Challenges and Recent Advances in Network Inference

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INFORMS, APS Tutorial
October 26, 2025

Statistical inference on graphs

- Detecting or estimating hidden structures in large network data

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  - Q1 Characterize statistical (information-theoretic) limit: What is possible/impossible?

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- Key challenges: Understanding the fundamental limits:
  - Q1 Characterize statistical (information-theoretic) limit: What is possible/impossible?
  - Q2 Can statistical limits be attained computationally efficiently, e.g., in polynomial time? If yes, how? If not, why?

Planted clique – graph view

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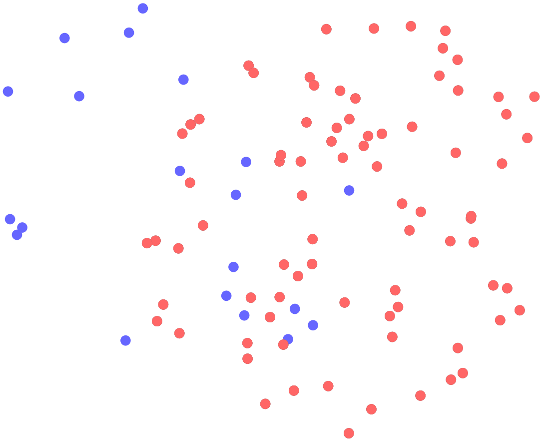
1. A set C of k vertices is chosen to form a clique



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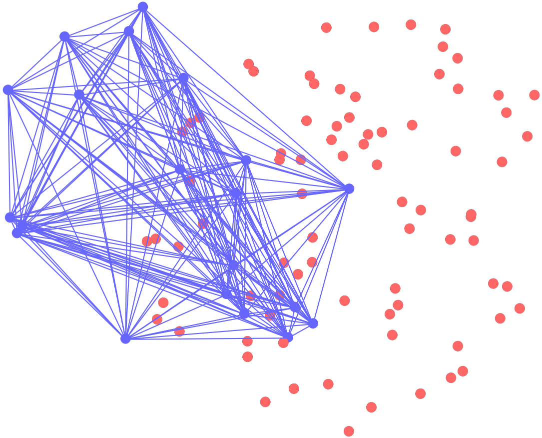

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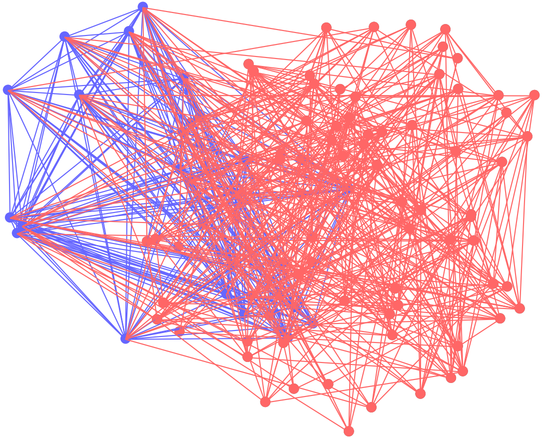

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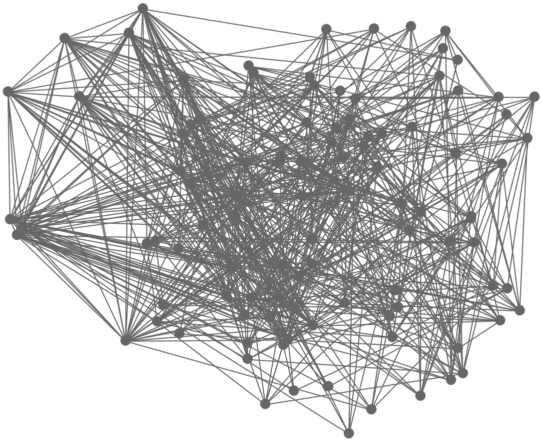
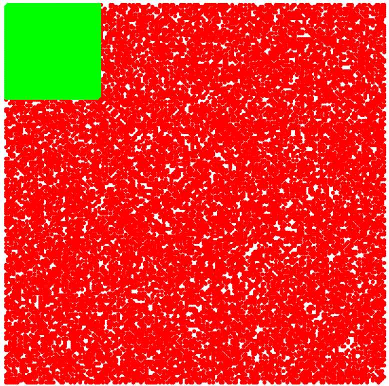


Planted clique – adjacency matrix view



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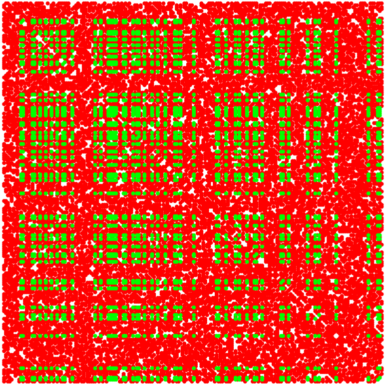

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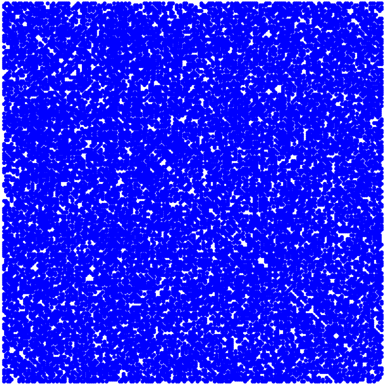


Community detection in networks

- Networks with community structures arise in many applications

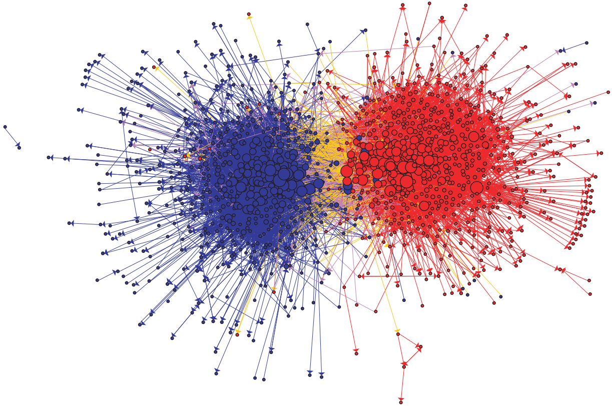
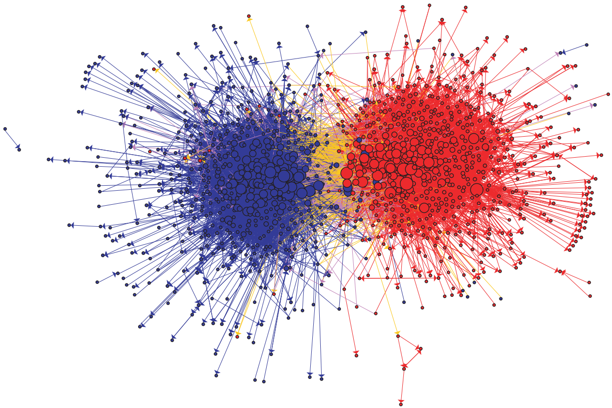


Figure: Political blogosphere and the 2004 U.S. election [Adamic-Glance '05]

Community detection in networks

- Networks with community structures arise in many applications
- Task: Discover underlying communities based on the network topology



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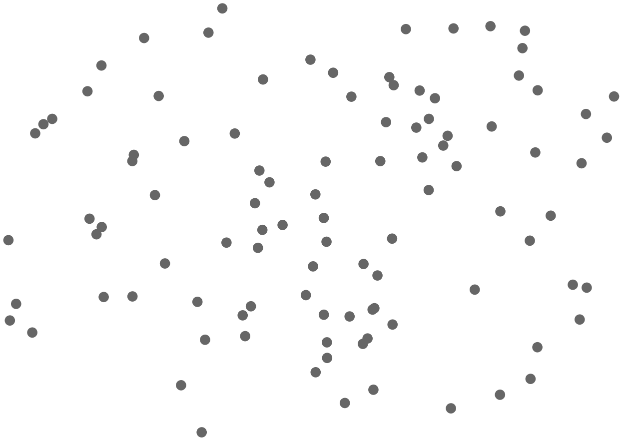
Stochastic block model – graph view



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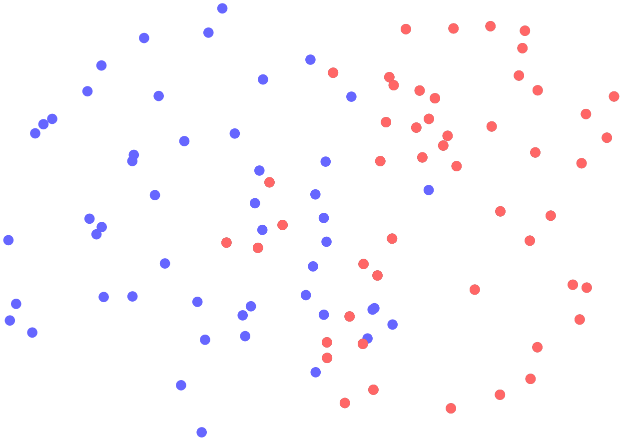

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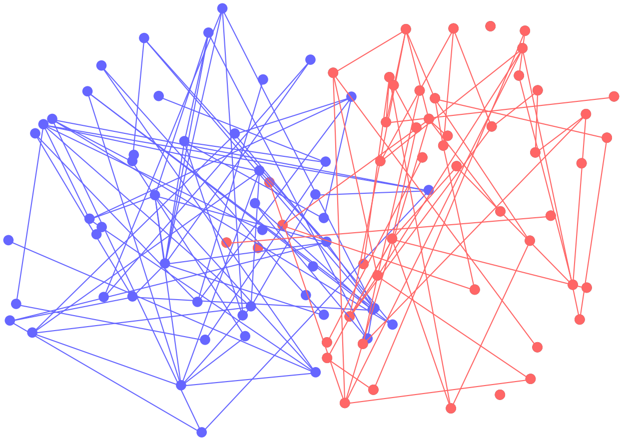

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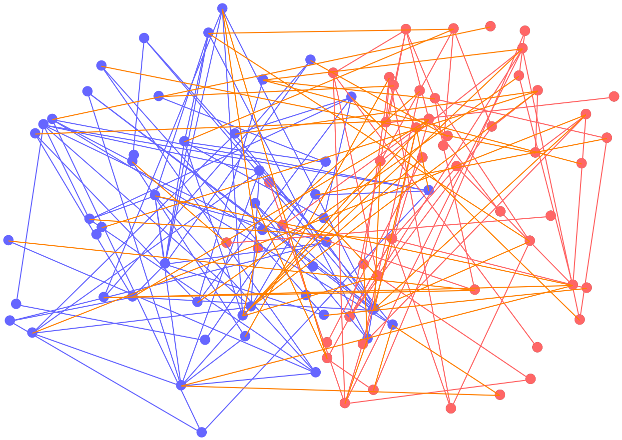

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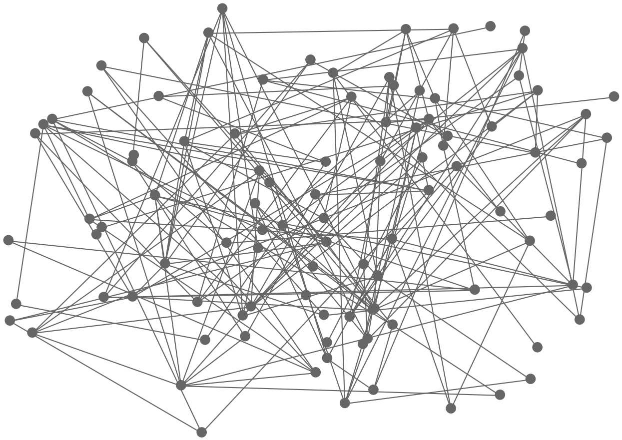

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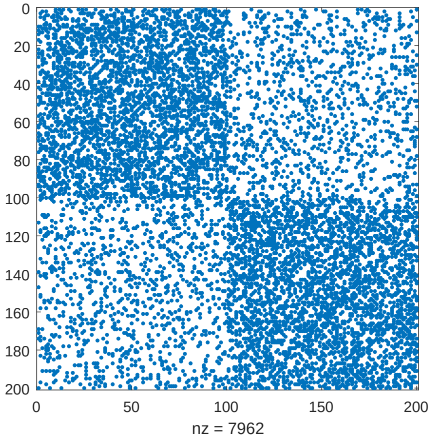

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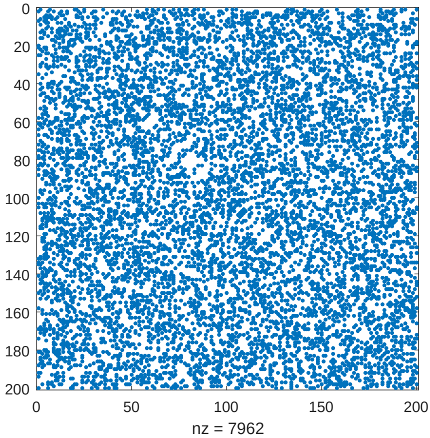
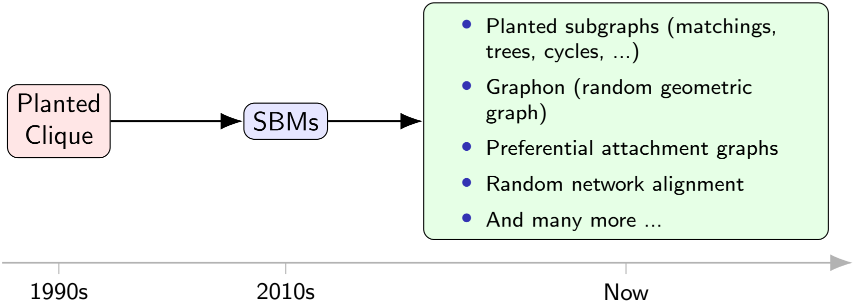


A flurry of network inference problems



Driven by both theoretical interests and practical applications

Significant methodological advances

Applied Probability

- Local weak convergence
- Random matrix & spectral methods

Optimization

- Relaxations (LP, SDP)
- Dual certificates & polyhedral combinatorics

Statistical Physics

- Belief propagation & message passing
- Interpolation method

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This tutorial: Low-degree polynomial method
Analog of drift method in stochastic networks

Outline of tutorial

- Introduction to low-degree polynomial method
- Three prototypical examples
  - Planted clique
  - Stochastic block model
  - Random network alignment
- Concluding remarks

Polynomials on graph

- Given a graph G represented by adjacency vector $A = (A_{ij})_{1 \le i < j \le n}$
- A multivariate polynomial $f : \{0,1\}^{\binom{n}{2}} \to \mathbb{R}$

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- # of closed walks: $\mathrm{Tr}(A^D) = \sum_{i_1, i_2, \ldots, i_D} A_{i_1 i_2} A_{i_2 i_3} \cdots A_{i_D i_1}$

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- Local algorithms: f depends on local neighborhood

Polynomial basis [Janson '90, '94]

- Consider the space $\mathcal{F}$ of real-valued functions on $\{0,1\}^{\binom{n}{2}}$ endowed with inner-product

$$\langle f, g \rangle \triangleq \mathbb{E}_{A_{ij} \overset{\text{iid}}{\sim} \mathsf{Bern}(q)}[f(A)g(A)]$$

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$$\Phi_S = \prod_{(i,j)\in S} \bar{A}_{ij}, \quad \bar{A}_{ij} = \frac{A_{ij} - q}{\sqrt{q(1-q)}}$$

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Question

How to design polynomial-based estimator?

Polynomial approximation of likelihood ratio

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- Instead, look for low-degree polynomial maximizing signal-to-noise ratio:

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- By Cauchy-Schwartz, optimum is $\|L_{\le D}\|$ and achieved by projection of L:

$$L_{\le D} = \underbrace{\sum_{S:|S|\le D} \langle L, \Phi_S\rangle \, \Phi_S}_{\text{weighted signed subgraph count}} \,, \text{ where } \Phi_S = \prod_{(i,j)\in S} \frac{A_{ij} - q}{\sqrt{q(1-q)}}$$

Low-degree polynomial prediction

Conjecture (Hopkins '18, informal)

For "sufficiently nice" planted problems,

- *If* $\|L_{\leq D}\| \to \infty$ *for* $D = O(\log n)$*, there exists degree-*D *polynomial succeeds in detecting or estimating the hidden structure*
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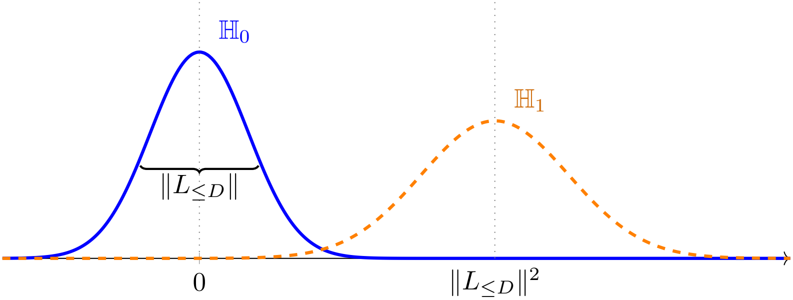

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Remark

- Remarkably, this prediction aligns with many proven algorithmic upper and lower bounds for a wide class of planted problems
- Need $O(\log n)$ degree to cover spectral method, and many $O(\log n)$-polynomials can be computed in poly-time
- Significant progress on proving low-degree polynomial lower bounds [Wein '25]
- Focus of this tutorial: low-degree polynomial as an algorithmic tool

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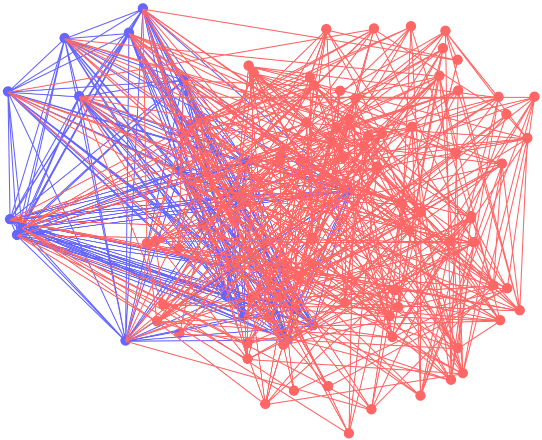

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$$\langle L, \Phi_S \rangle = \mathbb{E}_P[\Phi_S] = \mathbb{E}_C \mathbb{E}_{A|C}[\Phi_S] = \mathbb{E}_C[\mathbf{1}\{V(S) \subset C\}] \approx \left(\frac{k}{n}\right)^{|V(S)|}$$

where C is the vertex set of hidden clique and $V(S)$ is the vertex set of S

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- So, we get

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- If restricting S to be D-cycles with $D \to \infty$,

$$\|L_{D-\text{cycle}}\| \approx n^D \left(\frac{k}{n}\right)^{2D} \gg 1,$$

if $k^2 > n$ (limit of spectral method [Alon-Krivelevich-Sudakov '98])

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- If restricting S to be D-trees with $D \to \infty$,

$$\|L_{D-\text{tree}}\| \approx \binom{n}{D+1}(D+1)^{D-1}\left(\frac{k}{n}\right)^{2(D+1)},$$

  if $k^2 > n/e$ (limit of message passing [Deshpande-Montanari '15])

Planted clique problem: testing

$$\|L_{\le D}\|^2 = \sum_{S:|S|\le D} \langle L, \Phi_S \rangle^2 \approx \sum_{S:|S|\le D} \left(\frac{k}{n}\right)^{2|V(S)|}$$

- If $D = 1$, $\|L_{\le D}\|^2 \approx n^2 \left(\frac{k}{n}\right)^4 \gg 1$, if $k^2 \gg n$ $\Rightarrow$ counting edges succeeds
- If restricting S to be D-cycles with $D \to \infty$,

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- A complete proof of success also needs to bound the variance under $\mathbb{H}_1$

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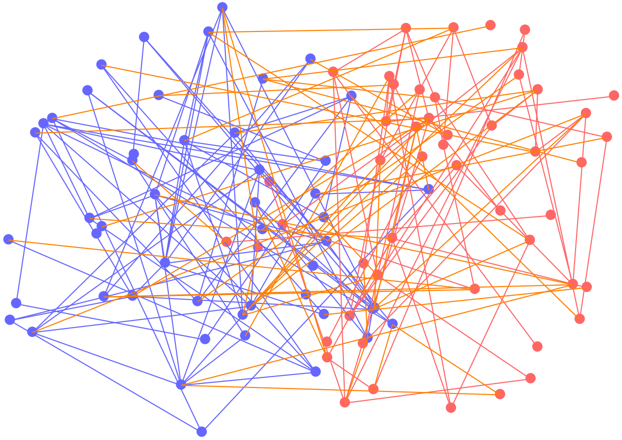
  - By Chebyshev's inequality, succeeds whp when $k^2 > n/e$ by choosing $D = \Theta(\log n)$

Community detection: Stochastic block model

1. n nodes are assigned to 2 communities uniformly at random
2. For every pair of nodes in same community, add an edge w.p. $\frac{a}{n}$
3. For every pair of nodes in diff. community, add an edge w.p. $\frac{b}{n}$



Stochastic block model: testing

$$\mathbb{H}_0 : A \sim \mathcal{G}\left(n, (a+b)/(2n)\right) \qquad \text{(Null model)}$$

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  - Mean separation:

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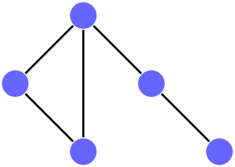
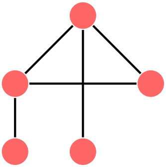
  - Attain the sharp estimation threshold $(a-b)^2 > 2(a+b)$, by choosing $D = \Theta(\log n)$ [Hopkins-Steurer '17]

Network alignment: Correlated Erdős-Rényi graphs



$A \sim \mathcal{G}(n, q)$



$B \sim \mathcal{G}(n, q)$

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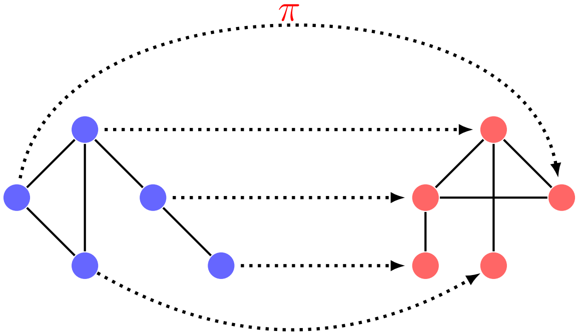

$A \sim \mathcal{G}(n, q)$ $B \sim \mathcal{G}(n, q)$

A and B are edge-wise correlated (ρ) under the hidden node correspondence π:

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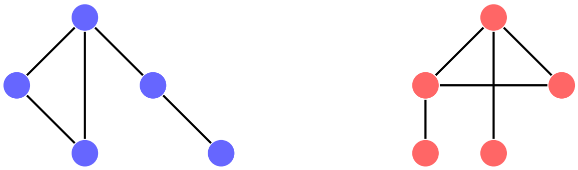

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Goal: observe A and B, recover the hidden node correspondence π

Random network alignment: testing

$\mathbb{H}_0$: A and B are two independent Erdős-Rényi graphs $\mathcal{G}(n, q)$

$\mathbb{H}_1$: A and B are two ρ-correlated Erdős-Rényi graphs $\mathcal{G}(n, q, \rho)$

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when $\rho^2 > \alpha,$ where $\alpha \approx 0.33833$ is Otter's constant [Otter '48]

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  - To control the variance, we restrict to a special family $\mathcal{T}^*$ of unlabeled rooted trees–chandeliers, where $|\mathcal{T}^*| = (1/\alpha - o(1))^D$ [Mao-Wu-X.-Yu '23]

Outline of tutorial

- Introduction to low-degree polynomial method
- Three prototypical examples
  - Planted clique
  - Stochastic block model
  - Random network alignment
- Concluding remarks

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- Tree- or cycle-based polynomials of degree D can be approximated in time $n^2 e^{O(D)}$ via color-coding [Alon-Yuster-Zwick '94]
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- To complete the analysis, we also need to bound the variance of the polynomial under $\mathbb{H}_1$. This can be quite challenging-sometimes special designs (e.g., counting chandeliers) help.
- The low-degree polynomial method extends to many other high-dimensional inference settings. For example, with i.i.d. Gaussian null model, the orthogonal basis is given by Hermite polynomials.

A partial and ever-growing list of successes

- Planted dense subgraph [Sohn-Wein '25]
- Planted dense cycles [Mao-Wein-Zhang '23]
- Dense stochastic block models [Banerjee-Ma '17, Banerjee '18]
- Degree-corrected stochastic block models [Gao-Lafferty '17, Jin-Ke-Luo '19]
- Mixed-membership stochastic block models [Hopkins-Steurer '17]
- Random network alignment: correlated stochastic block models [Chen-Ding-Gong-Li '24,25, Chai-Rácz 24]
- Attributed network alignment [Wang-Wang-Wang'24]
- Testing random geometric graph vs. Erdős-Rényi [Bubeck-Ding-Eldan-Rácz '16]
- Planted submatrix [Sohn-Wein '25]
- Spiked Wigner model [Hopkins-Steurer '17, Sohn-Wein '25]
- Tensor PCA [Hopkins '18, Li '25]
- Shuffled linear regression [Li '25, Gong-Wu-X. '25]
- Procrustes-Wasserstein matching [Niu-Schramm-X. '25]
- And many more...

Challenges and open problems

- The likelihood ratio projection can be hard to compute
  - Example: random geometric graph. Suppose x_i's are i.i.d. on the unit sphere in $\mathbb{R}^d$, and conditional on x_i's, $A_{ij} \overset{\text{iid}}{\sim} \text{Bern}(\kappa(x_i, x_j))$.
  - In this case, $\langle L, \Phi_S \rangle = \mathbb{E}_P[\Phi_S]$ is hard to compute except for simple subgraphs such as cycles. See recent progress [Bangachev-Bresler '25]
  - A key obstacle in resolving the long-standing conjecture on detection threshold for RGG vs Erdős-Rényi graph [Liu-Mohanty-Schramm-Yang '21]

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- Dynamic networks
  - Example: preferential attachment (PA) models. How can we design low-degree polynomial estimators for inference problems in PA graphs—such as community detection or network alignment?

Conclusions

- Network inference provides a rich family of problems that intertwine *applied probability, statistics, optimization, combinatorics, information theory, and more*.
- The low-degree polynomial method offers a simple yet principled framework for understanding the fundamental limits of high-dimensional inference.
- This tutorial has focused on low-degree "upper bounds"— showing how to design effective low-degree, polynomial-based estimators.
- A complementary perspective comes from low-degree "lower bounds", which characterize thresholds below which all low-degree polynomials fail. Under the low-degree conjecture, this further implies all polynomial-time algorithms fail

Further Reading

- A. Wein, *"Computational Complexity of Statistics: New Insights from Low-Degree Polynomials,"* June 2025.
- Y. Wu and J. Xu, *"Statistical Inference on Graphs: Selected Topics,"* https://people.duke.edu/~jx77/stats-graphs.pdf. Lecture notes