left[1-M_t\left(y_t|\pi_t\right)\right], \end{aligned}$$

which gives (note $H^e_{t,t+1}=(\pi_t,y^e_t)$ and $H^c_{t,t+1}=(\pi_t,y^c_t)$)

$$\begin{aligned} G_t'(y_t,\pi_t) &= C_t'(y_t,\pi_t)+G_{t+1}\left(s^*_{t+1}\left(0,H^e_{t,t+1}\right),H^e_{t,t+1}\right)m_t\left(y_t|\pi_t\right) \\ &\quad +\int_0^{y_t}\frac{dG_{t+1}\left(s^*_{t+1}\left(y_t-z_t,\pi_t,z^e_t\right),\pi_t,z^e_t\right)}{dy_t}m_t\left(z_t|\pi_t\right)dz_t \\ &\quad +\frac{d}{dy_t}\left\{G_{t+1}\left(s^*_{t+1}\left(0,H^c_{t,t+1}\right),H^c_{t,t+1}\right)\left[1-M_t\left(y_t|\pi_t\right)\right]\right\}. \end{aligned} \tag{36}$$

We study the term $\frac{d}{dy_t}\left\{G_{t+1}\left(s^*_{t+1}\left(0,H^c_{t,t+1}\right),H^c_{t,t+1}\right)\left[1-M_t\left(y_t|\pi_t\right)\right]\right\}$ in (36). We note that $s^*_{t+1}\left(0,H^c_{t,t+1}\right)$ is the optimal inventory level, which leads to $\frac{\partial G_{t+1}\left(s^*_{t+1}\left(0,H^c_{t,t+1}\right),H^c_{t,t+1}\right)}{\partial s^*_{t+1}\left(0,H^c_{t,t+1}\right)}=0$ (we apply the envelope theorem). Thus, in computing the term $\frac{d}{dy_t}\left\{G_{t+1}\left(s^*_{t+1}\left(0,H^c_{t,t+1}\right),H^c_{t,t+1}\right)\left[1-M_t\left(y_t|\pi_t\right)\right]\right\}$, we treat $s^*_{t+1}\left(0,H^c_{t,t+1}\right)$ as constant (i.e., fixed) and obtain

$$\begin{aligned} &\frac{d}{dy_t}\left\{G_{t+1}\left(s^*_{t+1}\left(0,H^c_{t,t+1}\right),H^c_{t,t+1}\right)\left[1-M_t\left(y_t|\pi_t\right)\right]\right\} \\ =\ &\frac{d}{dy_t}\int_0^{+\infty}\left[r\left(s^*_{t+1}\left(0,H^c_{t,t+1}\right),z_{t+1}\right)+G_{t+2}\left(s^*_{t+2}(x_{t+2},H^c_{t,t+2}),H^c_{t,t+2}\right)\right] \\ &\times m_{t+1}\left(z_{t+1}|H^c_{t,t+1}\right)\left[1-M_t\left(y_t|\pi_t\right)\right]dz_{t+1} \\ =\ &\frac{d}{dy_t}\int_0^{+\infty}r\left(s^*_{t+1}\left(0,H^c_{t,t+1}\right),z_{t+1}\right)m_{t+1}\left(z_{t+1}|H^c_{t,t+1}\right)\left[1-M_t\left(y_t|\pi_t\right)\right]dz_{t+1} \\ &+\frac{d}{dy_t}\int_0^{+\infty}G_{t+2}\left(s^*_{t+2}(x_{t+2},H^c_{t,t+2}),H^c_{t,t+2}\right)m_{t+1}\left(z_{t+1}|H^c_{t,t+1}\right)\left[1-M_t\left(y_t|\pi_t\right)\right]dz_{t+1}. \end{aligned}$$

Since $o_{t+1} = s^*_{t+1}\left(0, H^c_{t,t+1}\right) \otimes z_{t+1}$, we treat o_{t+1} as constant. For $s^*_{t+2}(x_{t+2}, H^c_{t,t+2})$, for any sample path, we note the following two facts:

(a) If $s^*_{t+2}(x_{t+2}, H^c_{t,t+2}) > s^*_{t+1}(0, H^c_{t,t+1}) - z_{t+1}$ (i.e., there is an order placed in period $t+2$), then we obtain $\frac{\partial G_{t+2}\left(s^*_{t+2}(x_{t+2}, H^c_{t,t+2}), H^c_{t,t+2}\right)}{\partial s^*_{t+2}(x_{t+2}, H^c_{t,t+2})} = 0$.

(b) If $s^*_{t+2}(x_{t+2}, H^c_{t,t+2}) = s^*_{t+1}(0, H^c_{t,t+1}) - z_{t+1}$ (i.e., there is no order placed in period $t+2$), then we obtain $G_{t+2}\left(s^*_{t+2}(x_{t+2}, H^c_{t,t+2}), H^c_{t,t+2}\right) = G_{t+2}\left(s^*_{t+1}(0, H^c_{t,t+1}) - z_{t+1}, H^c_{t,t+2}\right)$, where, as stated above, $s^*_{t+1}(0, H^c_{t,t+1})$ is treated as constant.

Thus, in computing $\frac{d}{dy_t}\int_0^{+\infty} G_{t+2}\left(s^*_{t+2}(x_{t+2}, H^c_{t,t+2}), H^c_{t,t+2}\right) m_{t+1}\left(z_{t+1}|H^c_{t,t+1}\right)\left[1 - M_t\left(y_t|\pi_t\right)\right] dz_{t+1}$, we treat both $s^*_{t+2}(x_{t+2}, H^c_{t,t+2})$ and o_{t+1} as constant and thus obtain

$$\begin{aligned}
& \frac{d}{dy_t}\int_0^{+\infty} G_{t+2}\left(s^*_{t+2}(x_{t+2}, H^c_{t,t+2}), H^c_{t,t+2}\right) m_{t+1}\left(z_{t+1}|H^c_{t,t+1}\right)\left[1 - M_t\left(y_t|\pi_t\right)\right] dz_{t+1} \\
= & \frac{d}{dy_t}\int_0^{+\infty}\int_0^{+\infty}\left[r\left(s^*_{t+2}(x_{t+2}, H^c_{t,t+2}), z_{t+2}\right) + G_{t+3}\left(s^*_{t+3}(x_{t+3}, H^c_{t,t+3}), H^c_{t,t+3}\right)\right] \\
& \times m_{t+2}\left(z_{t+2}|H^c_{t,t+2}\right) m_{t+1}\left(z_{t+1}|H^c_{t,t+1}\right)\left[1 - M_t\left(y_t|\pi_t\right)\right] dz_{t+2} dz_{t+1} \\
= & \frac{d}{dy_t}\int_0^{+\infty}\int_0^{+\infty} r\left(s^*_{t+2}(x_{t+2}, H^c_{t,t+2}), z_{t+2}\right) m_{t+2}\left(z_{t+2}|H^c_{t,t+2}\right) m_{t+1}\left(z_{t+1}|H^c_{t,t+1}\right)\left[1 - M_t\left(y_t|\pi_t\right)\right] dz_{t+2} dz_{t+1} \\
& + \frac{d}{dy_t}\int_0^{+\infty}\int_0^{+\infty} G_{t+3}\left(s^*_{t+3}(x_{t+3}, H^c_{t,t+3}), H^c_{t,t+3}\right) m_{t+2}\left(z_{t+2}|H^c_{t,t+2}\right) m_{t+1}\left(z_{t+1}|H^c_{t,t+1}\right) \\
& \times\left[1 - M_t\left(y_t|\pi_t\right)\right] dz_{t+2} dz_{t+1}.
\end{aligned}$$

Recursively, we obtain

$$\begin{aligned}
& \frac{d}{dy_t}\left\{G_{t+1}\left(s^*_{t+1}\left(0, H^c_{t,t+1}\right), H^c_{t,t+1}\right)\left[1 - M_t\left(y_t|\pi_t\right)\right]\right\} \\
= & \frac{d}{dy_t}\sum_{i=1}^{T-t}\int_0^{+\infty}\cdots\int_0^{+\infty} r\left(s^*_{t+i}(x_{t+i}, H^c_{t,t+i}), z_{t+i}\right) \Pi_{j=1}^{i} m_{t+j}\left(z_{t+j}|H^c_{t,t+j}\right) \\
& \times\left[1 - M_t\left(y_t|\pi_t\right)\right] dz_{t+i}\cdots dz_{t+1}.
\end{aligned}$$

where we treat $s^*_{t+i}(x_{t+i}, H^c_{t,t+i})$ and $\mathbf{o}_{t+1,t+i-1}$ as constant.

Continuing the derivation by following Lemma 19 yields

$$\begin{aligned}
& \frac{d}{dy_t}\left\{G_{t+1}\left(s^*_{t+1}\left(0,H^c_{t,t+1}\right),H^c_{t,t+1}\right)\left[1-M_t\left(y_t|\pi_t\right)\right]\right\} \\
= & \frac{d}{dy_t}\sum_{i=1}^{T-t}\int_0^{+\infty}\cdots\int_0^{+\infty} r\left(s^*_{t+i}(x_{t+i},H^c_{t,t+i}),z_{t+i}\right)\int_\Theta \Pi_{j=1}^i f\left(z_{t+j}|\theta\right)\left[1-F\left(y_t|\theta\right)\right]\pi_t\left(\theta\right)d\theta dz_{t+i}\cdots dz_{t+1} \\
= & \sum_{i=1}^{T-t}\int_0^{+\infty}\cdots\int_0^{+\infty} r\left(s^*_{t+i}(x_{t+i},H^c_{t,t+i}),z_{t+i}\right)\int_\Theta \Pi_{j=1}^i f\left(z_{t+j}|\theta\right)\frac{d}{dy_t}\left[1-F\left(y_t|\theta\right)\right]\pi_t\left(\theta\right)d\theta dz_{t+i}\cdots dz_{t+1} \\
= & -\sum_{i=1}^{T-t}\int_0^{+\infty}\cdots\int_0^{+\infty} r\left(s^*_{t+i}(x_{t+i},H^c_{t,t+i}),z_{t+i}\right)\int_\Theta \Pi_{j=1}^i f\left(z_{t+j}|\theta\right)f\left(y_t|\theta\right)\pi_t\left(\theta\right)d\theta dz_{t+i}\cdots dz_{t+1} \\
= & -J_{t+1}\left(s^*_{t+1}\left(0,H^c_{t,t+1}\right),H^e_{t,t+1}\right)m_t(y_t|\pi_t). \qquad (37)
\end{aligned}$$

Here we temporarily depart from the proof and provide an explanation of equation (37) above. The equation stands for the derivative of the expected cost due to a censored observation, i.e., the cost based on the censored observation of y^c_t. First of all, we see the sign of the derivative means that the marginal impact of being censored by a slightly higher inventory level, i.e., y_t+dy_t, is a cost reduction. Obviously, the increased probability of observing demand exactly due to the slightly increased inventory is $M_t\left(y_t+dy_t|\pi_t\right)-M_t\left(y_t|\pi_t\right)=m_t\left(y_t|\pi_t\right)dy_t$, which gives the term $m_t(y_t|\pi_t)$ on the right handside of (37). Note that the exact observation obtained due to the increased inventory is y^e_t. The term $J_{t+1}\left(s^*_{t+1}\left(0,H^c_{t,t+1}\right),H^e_{t,t+1}\right)$ represents the magnitude of the cost reduction. Recall $J_{t+1}\left(s^*_{t+1}\left(0,H^c_{t,t+1}\right),H^e_{t,t+1}\right)$ is the expected cost from period $t+1$ till the end of the planning horizon in which inventory decisions are made as if the obtained observation in period t was y^c_t but the demand realization is exactly y_t. With the increased inventory, this cost generated from the mismatch between decisions and observation can be eliminated.

From (36) and (37), we obtain

$$\begin{aligned}
G_t'\left(y_t,\pi_t\right) &= C_t'(y_t,\pi_t)+\left[G_{t+1}\left(s^*_{t+1}\left(0,H^e_{t,t+1}\right),H^e_{t,t+1}\right)-J_{t+1}\left(s^*_{t+1}\left(0,H^c_{t,t+1}\right),H^e_{t,t+1}\right)\right]m_t\left(y_t|\pi_t\right) \\
&\quad +\int_0^{y_t}\frac{dG_{t+1}\left(s^*_{t+1}\left(y_t-z_t,\pi_t,z^e_t\right),\pi_t,z^e_t\right)}{dy_t}m_t\left(z_t|\pi_t\right)dz_t \\
&= C_t'(y_t,\pi_t)+\varphi_{t,t}(y_t,\pi_t)+\int_0^{y_t}\frac{dG_{t+1}\left(s^*_{t+1}\left(y_t-z_t,\pi_t,z^e_t\right),\pi_t,z^e_t\right)}{dy_t}m_t\left(z_t|\pi_t\right)dz_t. \qquad (38)
\end{aligned}$$

Next, we analyze $\int_0^{y_t}\frac{dG_{t+1}\left(s^*_{t+1}(y_t-z_t,\pi_t,z^e_t),\pi_t,z^e_t\right)}{dy_t}m_t\left(z_t|\pi_t\right)dz_t$ in (36), which is caused by inventory carryover. We note that if $\tau(t,y_t)=t+1$ holds (i.e., an order is placed in period $t+1$), $s^*_{t+1}\left(y_t-z_t,\pi_t,z^e_t\right)$ is a reachable minimum of $G_{t+1}\left(\cdot,\pi_t,z^e_t\right)$, which leads to

$$\frac{dG_{t+1}\left(s^*_{t+1}\left(y_t-z_t,\pi_t,z^e_t\right),\pi_t,z^e_t\right)}{dy_t}=0. \qquad (39)$$

Thus, we obtain

$$\begin{aligned}
&\int_0^{y_t} \frac{dG_{t+1}\left(s_{t+1}^*\left(y_t - z_t, \pi_t, z_t^e\right), \pi_t, z_t^e\right)}{dy_t} m_t\left(z_t|\pi_t\right) dz_t \\
=& \int_0^{y_t} \frac{dG_{t+1}\left(y_t - z_t, \pi_t, z_t^e\right)}{dy_t} \mathbf{1}\left(\tau(t, y_t) > t+1\right) m_t\left(z_t|\pi_t\right) dz_t \\
=& E\left[G_{t+1}'\left(y_t - Z_t, \pi_t, Z_t^e\right) \mathbf{1}\left(\tau(t, y_t) > t+1\right)\right], \qquad (40)
\end{aligned}$$

where the first equality comes from (39) and the second one comes from the fact that $\tau(t, y_t) > t+1$ implies $z_t < y_t$.

From (38) and (40), we obtain

$$G_t'\left(y_t, \pi_t\right) = C_t'(y_t, \pi_t) + \varphi_{t,t}(y_t, \pi_t) + E\left[G_{t+1}'\left(y_t - Z_t, \pi_t, Z_t^e\right) \mathbf{1}\left(\tau(t, y_t) > t+1\right)\right]. \qquad (41)$$

Applying the above analysis to $G_{t+1}'\left(y_t - z_t, \pi_t, z_t^e\right)$ yields

$$\begin{aligned}
G_{t+1}'\left(y_t - z_t, \pi_t, z_t^e\right) =& C_{t+1}'\left(y_t - z_t, \pi_t, z_t^e\right) + [G_{t+2}\left(s_{t+2}^*\left(0, \pi_t, z_t^e, (y_t - z_t)^e\right), \pi_t, z_t^e, (y_t - z_t)^e\right) \\
&- J_{t+2}\left(s_{t+2}^*\left(0, \pi_t, z_t^e, (y_t - z_t)^c\right), \pi_t, z_t^e, (y_t - z_t)^e\right)] m_{t+1}\left(y_t - z_t|\pi_t, z_t^e\right) \\
&+ E\left[G_{t+2}'\left(y_t - z_t - Z_{t+1}, \pi_t, z_t^e, Z_{t+1}^e\right) \mathbf{1}\left(\tau(t, y_t) > t+2\right)\right],
\end{aligned}$$

which leads to

$$\begin{aligned}
G_t'\left(y_t, \pi_t\right) =& C_t'(y_t, \pi_t) + \varphi_{t,t}(y_t, \pi_t) + \varphi_{t,t+1}(y_t, \pi_t) \\
&+ \psi_{t,t+1}(y_t, \pi_t) + E\left[G_{t+2}'\left(y_t - Z_t - Z_{t+1}, \pi_t, Z_t^e, Z_{t+1}^e\right) \mathbf{1}\left(\tau(t, y_t) > t+2\right)\right].
\end{aligned}$$

Recursively, we have

$$G_t'\left(y_t, \pi_t\right) = C_t'(y_t, \pi_t) + \sum_{i=1}^{T-t} \psi_{t,t+i}(y_t, \pi_t) + \sum_{i=1}^{T-t} \varphi_{t,t+i-1}(y_t, \pi_t), \qquad (42)$$

which completes the proof. □

Proof of Theorem 4 We note

$$G_{t+1}\left(s_{t+1}^*(0, H_{t,t+1}^e), H_{t,t+1}^e\right) \leq J_{t+1}\left(s_{t+1}^*(0, H_{t,t+1}^c), H_{t,t+1}^e\right),$$

as we have shown in (15), because $G_{t+1}\left(s_{t+1}^*(0, H_{t,t+1}^e), H_{t,t+1}^e\right) = V_{t+1}(0, H_{t,t+1}^e)$ is the optimal cost-to-go and is no greater than the expected cost generated by any other policy. As a result, we have $\varphi_{t,t}(y_t, \pi_t) \leq 0$. In general, by a similar logic, we argue $\varphi_{t,t+i-1}(y_t, \pi_t) \leq 0$ for $i \geq 2$.

For (20) and (21), we claim

$$G_{t+1}'\left(y_t - z_t, \pi_t, z_t^e\right) \mathbf{1}\left(\tau(t, y_t) > t+1\right) \geq 0. \qquad (43)$$

That is, when $\tau(t,y_t) > t+1$ holds for a given sample path of z_t (i.e., $s^*_{t+1}(y_t - z_t, \pi_t, z^e_t) = y_t - z_t$, or equivalently, no order is placed in period $t+1$), we must have $G'_{t+1}(y_t - z_t, \pi_t, z^e_t) \geq 0$. The argument can be proved by contradiction. We assume $G'_{t+1}(y_t - z_t, \pi_t, z^e_t) < 0$ when $\tau(t,y_t) > t+1$ holds. Then, from $G'_{t+1}(y_t - z_t, \pi_t, z^e_t) < 0$, we obtain that there exists an inventory level $y'_t > y_t - z_t$ such that $G_{t+1}\left(y'_t, \pi_t, z^e_t\right) < G_{t+1}(y_t - z_t, \pi_t, z^e_t)$. In other words, $y_t - z_t$ cannot be a minimizer (i.e., $s^*_{t+1}(y_t - z_t, \pi_t, z^e_t) > y_t - z_t$, or equivalently, an order needs to be placed in period $t+1$), which contradicts with the fact that $\tau(t,y_t) > t+1$ holds. In general, we have

$$G'_{t+i}(y_t - z[t,t+i), \pi_t, \mathbf{z}^e_{t,t+i-1})\mathbf{1}(\tau(t,y_t) > t+i) \geq 0. \tag{44}$$

From (41), (42), and (43), we obtain

$$E\left[G'_{t+1}(y_t - Z_t, \pi_t, Z^e_t)\,\mathbf{1}(\tau(t,y_t) > t+1)\right] = \sum_{i=1}^{T-t}\psi_{t,t+i}(y_t,\pi_t) + \sum_{i=2}^{T-t}\varphi_{t,t+i-1}(y_t,\pi_t) \geq 0,$$

which, together with $\varphi_{t,t+i-1} \leq 0$, leads to

$$\sum_{i=1}^{T-t}\psi_{t,t+i}(y_t,\pi_t) \geq 0.$$

That way, we obtain the results stated in (20) and (21). □

Proof of Theorem 6 We first compare $V_1(0,\pi_1)$ and $V_1^{NI}(0,\pi_1)$. To this end, we claim

$$G_t(y_t,\pi_t) \leq C_t(y_t,\pi_t) + V_{t+1}(y_t,\pi_t)$$

for any t, $y_t \geq 0$, and π_t. To prove this claim, we define

$$\begin{aligned}
\Lambda(\Delta y_t) \;=\;& C_t(y_t,\pi_t) + \int_0^{y_t+\Delta y_t} V_{t+1}(y_t, \pi_{t+1}(\cdot|\pi_t, z^e_t))\, m_t(z_t|\pi_t)dz_t \\
&+V_{t+1}(y_t, \pi_{t+1}(\cdot|\pi_t, (y_t+\Delta y_t)^c))\,[1 - M_t(y_t+\Delta y_t|\pi_t)] \\
&-V_{t+1}(y_t, \pi_{t+1}(\cdot|\pi_t, (y_t+\Delta y_t)^c))\,[1 - M_t(y_t|\pi_t)] \\
&+V_{t+1}(0, \pi_{t+1}(\cdot|\pi_t, (y_t+\Delta y_t)^c))\,[1 - M_t(y_t|\pi_t)],
\end{aligned}$$

where $\Delta y_t \leq 0$. It is straightforward to see

$$\begin{aligned}
\Lambda(0) \;=\;& C_t(y_t,\pi_t) + \int_0^{y_t} V_{t+1}(y_t, \pi_{t+1}(\cdot|\pi_t, z^e_t))\, m_t(z_t|\pi_t)dz_t \\
&+V_{t+1}(0, \pi_{t+1}(\cdot|\pi_t, y^c_t))\,[1 - M_t(y_t|\pi_t)] \\
=\;& G_t(y_t,\pi_t).
\end{aligned}$$

We note that the last two terms in the definition of $\Lambda(\Delta y_t)$ satisfy

$$\begin{aligned}
&-V_{t+1}(y_t, \pi_{t+1}(\cdot|\pi_t, (y_t+\Delta y_t)^c))\,[1 - M_t(y_t|\pi_t)] \\
&+V_{t+1}(0, \pi_{t+1}(\cdot|\pi_t, (y_t+\Delta y_t)^c))\,[1 - M_t(y_t|\pi_t)] \leq 0
\end{aligned} \tag{45}$$

due to $V_{t+1}\left(y_t, \pi_{t+1}(\cdot|\pi_t, (y_t+\Delta y_t)^c)\right) = \max_{y_{t+1}\geq y_t} G_{t+1}\left(y_{t+1}, \pi_{t+1}(\cdot|\pi_t, (y_t+\Delta y_t)^c)\right)$ and $y_t \geq 0$. We also note that the sum of the first two terms in the definition of $\Lambda\left(\Delta y_t\right)$ is decreasing in Δy_t due to

$$\begin{aligned}
& \frac{d}{d\left(\Delta y_t\right)}\left\{\int_0^{y_t+\Delta y_t} V_{t+1}\left(y_t, \pi_{t+1}(\cdot|\pi_t, z_t^e)\right) m_t(z_t|\pi_t)dz_t \right. \\
& \left. +V_{t+1}\left(y_t, \pi_{t+1}(\cdot|\pi_t, (y_t+\Delta y_t)^c)\right)\left[1-M_t(y_t+\Delta y_t|\pi_t)\right]\right\} \\
= \; & \left[V_{t+1}\left(y_t, \pi_{t+1}(\cdot|\pi_t, (y_t+\Delta y_t)^e)\right)\right. \\
& \left. -J_{t+1}\left(s_{t+1}^*\left(y_t, \pi_t, (y_t+\Delta y_t)^c\right), \pi_t, (y_t+\Delta y_t)^e\right)\right] m_t(y_t+\Delta y_t|\pi_t) \\
\leq \; & 0, \qquad (46)
\end{aligned}$$

where the equality and the inequality come from arguments similar to those for (37) and (15), respectively.

So we obtain

$$\begin{aligned}
G_t\left(y_t, \pi_t\right) &= \Lambda\left(0\right) \\
&= C_t\left(y_t, \pi_t\right) + \int_0^{y_t+0} V_{t+1}\left(y_t, \pi_{t+1}(\cdot|\pi_t, z_t^e)\right) m_t(z_t|\pi_t)dz_t \\
&\quad +V_{t+1}\left(y_t, \pi_{t+1}(\cdot|\pi_t, (y_t+0)^c)\right)\left[1-M_t(y_t+0|\pi_t)\right] \\
&\quad -V_{t+1}\left(y_t, \pi_{t+1}(\cdot|\pi_t, (y_t+0)^c)\right)\left[1-M_t(y_t|\pi_t)\right] \\
&\quad +V_{t+1}\left(0, \pi_{t+1}(\cdot|\pi_t, (y_t+0)^c)\right)\left[1-M_t(y_t|\pi_t)\right] \\
&\leq C_t\left(y_t, \pi_t\right) + \int_0^{y_t+0} V_{t+1}\left(y_t, \pi_{t+1}(\cdot|\pi_t, z_t^e)\right) m_t(z_t|\pi_t)dz_t \\
&\quad +V_{t+1}\left(y_t, \pi_{t+1}(\cdot|\pi_t, (y_t+0)^c)\right)\left[1-M_t(y_t+0|\pi_t)\right] \\
&\leq C_t\left(y_t, \pi_t\right) + \int_0^{y_t+(-y_t)} V_{t+1}\left(y_t, \pi_{t+1}(\cdot|\pi_t, z_t^e)\right) m_t(z_t|\pi_t)dz_t \\
&\quad +V_{t+1}\left(y_t, \pi_{t+1}(\cdot|\pi_t, (y_t+(-y_t))^c)\right)\left[1-M_t(y_t+(-y_t)\,|\pi_t)\right] \\
&= C_t\left(y_t, \pi_t\right) + V_{t+1}\left(y_t, \pi_{t+1}(\cdot|\pi_t, 0^c)\right) \\
&= C_t\left(y_t, \pi_t\right) + V_{t+1}\left(y_t, \pi_t\right),
\end{aligned}$$

where the first and second inequalities come from (45) and (46), respectively. Therefore, the claim of $G_t\left(y_t, \pi_t\right) \leq C_t\left(y_t, \pi_t\right) + V_{t+1}\left(y_t, \pi_t\right)$ holds.

We now claim $V_t\left(x_t, \pi_1\right) \leq V_t^{NI}\left(x_t, \pi_1\right)$ for any t and $0 \leq x_t \leq s_1^m\left(0, \pi_1\right)$. We prove this claim by a backward induction. The claim holds trivially for $t = T+1$ due to $V_{T+1} = V_{T+1}^{NI} = 0$. Assume that the result holds for period $t+1$. Thus, we have

$$\begin{aligned} V_t(x_t,\pi_1) &\le G_t(s_1^m(0,\pi_1),\pi_1) \\ &\le C_t(s_1^m(0,\pi_1),\pi_1)+V_{t+1}(s_1^m(0,\pi_1),\pi_1) \\ &\le C_t(s_1^m(0,\pi_1),\pi_1)+V_{t+1}^{NI}(s_1^m(0,\pi_1),\pi_1) \\ &= V_t^{NI}(x_t,\pi_1), \end{aligned}$$

where the first inequality comes from $V_t(x_t,\pi_1)=\max_{y_t\ge x_t}G_t(y_t,\pi_1)$ and $x_t\le s_1^m(0,\pi_1)$, the second one is from the previous claim stated in this proof, the last one is due to the inductive assumption, and the equality is due to the optimality of the static order-up-to level $s_1^m(0,\pi_1)$ in the no information updating case.

For $V_1^{FI}(0,\pi_1)\le V_1(0,\pi_1)$, the proof is slightly different. In fact, we claim $V_t^{FI}(x_t,\pi_t)\le V_t(x_t,\pi_t)$ for any t, $x_t\ge 0$, and π_t and $G_t^{FI}(y_t,\pi_t)\le G_t(y_t,\pi_t)$ for any t, $y_t\ge 0$, and π_t. We can prove this claim by a backward induction. The claim holds trivially for $t=T+1$. Now we assume it holds for period $t+1$ and proceed to study period t. We introduce the following definition:

$$\begin{aligned} \Omega(\Delta y_t) &= C_t(y_t,\pi_t)+\int_0^{y_t}V_{t+1}(y_t-z_t,\pi_{t+1}(\cdot|\pi_t,z_t^e))\,m_t(z_t|\pi_t)dz_t \\ &\quad+\int_{y_t}^{y_t+\Delta y_t}V_{t+1}(0,\pi_{t+1}(\cdot|\pi_t,z_t^e))\,m_t(z_t|\pi_t)dz_t \\ &\quad+V_{t+1}(0,\pi_{t+1}(\cdot|\pi_t,(y_t+\Delta y_t)^c))\,[1-M_t(y_t+\Delta y_t|\pi_t)], \end{aligned}$$

where $\Delta y_t\ge 0$. Obviously, $\Omega(0)=G_t(y_t,\pi_t)$. Furthermore, we can show

$$\begin{aligned} \frac{d\Omega(\Delta y_t)}{d(\Delta y_t)} &= [V_{t+1}(0,\pi_{t+1}(\cdot|\pi_t,(y_t+\Delta y_t)^e)) \\ &\quad -J_{t+1}\left(s_{t+1}^*(0,\pi_t,(y_t+\Delta y_t)^c),\pi_t,(y_t+\Delta y_t)^e\right)]\,m_t(y_t+\Delta y_t|\pi_t) \\ &\le 0, \end{aligned}$$

where the equality and the inequality come from arguments similar to those for (37) and (15), respectively. As a result, we obtain $\Omega(0)\ge\Omega(\infty)$ (note that as shown above, V_{t+1} is bounded above by V_{t+1}^{NI}), which, together with the inductive assumption, leads to

$$\begin{aligned} G_t(y_t,\pi_t) &\ge C_t(y_t,\pi_t)+E\left[V_{t+1}\left((y_t-Z_t)^+,\pi_t,z_t^e\right)\right] \\ &\ge C_t(y_t,\pi_t)+E\left[V_{t+1}^{FI}\left((y_t-Z_t)^+,\pi_t,z_t^e\right)\right] \\ &= G_t^{FI}(y_t,\pi_t). \end{aligned}$$

Due to $V_t(x_t,\pi_t)=\max_{y_t\ge x_t}G_t(y_t,\pi_t)$ and $V_t^{FI}(x_t,\pi_t)=\max_{y_t\ge x_t}G_t^{FI}(y_t,\pi_t)$, we conclude

$$V_t(x_t,\pi_t)\ge V_t^{FI}(x_t,\pi_t),$$

which completes the inductive proof of the claim. □

Proof of Theorem 5 We use induction to show $s_t^*(x_t,\pi_t) \geq s_t^m(x_t,\pi_t)$ for any $x_t \geq 0$ and π_t. The result holds for period T, because we know that the myopic inventory level is optimal. Now, assume this is true for $t+i$ where $i \geq 1$, that is, $s_{t+i}^*(x_{t+i},\pi_{t+i}) \geq s_{t+i}^m(x_{t+i},\pi_{t+i})$. We proceed to show that $s_t^*(x_t,\pi_t) \geq s_t^m(x_t,\pi_t)$. Note that according to the given condition, we have $s_t^m(0,\pi_t) - z_t \leq s_{t+1}^m(0,\pi_{t+1})$. Furthermore, according to the induction assumption, we know that $s_{t+1}^m(0,\pi_{t+1}) \leq s_{t+1}^*(0,\pi_{t+1})$ by setting $x_{t+1}=0$. Thus, we have

$$s_t^m(0,\pi_t) - z_t \leq s_{t+1}^m(0,\pi_{t+1}) \leq s_{t+1}^*(0,\pi_{t+1}).$$

So, for any $y_t < s_t^m(0,\pi_t)$, we have $\psi_t(y_t,\pi_t)=0$, which implies

$$G_t'(y_t,\pi_t) = C_t'(y_t,\pi_t) + \psi_t(y_t,\pi_t) + \varphi_t(y_t,\pi_t) = C_t'(y_t,\pi_t) + \varphi_t(y_t,\pi_t) \leq 0. \tag{47}$$

Now we discuss two cases.

Case 1: $x_t \leq s_t^m(0,\pi_t)$. The myopic policy leads to $s_t^m(x_t,\pi_t) = s_t^m(0,\pi_t)$. For the optimal policy, that (47) holds for $y_t < s_t^m(0,\pi_t)$ implies $s_t^*(x_t,\pi_t) \geq s_t^m(0,\pi_t)$. Thus, we obtain $s_t^m(x_t,\pi_t) \leq s_t^*(x_t,\pi_t)$.

Case 2: $x_t > s_t^m(0,\pi_t)$. The myopic policy leads to $s_t^m(x_t,\pi_t) = x_t$. For the optimal policy, $s_t^*(x_t,\pi_t) = \arg\min_{y_t \geq x_t} G_t(y_t,\pi_t)$ implies $s_t^*(x_t,\pi_t) \geq x_t$. Thus, we still have $s_t^m(x_t,\pi_t) \leq s_t^*(x_t,\pi_t)$.

In short, $s_t^m(x_t,\pi_t) \leq s_t^*(x_t,\pi_t)$ holds for any $x_t \geq 0$, which completes the inductive proof. □

Proof of Lemma 7 Based on (16), we obtain $G_t'(y_t,\pi_t) \leq C_t'(y_t,\pi_t) + \psi_t(y_t,\pi_t)$ by applying (17) and $G_t'(y_t,\pi_t) \geq C_t'(y_t,\pi_t) + \varphi_{t,t}(y_t,\pi_t)$ by setting $i=1$ in (20). □

Proof of Lemma 8 Note the definition of J_{t+1} given by (14), that is,

$$J_{t+1}\left(s_{t+1}^*\left(x_{t+1},\pi_t,y_t^c\right),\pi_t,y_t^e\right) = \sum_{i=1}^{T-t} EC_{t+i}\left(s_{t+i}^*(x_{t+i},\pi_t,y_t^c,\mathbf{o}_{t+1,t+i-1}),\pi_t,y_t^e,\mathbf{o}_{t+1,t+i-1}\right).$$

We next claim that each term on the right handside of the above satisfies

$$\begin{aligned} & EC_{t+i}\left(s_{t+i}^*(x_{t+i},\pi_t,y_t^c,\mathbf{o}_{t+1,t+i-1}),\pi_t,y_t^e,\mathbf{o}_{t+1,t+i-1}\right) \\ = & \int_0^{+\infty}\cdots\int_0^{+\infty} r\left(s_{t+i}^*(x_{t+i},\pi_t,y_t^c,\mathbf{o}_{t+1,t+i-1}),z_{t+i}\right) \frac{\int_\Theta f\left(z_{t+i}|\theta\right)\Pi_{j=1}^{i-1} l\left(o_{t+j}|\theta\right) f\left(y_t|\theta\right)\pi_t\left(\theta\right)d\theta}{\int_\Theta f\left(z_{t+i}|\theta\right)\Pi_{j=1}^{i-1} l\left(o_{t+j}|\theta\right)\left[1-F\left(y_t|\theta\right)\right]\pi_t\left(\theta\right)d\theta} \\ & \times\Pi_{j=1}^{i} m_{t+j}\left(z_{t+j}|\pi_t,y_t^c,\mathbf{o}_{t+1,t+j-1}\right)dz_{t+i}\cdots dz_{t+1}\frac{\left[1-M_t\left(y_t|\pi_t\right)\right]}{m_t\left(y_t|\pi_t\right)}. \end{aligned}$$

Specifically, we have

$$
\begin{aligned}
& \int_0^{+\infty} \cdots \int_0^{+\infty} r\left(s_{t+i}^*(x_{t+i}, \pi_t, y_t^c, \mathbf{o}_{t+1,t+i-1}), z_{t+i}\right) \frac{\int_\Theta f\left(z_{t+i}|\theta\right) \Pi_{j=1}^{i-1} l\left(o_{t+j}|\theta\right) f\left(y_t|\theta\right) \pi_t\left(\theta\right) d\theta}{\int_\Theta f\left(z_{t+i}|\theta\right) \Pi_{j=1}^{i-1} l\left(o_{t+j}|\theta\right) \left[1 - F\left(y_t|\theta\right)\right] \pi_t\left(\theta\right) d\theta} \\
& \times \Pi_{j=1}^{i} m_{t+j}\left(z_{t+j}|\pi_t, y_t^c, \mathbf{o}_{t+1,t+j-1}\right) dz_{t+i} \cdots dz_{t+1} \\
= & \int_0^{+\infty} \cdots \int_0^{+\infty} r\left(s_{t+i}^*(x_{t+i}, \pi_t, y_t^c, \mathbf{o}_{t+1,t+i-1}), z_{t+i}\right) \frac{\int_\Theta f\left(z_{t+i}|\theta\right) \Pi_{j=1}^{i-1} l\left(o_{t+j}|\theta\right) f\left(y_t|\theta\right) \pi_t\left(\theta\right) d\theta}{\int_\Theta f\left(z_{t+i}|\theta\right) \Pi_{j=1}^{i-1} l\left(o_{t+j}|\theta\right) \left[1 - F\left(y_t|\theta\right)\right] \pi_t\left(\theta\right) d\theta} \\
& \times \frac{\int_\Theta f\left(z_{t+i}|\theta\right) \Pi_{j=1}^{i-1} l\left(o_{t+j}|\theta\right) \left[1 - F\left(y_t|\theta\right)\right] \pi_t\left(\theta\right) d\theta}{\int_\Theta \Pi_{j=1}^{i-1} l\left(o_{t+j}|\theta\right) \left[1 - F\left(y_t|\theta\right)\right] \pi_t\left(\theta\right) d\theta} \Pi_{j=1}^{i-1} m_{t+j}\left(z_{t+j}|\pi_t, y_t^c, \mathbf{o}_{t+1,t+j-1}\right) dz_{t+i} \cdots dz_{t+1} \\
& \text{(applying Lemma 18(b) to } m_{t+i}\left(z_{t+i}|\pi_t, y_t^c, \mathbf{o}_{t+1,t+i-1}\right)\text{)} \\
= & \int_0^{+\infty} \cdots \int_0^{+\infty} r\left(s_{t+i}^*(x_{t+i}, \pi_t, y_t^c, \mathbf{o}_{t+1,t+i-1}), z_{t+i}\right) \frac{\int_\Theta f\left(z_{t+i}|\theta\right) \Pi_{j=1}^{i-1} l\left(o_{t+j}|\theta\right) f\left(y_t|\theta\right) \pi_t\left(\theta\right) d\theta}{\int_\Theta \Pi_{j=1}^{i-1} l\left(o_{t+j}|\theta\right) \left[1 - F\left(y_t|\theta\right)\right] \pi_t\left(\theta\right) d\theta} \\
& \times \Pi_{j=1}^{i-1} m_{t+j}\left(z_{t+j}|\pi_t, y_t^c, \mathbf{o}_{t+1,t+j-1}\right) dz_{t+i} \cdots dz_{t+1} \\
= & \int_0^{+\infty} \cdots \int_0^{+\infty} r\left(s_{t+i}^*(x_{t+i}, \pi_t, y_t^c, \mathbf{o}_{t+1,t+i-1}), z_{t+i}\right) \frac{\int_\Theta f\left(z_{t+i}|\theta\right) \Pi_{j=1}^{i-1} l\left(o_{t+j}|\theta\right) f\left(y_t|\theta\right) \pi_t\left(\theta\right) d\theta}{\int_\Theta \Pi_{j=1}^{i-1} l\left(o_{t+j}|\theta\right) f\left(y_t|\theta\right) \pi_t\left(\theta\right) d\theta} \\
& \times \frac{\int_\Theta \Pi_{j=1}^{i-1} l\left(o_{t+j}|\theta\right) f\left(y_t|\theta\right) \pi_t\left(\theta\right) d\theta}{\int_\Theta \Pi_{j=1}^{i-1} l\left(o_{t+j}|\theta\right) \left[1 - F\left(y_t|\theta\right)\right] \pi_t\left(\theta\right) d\theta} \Pi_{j=1}^{i-1} m_{t+j}\left(z_{t+j}|\pi_t, y_t^c, \mathbf{o}_{t+1,t+j-1}\right) dz_{t+i} \cdots dz_{t+1} \\
= & \int_0^{+\infty} \cdots \int_0^{+\infty} \int_0^{+\infty} r\left(s_{t+i}^*(x_{t+i}, \pi_t, y_t^c, \mathbf{o}_{t+1,t+i-1}), z_{t+i}\right) m_{t+i}\left(z_{t+i}|\pi_t, y_t^e, \mathbf{o}_{t+1,t+i-1}\right) dz_{t+i} \\
& \times \frac{\int_\Theta \Pi_{j=1}^{i-1} l\left(o_{t+j}|\theta\right) f\left(y_t|\theta\right) \pi_t\left(\theta\right) d\theta}{\int_\Theta \Pi_{j=1}^{i-1} l\left(o_{t+j}|\theta\right) \left[1 - F\left(y_t|\theta\right)\right] \pi_t\left(\theta\right) d\theta} \Pi_{j=1}^{i-1} m_{t+j}\left(z_{t+j}|\pi_t, y_t^c, \mathbf{o}_{t+1,t+j-1}\right) dz_{t+i-1} \cdots dz_{t+1} \\
& \text{(applying Lemma 18(b) to } m_{t+i}\left(z_{t+i}|\pi_t, y_t^e, \mathbf{o}_{t+1,t+i-1}\right)\text{)} \\
= & \int_0^{+\infty} \cdots \int_0^{+\infty} \int_0^{+\infty} r\left(s_{t+i}^*(x_{t+i}, \pi_t, y_t^c, \mathbf{o}_{t+1,t+i-1}), z_{t+i}\right) m_{t+i}\left(z_{t+i}|\pi_t, y_t^e, \mathbf{o}_{t+1,t+i-1}\right) dz_{t+i} \\
& \times \frac{\int_\Theta f\left(z_{t+i-1}|\theta\right) \Pi_{j=1}^{i-2} l\left(o_{t+j}|\theta\right) f\left(y_t|\theta\right) \pi_t\left(\theta\right) d\theta}{\int_\Theta f\left(z_{t+i-1}|\theta\right) \Pi_{j=1}^{i-2} l\left(o_{t+j}|\theta\right) \left[1 - F\left(y_t|\theta\right)\right] \pi_t\left(\theta\right) d\theta} \Pi_{j=1}^{i-1} m_{t+j}\left(z_{t+j}|\pi_t, y_t^c, \mathbf{o}_{t+1,t+j-1}\right) dz_{t+i-1} \cdots dz_{t+1},
\end{aligned}
$$

where the last equality comes from

$$
\begin{aligned}
& \int_0^{+\infty} \int_0^{+\infty} r\left(s_{t+i}^*(x_{t+i}, \pi_t, y_t^c, \mathbf{o}_{t+1,t+i-1}), z_{t+i}\right) m_{t+i}\left(z_{t+i}|\pi_t, y_t^e, \mathbf{o}_{t+1,t+i-1}\right) dz_{t+i} \\
& \times \frac{\int_\Theta \Pi_{j=1}^{i-1} l\left(o_{t+j}|\theta\right) f\left(y_t|\theta\right) \pi_t\left(\theta\right) d\theta}{\int_\Theta \Pi_{j=1}^{i-1} l\left(o_{t+j}|\theta\right) \left[1 - F\left(y_t|\theta\right)\right] \pi_t\left(\theta\right) d\theta} m_{t+i-1}\left(z_{t+i-1}|\pi_t, y_t^c, \mathbf{o}_{t+1,t+i-2}\right) dz_{t+i-1} \\
= & \int_0^{+\infty} \int_0^{+\infty} r\left(s_{t+i}^*(x_{t+i}, \pi_t, y_t^c, \mathbf{o}_{t+1,t+i-1}), z_{t+i}\right) m_{t+i}\left(z_{t+i}|\pi_t, y_t^e, \mathbf{o}_{t+1,t+i-1}\right) dz_{t+i} \\
& \times \frac{\int_\Theta f\left(z_{t+i-1}|\theta\right) \Pi_{j=1}^{i-2} l\left(o_{t+j}|\theta\right) f\left(y_t|\theta\right) \pi_t\left(\theta\right) d\theta}{\int_\Theta f\left(z_{t+i-1}|\theta\right) \Pi_{j=1}^{i-2} l\left(o_{t+j}|\theta\right) \left[1 - F\left(y_t|\theta\right)\right] \pi_t\left(\theta\right) d\theta} m_{t+i-1}\left(z_{t+i-1}|\pi_t, y_t^c, \mathbf{o}_{t+1,t+i-2}\right) dz_{t+i-1},
\end{aligned}
$$

which in turn can be shown by using a logic similar to that used in proving Lemma 19.

Carrying on the above recursively, we obtain

$$\int_0^{+\infty}\cdots\int_0^{+\infty} r\left(s_{t+i}^*(x_{t+i},\pi_t,y_t^c,\mathbf{o}_{t+1,t+i-1}),z_{t+i}\right)\frac{\int_\Theta f\left(z_{t+i}|\theta\right)\Pi_{j=1}^{i-1}l\left(o_{t+j}|\theta\right)f\left(y_t|\theta\right)\pi_t\left(\theta\right)d\theta}{\int_\Theta f\left(z_{t+i}|\theta\right)\Pi_{j=1}^{i-1}l\left(o_{t+j}|\theta\right)\left[1-F\left(y_t|\theta\right)\right]\pi_t\left(\theta\right)d\theta}$$
$$\times\Pi_{j=1}^{i}m_{t+j}\left(z_{t+j}|\pi_t,y_t^c,\mathbf{o}_{t+1,t+j-1}\right)dz_{t+i}\cdots dz_{t+1}$$
$$=\int_0^{+\infty}\cdots\int_0^{+\infty} r\left(s_{t+i}^*(x_{t+i},\pi_t,y_t^c,\mathbf{o}_{t+1,t+i-1}),z_{t+i}\right)\Pi_{j=1}^{i}m_{t+j}\left(z_{t+j}|\pi_t,y_t^e,\mathbf{o}_{t+1,t+j-1}\right)dz_{t+i}\cdots dz_{t+1}$$
$$\times\frac{\int_\Theta f\left(y_t|\theta\right)\pi_t\left(\theta\right)d\theta}{\int_\Theta\left[1-F\left(y_t|\theta\right)\right]\pi_t\left(\theta\right)d\theta}$$
$$= EC_{t+i}\left(s_{t+i}^*(x_{t+i},\pi_t,y_t^c,\mathbf{o}_{t+1,t+i-1}),\pi_t,y_t^e,\mathbf{o}_{t+1,t+i-1}\right)\frac{m_t\left(y_t|\pi_t\right)}{1-M_t\left(y_t|\pi_t\right)}.$$

That is, the above claim holds.

Based on the claim, we can show (note $H_{t,t+i}^c=(\pi_t,y_t^c,\mathbf{o}_{t+1,t+i-1})$)

$$J_{t+1}\left(s_{t+1}^*\left(0,\pi_t,y_t^c\right),\pi_t,y_t^e\right)$$
$$=\sum_{i=1}^{T-t}\int_0^{+\infty}\cdots\int_0^{+\infty} r\left(s_{t+i}^*(x_{t+i},H_{t,t+i}^c),z_{t+i}\right)\frac{\int_\Theta f\left(z_{t+i}|\theta\right)\Pi_{j=1}^{i-1}l\left(o_{t+j}|\theta\right)f\left(y_t|\theta\right)\pi_t\left(\theta\right)d\theta}{\int_\Theta f\left(z_{t+i}|\theta\right)\Pi_{j=1}^{i-1}l\left(o_{t+j}|\theta\right)\left[1-F\left(y_t|\theta\right)\right]\pi_t\left(\theta\right)d\theta}$$
$$\times\Pi_{j=1}^{i}m_{t+j}\left(z_{t+j}|H_{t,t+j}^c\right)dz_{t+i}\cdots dz_{t+1}\frac{\left[1-M_t\left(y_t|\pi_t\right)\right]}{m_t\left(y_t|\pi_t\right)}$$
$$\leq\sum_{i=1}^{T-t}\int_0^{+\infty}\cdots\int_0^{+\infty} r\left(s_{t+i}^*(x_{t+i},H_{t,t+i}^c),z_{t+i}\right)\frac{\int_\Theta f\left(z_{t+i}|\theta\right)\Pi_{j=1}^{i-1}l\left(o_{t+j}|\theta\right)\left[1-F\left(y_t|\theta\right)\right]\pi_t\left(\theta\right)d\theta}{\int_\Theta f\left(z_{t+i}|\theta\right)\Pi_{j=1}^{i-1}l\left(o_{t+j}|\theta\right)\left[1-F\left(y_t|\theta\right)\right]\pi_t\left(\theta\right)d\theta}$$
$$\times\max_{\theta\in\Theta}\left\{\frac{f\left(y_t|\theta\right)}{1-F\left(y_t|\theta\right)}\right\}\Pi_{j=1}^{i}m_{t+j}\left(z_{t+j}|H_{t,t+j}^c\right)dz_{t+i}\cdots dz_{t+1}\frac{\left[1-M_t\left(y_t|\pi_t\right)\right]}{m_t\left(y_t|\pi_t\right)}$$
$$=\sum_{i=1}^{T-t}\int_0^{+\infty}\cdots\int_0^{+\infty} r\left(s_{t+i}^*(x_{t+i},H_{t,t+i}^c),z_{t+i}\right)\Pi_{j=1}^{i}m_{t+j}\left(z_{t+j}|H_{t,t+j}^c\right)dz_{t+i}\cdots dz_{t+1}\xi(y_t|\pi_t)$$
$$=\sum_{i=1}^{T-t}EC_{t+i}\left(s_{t+i}^*(x_{t+i},H_{t,t+i}^c),H_{t,t+i}^c\right)\xi(y_t|\pi_t)$$
$$=G_{t+1}\left(s_{t+1}^*(0,H_{t,t+1}^c),H_{t,t+1}^c\right)\xi(y_t|\pi_t).$$

The key in this proof is the inequality, which we obtain essentially by applying a change of measure from y_t^e to y_t^c. □

Proof of Lemma 9 We note $V_{t+1}\left(0,H_{t,t+1}\right)=G_{t+1}\left(s_{t+1}^*\left(0,H_{t,t+1}\right),H_{t,t+1}\right)$. So this lemma comes from Theorem 6. □

Proof of Lemma 10 Applying Lemmas 8 and 9, we can obtain the result of this lemma. □

Proof of Lemma 11 We note

$$\mathbf{1}(\tau^\ell(t,y_t)>t+i)\geq\mathbf{1}(\tau(t,y_t)>t+i)\geq 0.$$

We also note

$$\begin{aligned}
& C'_{t+i}(y_t - z[t,t+i), \pi_t, \mathbf{z}^e_{t,t+i-1})\mathbf{1}(A^m_{t+i}) \\
= \; & \max\left\{C'_{t+i}(y_t - z[t,t+i), \pi_t, \mathbf{z}^e_{t,t+i-1}), 0\right\} \\
\geq \; & C'_{t+i}(y_t - z[t,t+i), \pi_t, \mathbf{z}^e_{t,t+i-1})\mathbf{1}(\tau(t,y_t) > t+i).
\end{aligned}$$

Applying the above two facts, we obtain

$$C'_{t+1}(y_t - z_t, \pi_t, z^e_t)\mathbf{1}(A^m_{t+1}) \geq C'_{t+1}(y_t - z_t, \pi_t, z^e_t)\mathbf{1}(\tau(t,y_t) > t+1),$$

and, for $2 \leq i \leq T-t$,

$$\begin{aligned}
& C'_{t+i}(y_t - z[t,t+i), \pi_t, \mathbf{z}^e_{t,t+i-1})\mathbf{1}(\tau^\ell(t,y_t) > t+i-1)\mathbf{1}(A^m_{t+i}) \\
\geq \; & C'_{t+i}(y_t - z[t,t+i), \pi_t, \mathbf{z}^e_{t,t+i-1})\mathbf{1}(\tau(t,y_t) > t+i-1)\mathbf{1}(A^m_{t+i}) \\
\geq \; & C'_{t+i}(y_t - z[t,t+i), \pi_t, \mathbf{z}^e_{t,t+i-1})\mathbf{1}(\tau(t,y_t) > t+i-1)\mathbf{1}(\tau(t,y_t) > t+i) \\
= \; & C'_{t+i}(y_t - z[t,t+i), \pi_t, \mathbf{z}^e_{t,t+i-1})\mathbf{1}(\tau(t,y_t) > t+i),
\end{aligned}$$

which in turn leads to

$$\psi_t(y_t,\pi_t) \leq \sum_{i=1}^{T-t} E[C'_{t+i}(y_t - Z[t,t+i), \pi_t, \mathbf{Z}^e_{t,t+i-1})\mathbf{1}(\tau^\ell(t,y_t) > t+i-1)\mathbf{1}(A^m_{t+i})].$$

□

Proof of Theorem 12 We note ${C_t}'(y_t,\pi_t) + \varphi_{t,t}(y_t,\pi_t) \leq {G_t}'(y_t,\pi_t) \leq {C_t}'(y_t,\pi_t) + \psi_t(y_t,\pi_t)$ as stated in Lemma 7. Then, applying Lemmas 10 and 11 leads to the result stated in this theorem. □

Proof of Corollary 13 This corollary comes immediately from Theorem 12. □

Proof of Corollary 14 We prove by a backward induction. It holds trivially for $t = T$ as $G_T(y_T,\pi_T) = G_T^{FI}(y_T,\pi_T) = C_T(y_T,\pi_T)$. Assume that the result holds for period $t+1$. Now we analyze period t. For period t, we have (see equation (36))

$$\begin{aligned}
{G_t}'(y_t,\pi_t) &= {C_t}'(y_t,\pi_t) + \varphi_{t,t}(y_t,\pi_t) + \int_0^{y_t} \frac{dG_{t+1}\left(s^*_{t+1}(y_t - z_t, \pi_t, z^e_t), \pi_t, z^e_t\right)}{dy_t} m_t(z_t|\pi_t)\, dz_t \\
&= {C_t}'(y_t,\pi_t) + \varphi_{t,t}(y_t,\pi_t) + \int_0^{y_t} \frac{dG_{t+1}(y_t - z_t, \pi_t, z^e_t)}{dy_t} m_t(z_t|\pi_t) \\
&\quad \times \left[1 - \mathbf{1}\{s^*_{t+1}(y_t - z_t, \pi_t, z^e_t) > y_t - z_t\}\right] dz_t,
\end{aligned}$$

and

$$\begin{aligned}\frac{dG_t^{FI}(y_t,\pi_t)}{dy_t} &= C_t{}'(y_t,\pi_t)+\int_0^{y_t}\frac{dV_{t+1}^{FI}(y_t-z_t,\pi_t,z_t^e)}{dy_t}m_t(z_t|\pi_t)\,dz_t\\ &= C_t{}'(y_t,\pi_t)+\int_0^{y_t}\frac{dG_{t+1}^{FI}(y_t-z_t,\pi_t,z_t^e)}{dy_t}m_t(z_t|\pi_t)\\ &\quad\times\left[1-\mathbf{1}\{s_{t+1}^{FI}(y_t-z_t,\pi_t,z_t^e)>y_t-z_t\}\right]dz_t.\end{aligned}$$

Note the following facts:

1) $\varphi_{t,t}(y_t,\pi_t)\le 0$.

2) $\frac{dG_{t+1}(y_t-z_t,\pi_t,z_t^e)}{dy_t}\le\frac{dG_{t+1}^{FI}(y_t-z_t,\pi_t,z_t^e)}{dy_t}$ by the inductive assumption.

3) $1-\mathbf{1}\{s_{t+1}^*(y_t-z_t,\pi_t,z_t^e)>y_t-z_t\}\le 1-\mathbf{1}\{s_{t+1}^{FI}(y_t-z_t,\pi_t,z_t^e)>y_t-z_t\}$ due to the inductive assumption.

4) When $1-\mathbf{1}\{s_{t+1}^{FI}(y_t-z_t,\pi_t,z_t^e)>y_t-z_t\}=1$ (i.e., $s_{t+1}^{FI}(y_t-z_t,\pi_t,z_t^e)=y_t-z_t$ does hold, or equivalently, no order is placed in period $t+1$ in the model of observable lost sales), we must have $\frac{dG_{t+1}^{FI}(y_t-z_t,\pi_t,z_t^e)}{dy_t}\ge 0$.

Therefore, $G_t{}'(y_t,\pi_t)\le\frac{dG_t^{FI}(y_t,\pi_t)}{dy_t}$, which immediately leads to $s_t^{FI}(x_t,\pi_t)\le s_t^*(x_t,\pi_t)$, that is, $s_t^{FI}(x_t,\pi_t)$ is a lower bound on $s_t^*(x_t,\pi_t)$.

To show $\psi_t(y_t,\pi_t)\le\psi_t^{FI}(y_t,\pi_t)$, we use a proof that is similar to that used for Lemma 11. In particular, let $\tau^u(t,y)=\min\{t+j|(y-Z[t,t+j))^+<s_{t+j}^u((y-Z[t,t+j))^+,\pi_{t+j}),j\ge 1\}$. Then, it is straightforward that

$$\mathbf{1}(\tau(t,y)>t+i)\ge\mathbf{1}(\tau^u(t)>t+i)\ge 0. \tag{48}$$

We have, for given $\mathbf{z}_{t,t+i-1}^e$,

$$\begin{aligned} &C_{t+i}{}'(y-z[t,t+i),\pi_t,\mathbf{z}_{t,t+i-1}^e)\mathbf{1}(\tau(t,y) > t+i)\\ +\sum_{k=i+1}^{T-t}E[C_{t+k}{}'(y-\sum_{j=0}^{i-1}z_{t+j}-\sum_{j=i}^{k-1}Z_{t+j},\pi_t,\mathbf{z}_{t,t+i-1}^e,\mathbf{Z}_{t+i,t+k-1}^e)&\mathbf{1}(\tau(t,y) > t+k)]\ge 0,\end{aligned} \tag{49}$$

which comes from inductively using (44). We obtain

$$\begin{aligned} &\psi_t(y,\pi_t)\\ =\ &\sum_{i=1}^{T-t}E[C_{t+i}{}'(y-Z[t,t+i),\pi_t,\mathbf{Z}_{t,t+i-1}^e)\mathbf{1}(\tau(t,y)>t+i)]\\ =\ &E[\{C_{t+1}{}'(y-Z_t,\pi_t,Z_t^e)\mathbf{1}(\tau(t,y)>t+1)+\sum_{i=2}^{T-t}E[C_{t+i}'(y-Z[t,t+i),\pi_t,\mathbf{Z}_{t,t+i-1}^e)\mathbf{1}(\tau(t,y)>t+i)]\}\\ &\times\mathbf{1}(\tau(t,y)>t+1)]\\ \ge\ &E[\{C_{t+1}{}'(y-Z_t,\pi_t,Z_t^e)\mathbf{1}(\tau^u(t)>t+1)+\sum_{i=2}^{T-t}E[C_{t+i}'(y-Z[t,t+i),\pi_t,\mathbf{Z}_{t,t+i-1}^e)\mathbf{1}(\tau(t,y)>t+i)]\}\\ &\times\mathbf{1}(\tau^u(t)>t+1)]\end{aligned}$$

where the inequality comes from (48) and (49). Recursively, we obtain

$$\psi_t(y,\pi_t) \geq \sum_{i=1}^{T-t} E[C'_{t+i}(y - Z[t,t+i), \pi_t, \mathbf{Z}^e_{t,t+i-1})\mathbf{1}(\tau^u(t) > t+i)].$$

Note the same technique shown above can be used to analyze $\psi_t^{FI}(y,\pi_t)$. Also note that we have shown $s_t^{FI}(x_t,\pi_t) \leq s_t^*(x_t,\pi_t)$, that is, $s_t^*(x_t,\pi_t)$ is an upper solution bound on $s_t^{FI}(x_t,\pi_t)$. Then, the technique will give $\psi_t^{FI}(y,\pi_t) \geq \psi_t(y,\pi_t)$, which completes the proof. □

Proof of Theorem 15 (a) We have proved $G_t^{FI}(y_t,\pi_t) \leq G_t(y_t,\pi_t)$ in the proof of Theorem 6. Now we study the upper bound on $G_t(y_t,\pi_t)$. We can write

$$\begin{aligned} & G_t(y_t,\pi_t) = C_t(y_t,\pi_t) + EV_{t+1}\left((y_t - Z_t)^+, \pi_t, y_t \otimes Z_t\right) \\ = \;& C_t(y_t,\pi_t) + EV_{t+1}(0,\pi_t, y_t\otimes Z_t) + \left[EV_{t+1}\left((y_t - Z_t)^+, \pi_t, y_t\otimes Z_t\right) - EV_{t+1}(0,\pi_t,y_t\otimes Z_t)\right]. \end{aligned}$$

On the right handside of the above equation, we can see that the first term, $C_t(y_t,\pi_t)$, is the expected cost in the current period t and hence is a convex function in the inventory decision y_t.

For the second term, we have

$$EV_{t+1}(0,\pi_t,y_t\otimes Z_t) = EG_{t+1}\left(s^*_{t+1}(0,\pi_t,y_t\otimes Z_t), \pi_t, y_t\otimes Z_t\right).$$

We note

$$\begin{aligned} & \frac{dEG_{t+1}\left(s^*_{t+1}(0,\pi_t,y_t\otimes Z_t),\pi_t,y_t\otimes Z_t\right)}{dy_t} \\ = \;& \frac{d}{dy_t}\left\{\int_0^{y_t} G_{t+1}\left(s^*_{t+1}(0,\pi_t,z_t^e),\pi_t,z_t^e\right)m_t(z_t|\pi_t)\,dz_t + G_{t+1}\left(s^*_{t+1}\left(0,H^c_{t,t+1}\right),H^c_{t,t+1}\right)\left[1 - M_t(y_t|\pi_t)\right]\right\} \\ = \;& G_{t+1}\left(s^*_{t+1}\left(0,H^e_{t,t+1}\right),H^e_{t,t+1}\right)m_t(y_t|\pi_t) + \frac{d}{dy_t}\left\{G_{t+1}\left(s^*_{t+1}\left(0,H^c_{t,t+1}\right),H^c_{t,t+1}\right)\left[1-M_t(y_t|\pi_t)\right]\right\} \\ = \;& \left[G_{t+1}\left(s^*_{t+1}\left(0,H^e_{t,t+1}\right),H^e_{t,t+1}\right) - J_{t+1}\left(s^*_{t+1}\left(0,H^c_{t,t+1}\right),H^e_{t,t+1}\right)\right]m_t(y_t|\pi_t) \leq 0, \end{aligned}$$

where the last equality comes from (37). Therefore, we obtain

$$EV_{t+1}(0,\pi_t,y_t\otimes Z_t) \leq EV_{t+1}(0,\pi_t,0^c) = EV_{t+1}(0,\pi_t),$$

which, together with Theorem 6, leads to

$$EV_{t+1}(0,\pi_t,y_t\otimes Z_t) \leq V^{NI}_{t+1}(0,\pi_t).$$

For the last term of $EV_{t+1}((y_t - Z_t)^+,\pi_t,y_t\otimes Z_t) - EV_{t+1}(0,\pi_t,y_t\otimes Z_t)$, it represents the additional cost caused by overstock. Obviously, $EV_{t+1}((y_t - Z_t)^+,\pi_t,y_t\otimes Z_t)$ is no greater than the expected cost generated the following heuristic: We set aside $(y_t - z_t)^+$ and hold such units till the end of the planning horizon. Then, on each sample path, we follow the same decisions that are optimal for $EV_{t+1}(0, \pi_t, y_t \otimes Z_t)$. That is, we have

$$EV_{t+1}\left((y_t - Z_t)^+, \pi_t, y_t \otimes Z_t\right) \leq (T-t)\, hE(y_t - Z_t)^+ + EV_{t+1}(0, \pi_t, y_t \otimes Z_t),$$

which leads to

$$EV_{t+1}\left((y_t - Z_t)^+, \pi_t, y_t \otimes Z_t\right) - EV_{t+1}(0, \pi_t, y_t \otimes Z_t) \leq (T-t)\, hE(y_t - Z_t)^+.$$

Therefore, we obtain

$$G_t(y_t, \pi_t) \leq C_t(y_t, \pi_t) + (T-t)\, hE(y_t - Z_t)^+ + V_{t+1}^{NI}(0, \pi_t),$$

in which the right handside is a convex function of y_t.

(b) Note that two bounding functions derived above are both convex in y_t.

For given x_t, let $\tilde{y}_t = \arg\min_{y'_t \geq x_t} \left\{C_t(y'_t, \pi_t) + (T-t)\, hE(y'_t - Z_t)^+ + V_{t+1}^{NI}(0, \pi_t)\right\}$. Thus, we have $Q = \left\{C_t(\tilde{y}_t, \pi_t) + (T-t)\, hE(\tilde{y}_t - Z_t)^+ + V_{t+1}^{NI}(0, \pi_t)\right\}$. Then, let $\hat{y}_t$ be the solution to $G_t^{FI}(\hat{y}_t, \pi_t) = Q$ over $y_t > s_t^{FI}(0, \pi_t)$. It is straightforward that such a solution exists. For any $y_t > \hat{y}_t$, we obtain

$$\begin{aligned}
G_t(y_t, \pi_t) &> G_t^{FI}(y_t, \pi_t) \\
&\geq G_t^{FI}(\hat{y}_t, \pi_t) \\
&= Q \\
&= C_t(\tilde{y}_t, \pi_t) + (T-t)\, hE(\tilde{y}_t - Z_t)^+ + V_{t+1}^{NI}(0, \pi_t) \\
&\geq G_t(\tilde{y}_t, \pi_t),
\end{aligned}$$

where the first and the last inequalities come from part 1), and the second inequality is due to the fact that $G_t^{FI}(y_t, \pi_t)$ achieves the minimum at $s_t^{FI}(0, \pi_t)$ and is increasing in y_t for $y_t > s_t^{FI}(0, \pi_t)$. So y_t cannot be the minimizer of G_t over $[x_t, +\infty)$. Hence $\hat{y}_t$ is an upper solution bound. □

Proof of Theorem 16 We note

$$\begin{aligned}
& V_t^H(x_t, \pi_t) - V_t(x_t, \pi_t) = V_t^H(x_t, \pi_t) - G_t(s_t^*(x_t, \pi_t), \pi_t) \\
= \; & G_t(s_t^H(x_t, \pi_t), \pi_t) - G_t(s_t^*(x_t, \pi_t), \pi_t) + V_t^H(x_t, \pi_t) - G_t(s_t^H(x_t, \pi_t), \pi_t) \\
= \; & G_t(s_t^H(x_t, \pi_t), \pi_t) - G_t(s_t^*(x_t, \pi_t), \pi_t) + \left\{C_t(s_t^H(x_t, \pi_t), \pi_t) + E[V_{t+1}^H\left(\left(s_t^H(x_t, \pi_t) - Z_t\right)^+, \pi_{t+1}\right)]\right\} \\
& - \left\{C_t(s_t^H(x_t, \pi_t), \pi_t) + E[V_{t+1}\left(\left(s_t^H(x_t, \pi_t) - Z_t\right)^+, \pi_{t+1}\right)]\right\} \\
= \; & G_t(s_t^H(x_t, \pi_t), \pi_t) - G_t(s_t^*(x_t, \pi_t), \pi_t) \\
& + E[V_{t+1}^H\left(\left(s_t^H(x_t, \pi_t) - Z_t\right)^+, \pi_{t+1}\right) - V_{t+1}\left(\left(s_t^H(x_t, \pi_t) - Z_t\right)^+, \pi_{t+1}\right)], \qquad (50)
\end{aligned}$$

where both π_{t+1}'s in (50) refer to the same posterior distribution obtained from the observation $s_t^H(x_t, \pi_t) \otimes Z_t$ in period t. We then develop an upper bound on $G_t(s_t^H(x_t, \pi_t), \pi_t) - G_t(s_t^*(x_t, \pi_t), \pi_t)$, for which we discuss two cases, $s_t^H(x_t, \pi_t) \geq s_t^*(x_t, \pi_t)$ and $s_t^H(x_t, \pi_t) < s_t^*(x_t, \pi_t)$. For $s_t^H(x_t, \pi_t) \geq s_t^*(x_t, \pi_t)$, we have

$$\begin{aligned} & G_t(s_t^H(x_t, \pi_t), \pi_t) - G_t(s_t^*(x_t, \pi_t), \pi_t) \\ = \; & G_t'(y_t, \pi_t) \mid_{s_t^*(x_t,\pi_t) \leq y \leq s_t^H(x_t,\pi_t)} \left(s_t^H(x_t, \pi_t) - s_t^*(x_t, \pi_t)\right) \\ \leq \; & \max_{s_t^\ell(x_t,\pi_t) \leq y \leq s_t^H(x_t,\pi_t)} \left\{G_t'(y_t, \pi_t)\right\} \left(s_t^H(x_t, \pi_t) - s_t^\ell(x_t, \pi_t)\right) \\ \leq \; & \max_{s_t^\ell(x_t,\pi_t) \leq y \leq s_t^H(x_t,\pi_t)} \left\{g_t^u(y_t, \pi_t)\right\} \left(s_t^H(x_t, \pi_t) - s_t^\ell(x_t, \pi_t)\right) \text{(from Theorem 12)} \\ \leq \; & \max_{s_t^\ell(0,\pi_t) \leq y \leq s_t^H(0,\pi_t)} \left\{g_t^u(y_t, \pi_t)\right\} \left(s_t^H(0, \pi_t) - s_t^\ell(0, \pi_t)\right), \end{aligned}$$

where the last inequality comes from the fact that $s_t^H(x_t, \pi_t) - s_t^\ell(x_t, \pi_t) = s_t^H(0, \pi_t) - s_t^\ell(0, \pi_t)$ when $x_t < s_t^u(0, \pi_t)$ and $s_t^H(x_t, \pi_t) - s_t^\ell(x_t, \pi_t) = 0$ when $x_t \geq s_t^u(0, \pi_t)$. For $s_t^H(x_t, \pi_t) < s_t^*(x_t, \pi_t)$, by a similar analysis, we can obtain

$$G_t(s_t^H(x_t, \pi_t), \pi_t) - G_t(s_t^*(x_t, \pi_t), \pi_t) \leq -\min_{s_t^H(0,\pi_t) \leq y \leq s_t^u(0,\pi_t)} \left\{g_t^\ell(y_t, \pi_t)\right\} \left(s_t^u(0, \pi_t) - s_t^H(0, \pi_t)\right).$$

The same analysis as above can be recursively applied to the last term on the right handside of (50), $E[V_{t+1}^H\left(\left(s_t^H(x_t, \pi_t) - Z_t\right)^+, \pi_{t+1}\right) - V_{t+1}\left(\left(s_t^H(x_t, \pi_t) - Z_t\right)^+, \pi_{t+1}\right)]$.

On the other hand, $V_1^H(0, \pi_1) - V_1(0, \pi_1) \leq V_1^H(0, \pi_1) - V_1^{FI}(0, \pi_1)$ comes from $V_1(0, \pi_1) \geq V_1^{FI}(0, \pi_1)$, as shown in Theorem 6. □

Proof of Proposition 17 For the capacitated myopic policy, we can see that the starting inventory level in each period is no greater than $s_1^m(0, \pi_1)$, i.e., $x_t \leq s_1^m(0, \pi_1)$. Note $s_t^m(x_t, \pi_t) = \max\{s_t^m(0, \pi_t), x_t\}$. Let $y_t^m = \min\{s_t^m(x_t, \pi_t), s_1^m(0, \pi_1)\}$. We have the following cases to discuss:

Case 1a: $x_t \leq s_t^m(0, \pi_t)$ and $s_t^m(0, \pi_t) \leq s_1^m(0, \pi_1)$. We have $y_t^m = s_t^m(0, \pi_t) \leq s_1^m(0, \pi_1)$.

Case 1b: $x_t \leq s_t^m(0, \pi_t)$ and $s_t^m(0, \pi_t) > s_1^m(0, \pi_1)$. We have $y_t^m = s_1^m(0, \pi_1)$.

Case 2a: $x_t > s_t^m(0, \pi_t)$ and $x_t \leq s_1^m(0, \pi_1)$. We have $s_t^m(0, \pi_t) < y_t^m = x_t \leq s_1^m(0, \pi_1)$.

Case 2b: $x_t > s_t^m(0, \pi_t)$ and $x_t > s_1^m(0, \pi_1)$. This is not possible.

Thus, we obtain, due to the convexity of $C_t(y_t, \pi_t)$ in y_t,

$$C_t(y_t^m, \pi_t) \leq C_t(s_1^m(0, \pi_1), \pi_t).$$

Note that $s_1^m(0, \pi_1)$ is independent of demand updating. We now claim

$$E_{Z_{t-1}} C_t(s_1^m(0, \pi_1), \pi_t) = C_{t-1}(s_1^m(0, \pi_1), \pi_{t-1}).$$

We can prove the claim by

$$\begin{aligned}
& E_{Z_{t-1}} C_t\left(s_1^m(0,\pi_1),\pi_t\right) \\
= & \int_0^{y_{t-1}^m} C_t\left(s_1^m(0,\pi_1),\pi_t\left(\cdot|\pi_{t-1},z_{t-1}^e\right)\right) m_{t-1}\left(z_{t-1}|\pi_{t-1}\right) dz_{t-1} \\
& + C_t\left(s_1^m(0,\pi_1),\pi_t\left(\cdot|\pi_{t-1},\left(y_{t-1}^m\right)^c\right)\right)\left[1-M_{t-1}\left(y_{t-1}^m|\pi_{t-1}\right)\right],
\end{aligned}$$

where

$$\begin{aligned}
& C_t\left(s_1^m(0,\pi_1),\pi_t\left(\cdot|\pi_{t-1},z_{t-1}^e\right)\right) m_{t-1}\left(z_{t-1}|\pi_{t-1}\right) \\
= & \int_0^{+\infty} r\left(s_1^m(0,\pi_1),z_t\right)\int_\Theta f\left(z_t|\theta\right)\pi_t\left(\theta|\pi_{t-1},z_{t-1}^e\right) d\theta dz_t m_{t-1}\left(z_{t-1}|\pi_{t-1}\right) \\
= & \int_0^{+\infty} r\left(s_1^m(0,\pi_1),z_t\right)\int_\Theta f\left(z_t|\theta\right)\frac{f\left(z_{t-1}|\theta\right)\pi_{t-1}\left(\theta\right)}{\int_\Theta f\left(z_{t-1}|\theta'\right)\pi_{t-1}\left(\theta'\right)d\theta'} d\theta dz_t \int_\Theta f(z_{t-1}|\theta')\pi_{t-1}(\theta')d\theta' \\
= & \int_\Theta\int_0^{+\infty} r\left(s_1^m(0,\pi_1),z_t\right) f\left(z_t|\theta\right) f\left(z_{t-1}|\theta\right)\pi_{t-1}\left(\theta\right) dz_t d\theta,
\end{aligned}$$

which leads to

$$\begin{aligned}
& \int_0^{y_{t-1}^m} C_t\left(s_1^m(0,\pi_1),\pi_t\left(\cdot|\pi_{t-1},z_{t-1}^e\right)\right) m_{t-1}\left(z_{t-1}|\pi_{t-1}\right) dz_{t-1} \\
= & \int_\Theta\int_0^{+\infty} r\left(s_1^m(0,\pi_1),z_t\right) f\left(z_t|\theta\right)\int_0^{y_{t-1}^m} f\left(z_{t-1}|\theta\right) dz_{t-1}\pi_{t-1}\left(\theta\right) dz_t d\theta.
\end{aligned}$$

In a similar way, we can show

$$\begin{aligned}
& C_t\left(s_1^m(0,\pi_1),\pi_t\left(\cdot|\pi_{t-1},\left(y_{t-1}^m\right)^c\right)\right)\left[1-M_{t-1}\left(y_{t-1}^m|\pi_{t-1}\right)\right] \\
= & \int_\Theta\int_0^{+\infty} r\left(s_1^m(0,\pi_1),z_t\right) f\left(z_t|\theta\right)\int_{y_{t-1}^m}^{+\infty} f\left(z_{t-1}|\theta\right) dz_{t-1}\pi_{t-1}\left(\theta\right) dz_t d\theta.
\end{aligned}$$

Thus, we have

$$\begin{aligned}
E_{Z_{t-1}} C_t\left(s_1^m(0,\pi_1),\pi_t\right) & = \int_\Theta\int_0^{+\infty} r\left(s_1^m(0,\pi_1),z_t\right) f\left(z_t|\theta\right)\pi_{t-1}\left(\theta\right) dz_t d\theta \\
& = \int_0^{+\infty} r\left(s_1^m(0,\pi_1),z_t\right)\int_\Theta f\left(z_t|\theta\right)\pi_{t-1}\left(\theta\right) d\theta dz_t \\
& = C_{t-1}\left(s_1^m(0,\pi_1),\pi_{t-1}\right).
\end{aligned}$$

Recursively, we have

$$E_{Z_1,Z_2,,\ldots,Z_{t-1}} C_t\left(s_1^m(0,\pi_1),\pi_t\right) = C_1\left(s_1^m(0,\pi_1),\pi_1\right).$$

Therefore, the capacitated myopic policy yields a total cost of no more than $C_1\left(s_1^m(0,\pi_1),\pi_1\right)\times T$, which is equal to $V^{NI}\left(0,\pi_1\right)$. □