Semidefinite Programming Relaxations for Recovering Hidden Communities

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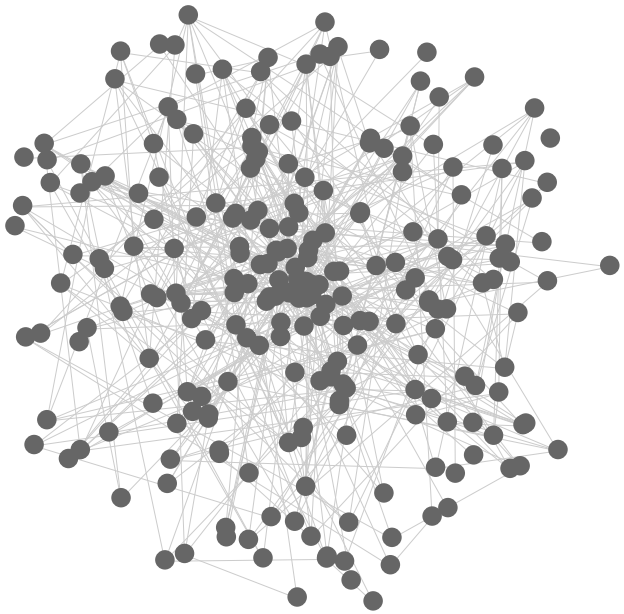
Joint work with Bruce Hajek (Illinois) and Yihong Wu (Yale)

December 17, 2016

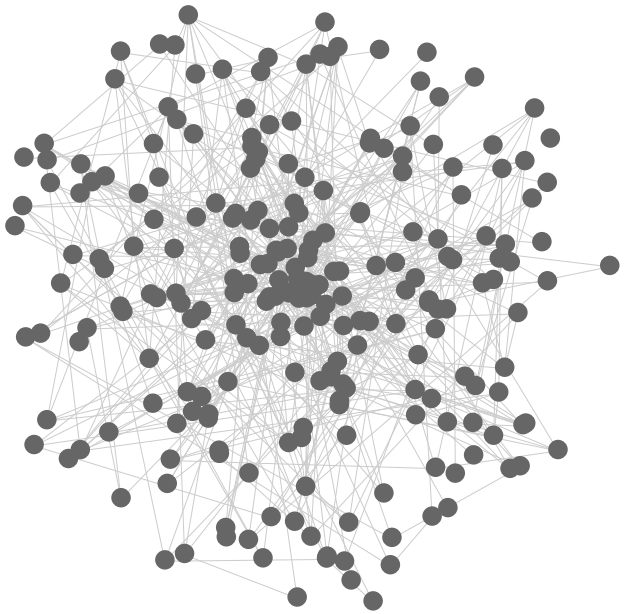
Community detection in networks

- Observe local pairwise interactions between objects, e.g., social networks, biological networks ...



Community detection in networks

- Observe local pairwise interactions between objects, e.g., social networks, biological networks ...
- Interested in global properties of objects, e.g., similarity



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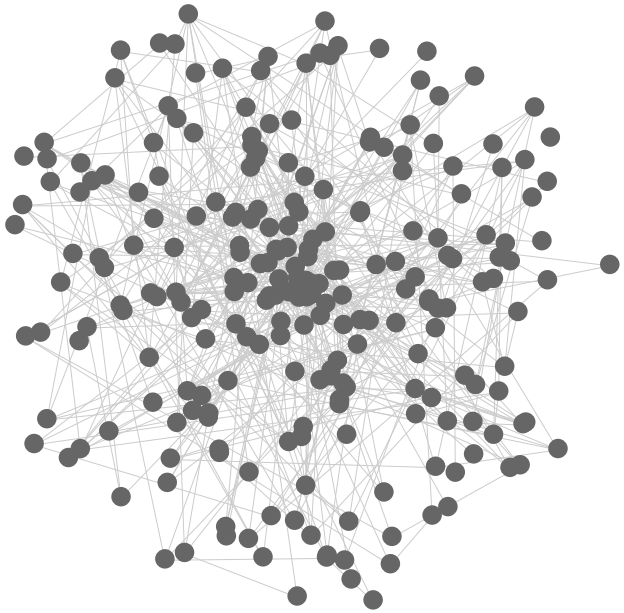
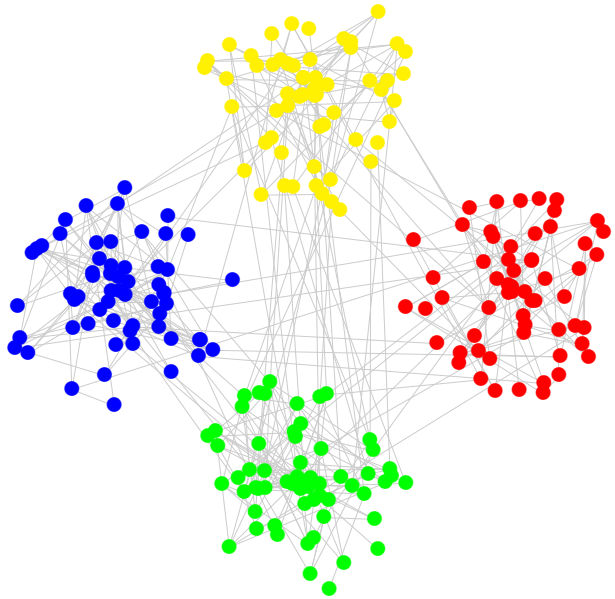
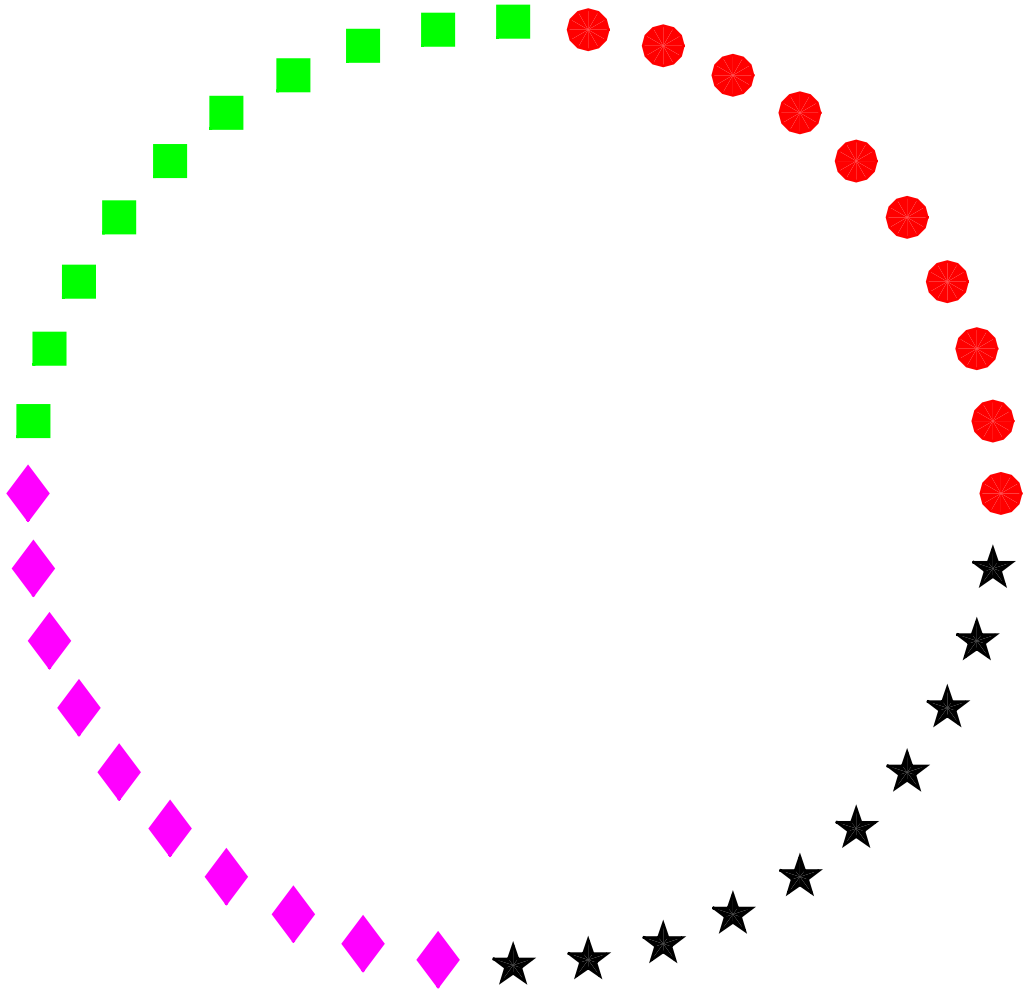




Goal: identify communities of similar objects, related to clustering and graph partitioning.

Stochastic block model [Holland-Laskey-Leinhardt '83]
Planted partition model [Condon-Karp 01']



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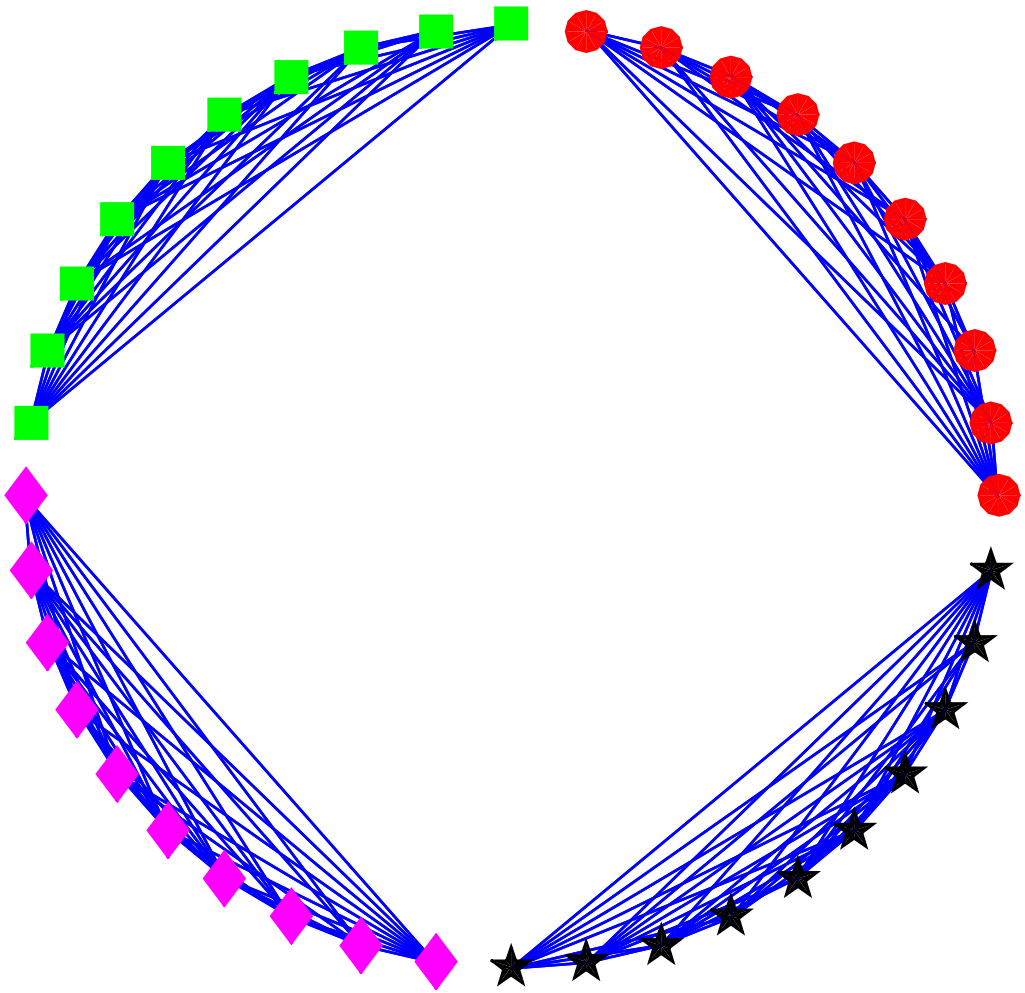

$p = 0.8$

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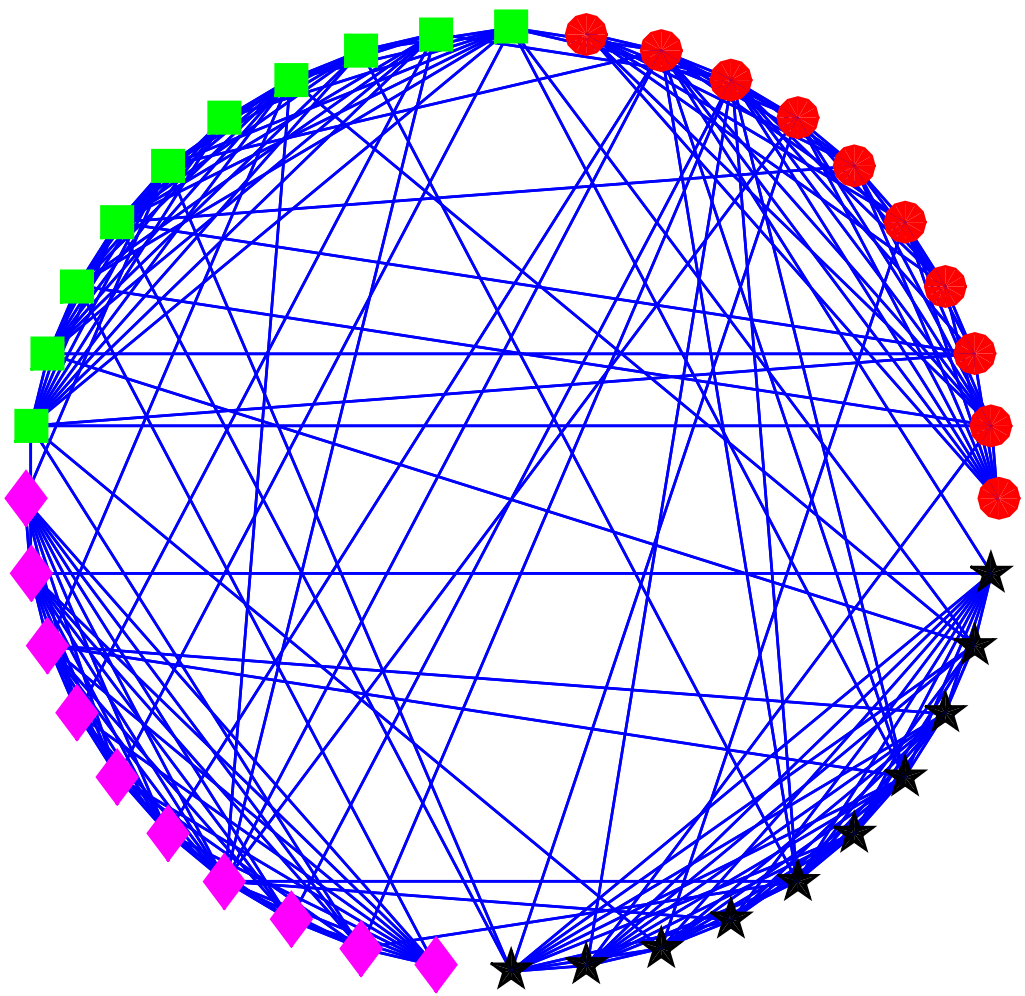

$p = 0.8 \quad q = 0.09$

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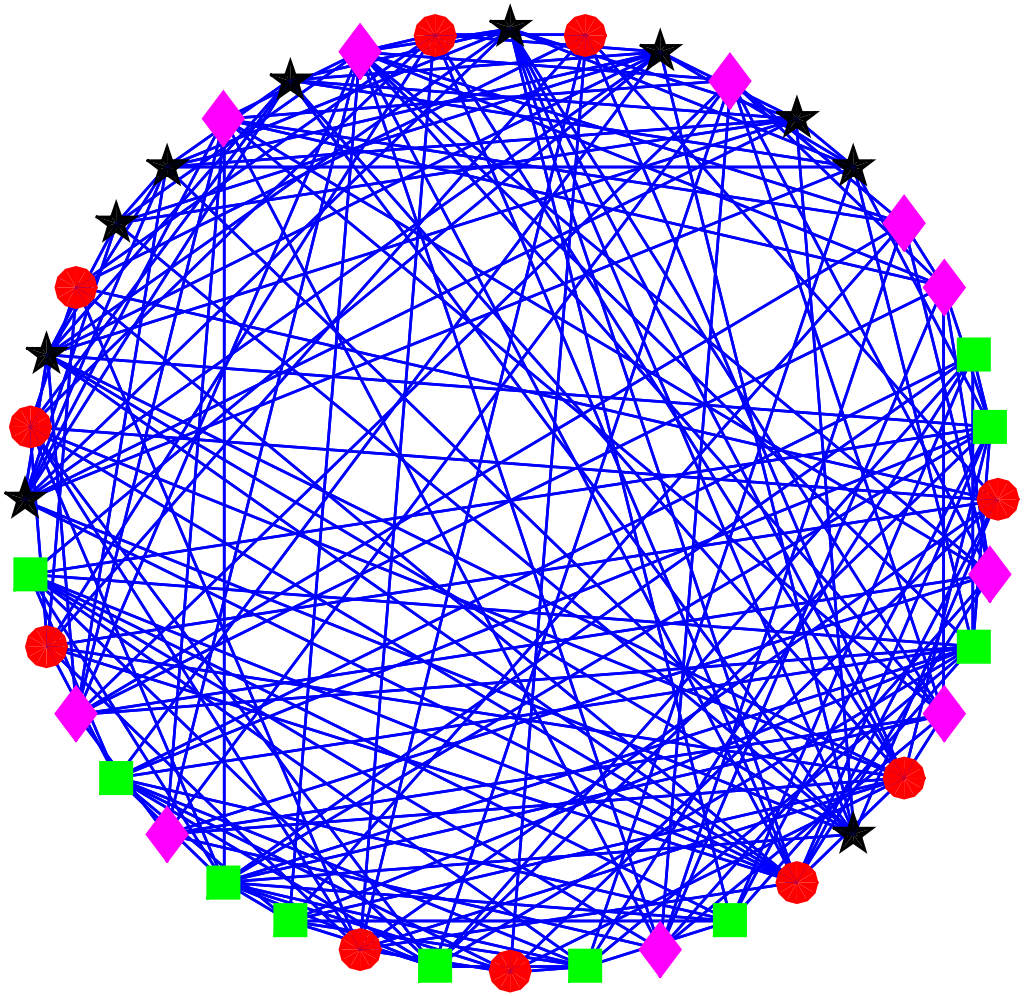

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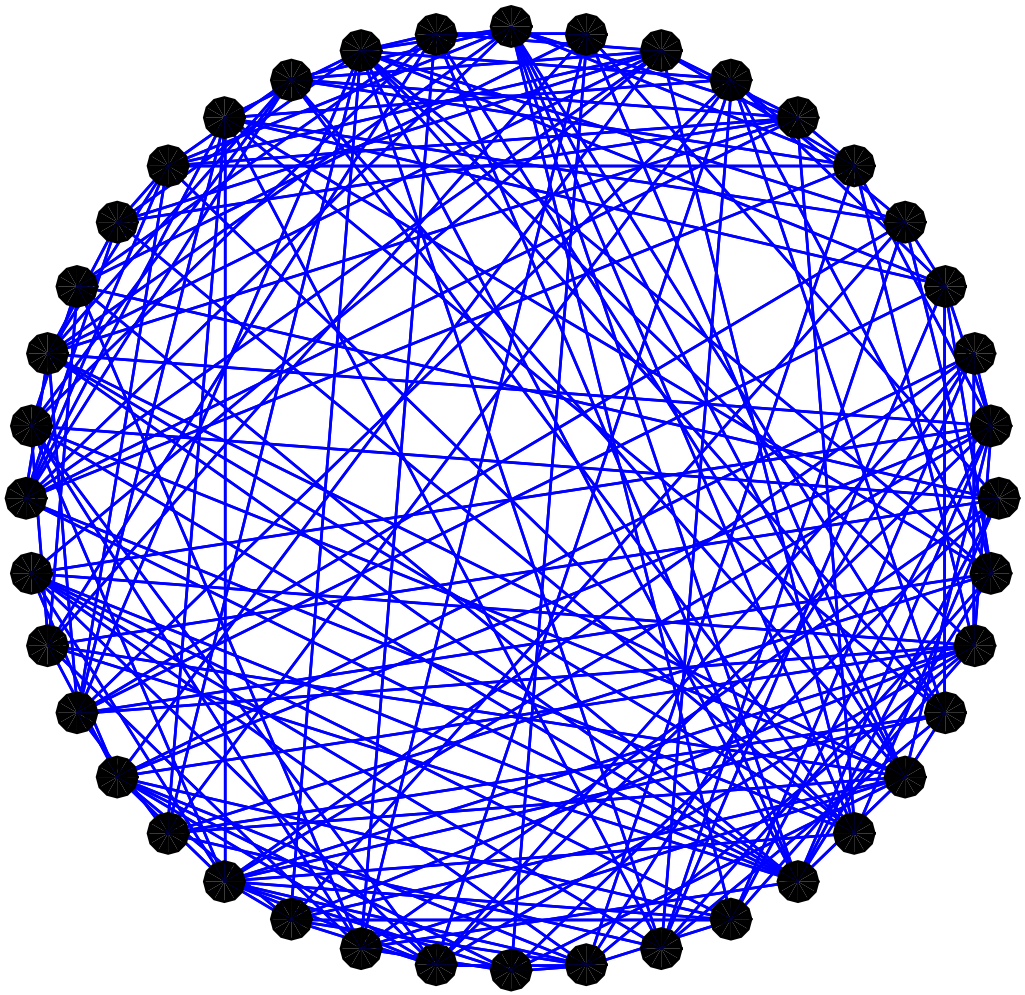

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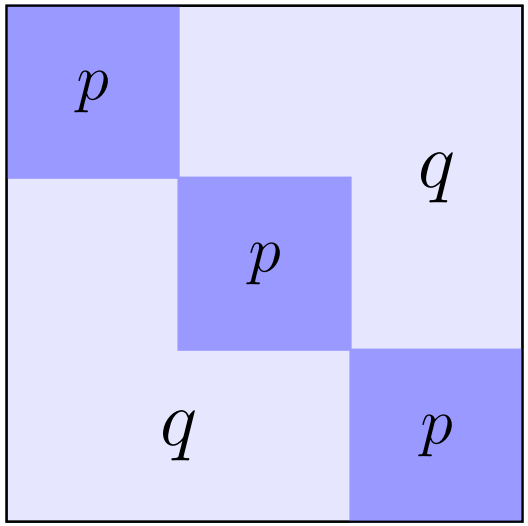
SBM - adjacency matrix view



- n: total number of nodes
- k: number of communities
- p: **within-community** edge prob. q: **across-community** edge prob.

Exact recovery

$$C^* \longrightarrow A \longrightarrow \widehat{C}$$

- Goal: exact recovery

$$\mathbb{P}\{\widehat{C} = C^*\} \xrightarrow{n\to\infty} 1$$

- Alternatives
  - almost exact recovery:
    [Mossel-Neeman-Sly '14, Abbe-Sandon '15, Montanari '15, Zhang-Zhou'15, Yun-Proutiere '15]...
  - correlated recovery:
    [Decelle-Krzakala-Moore-Zdeborova '11, Mossel-Neeman-Sly '12 '13, Massoulie '13]...

Objectives of this talk

Exact recovery:

$$\mathbb{P}\{\widehat{C} = C^*\} \xrightarrow{n\to\infty} 1$$

- Information limit: When is exact recovery possible (impossible)?
- Is the information limit achievable in polynomial time, e.g., via semidefinite programming?

Remainder of the talk

1. Two equal-sized communities
2. Multiple equal-sized communities
3. Conclusions

Two equal-sized communities: Binary symmetric SBM

Model:

- n nodes partitioned into two communities of size $\frac{n}{2}$ $(\sigma_i^* = \pm 1)$.
- $i \sim j$ independently w.p. $\begin{cases} p = \frac{a \log n}{n} & \sigma_i^* = \sigma_j^* \\ q = \frac{b \log n}{n} & \sigma_i^* \neq \sigma_j^* \end{cases}$

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Remarks

- $a + b > 2$ is the connectivity threshold and necessary for exact recovery

Two equal-sized communities: MLE $\Rightarrow$ SDP relaxation

- Maximum likelihood estimator (MLE): Assume $p \geq q$

$$\begin{aligned} \max_{\sigma} \ & \langle A, \sigma\sigma^\top \rangle \quad \rightarrow \# \text{ of in-cluster edges} \\ \text{s.t.} \ & \sigma_i \in \{\pm 1\} \quad i \in [n] \\ & \sigma^\top \mathbf{1} = 0 \end{aligned}$$

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$$\begin{aligned} \max_{Y} \ & \langle A, Y \rangle \\ \text{s.t.} \ & \mathsf{rank}(Y) = 1 \\ & Y_{ii} = 1 \quad i \in [n] \\ & \langle \mathbf{J}, Y \rangle = 0 \end{aligned}$$

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- Goal: $\mathbb{P}\left\{ \widehat{Y}_{\text{SDP}} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \right\} \rightarrow 1$

Two equal-sized communities: Optimal recovery via SDP

Theorem (Abbe-Bandeira-Hall '14, Mossel-Neeman-Sly '14)

For two equal-sized communities with $p = a \log n/n$ and $q = b \log n/n$:

- *If $(\sqrt{a} - \sqrt{b})^2 > 2$, recovery is achievable in polynomial-time.*
- *If $(\sqrt{a} - \sqrt{b})^2 < 2$, recovery is impossible.*

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SDP achieves the optimal recovery threshold $(\sqrt{a} - \sqrt{b})^2 > 2$.

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Remarks

- originally conjectured in [Abbe-Bandeira-Hall '14]
- independently proved by [Bandeira '15]
- $\mathbb{P}\left\{\widehat{Y}_{\mathrm{SDP}} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\right\} = 1 - n^{-\Omega(1)}$

Two equal-sized communities: Dual certificate argument

$$\widehat{Y}_{\mathrm{SDP}} = \arg\max_{Y} \langle A, Y \rangle$$
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$$\widehat{Y}_{\mathrm{SDP}} = \arg\max_{Y} \langle A, Y \rangle \qquad \text{dual variables}$$

$$\text{s.t.} \quad Y \succeq 0 \qquad S \succeq 0$$

$$Y_{ii} = 1 \qquad D = \mathsf{diag}\{d_i\}$$

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- $\min d_i = \Omega_P(\log n)$ if $\sqrt{a} - \sqrt{b} > \sqrt{2}$

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- $\min d_i = \Omega_P(\log n)$ if $\sqrt{a} - \sqrt{b} > \sqrt{2}$
- $\|A - \mathbb{E}[A]\| = O_P(\sqrt{\log n})$: 2nd-order stochastic dominance [Tomozei-Massoulié '14] + result for iid matrix [Seginer '00]

k equal-sized communities: MLE $\Rightarrow$ SDP relaxation

$$\max \sum_{\ell=1}^{k} \langle A, \boldsymbol{\theta}_{\boldsymbol{\ell}} \boldsymbol{\theta}_{\boldsymbol{\ell}}^{\top} \rangle$$

$$\text{s.t.} \quad \boldsymbol{\theta}_{\boldsymbol{\ell}} \in \{0,1\}^n$$

$$\langle \boldsymbol{\theta}_{\boldsymbol{\ell}}, \mathbf{1} \rangle = n/k$$

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$$\xleftrightarrow{\text{lift: } Z = \sum_{\ell=1}^{k} \boldsymbol{\theta}_{\boldsymbol{\ell}} \boldsymbol{\theta}_{\boldsymbol{\ell}}^{\top}}$$

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$$\text{s.t.} \quad \text{rank}(Z) = k$$

$$Z_{ii} = 1 \quad \forall i \in [n]$$

$$Z_{ij} \geq 0, \quad \sum_{j} Z_{ij} = n/k$$

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Goal: $\mathbb{P}\left\{ \widehat{Z}_{\mathrm{SDP}} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \right\} \to 1$

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Theorem (Hajek-Wu-X. '15)

For a fixed k communities with $p = a \log n/n$ and $q = b \log n/n$.

- *If $\sqrt{a} - \sqrt{b} > \sqrt{k}$, exact recovery is attained via SDP in poly-time.*
- *If $\sqrt{a} - \sqrt{b} < \sqrt{k}$, exact recovery is impossible.*

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Remarks

- Extended to $k = o(\log n)$ in [Agarwal-Bandeira-Koiliaris-Kolla '15]
- Extended to the case with multiple unequal-sized clusters [Perry-Wein '15]
- Heterogeneous setting: [Yun-Proutiere '14] and [Abbe-Sandon '15]

When does SDP cease to be optimal?

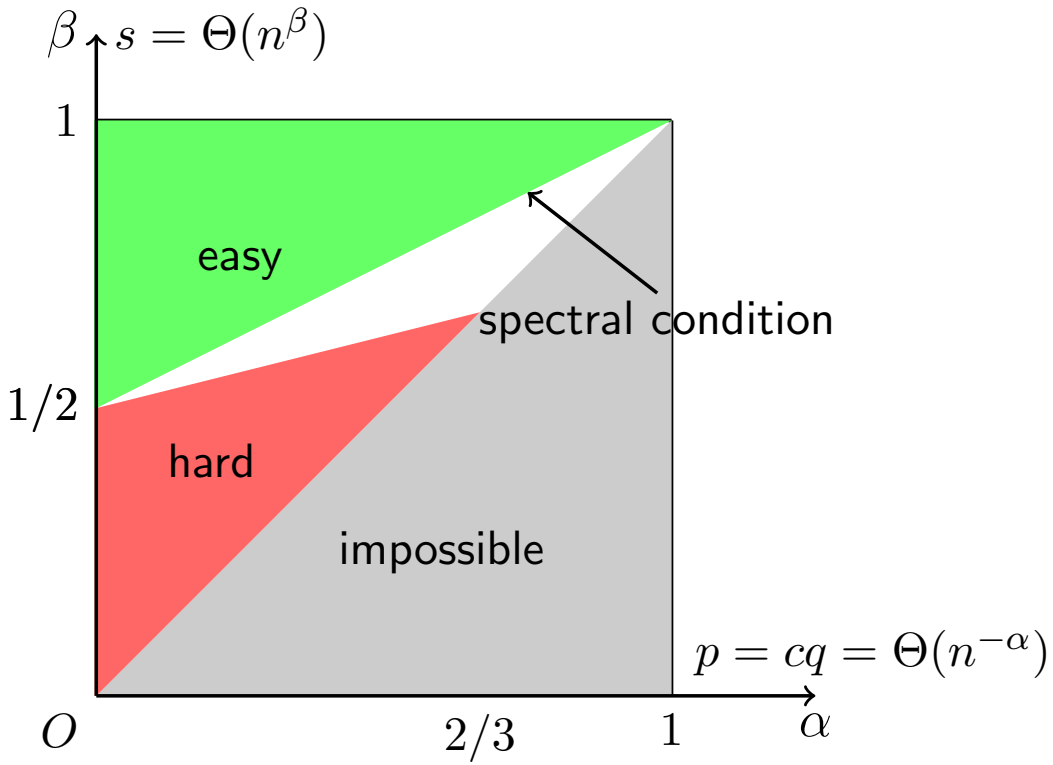
Theorem (Hajek-Wu-X. 'COLT16)

- *If $k \ll \log n$, SDP achieves the optimal exact recovery threshold.*
- *If $k \geq c \log n$, SDP is suboptimal by a constant factor.*
- *If $k \gg \log n$, SDP is order-suboptimal.*

Remarks

- A "hard but informationally possible" regime is conjectured to exist for exact recovery when $k \gg \log n$ [Chen-X. '14]

Concluding remarks



References

- B. Hajek, Y. Wu & J. X. *Achieving exact cluster recovery threshold via semidefinite programming*. (Transactions on IT '16)
- B. Hajek, Y. Wu & J. X. *Achieving exact cluster recovery threshold via semidefinite programming: Extensions*. (Transactions on IT '16)
- B. Hajek, Y. Wu & J. X. *Semidefinite programs for exact recovery of a hidden community*. (COLT'16)

SDP in real networks

- Y. Chen, X. Li, and J. X. (2015), Convexified modularity maximization for degree-corrected stochastic block models. arXiv:1512.08425.
- Code available at http://people.orie.cornell.edu/yudong.chen/cmm