# Optimality of Random Regular Graphs for Sparse Network Designs

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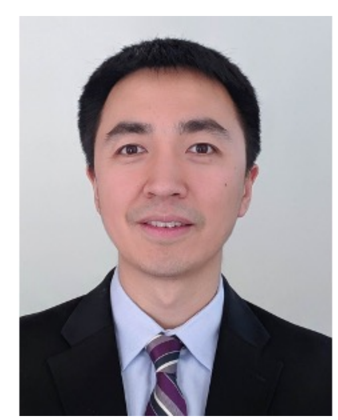
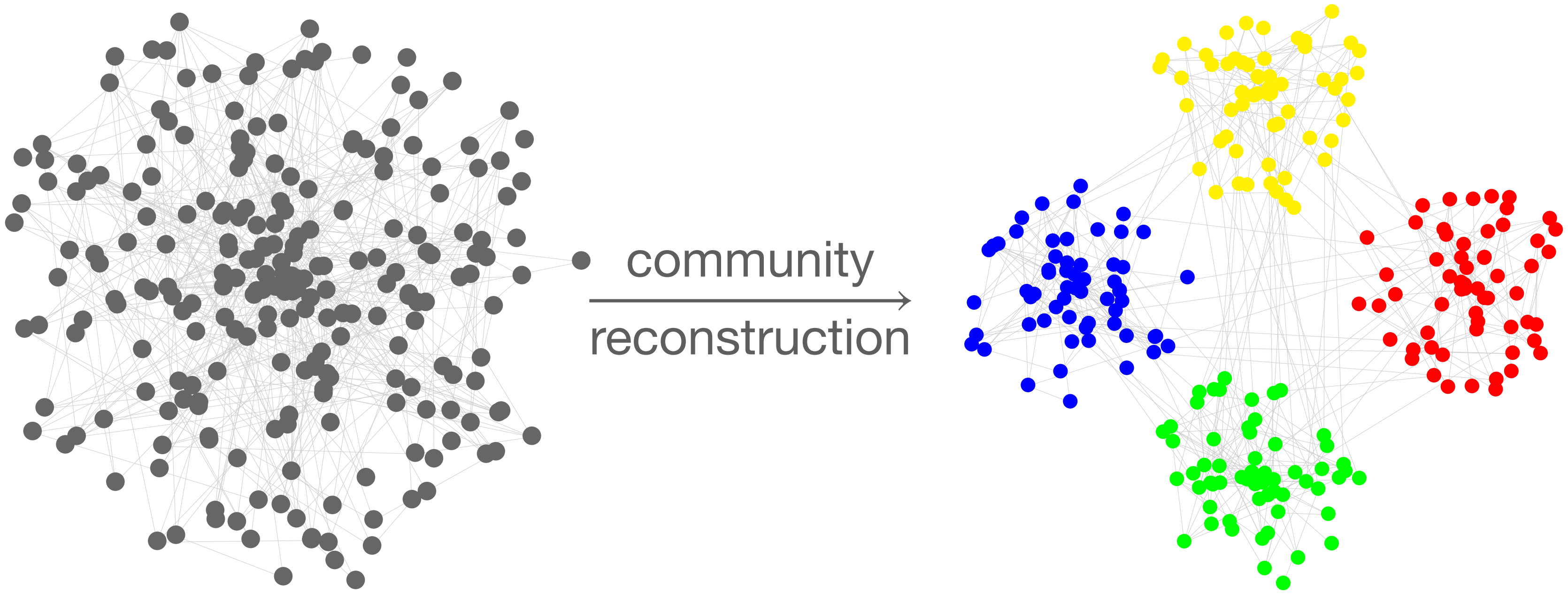
Joint work with
Weijia Li (Tsinghua), Xiaochun (Nora) Niu (Duke), and Yehua Wei (Duke)





# Network Theory

- Networked systems and network data are everywhere
  - Statistical network analysis: Observe network data $\longrightarrow$ Infer hidden structures



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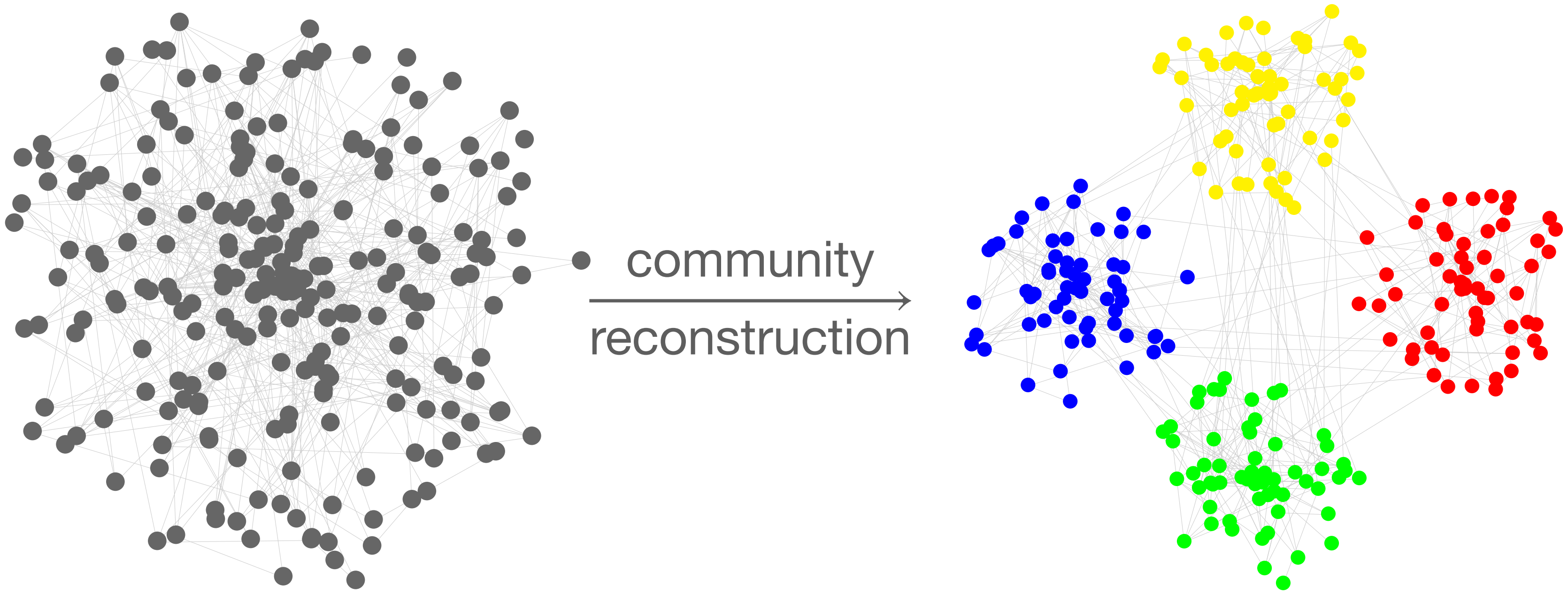
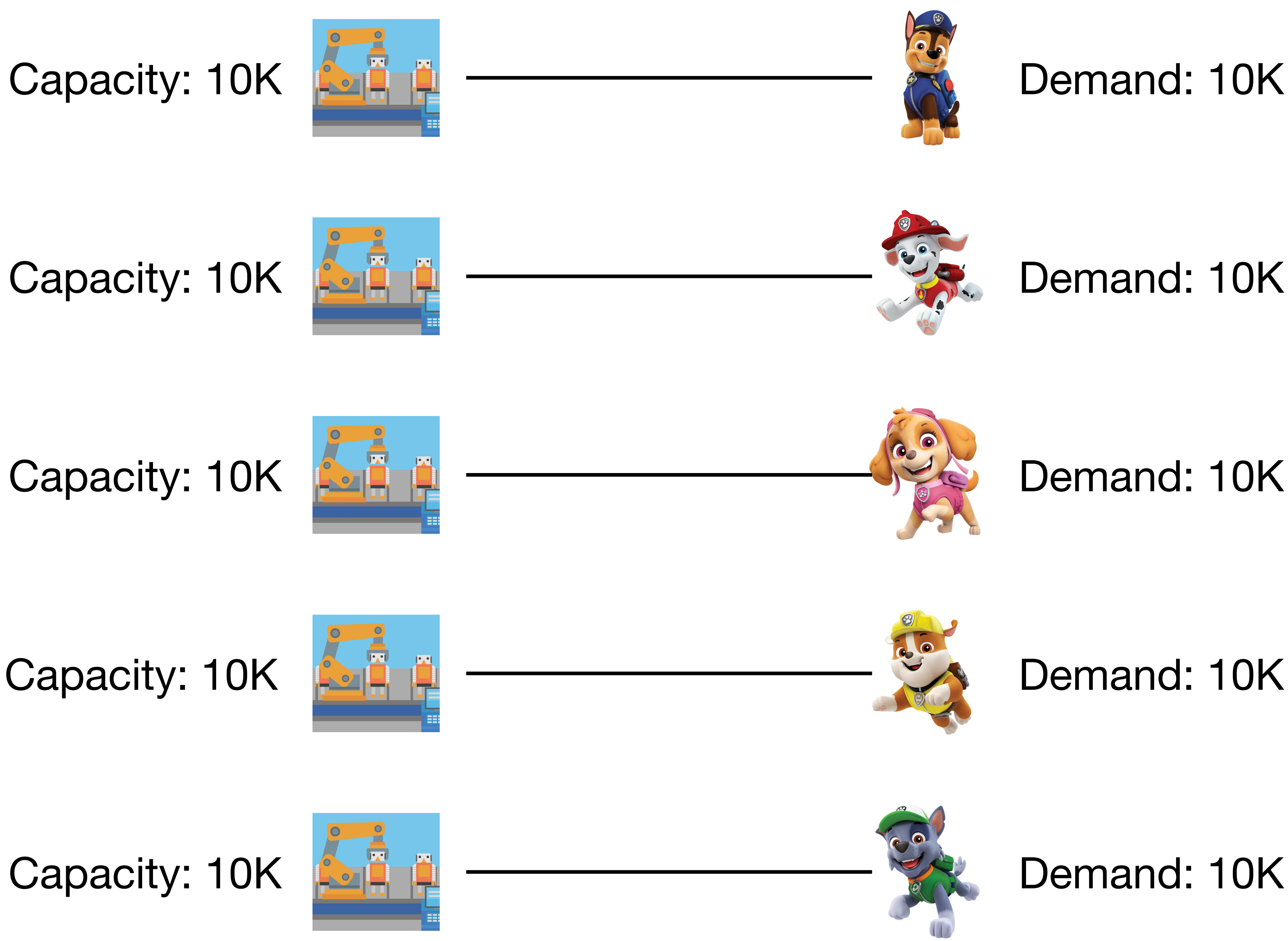



  - Network design: Design network structure $\longrightarrow$ Enable efficient operations (This Talk)

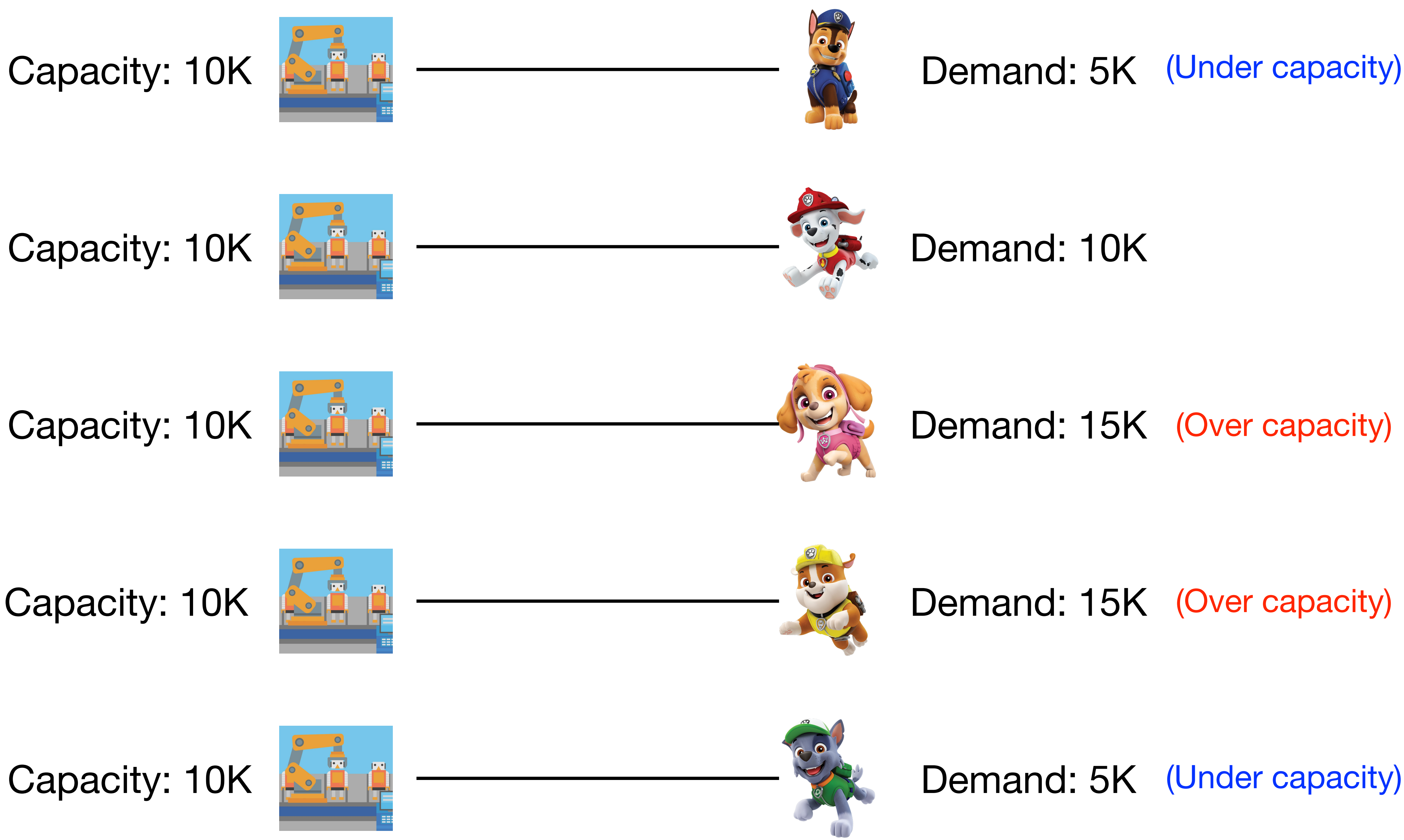
# Flexibility Network Design Under Uncertainty

A Toy Example: Ideal setting



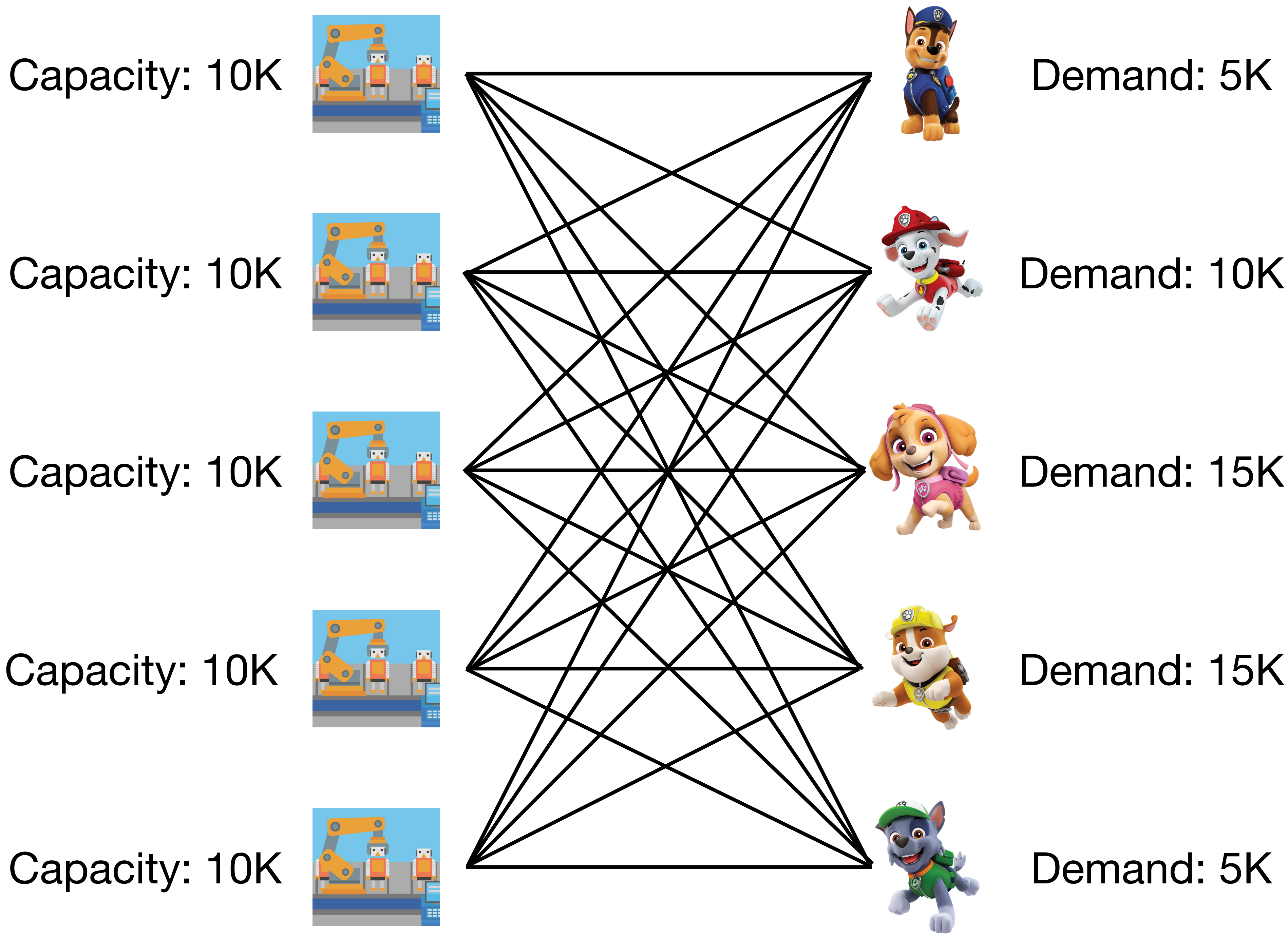
# Flexibility Network Design Under Uncertainty

A Toy Example: Actual demand may change dramatically depending on popularity



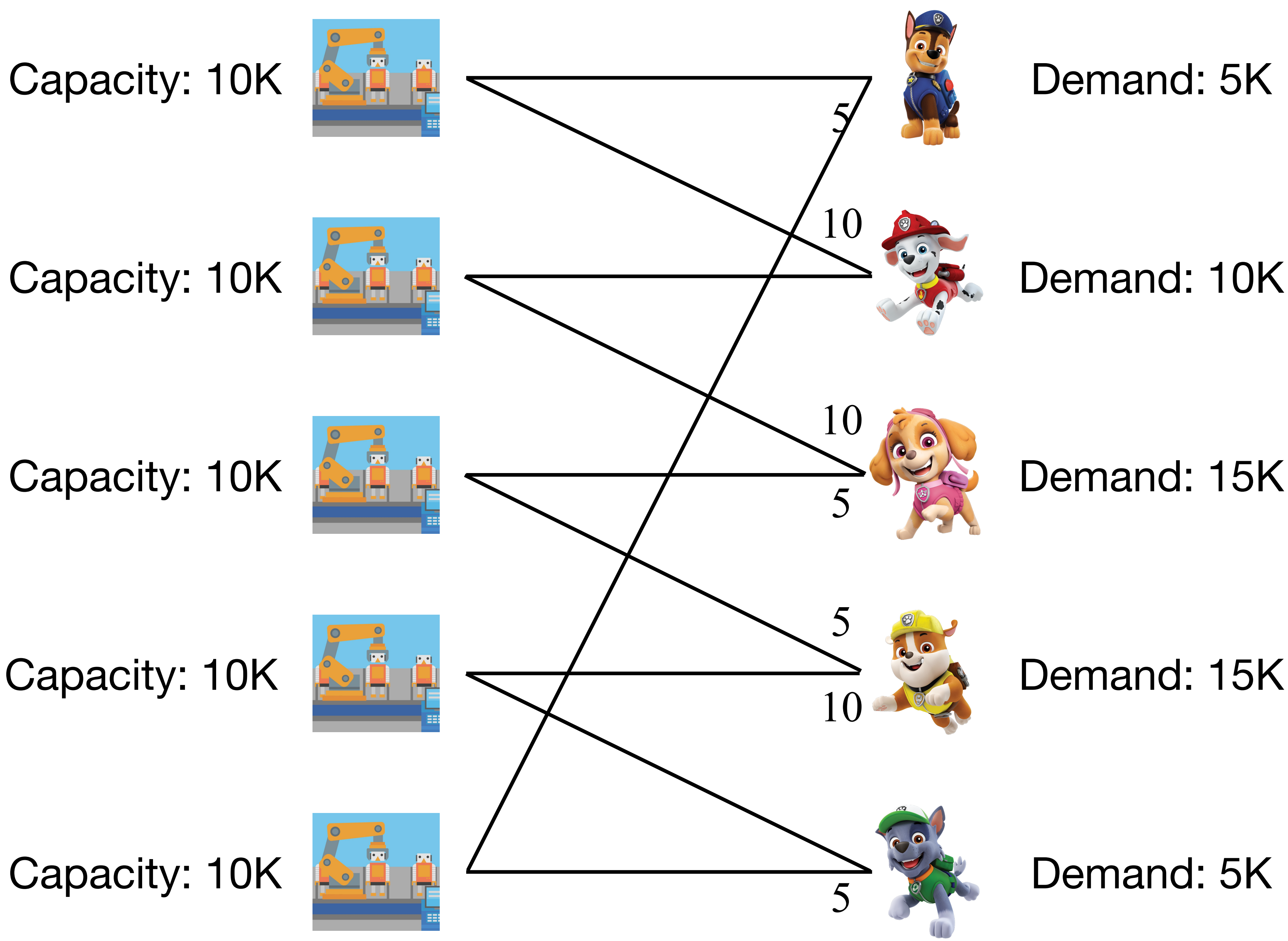
# Flexibility Network Design Under Uncertainty

A Toy Example: Configure all production lines to be fully flexible, but too costly



# Flexibility Network Design Under Uncertainty

A Toy Example: Add a few flexible edges (long-chain design)



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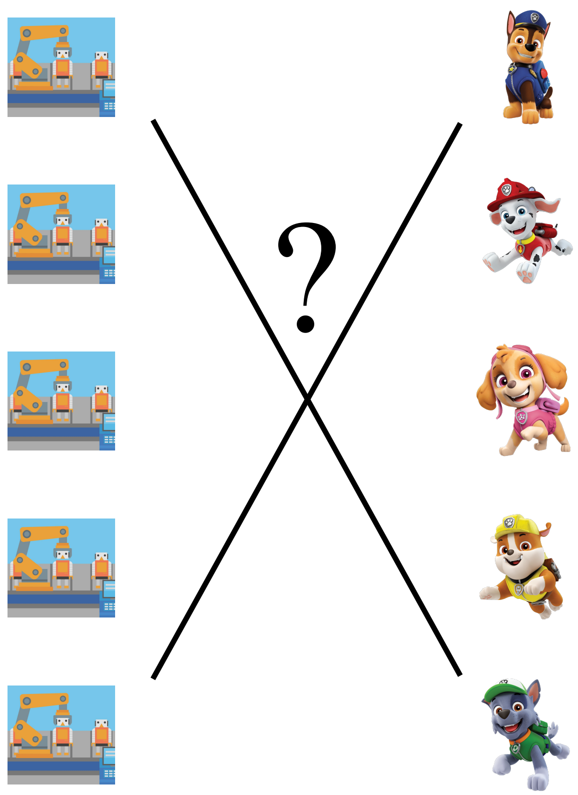


**Question**: How to design a sparse graph that achieves efficiency as close as the complete graph?

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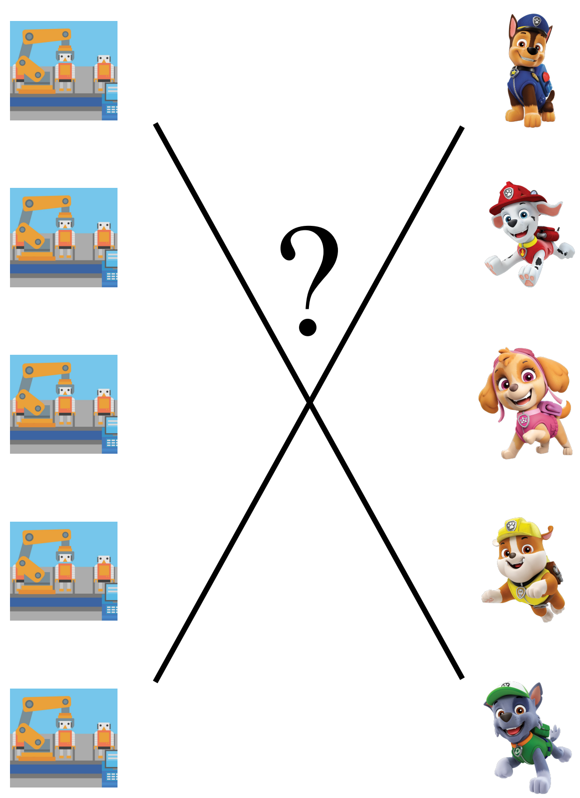
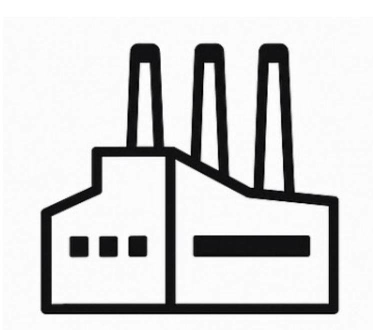
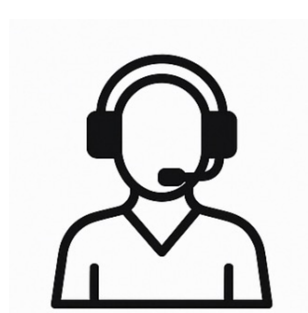



**Question**: How to design a sparse graph that achieves efficiency as close as the complete graph?

Many real-world applications: match capacitated resources with uncertain demand

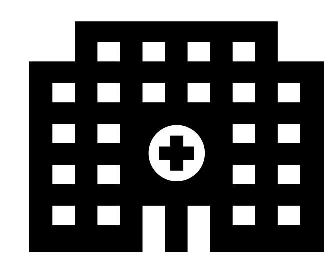


Manufacturing



Service operations

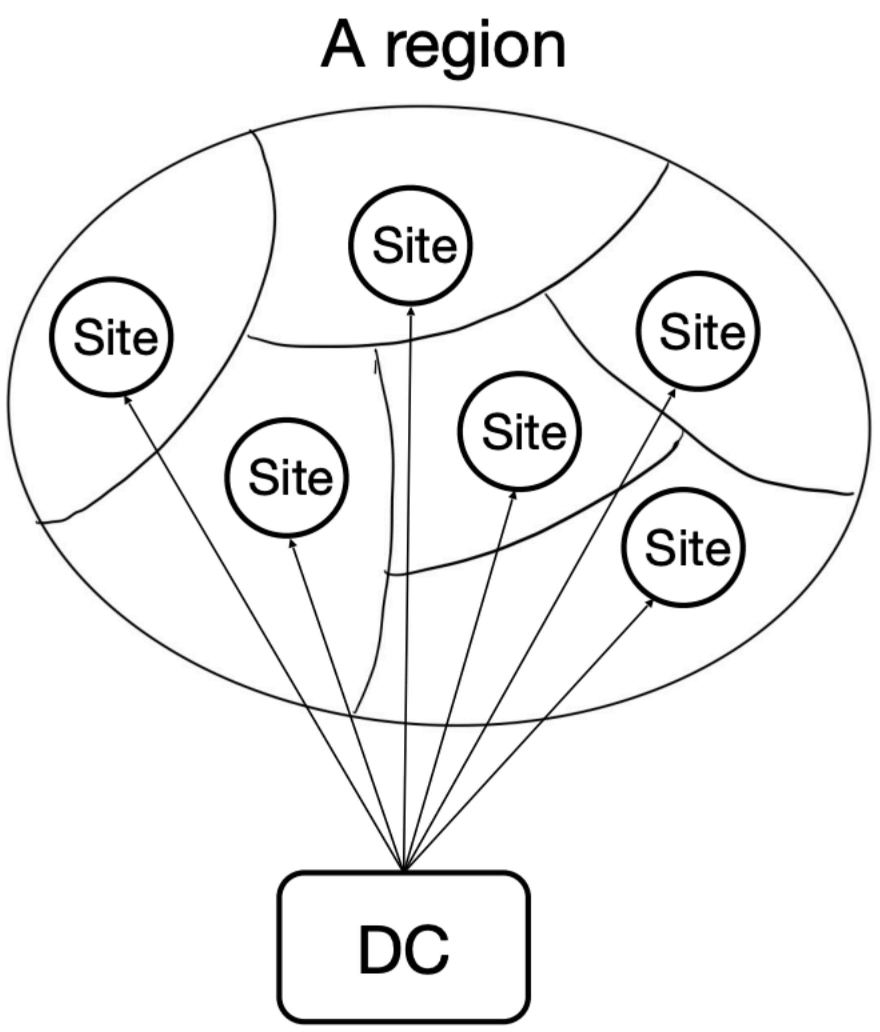
Transportation



Healthcare

# Motivation: Transportation flexibility

- Proposed in [Feng-Caldentey-Xin-Zhong-Wang-Hu ’24]
- Logistics networks are divided into regions
- With each region:
  - 1 distribution center (DC)
  - Multiple delivery stations (Sites)



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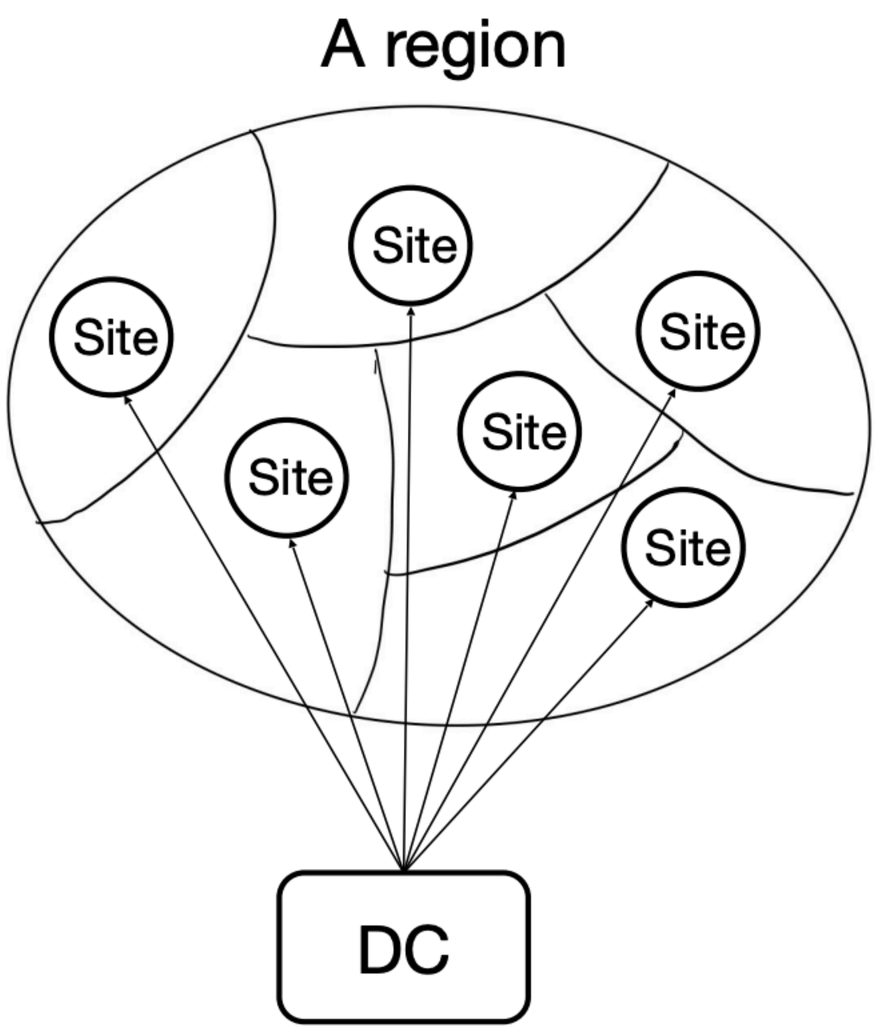

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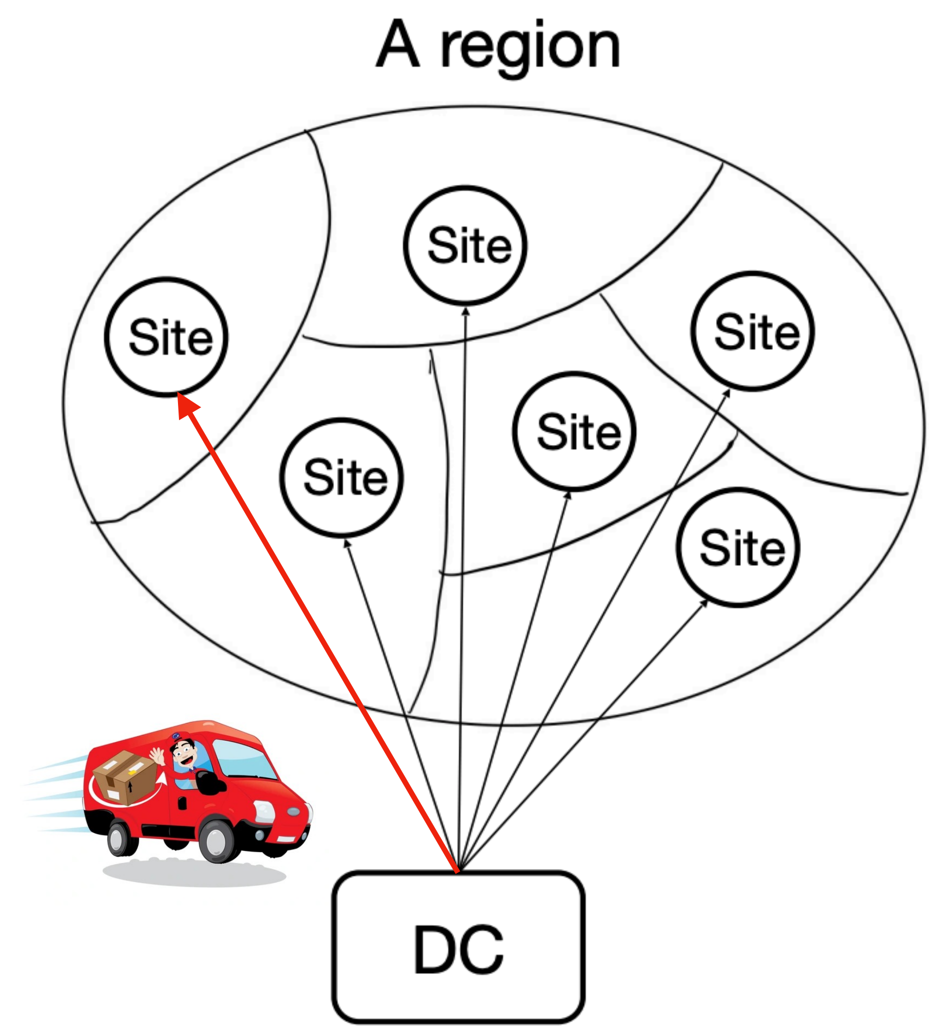
  - Direct shipping (mostly): one truck one site



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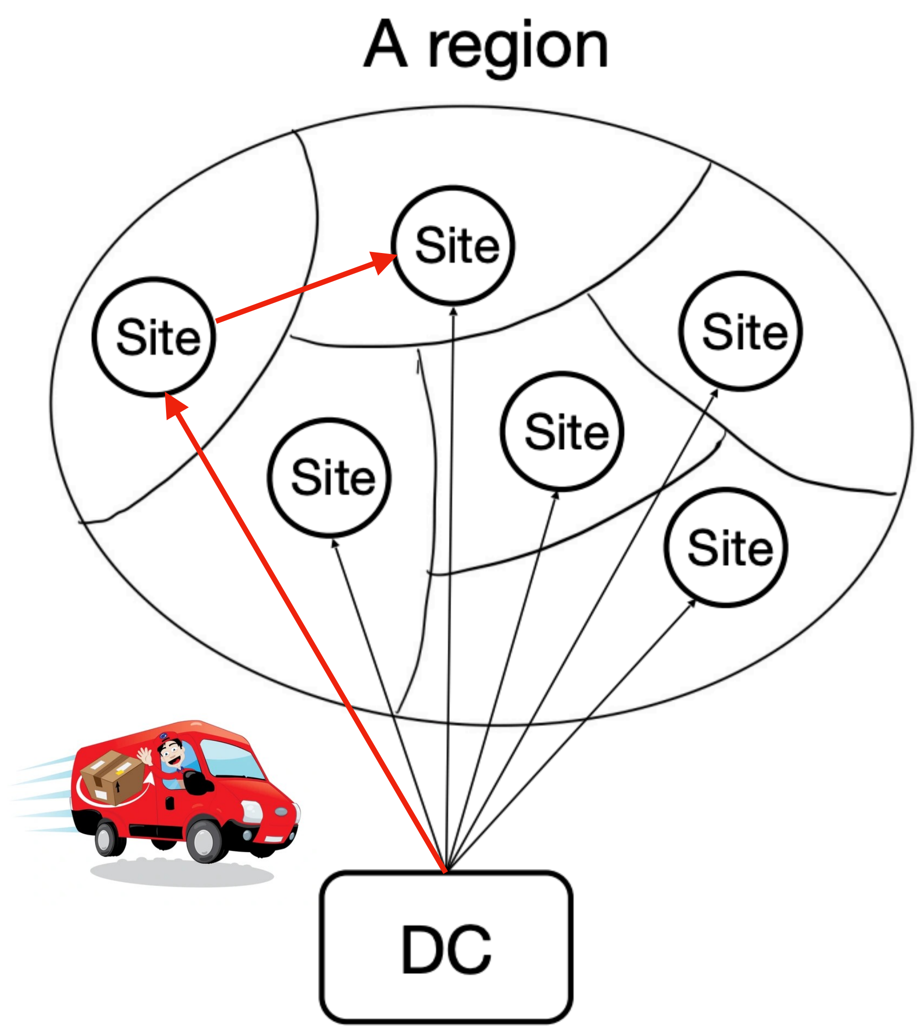

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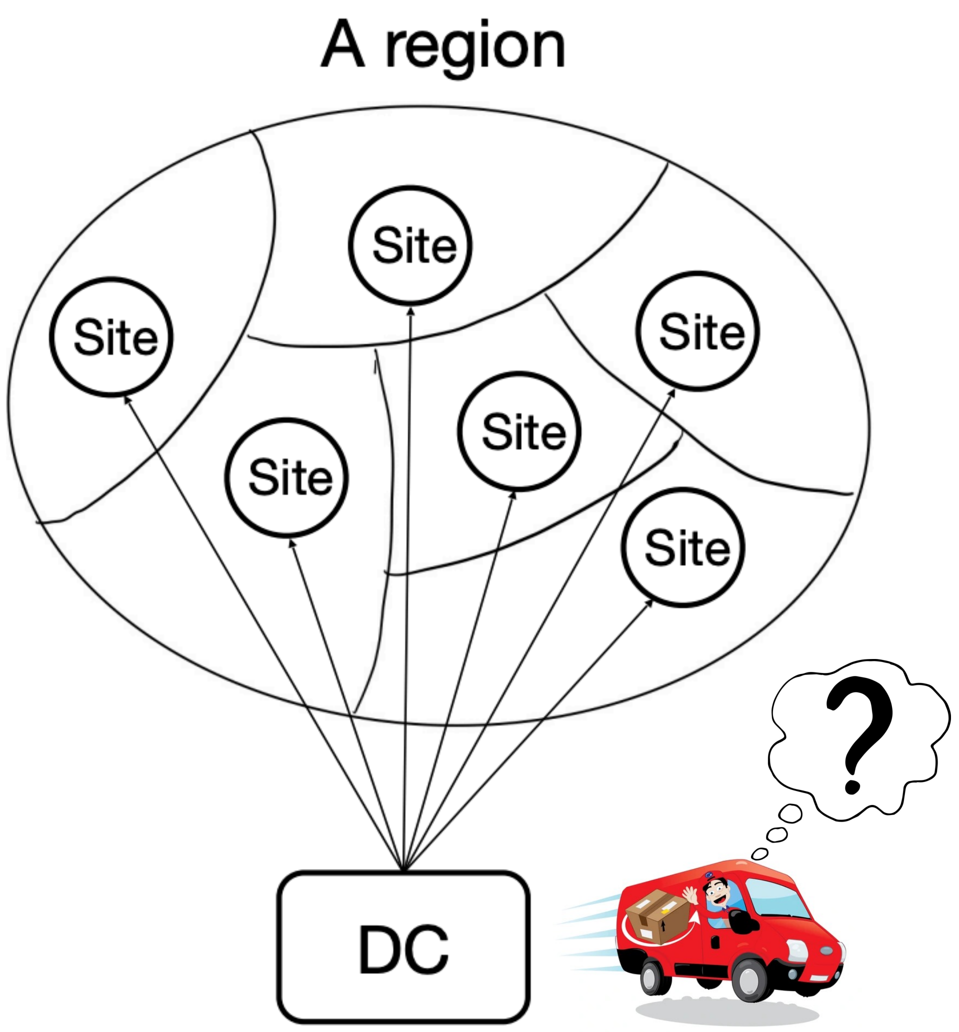
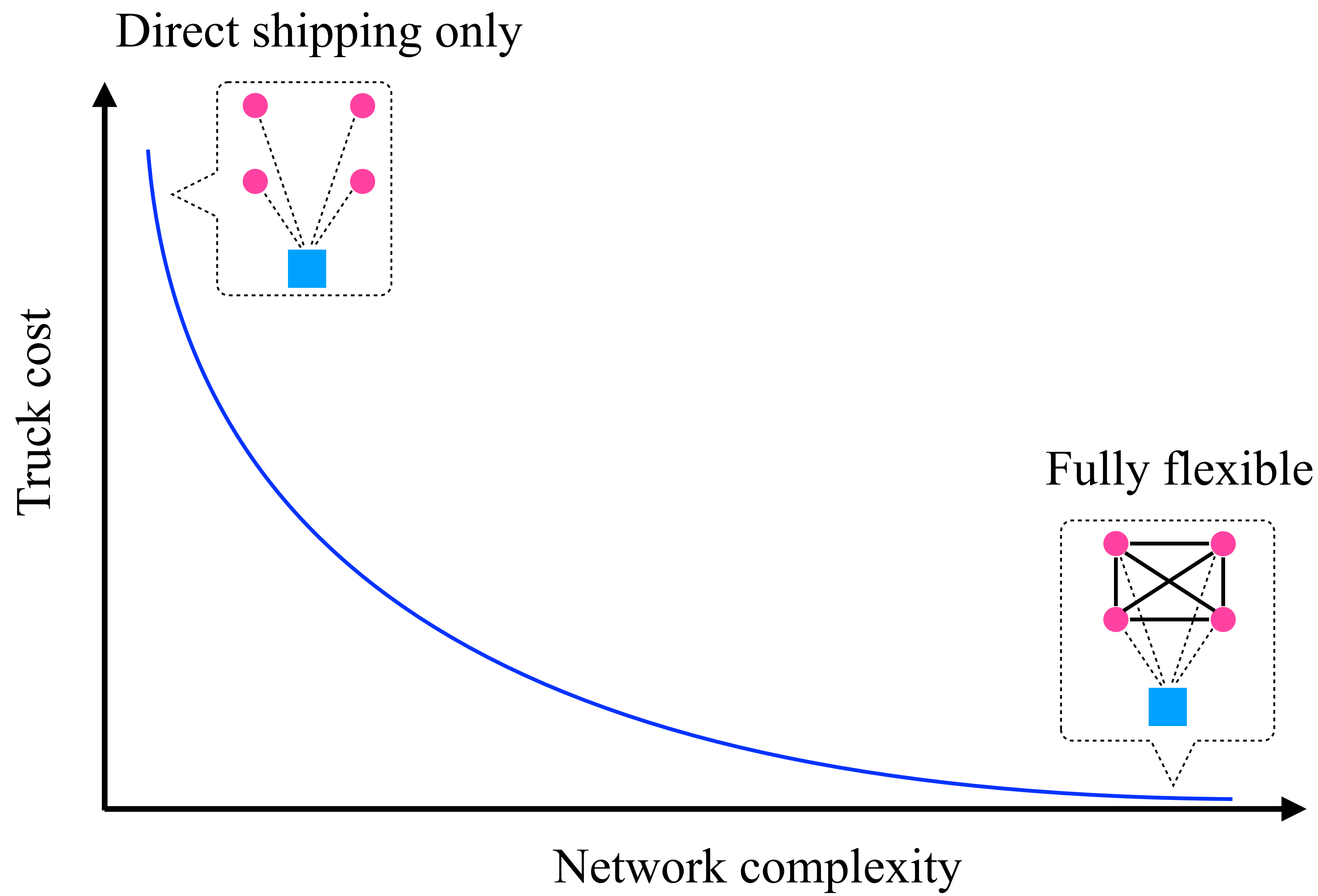
- Question
  - How to design delivery routes for indirect shipping under uncertain demands?



# Tradeoff between truck cost and network complexity



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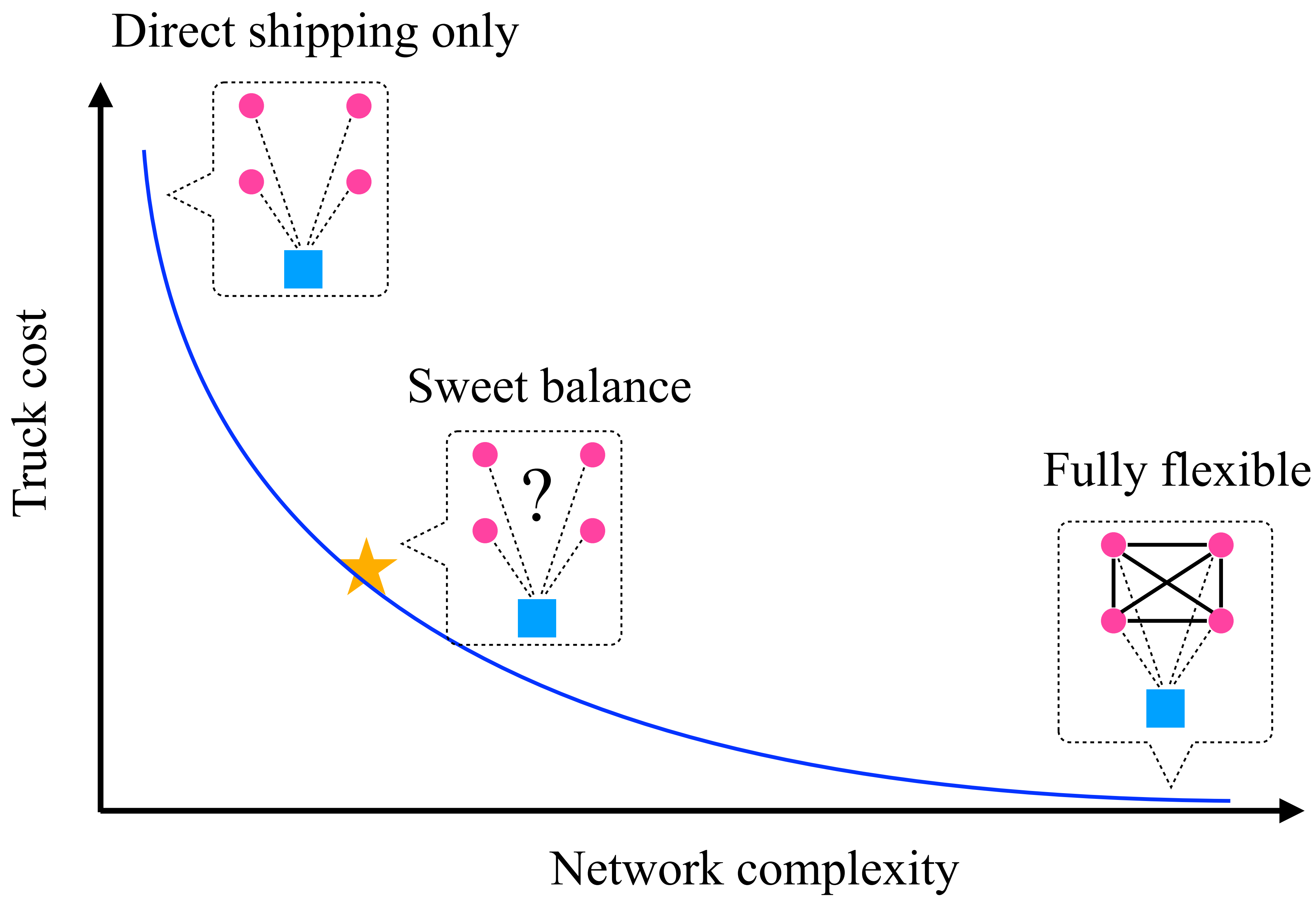


How to design a sparse network that achieves low truck cost?

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- Similar principle has been shown in many other related flexible systems: queueing systems [Tsitsiklis-Xu'17, Rutten-Mukherjee'23], dynamic resource allocation [Dong-Hu-Wang'24], online platforms [Freund-Martin-Zhao'24]

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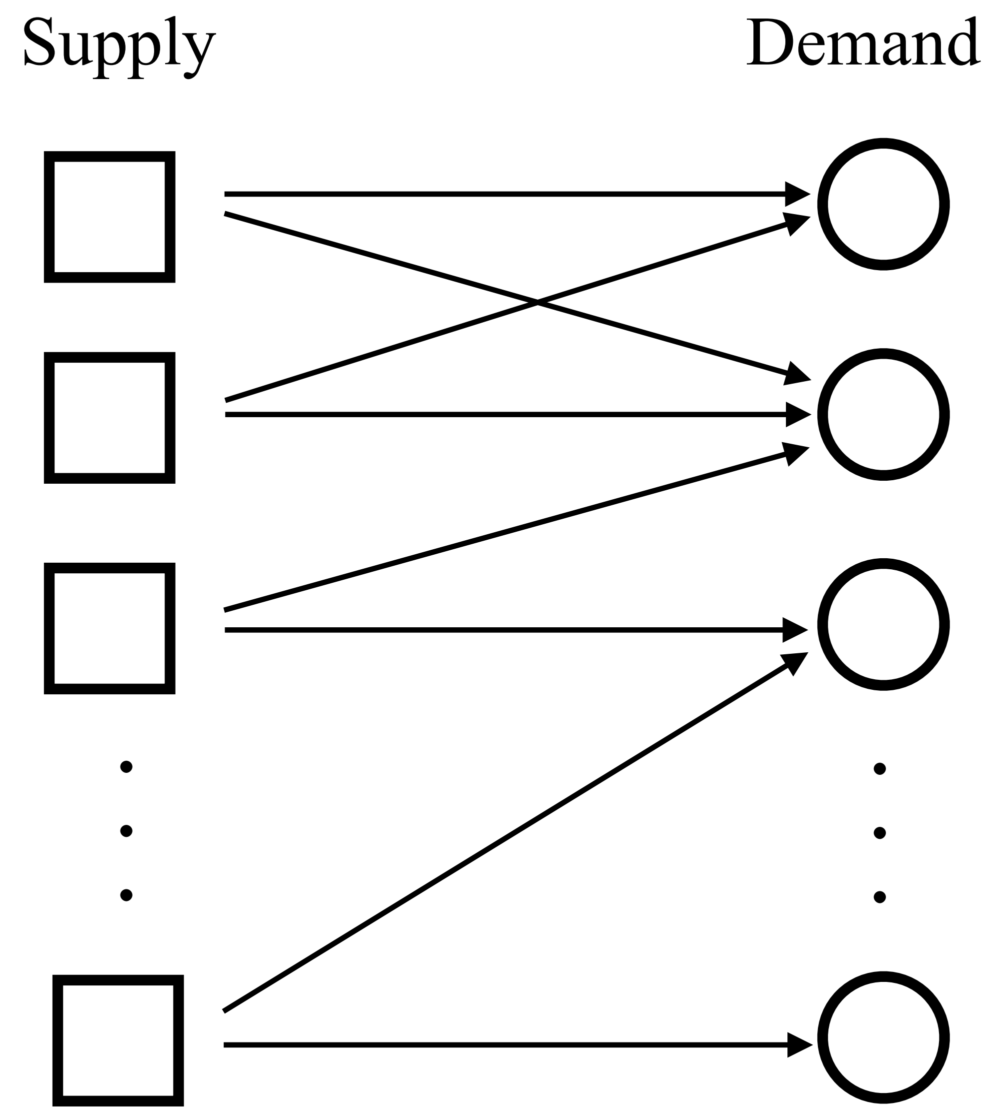
**Open Question**: What is the precise limit of flexibility needed for achieving $(1-\epsilon)$ performance ratio?

# Model

# Process Flexibility Model

- Bipartite graph $G = ([n], [n], E)$



# Process Flexibility Model

- Bipartite graph $G = ([n], [n], E)$
- The total fulfilled demand

$$z(G,D) = \max_{x_{ij} \geq 0} \sum_{(i,j)\in E} x_{ij}$$

$$\text{s.t.} \quad \sum_{j:\,(i,j)\in E} x_{ij} \leq 1 \qquad \forall i$$

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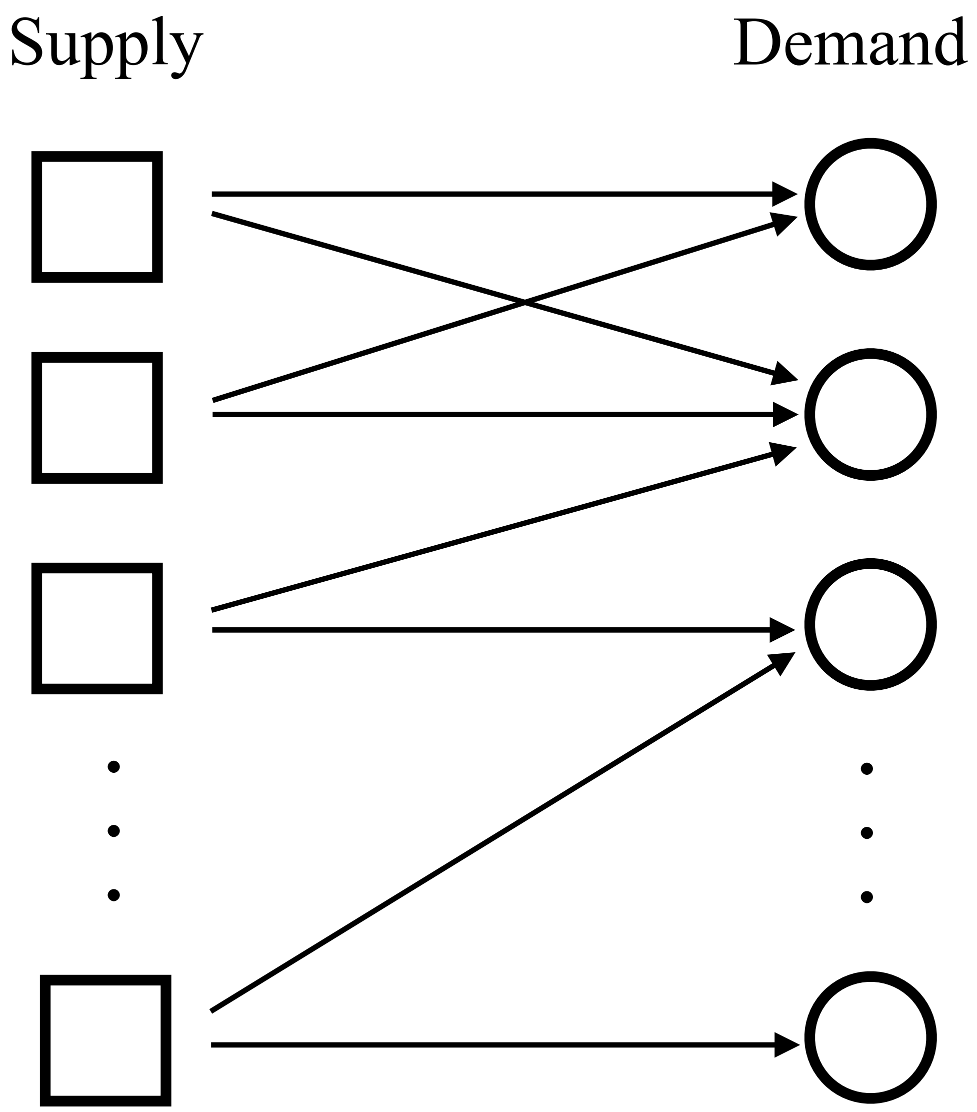

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- The expected loss of $G$:

Total fulfilled demand under complete graph

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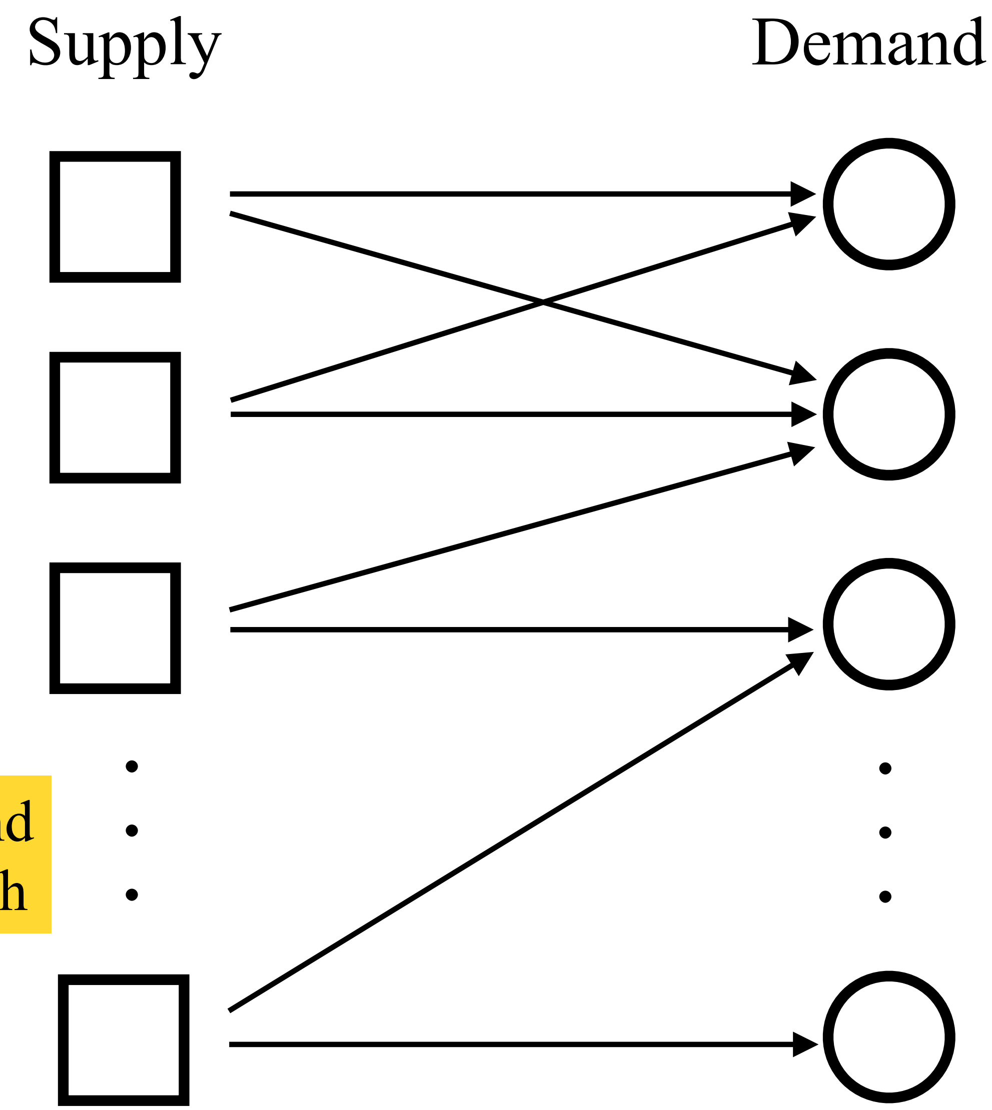

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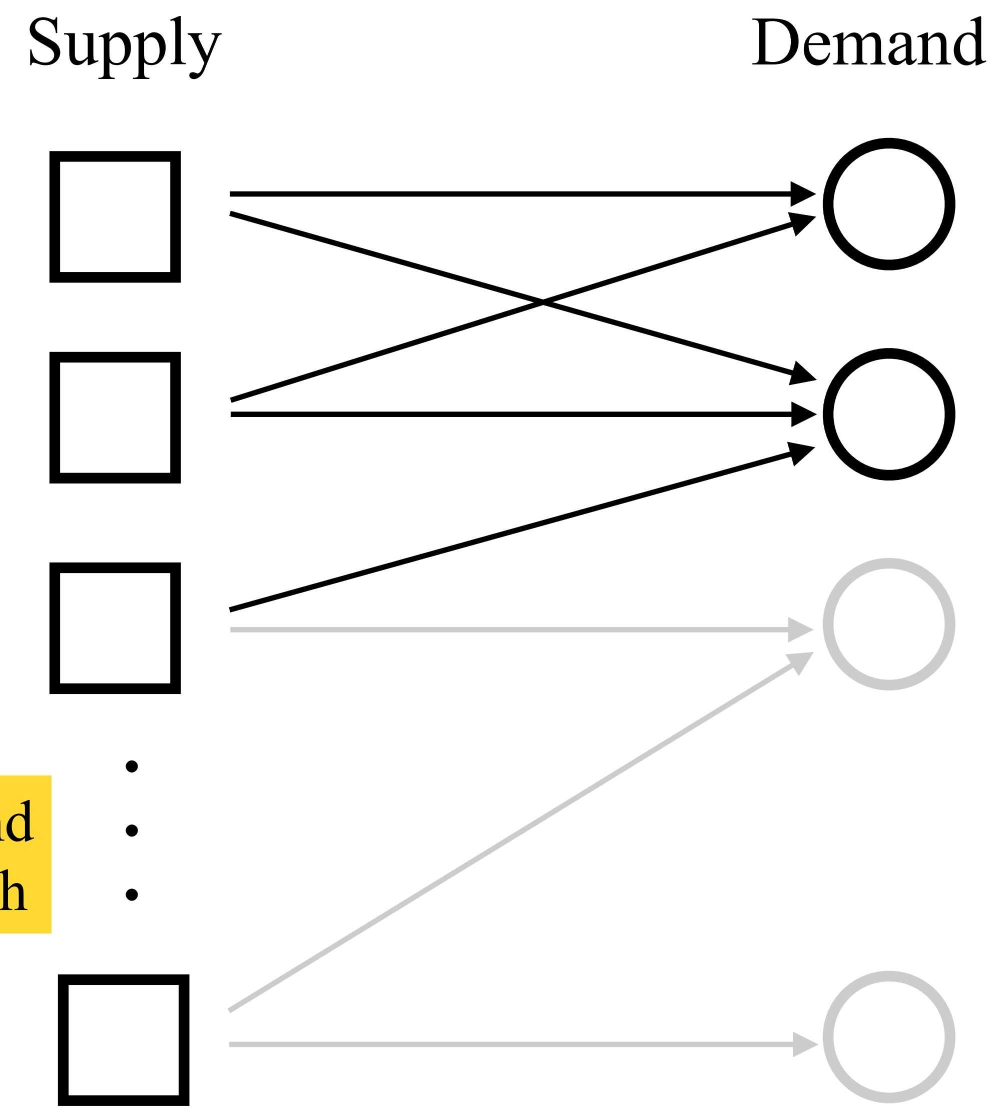
- IID Bernoulli demand with mean 1:

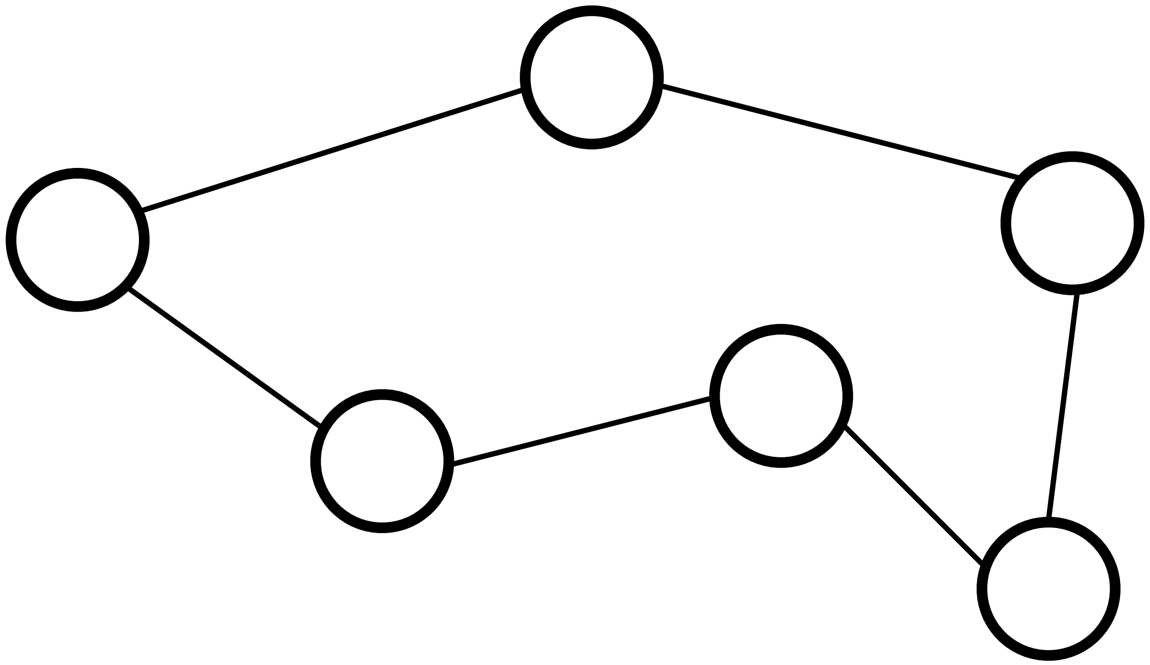
$$D_j = \begin{cases} 0 & \text{w.p. } q \\ 1/p & \text{w.p. } p = 1 - q \end{cases}$$

Random demand node deletion with probability $q$



# Transportation Flexibility Model

- Given a graph design $G = ([n], E)$
  - Vertices = sites
  - Edges = delivery routes



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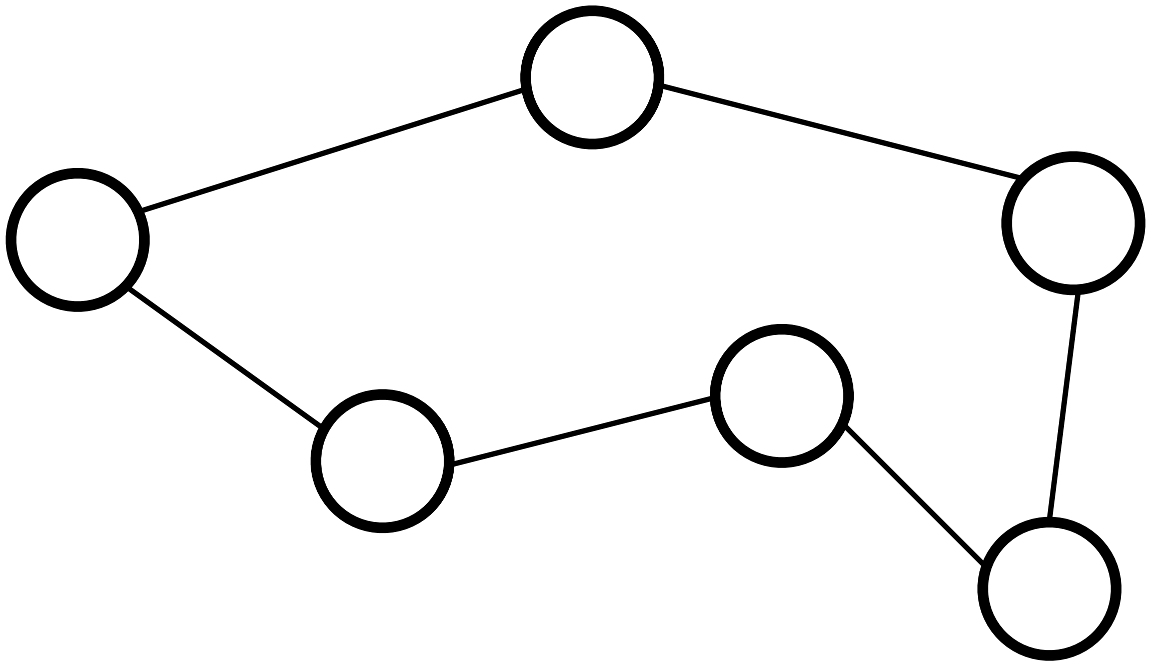

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$$y_{ij} \leq \mathbf{1}_{D_i + D_j \leq 1} \qquad \forall (i, j) \in E \longleftarrow$$

Two connected sites can be served by a single truck if their total demand does not exceed the truck capacity

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- The expected loss is

Total fulfilled demand under complete graph

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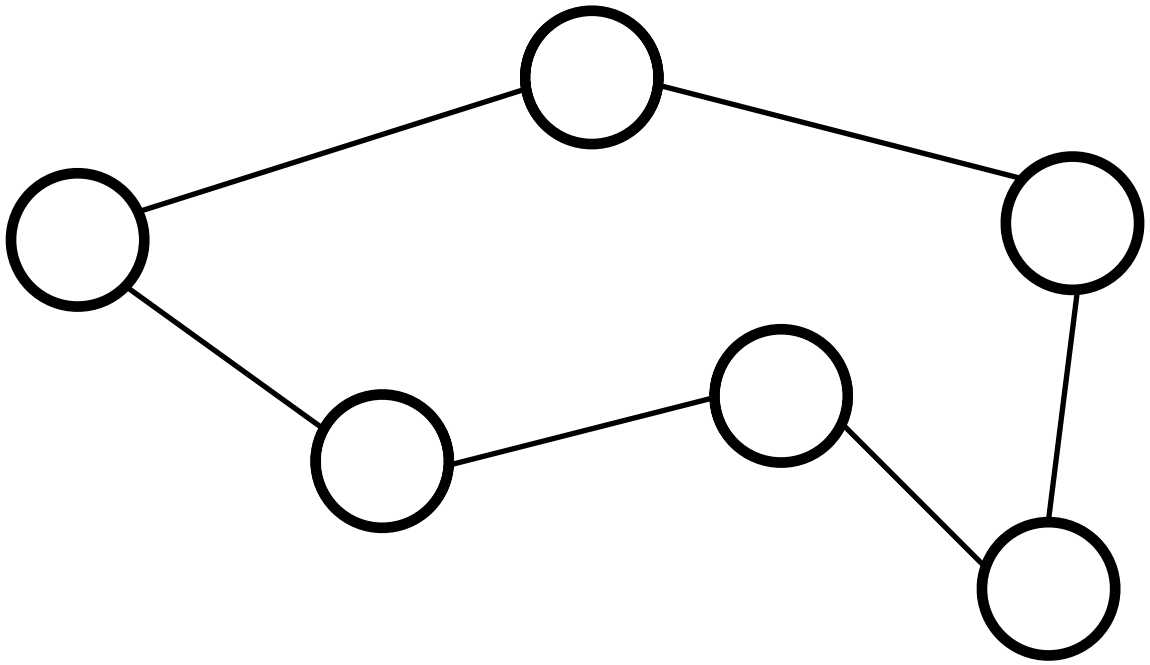

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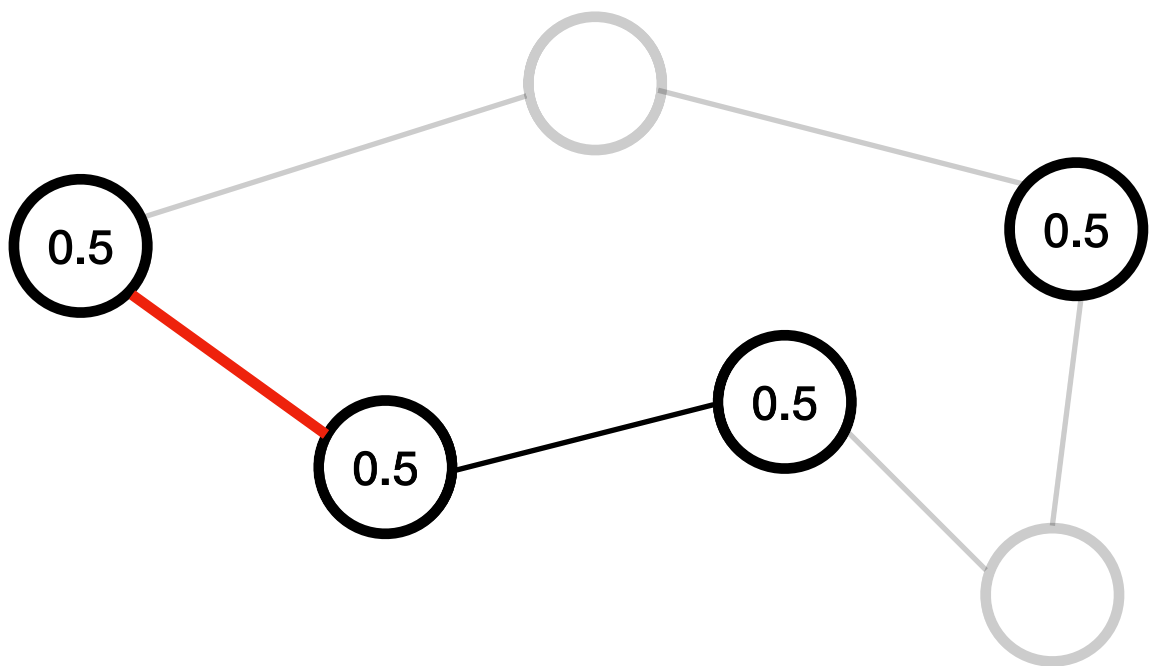



- The expected loss is

$$\mathbb{E}_D[L(G, D)] = 2\mathbb{E}_D\left[\mu(K_n, D) - \mu(G, D)\right] \longleftarrow$$

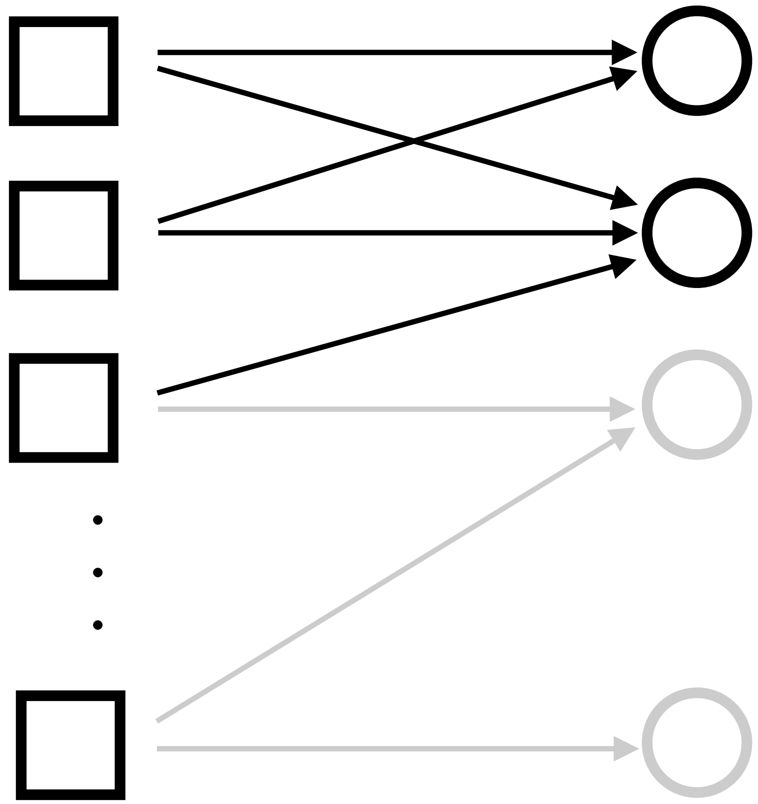
Expected number of unmatched sites

- IID Bernoulli demand with mean 1:

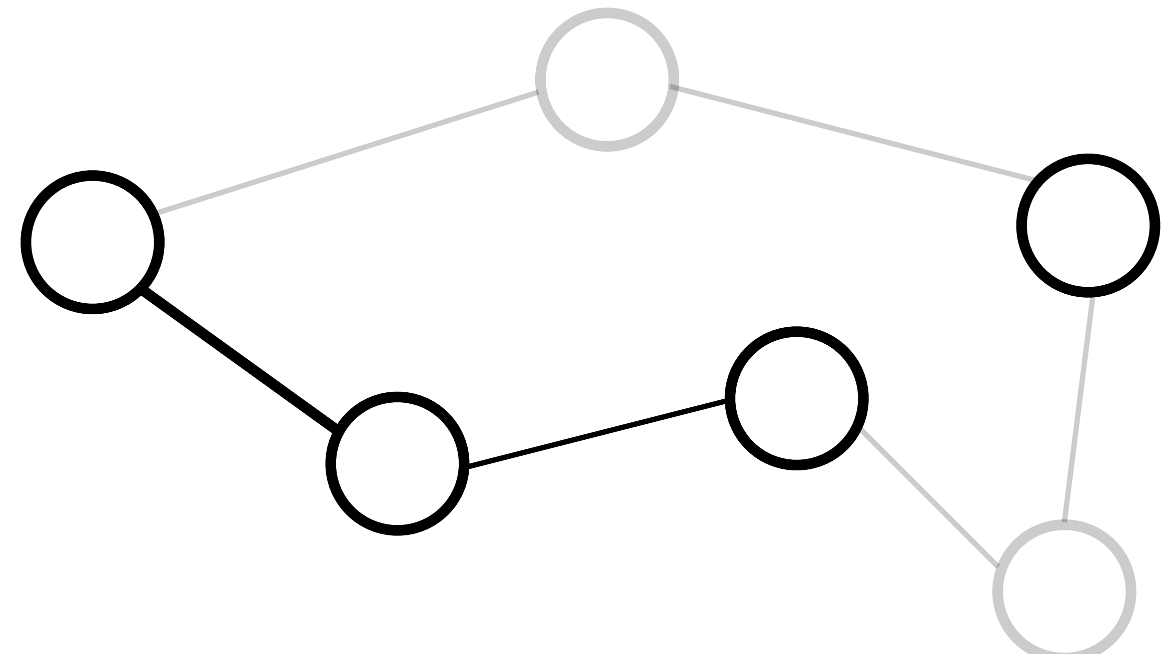
$$D_j = \begin{cases} 1 & \text{w.p. } q \\ 0.5 & \text{w.p. } p = 1 - q \end{cases} \longleftarrow$$

Random node deletion with probability $q$

# Stochastic matching with random node deletion



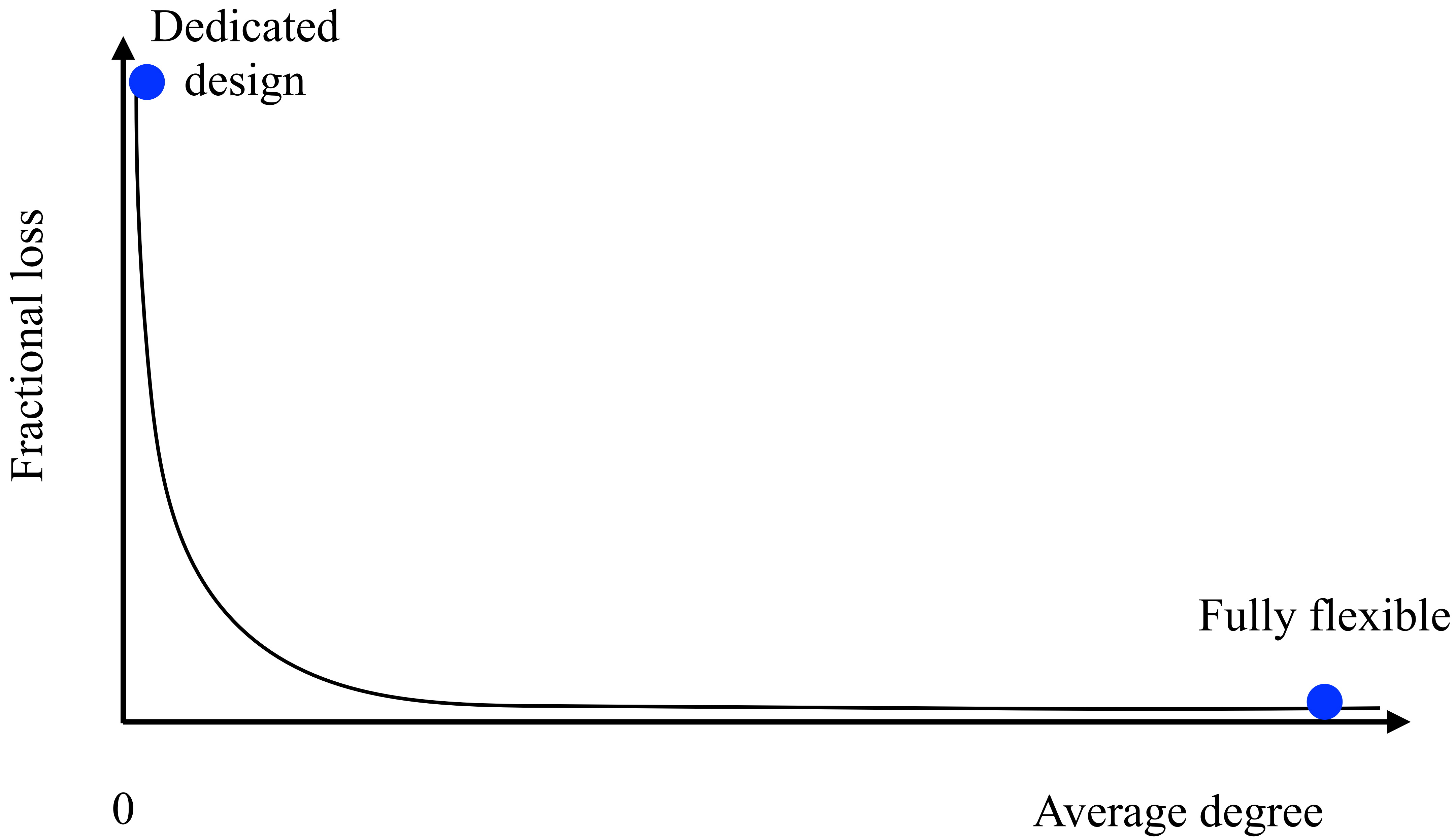
Process flexibility



Transportation flexibility

- Both problems reduce to a “matching” problem in the residual graph $G_{\mathrm{res}}$ obtained after each demand node is independently deleted with probability $q$
- Goal: Design a graph that sustains a large “matching” after random node deletions

# Tradeoff: Demand Loss vs Network Complexity



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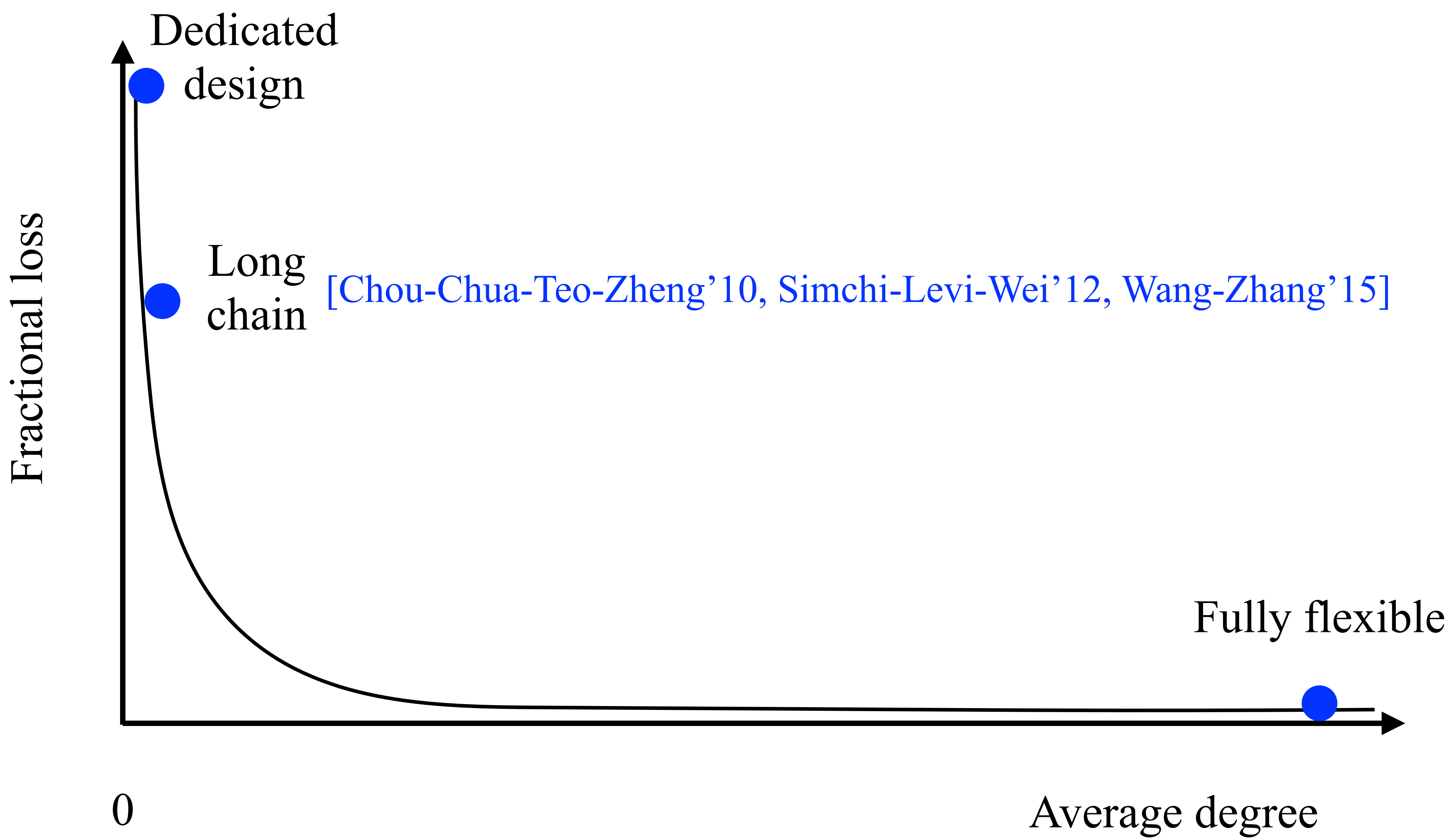

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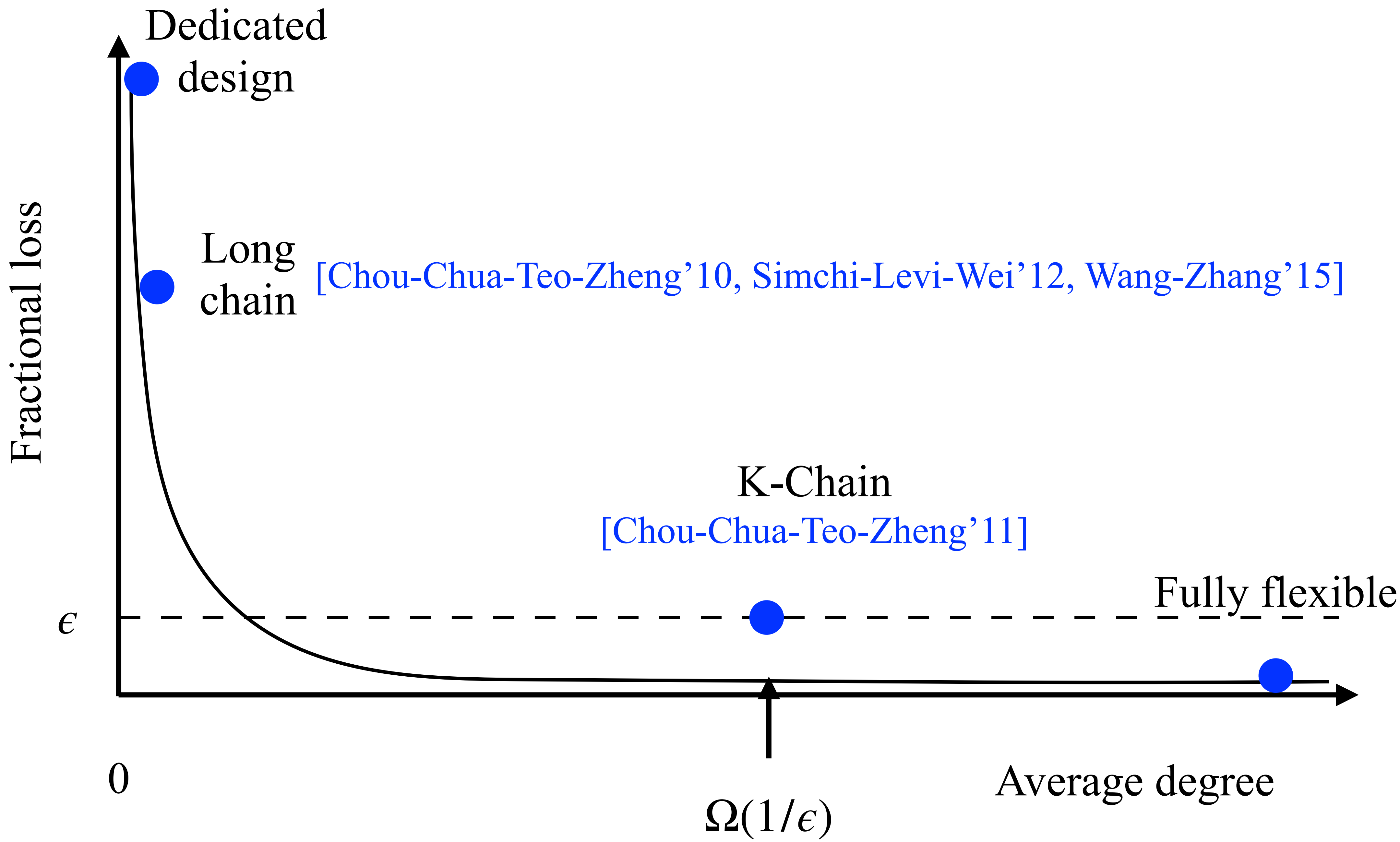

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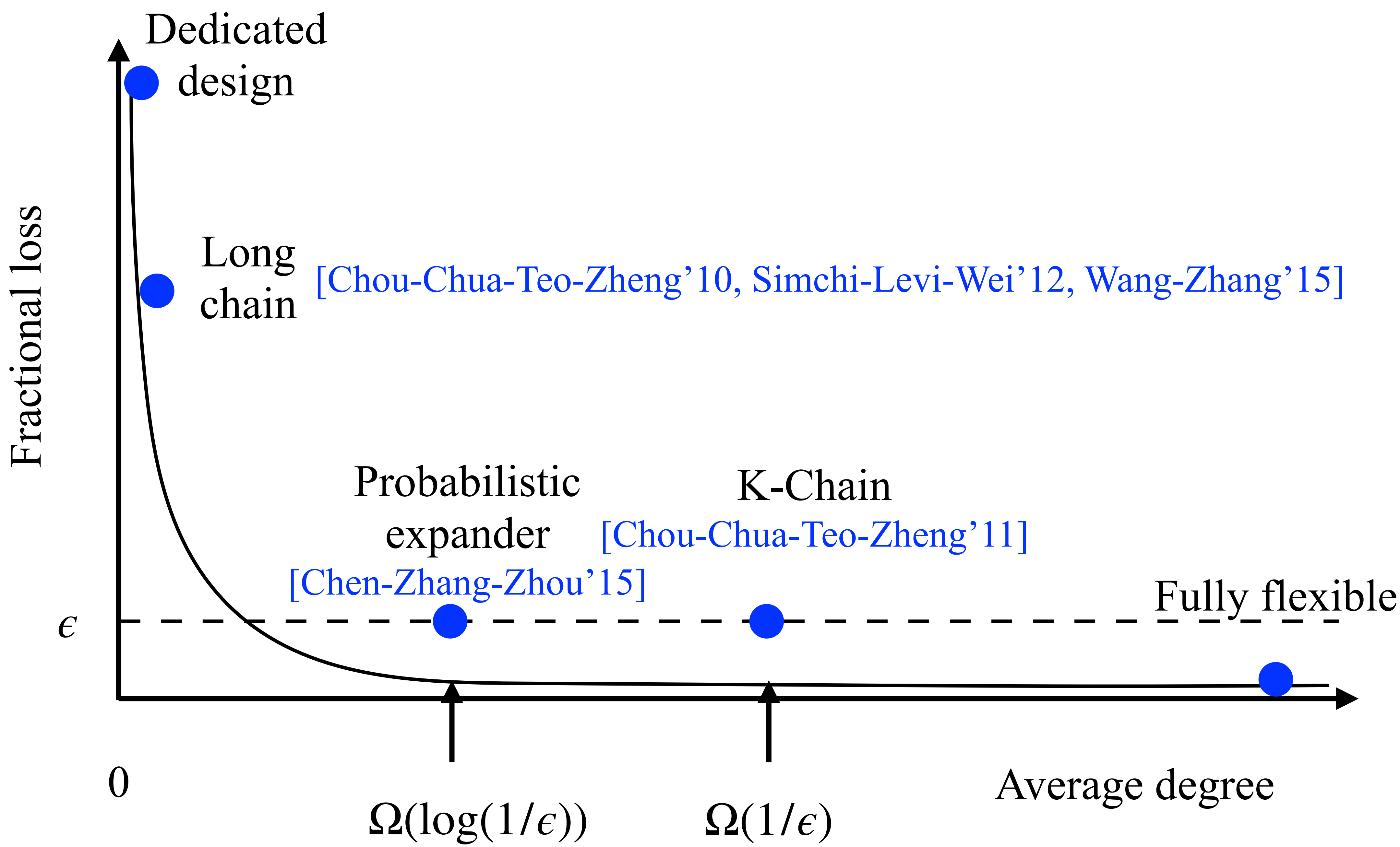

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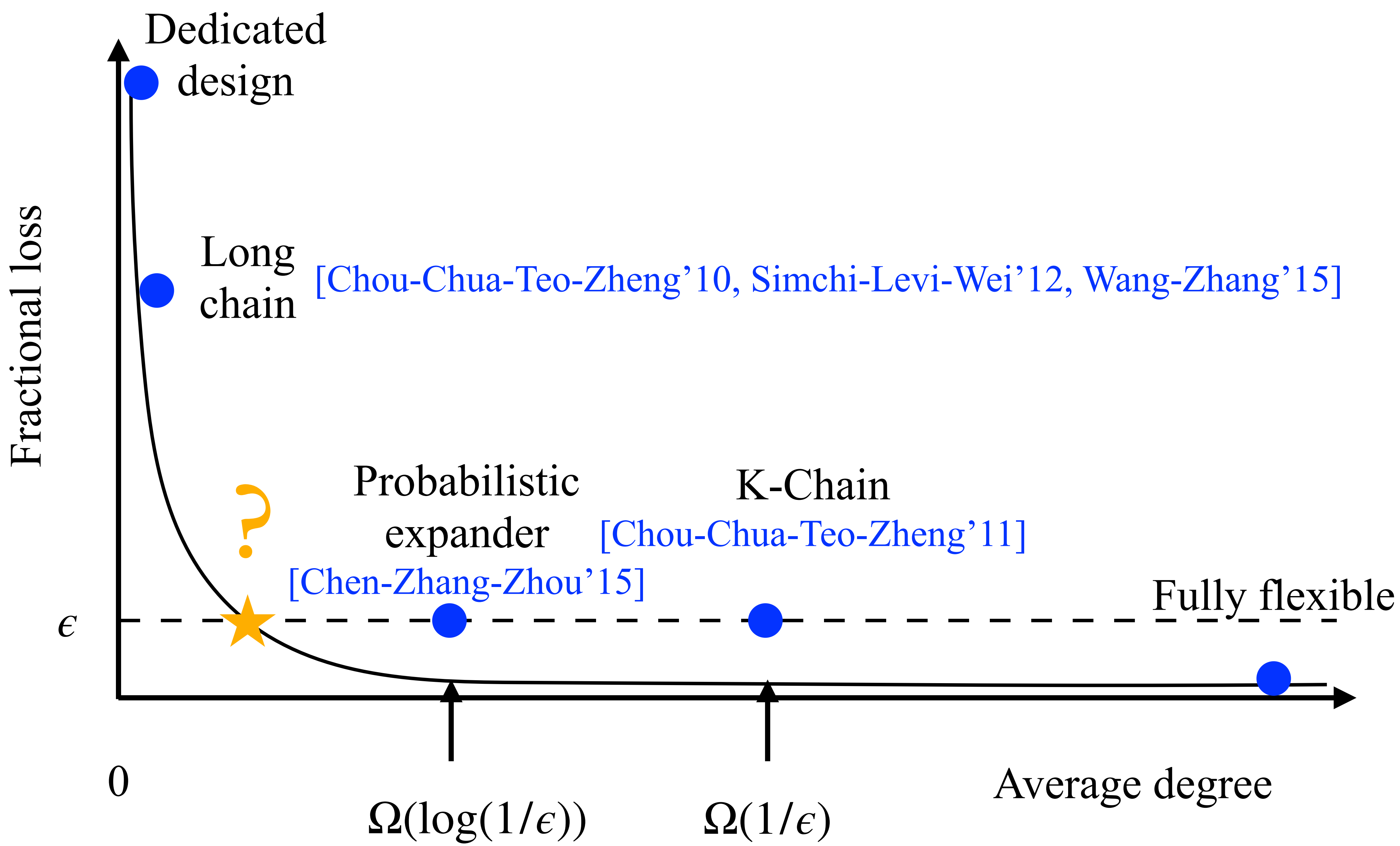

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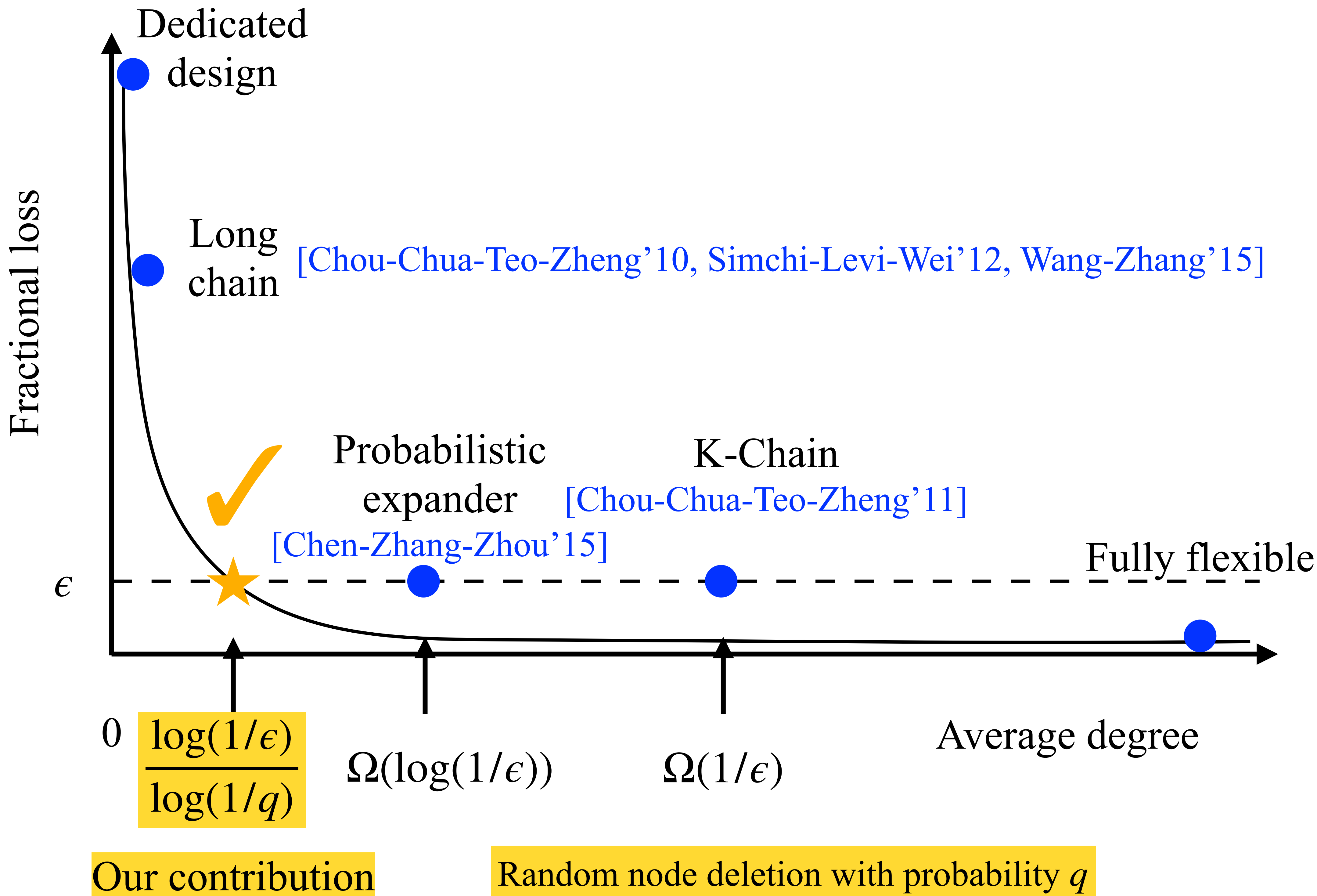


# Lower Bound

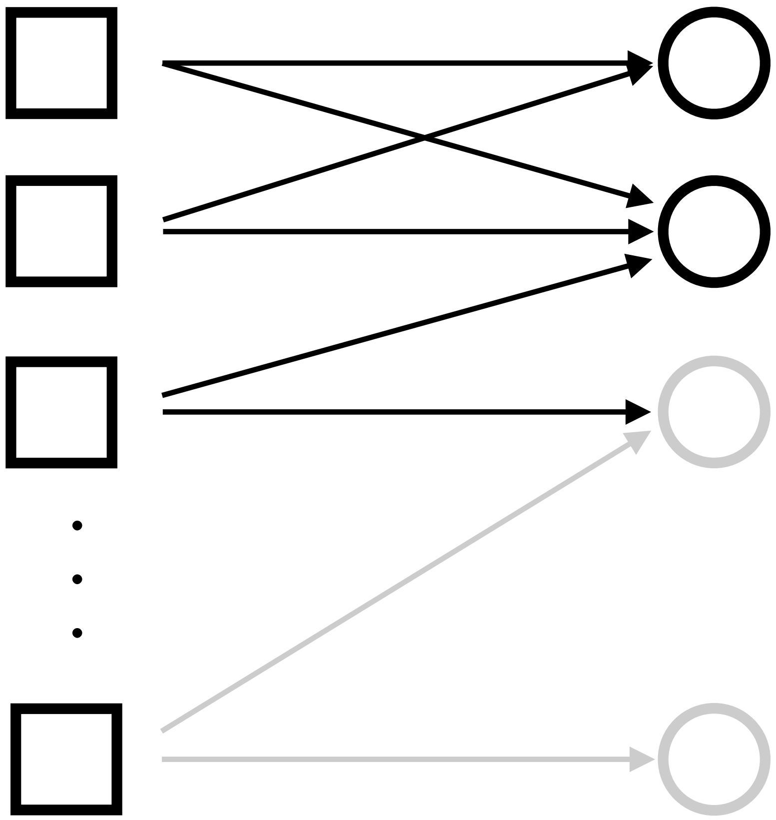
# Process Flexibility: A simple lower bound

**Theorem**

For any bipartite graph $G$ with an average degree $d$,

$$\mathbb{E}_D[L(G,D)] \gtrsim nq^d.$$

Expected demand loss



Demand node deletion w.p. $q$

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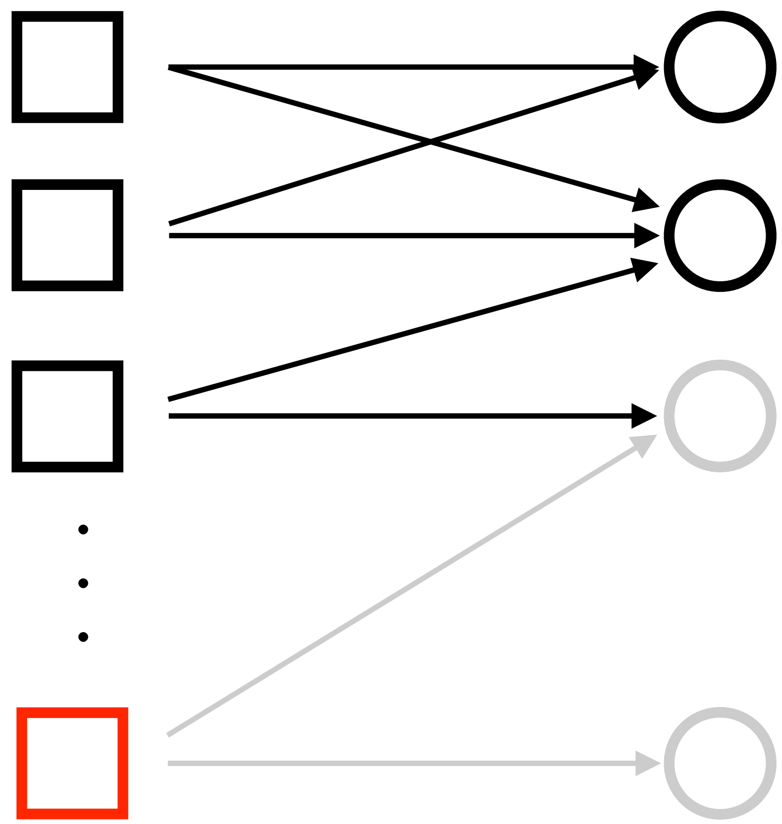
- Proof Ideas:
  - $L(G, D) \gtrsim$ # of isolated supply nodes in $G_{\text{res}}$



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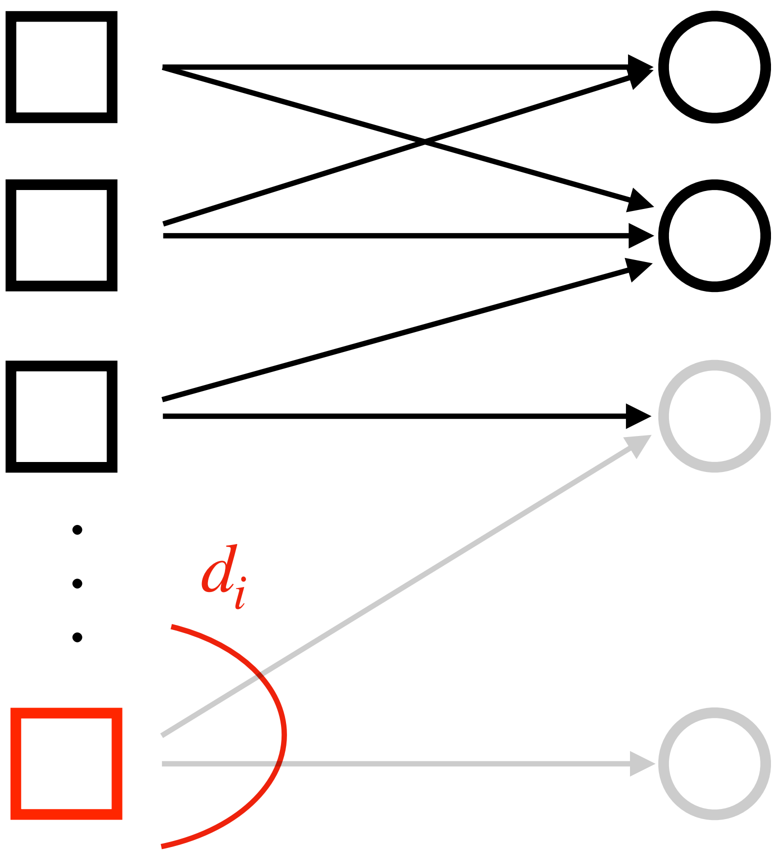


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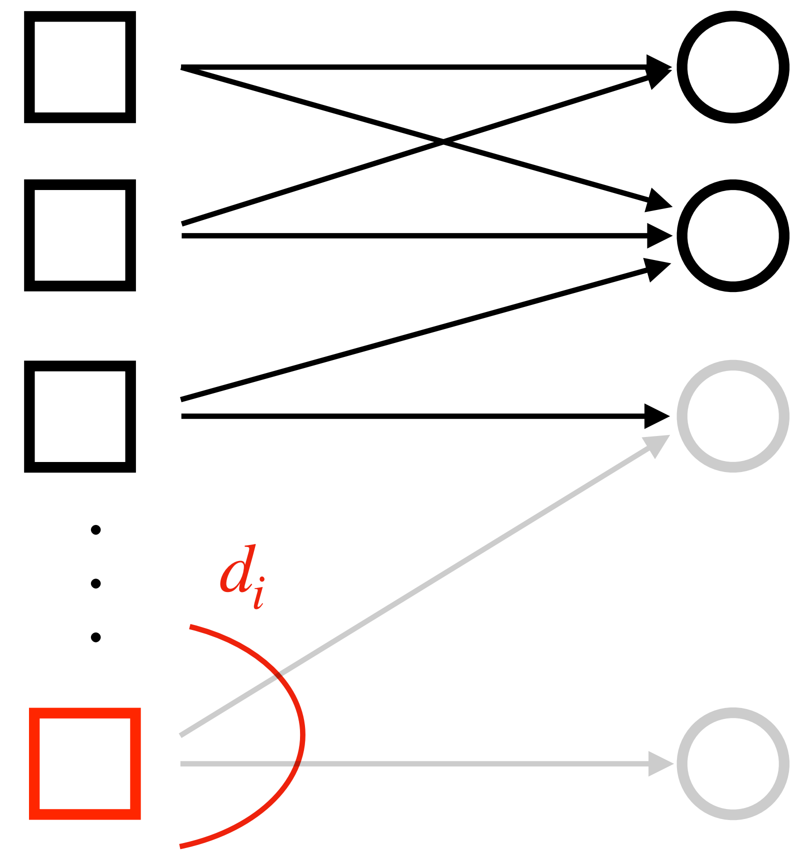
Jensen's; Equality holds for regular graphs

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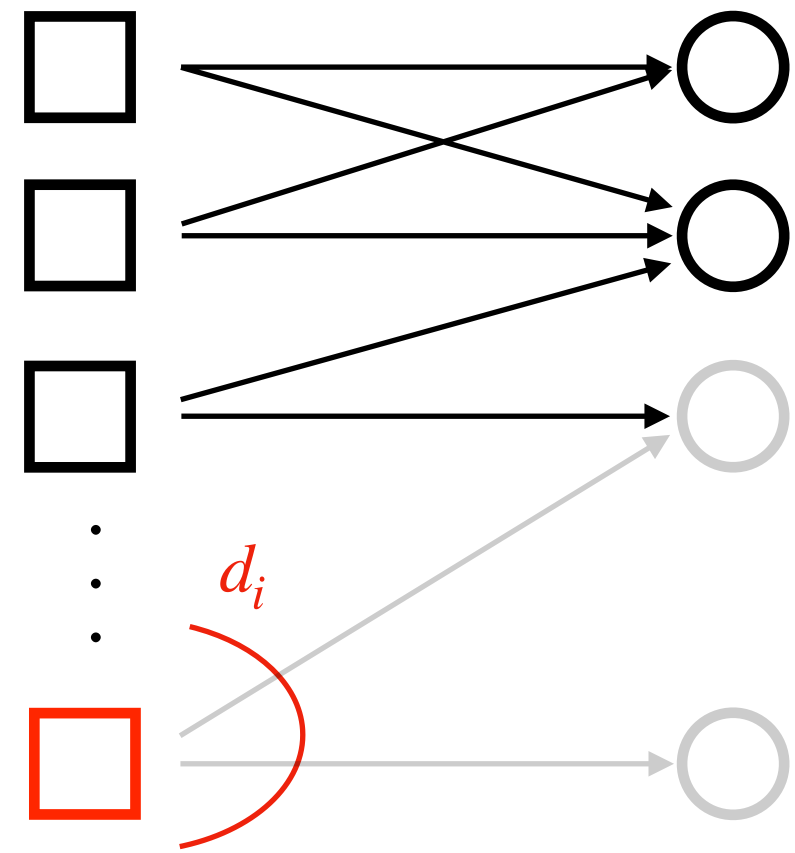



- Proof Ideas:
  - $L(G,D) \gtrsim$ # of isolated supply nodes in $G_{\text{res}}$
  - $\mathbb{E}[\text{ \# of isolated supply nodes in } G_{\text{res}}] = \sum_{i=1}^{n} q^{d_i} \geq nq^d$

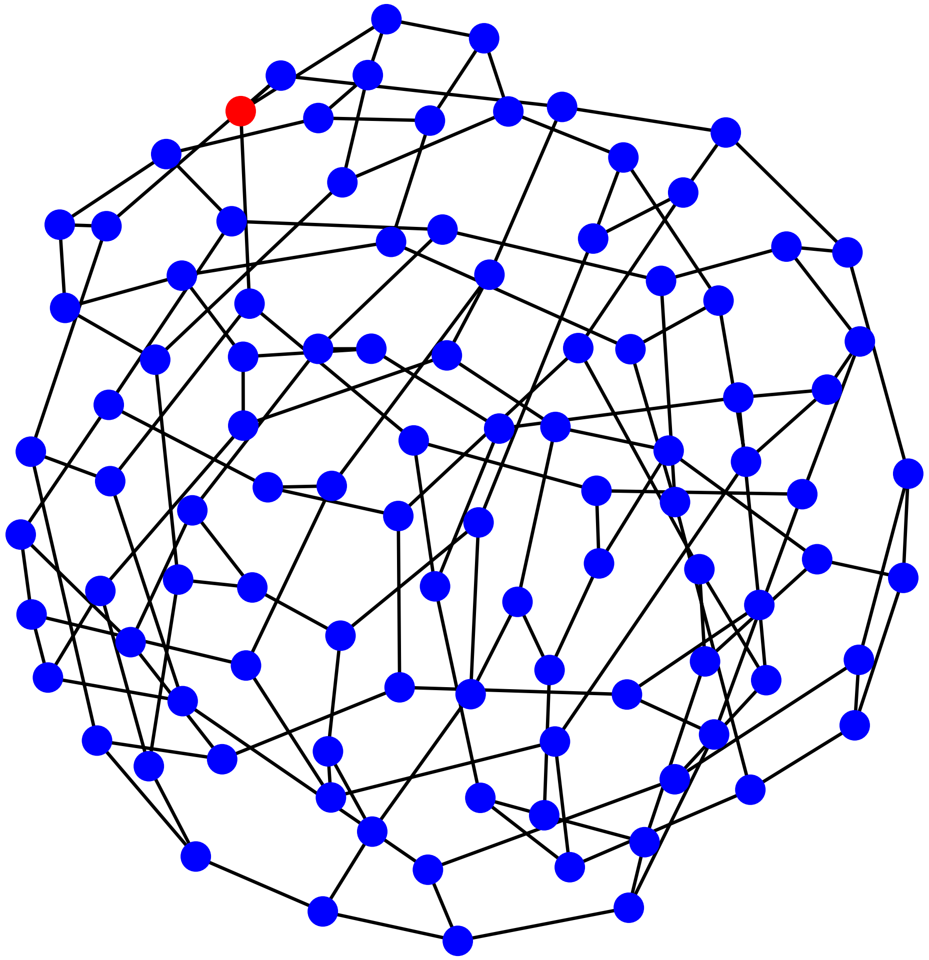
Jensen's; Equality holds for regular graphs

- Implications:
  - $\epsilon n$-loss requires $d \geq \log(1/\epsilon)/\log(1/q)$ for any graph;
  - An optimal graph design needs to be regular
  - K-Chain is also regular, but it still requires $d \geq \Omega(1/\epsilon)$ [Chou-Chua-Teo-Zheng'11]

# Our solution: Random regular graphs

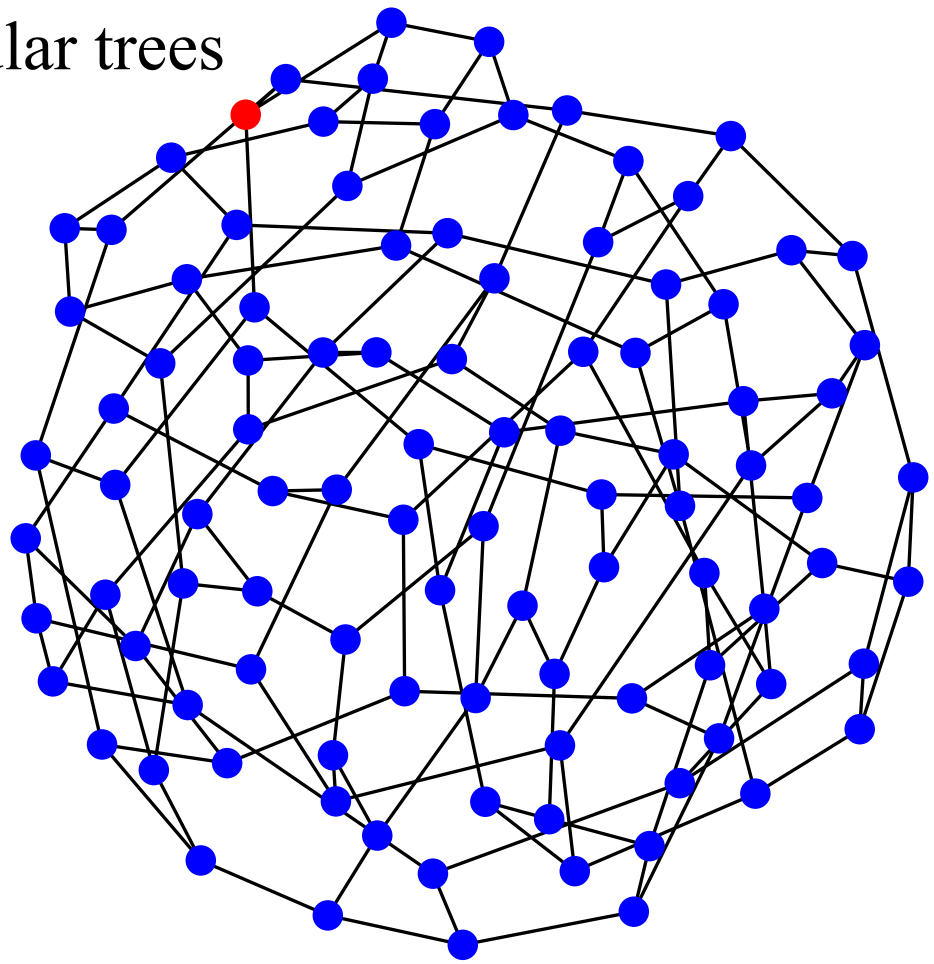
# Random Regular Graphs

- An object widely study in probability and TCS



# Random Regular Graphs

- An object widely study in probability and TCS
- Locally tree-like: local neighborhoods look like regular trees



# Random Regular Graphs

- An object widely study in probability and TCS
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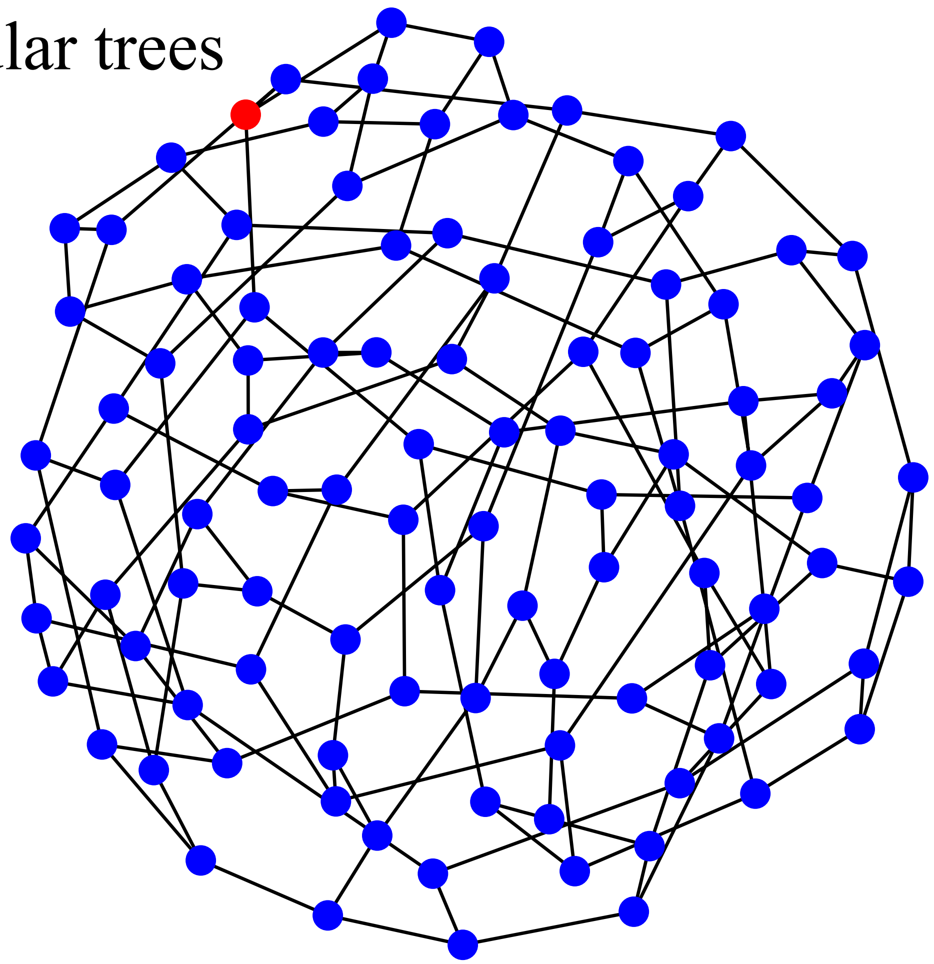
- Spectral expansion: Almost Ramanujan graphs

**Freedman's Theorem (2003)**

For random $d$-regular graph with eigenvalues $\lambda_n \leq \ldots \leq \lambda_2 \leq \lambda_1 = d$ ,

$$\max\{\lambda_2, |\lambda_n|\} \leq 2\sqrt{d-1} + o(1)$$

Optimal among all $d$-regular graphs



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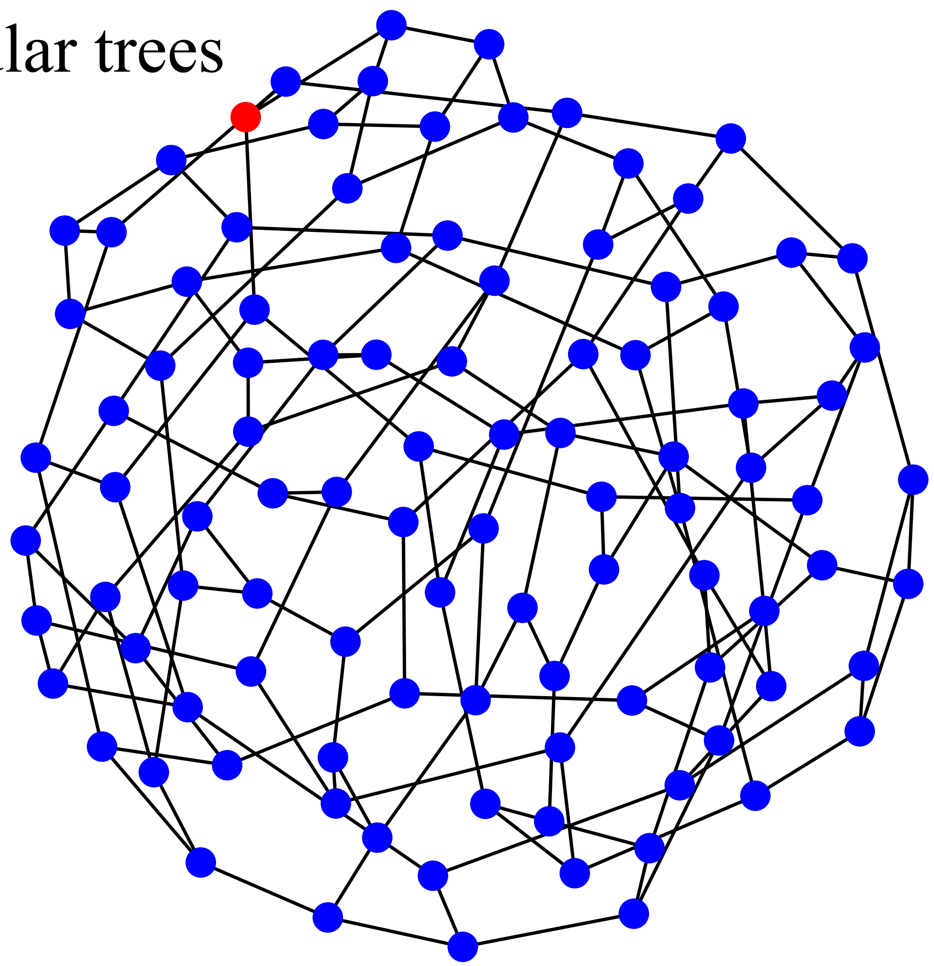

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Optimal among all $d$-regular graphs

**Question**: Do random regular graphs enjoy optimality in flexibility design?

# Process Flexibility: Random Regular Graphs Are Optimal

**Theorem** [Li-Niu-Wei-Xu '25]

For any constant $\delta > 0$, a random $d$-regular bipartite graph $G$ achieves whp:

- $\epsilon$-fractional loss ($\mathbb{E}_D[L(G, D)] \leq \epsilon n$), if

$$d \geq \frac{(1+\delta)\log(1/\epsilon)}{\log(1/q)}$$

- Constant loss ($\mathbb{E}_D[L(G, D)] = O(1)$), if

$$d \geq \frac{(1+\delta)\log n}{\log(1/q)}$$

Remarks

- Match the lower bounds with the sharp constant;
- Previous best-known upper bound $d \geq \Omega(\log(1/\epsilon))$ [Chen-Zhang-Zhou'15];
- $K$-chain needs $d \geq \Omega(1/\epsilon)$ [Chou-Chua-Teo-Zheng'11].

# Transportation Flexibility

**Theorem** [Li-Niu-Wei-Xu '25]

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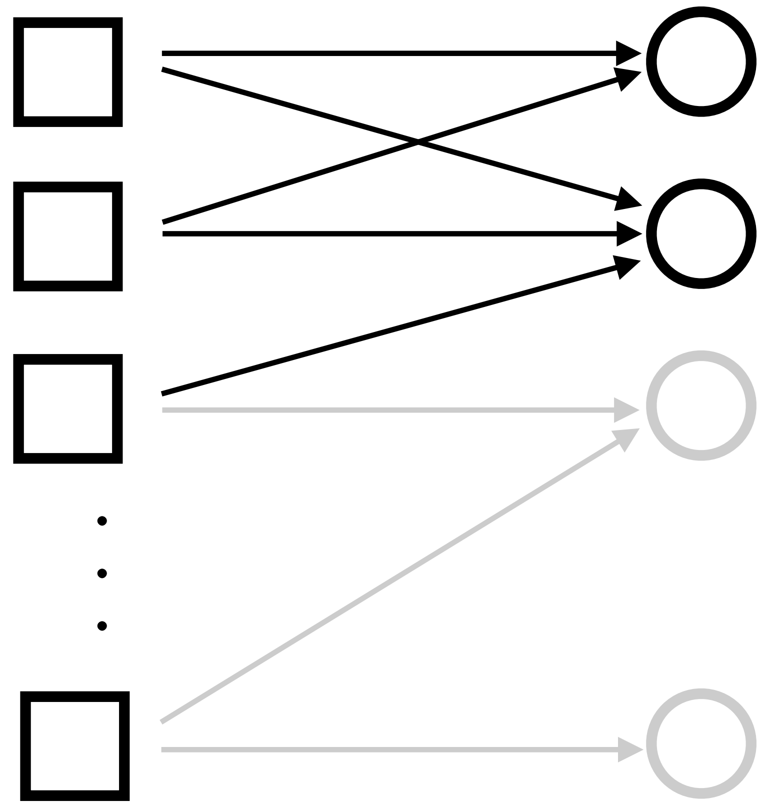
Remarks

- Match the lower bounds with the sharp constant;
- Erdős–Rényi random graph needs $d > \log(1/\epsilon)/(1-q)$ [Li-Niu-Wei-Xu'25];
- $K$-chain needs $d \geq 2\log(1/\epsilon)/\log(1/q)$ [Li-Niu-Wei-Xu'25].

# Proof ideas for process flexibility

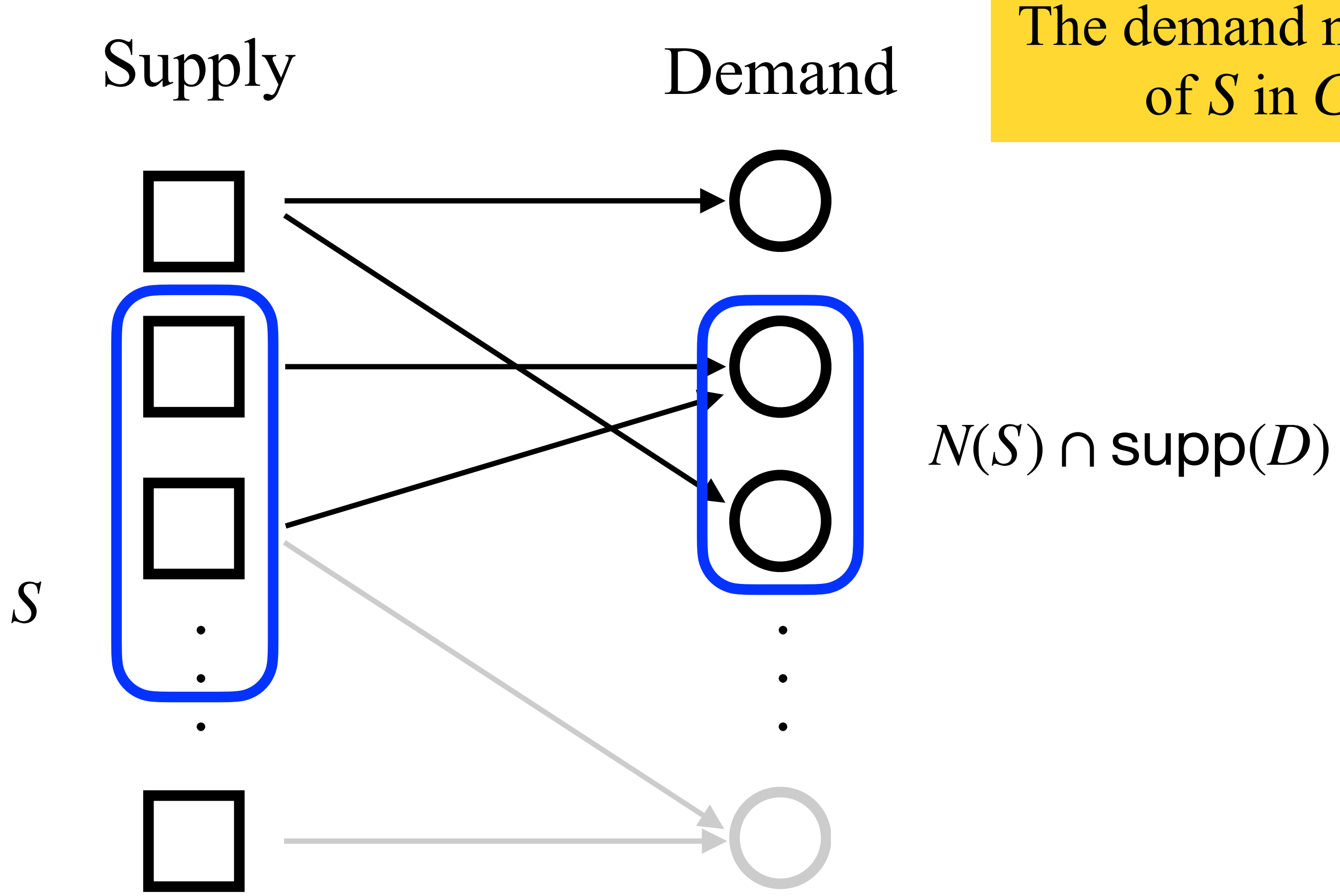


# Strong Connectivity Property

By max-flow min-cut theorem:

$$Z(G, D) = \min_{S \subset [n]} \{n - |S| + |N(S) \cap \mathrm{supp}(D)|/p\}$$

The demand neighbors of $S$ in $G_{\mathrm{res}}$



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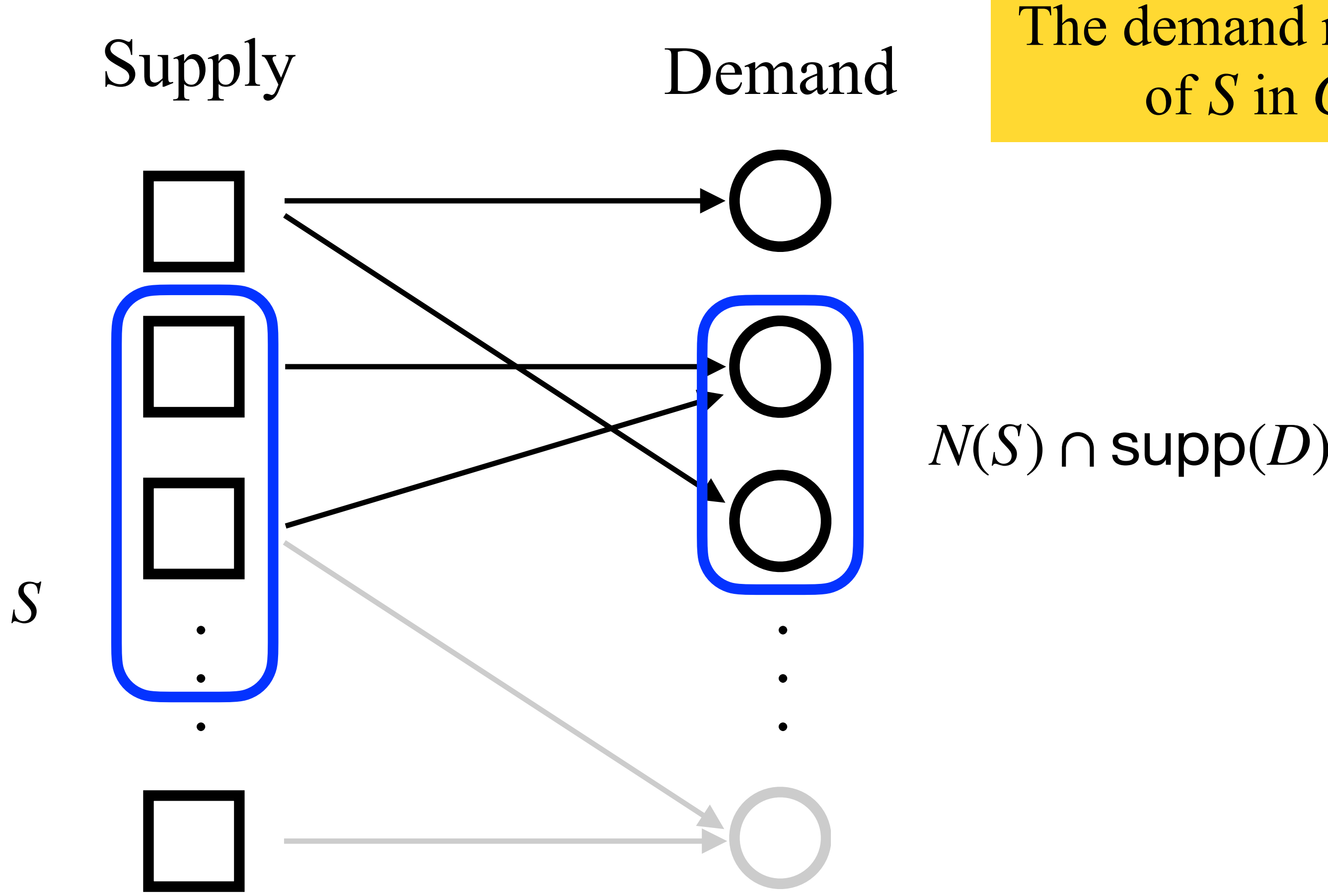


To ensure $\epsilon n$-loss:

Strong "$\epsilon n$-connectivity"

$$L(G, D) \geq (1 - \epsilon)n \iff \forall S \subset [n] : \ |N(S) \cap \mathrm{supp}(D)| \geq p(|S| - \epsilon n)$$

# Ruling out Large Bottleneck $S$

- $S$ is called a bottleneck, if $|N(S) \cap \text{supp}(D)| < p(|S| - \epsilon n)$
- A bottleneck $S$ is called "large", if $|S| \geq \epsilon^c n$ for some choice of $c \in (0,1)$

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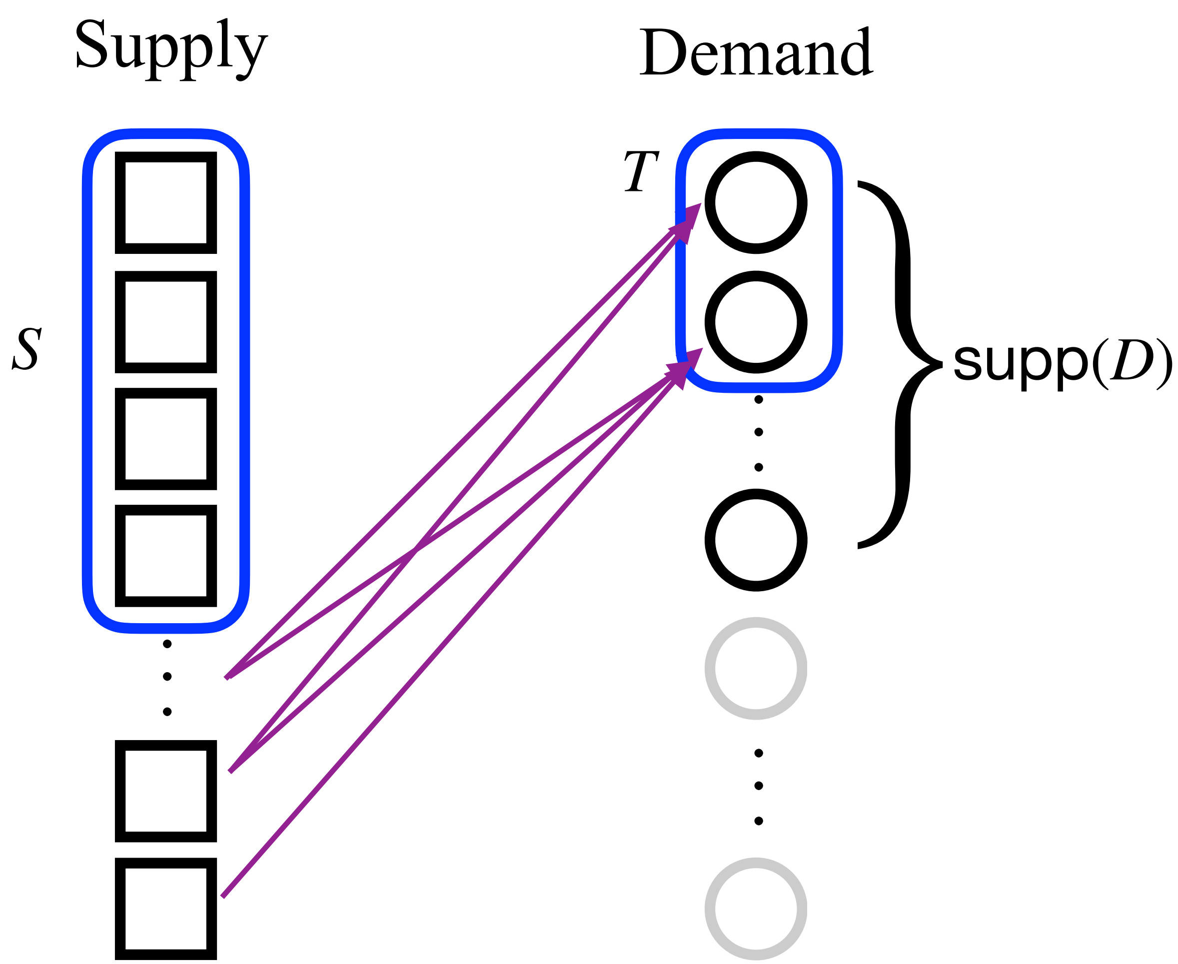


Large Bottleneck $S$

- $S$ is a "large" bottleneck $\Longrightarrow$
- $\exists$ demand subset $T \subset \mathrm{supp}(D)$ with $|T| = |\mathrm{supp}(D)| - p(|S| - \epsilon n)$ s.t. there is no cross-edge between $S$ and $T$, i.e., $e(S, T) = 0$

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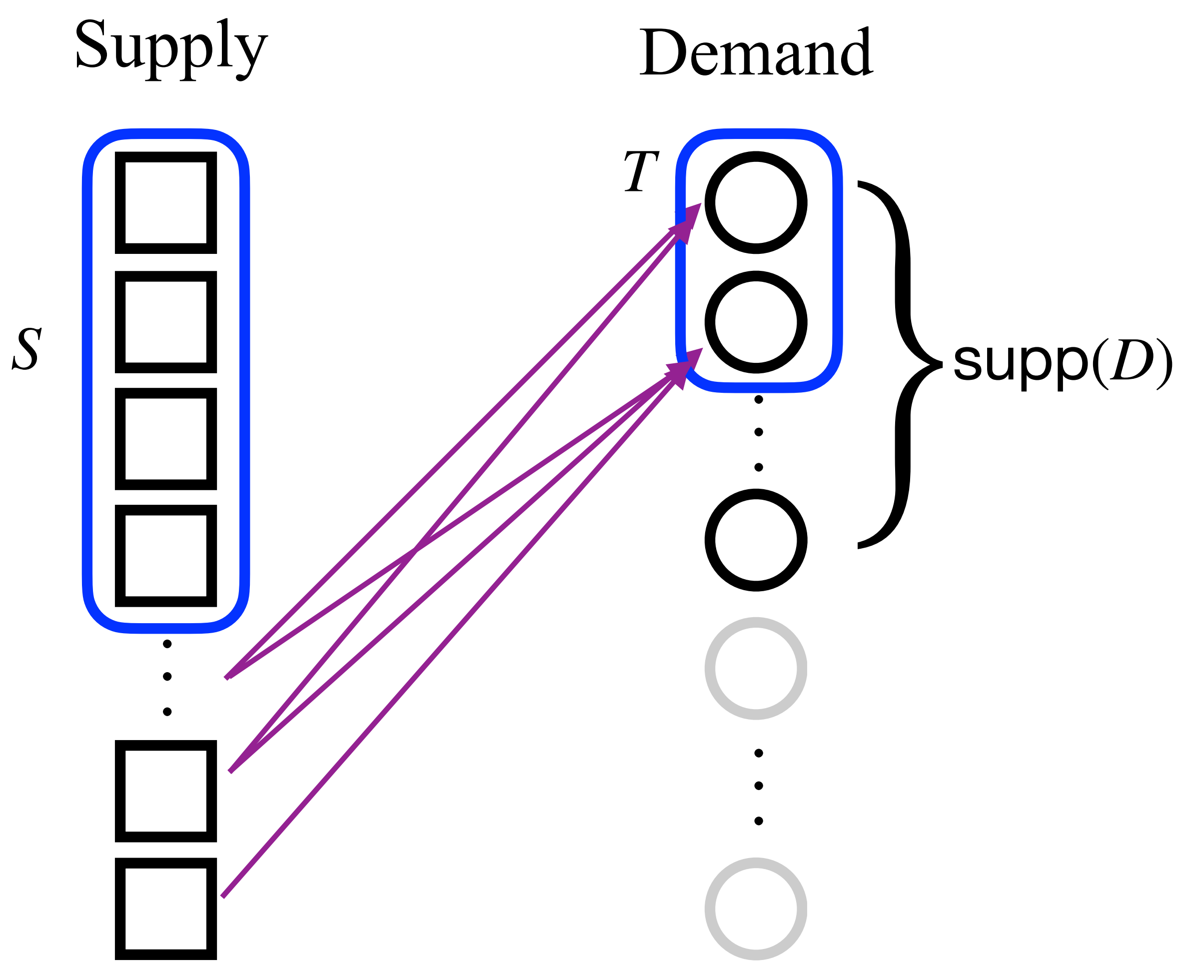


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- A simple union bound over $S$ and $T$ shows that there is no such big cut whp
- Regularity on the demand side and random connections are important

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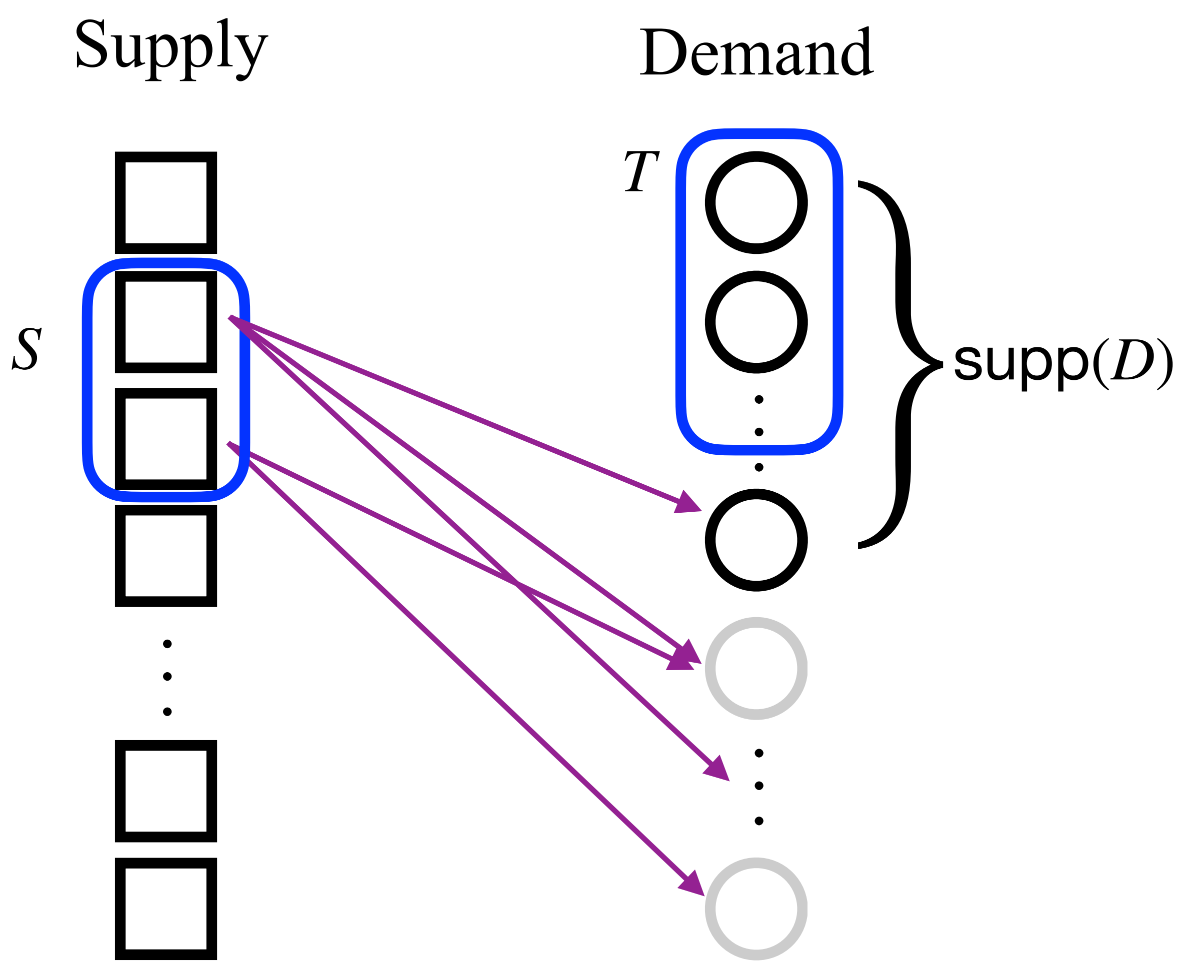


Small Bottleneck $S$

Key challenge:

$\{e(S, T)\}$ are highly correlated across different $T \subset \mathrm{supp}(D)$, so a simple union bound over $S$ and $T$ does not work

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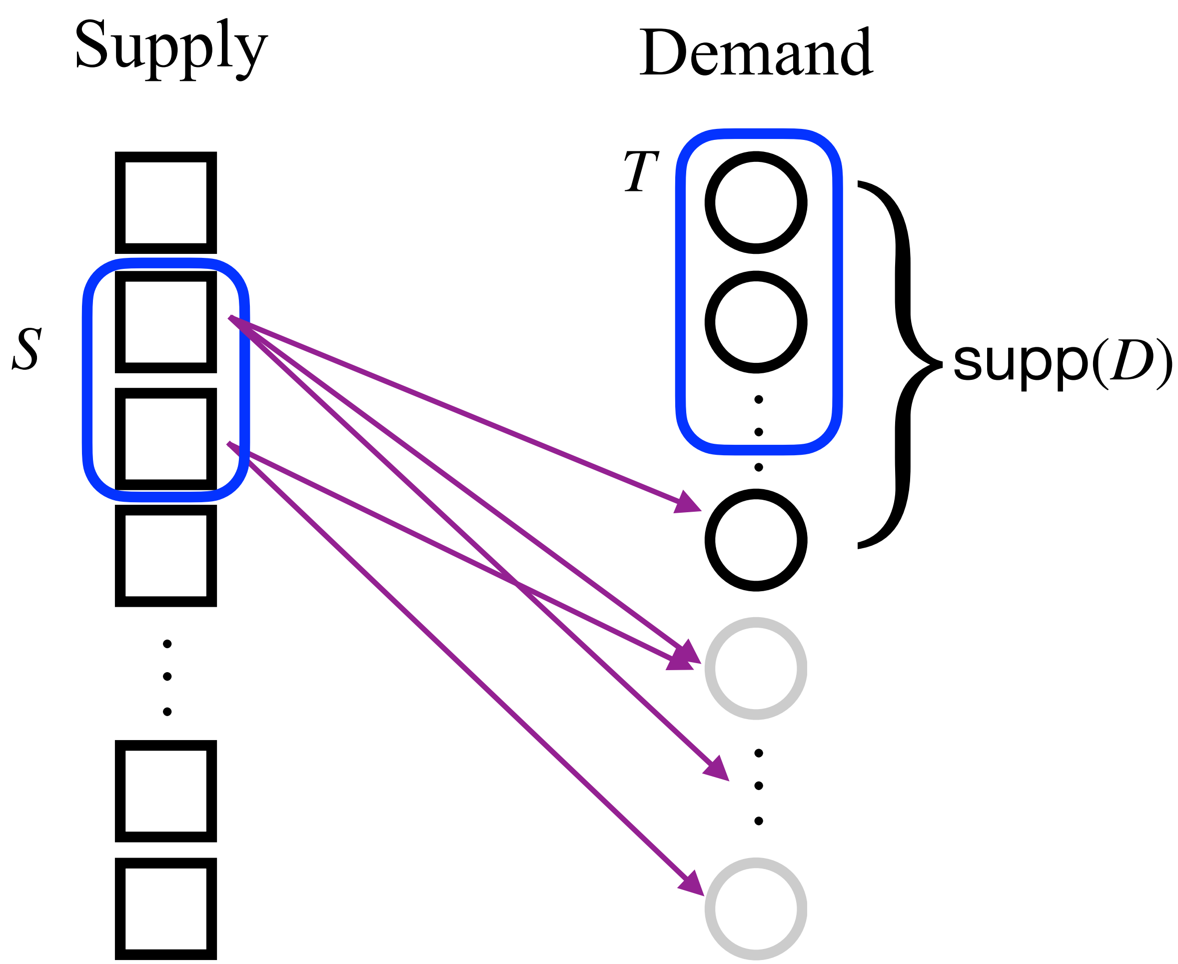


Small Bottleneck $S$
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(1) *Little waste of supply*:
$e(S, \mathrm{supp}(D)) > \gamma |S|$ for all $S$ with $|S| \geq \epsilon n$, for some $\gamma > 0$

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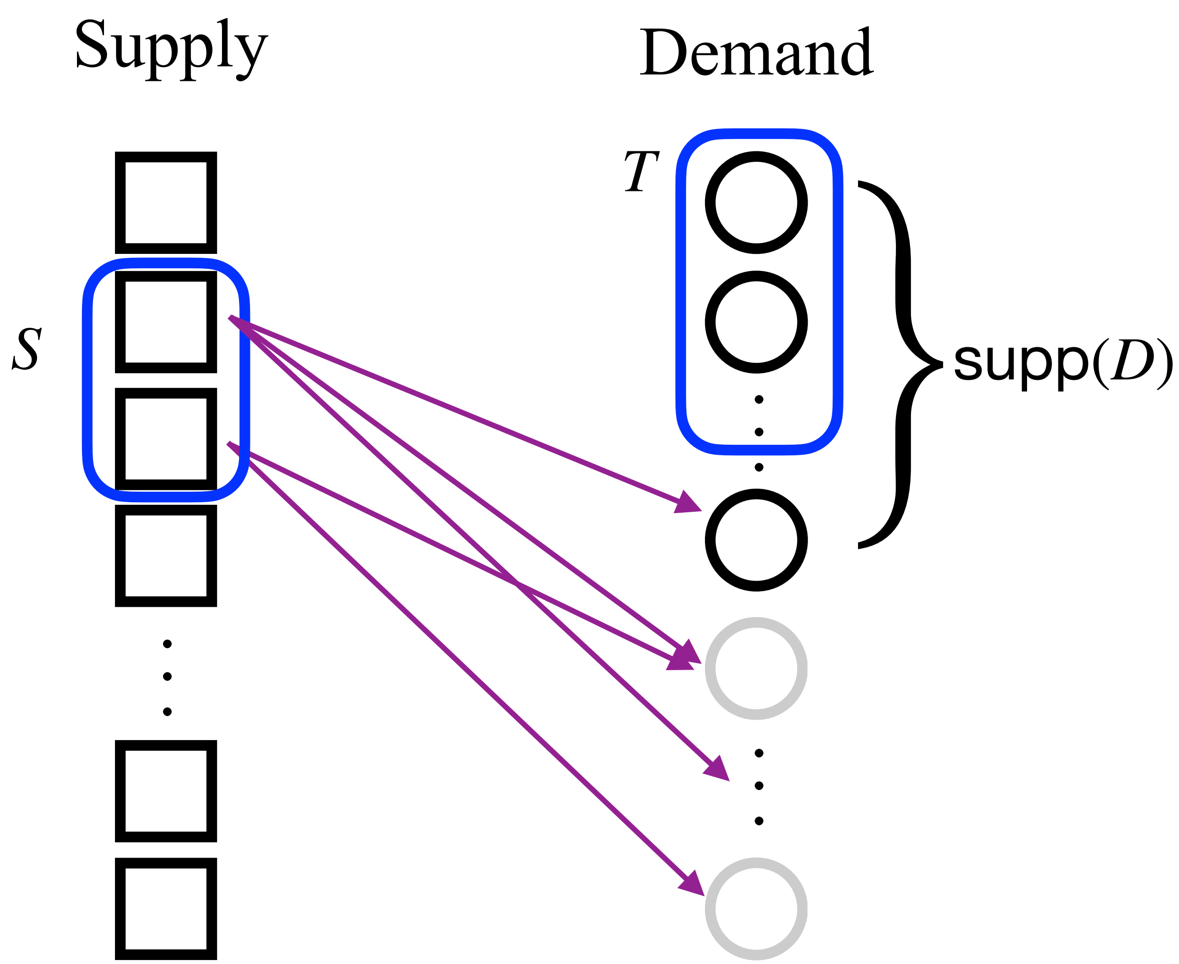


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A stronger requirement than the bound on the # isolated supplier nodes

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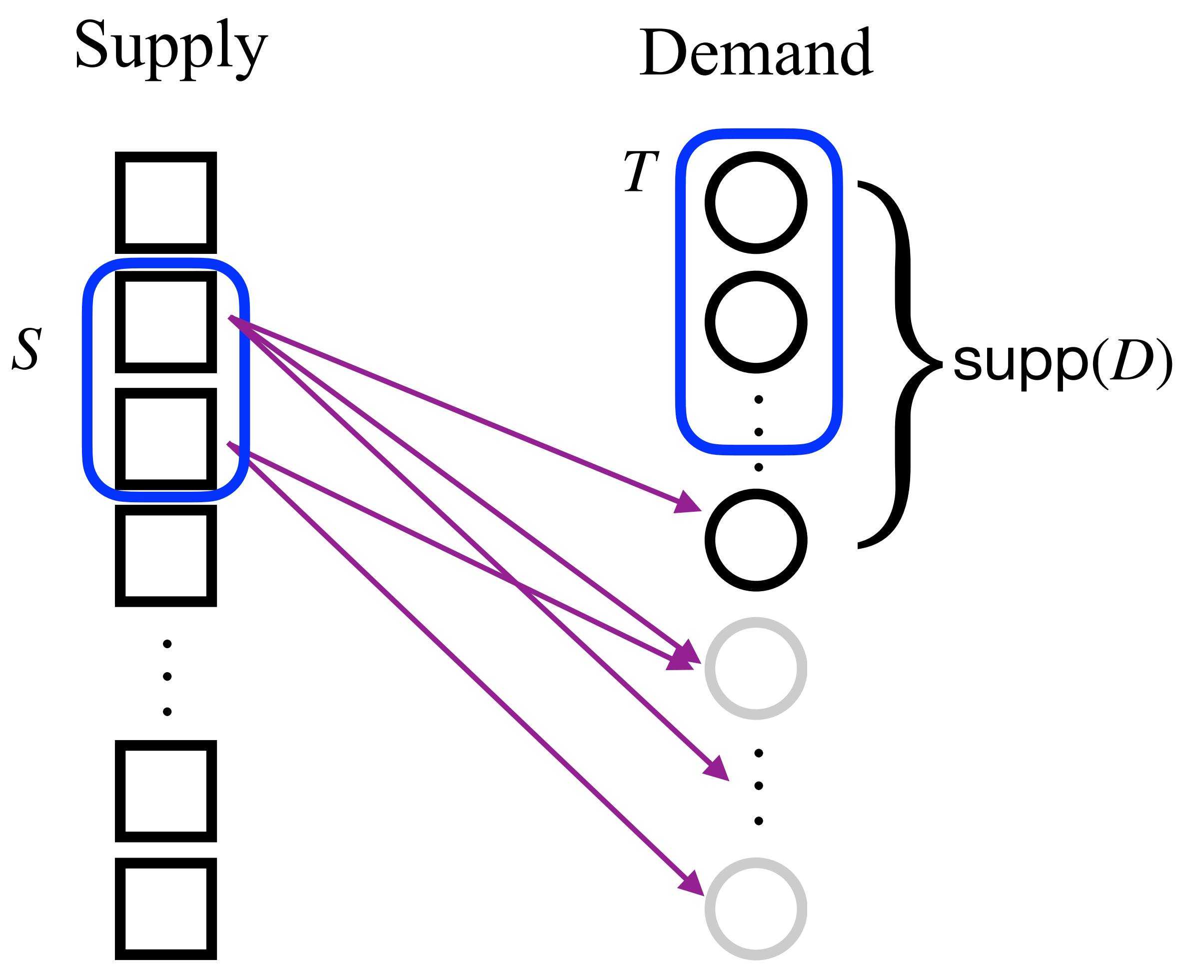


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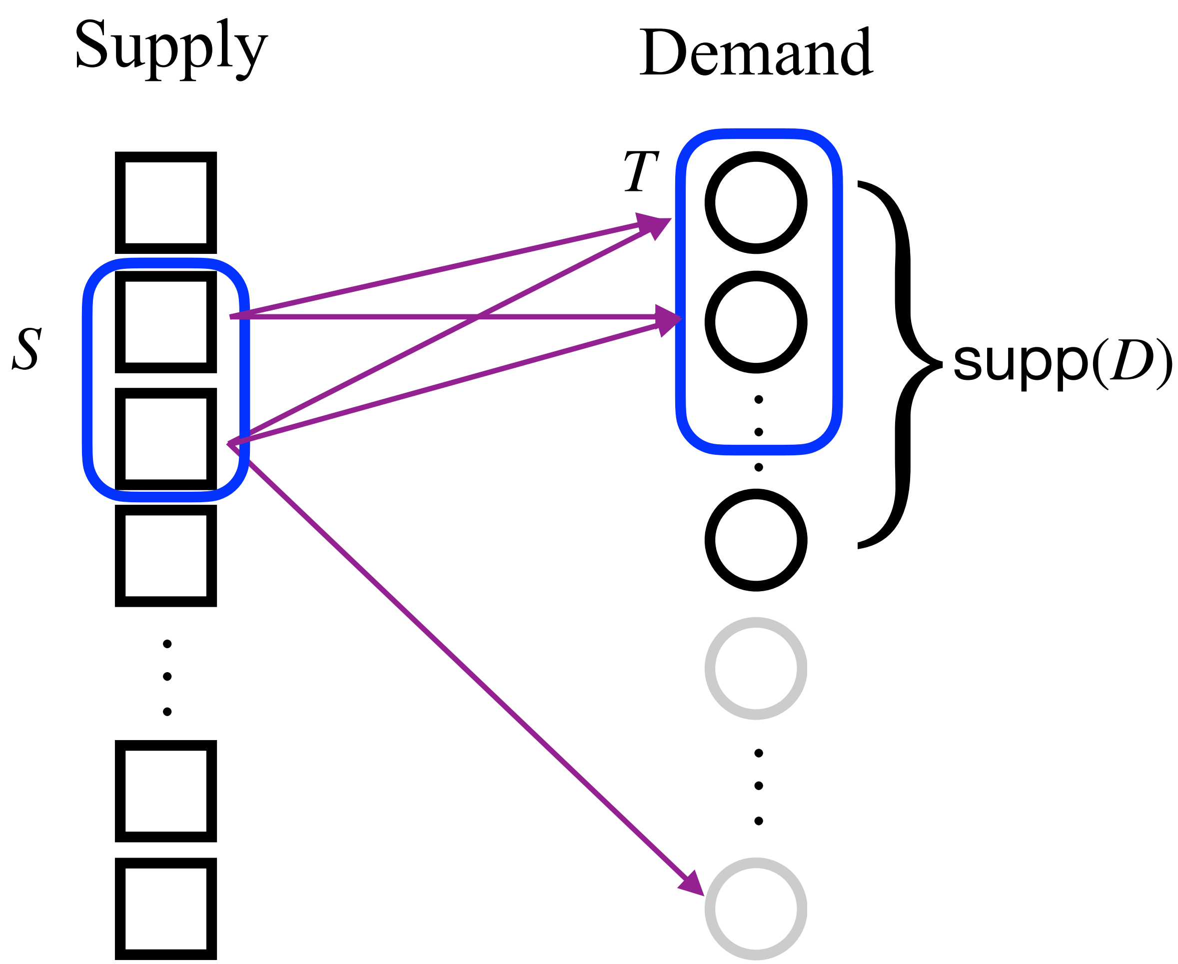


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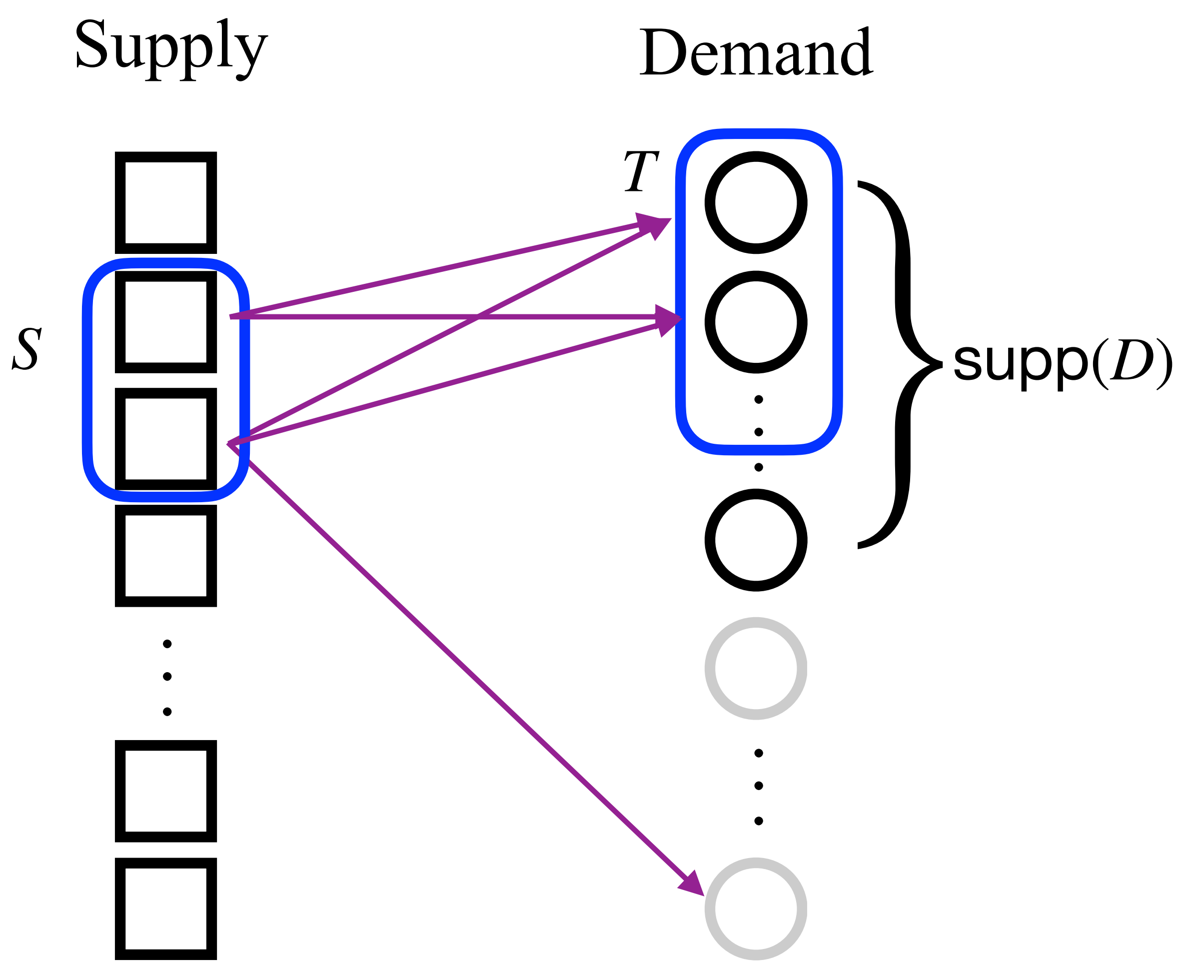


Small Bottleneck $S$

Rule out by (2)

(1) *Little waste of supply*:
$e(S, \mathrm{supp(D)}) > \gamma |S|$ for all $S$ with $|S| \geq \epsilon n$, for some $\gamma > 0$

(2) *Little congestion*: $e(S, T) < \gamma |S|$
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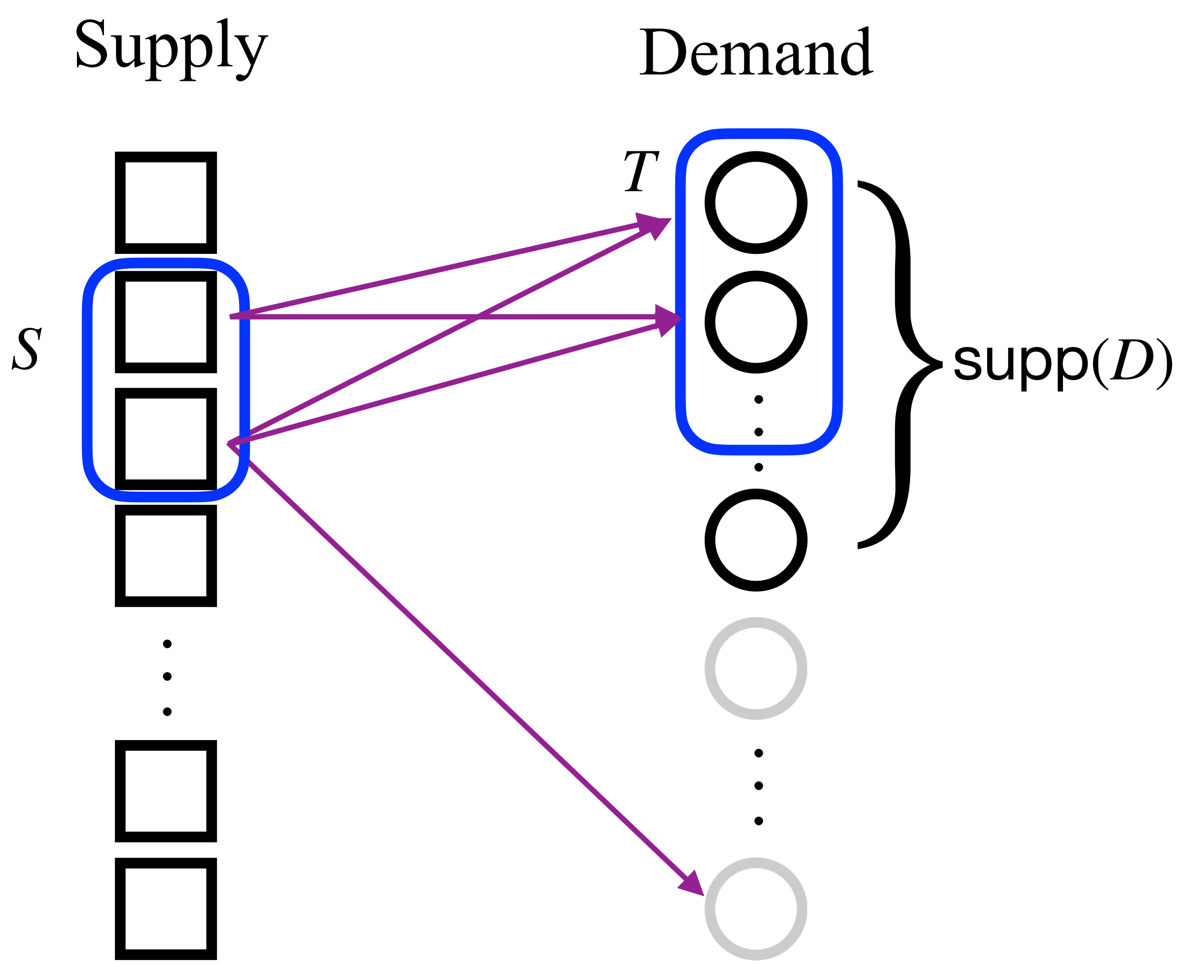


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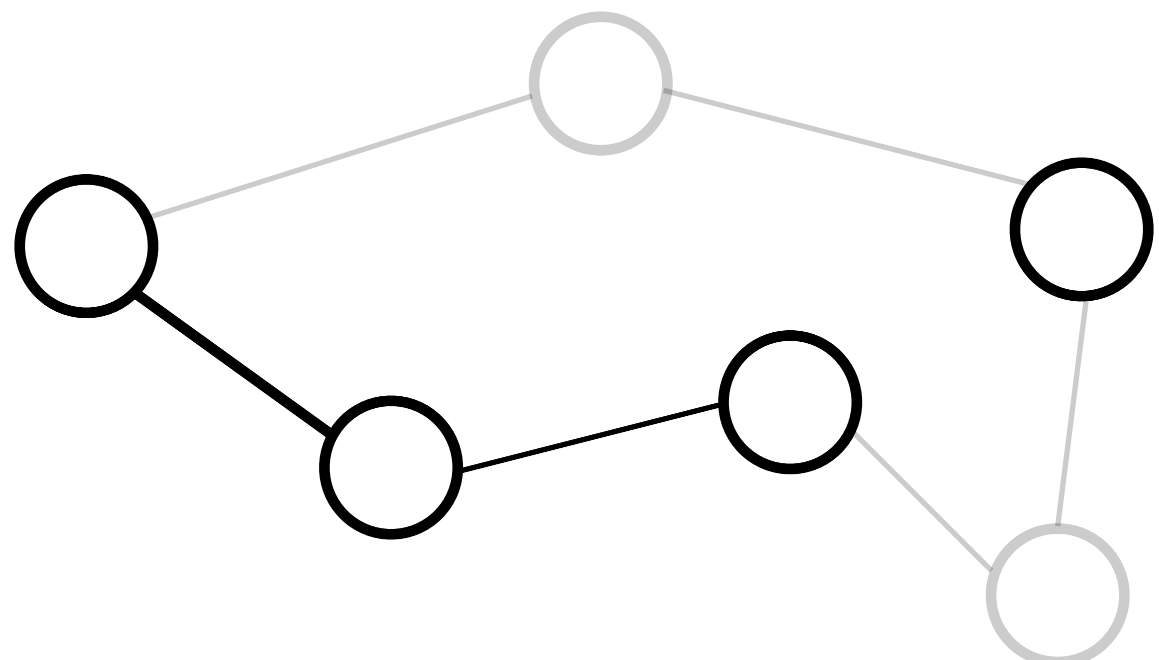
- (1) + (2) $\Rightarrow$ small bottleneck $S$ cannot occur
- Regularity on the supply side and random connections are important

# Proof ideas for transportation flexibility



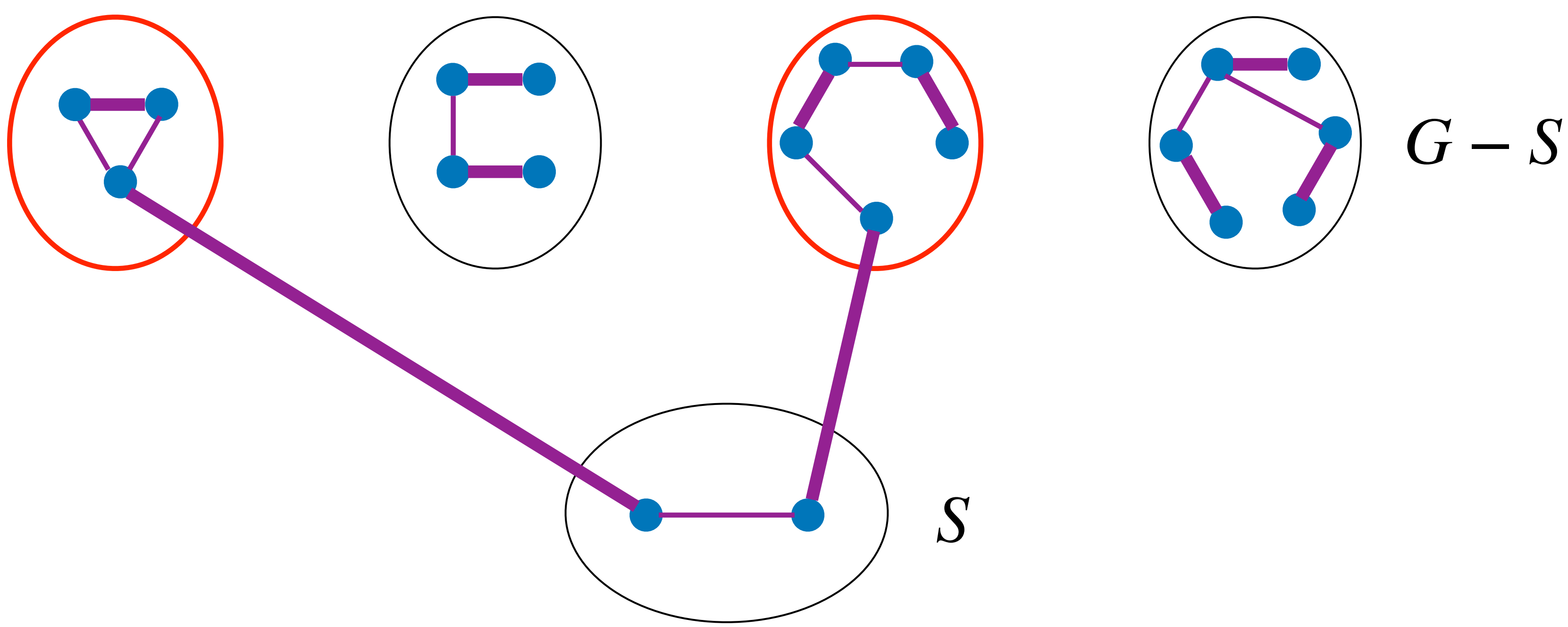
# Tutte-Berge theorem

**Theorem** [Tutte '47, Berge'58]

The number of unmatched vertices in a maximum matching of graph $G$ is given by

$$\max_{S \subseteq V(G)} \left( \mathrm{odd}(G - S) - |S| \right)$$

# of odd components when $S$ removed



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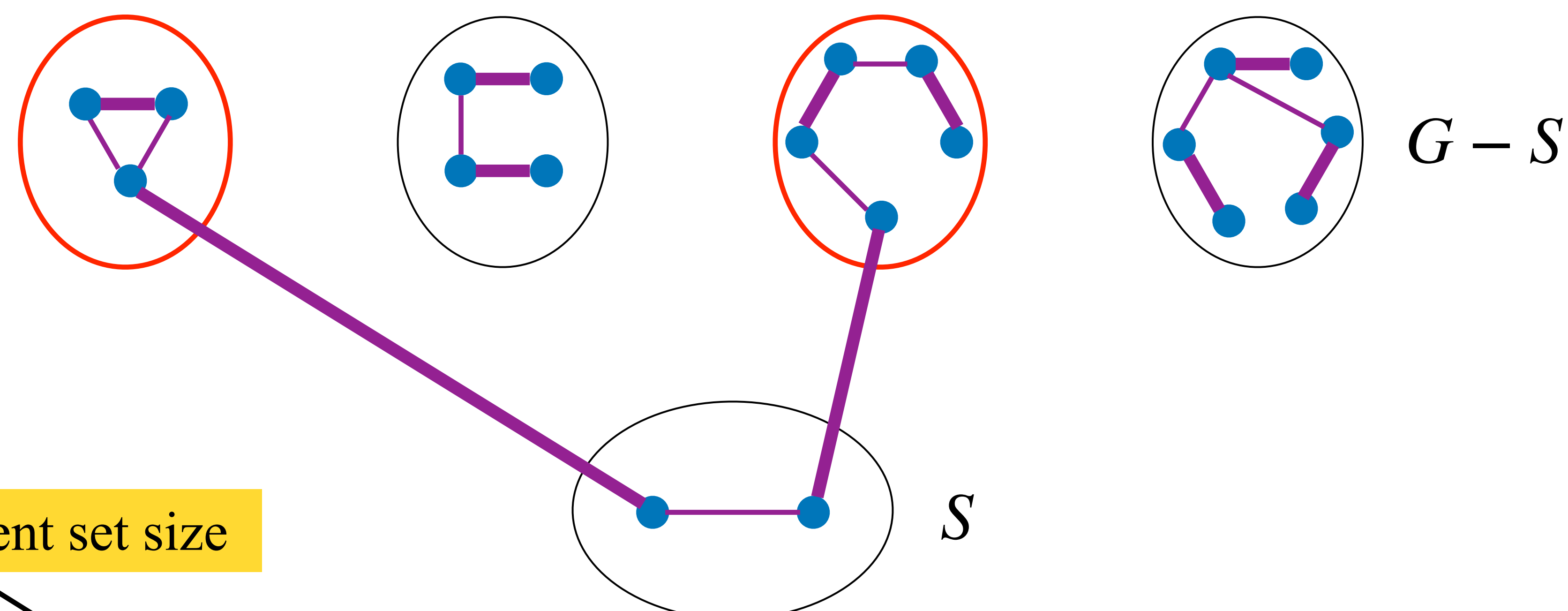


Maximum independent set size

Remark: $\mathrm{odd}(G - S) \leq \mathrm{Ind}(G) \Rightarrow$ It suffices to check $S$ with $|S| \leq \mathrm{Ind}(G)$

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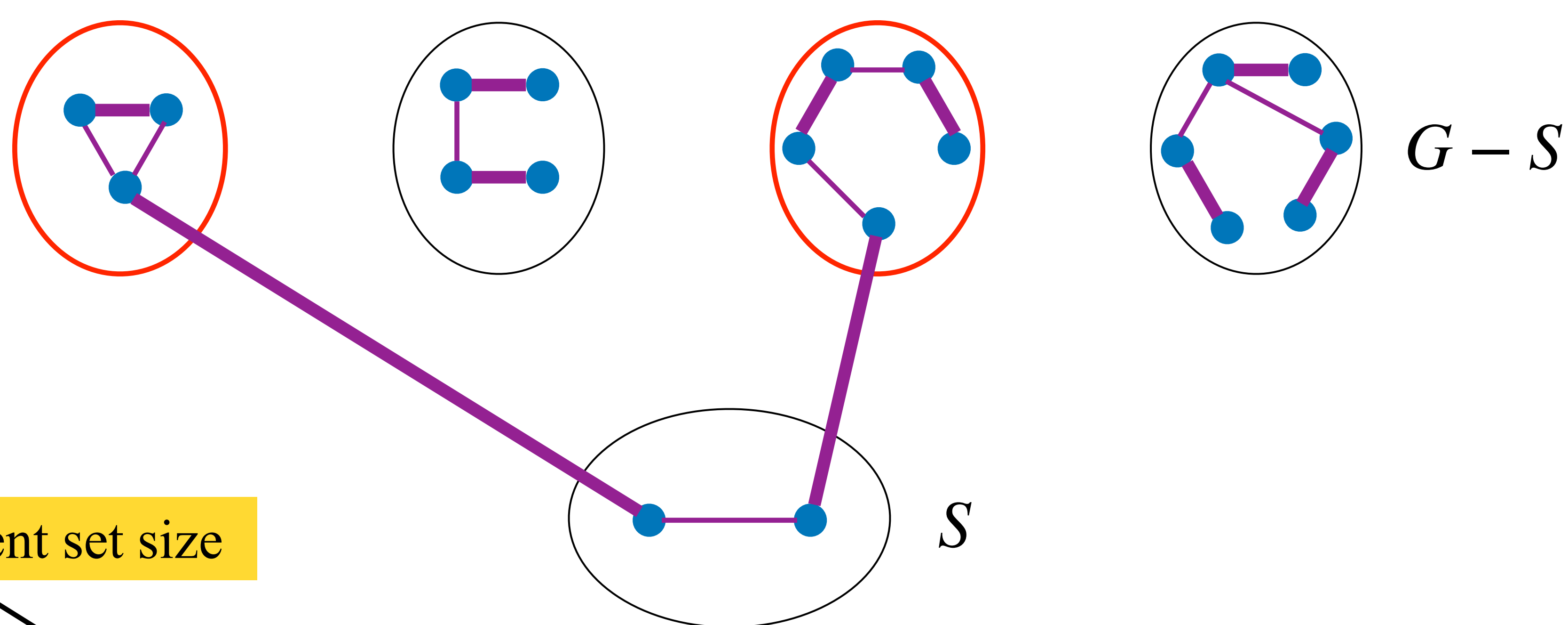


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For a random $d$-regular graph $G$, $\mathrm{Ind}(G) \leq 2n \log(d)/d$

# Constant Loss Regime: Ruling out Small Bottleneck $S$

- $S$ is called a bottleneck, if $\text{odd}(G^* - S) \geq |S| + 2$ , where $G^* := G[\text{supp}(D)]$
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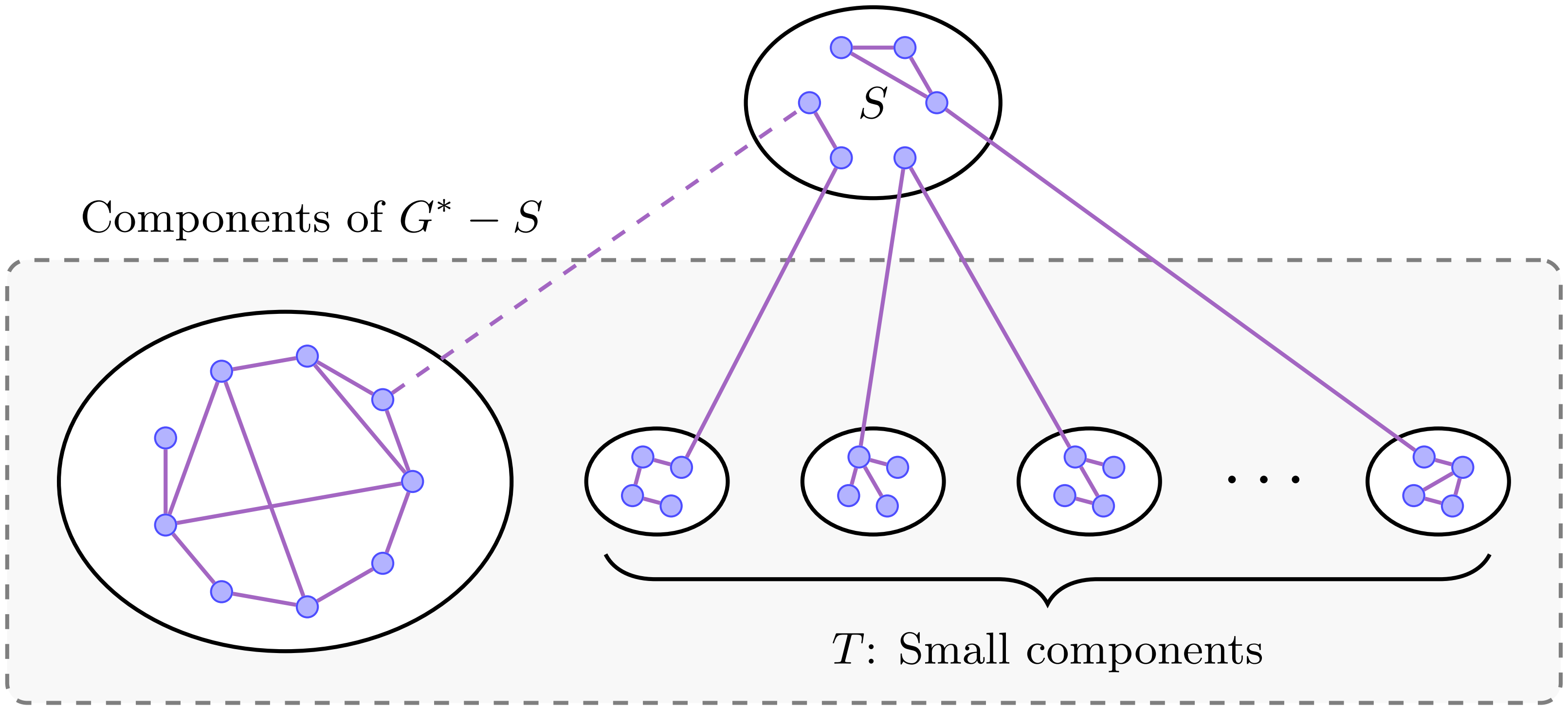

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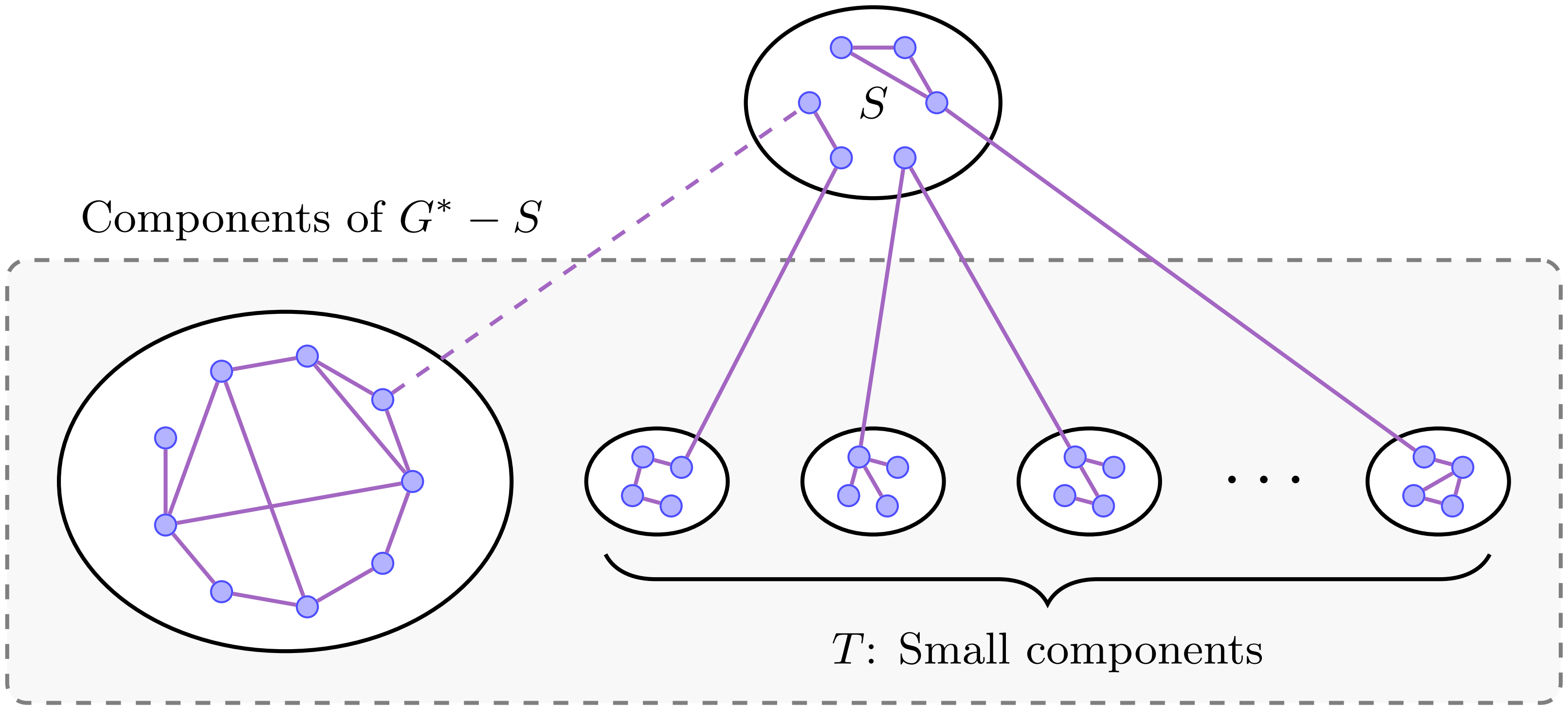


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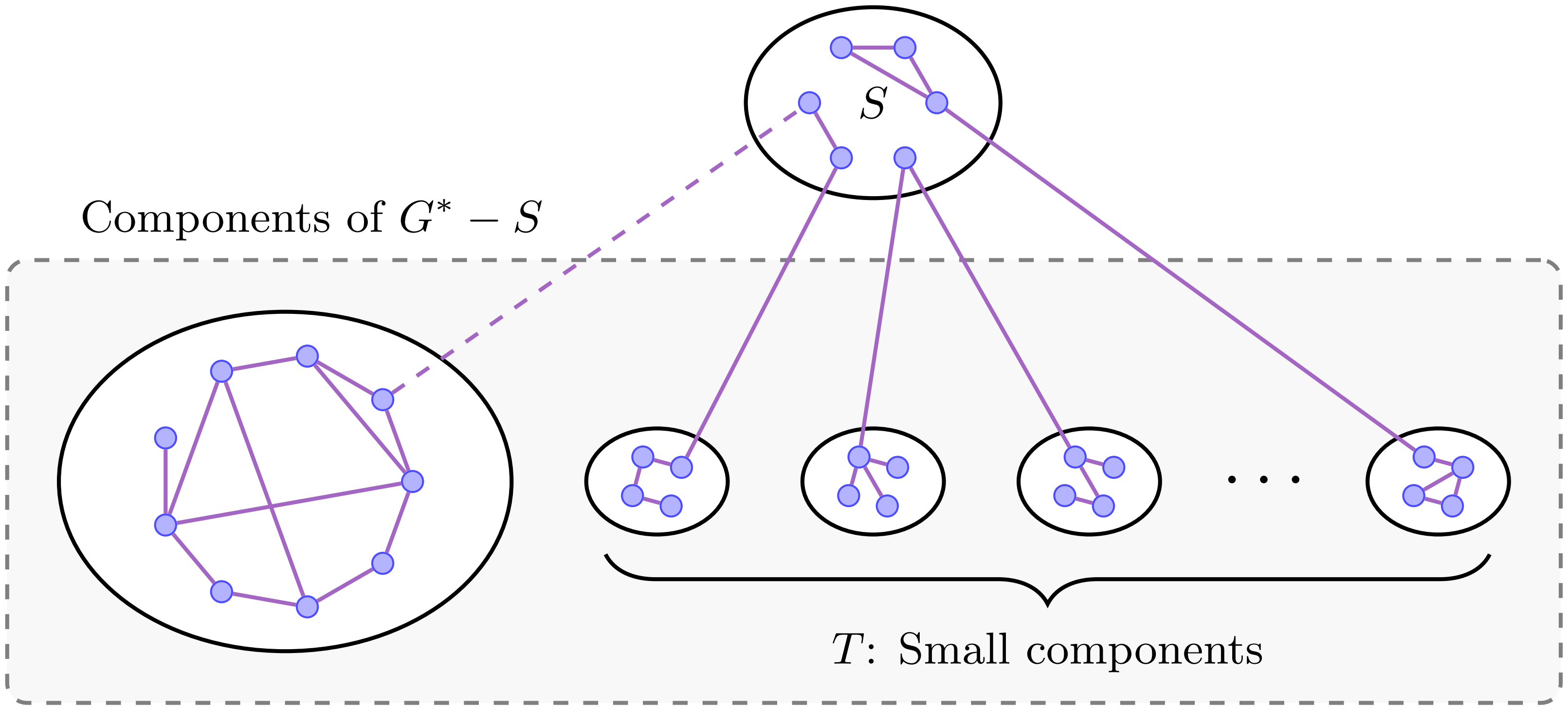


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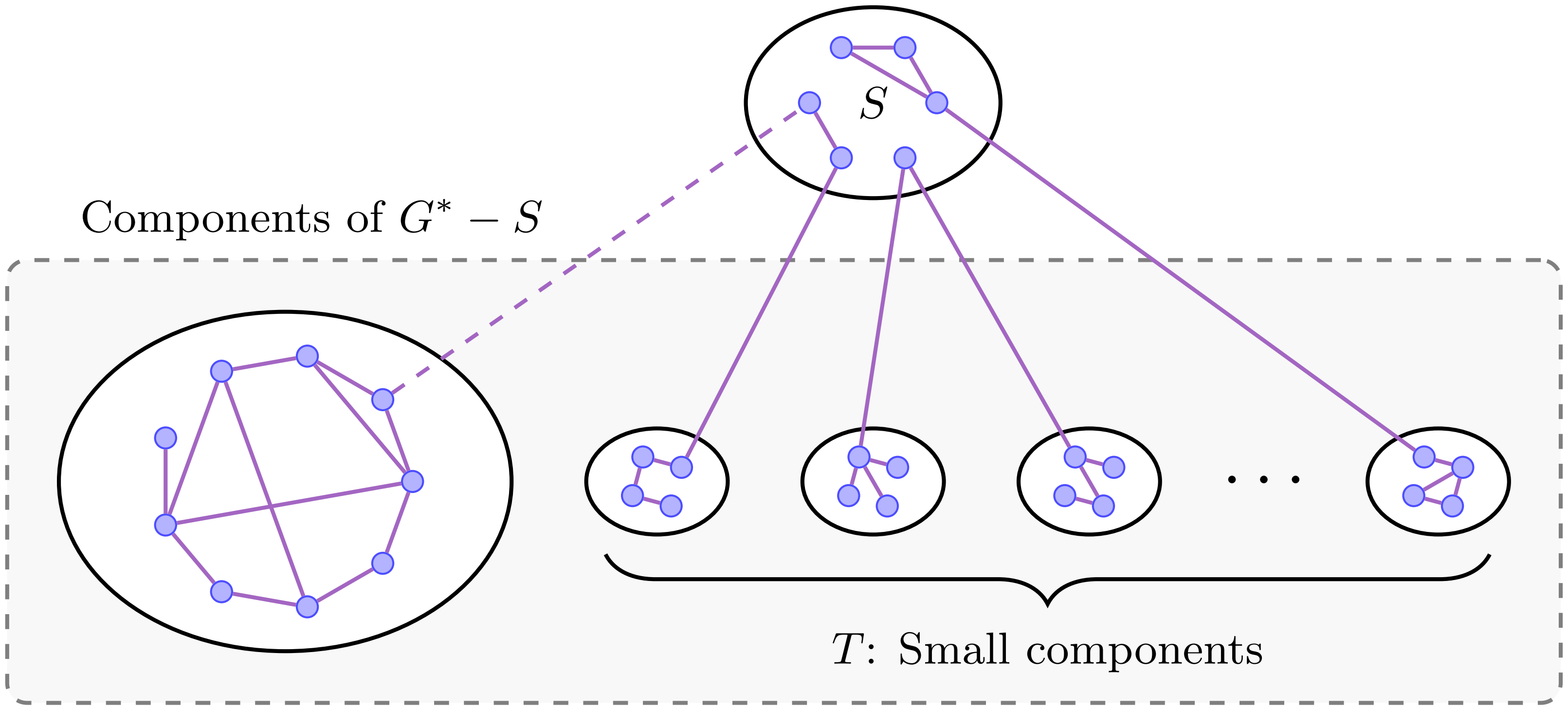


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- (Little congestion) $\Rightarrow$ since both $S$ and $T$ are small, $e(S, T) \leq |S|$ (contradiction)

# Fractional Loss Regime: Local Weak Convergence

Asymptotic size of maximum matching can be determined locally

**Theorem** [Bordenave-Lelarge-Salez '13]

Suppose $G_n$ admits a local weak convergence to a Galton-Watson tree with degree distribution $\pi$. The fraction of unmatched vertices in a maximum matching

$$\alpha(G_n) \xrightarrow{\mathbb{P}} \max_{t \in [0,1]} F(t),$$

where $F(t) = t\phi'(1-t) + \phi(1-t) + \phi\left(1 - \frac{\phi'(1-t)}{\phi'(1)}\right) - 1$ and $\phi(t) = \sum_k \pi_k t^k$.

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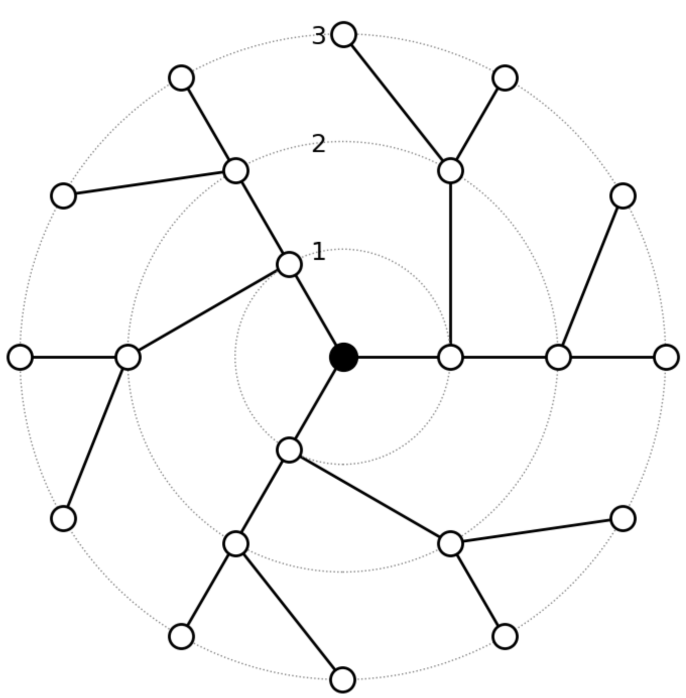


Local neighborhood of 3-regular graph

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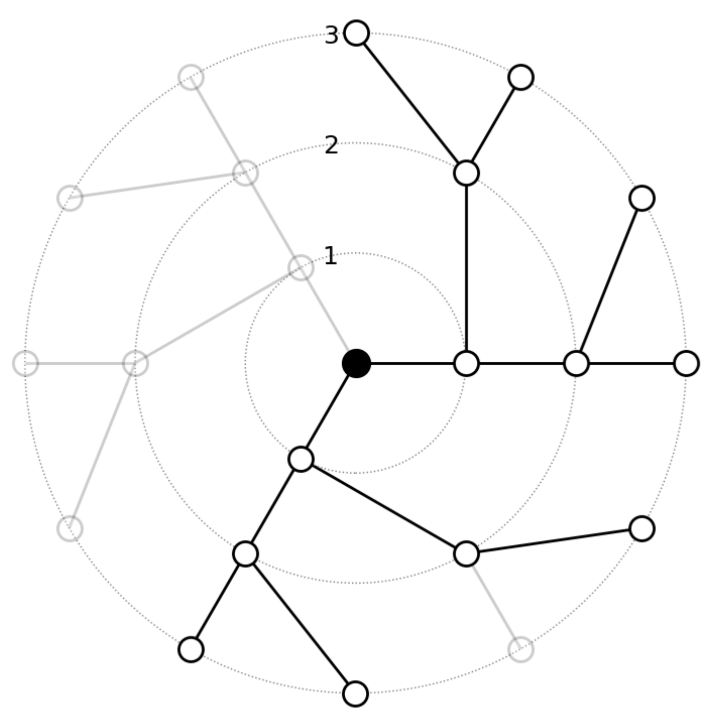


Local neighborhood of 3-regular graph after vertex deletion

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- In our setting, $G_n$ is the random $d$-regular graph with random vertex removal (with probability $q$). We show it converges locally to GW tree with degree distribution $\pi = \mathrm{Binom}(d, p)$ with $p = 1 - q$

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- Thus, $\phi(t) = (pt + 1 - p)^d$ and we show $F(t)$ has three extremum points with the first one being the maximum and $\max_{t \in [0,1]} F(t) = (1 + o_d(1))q^d$

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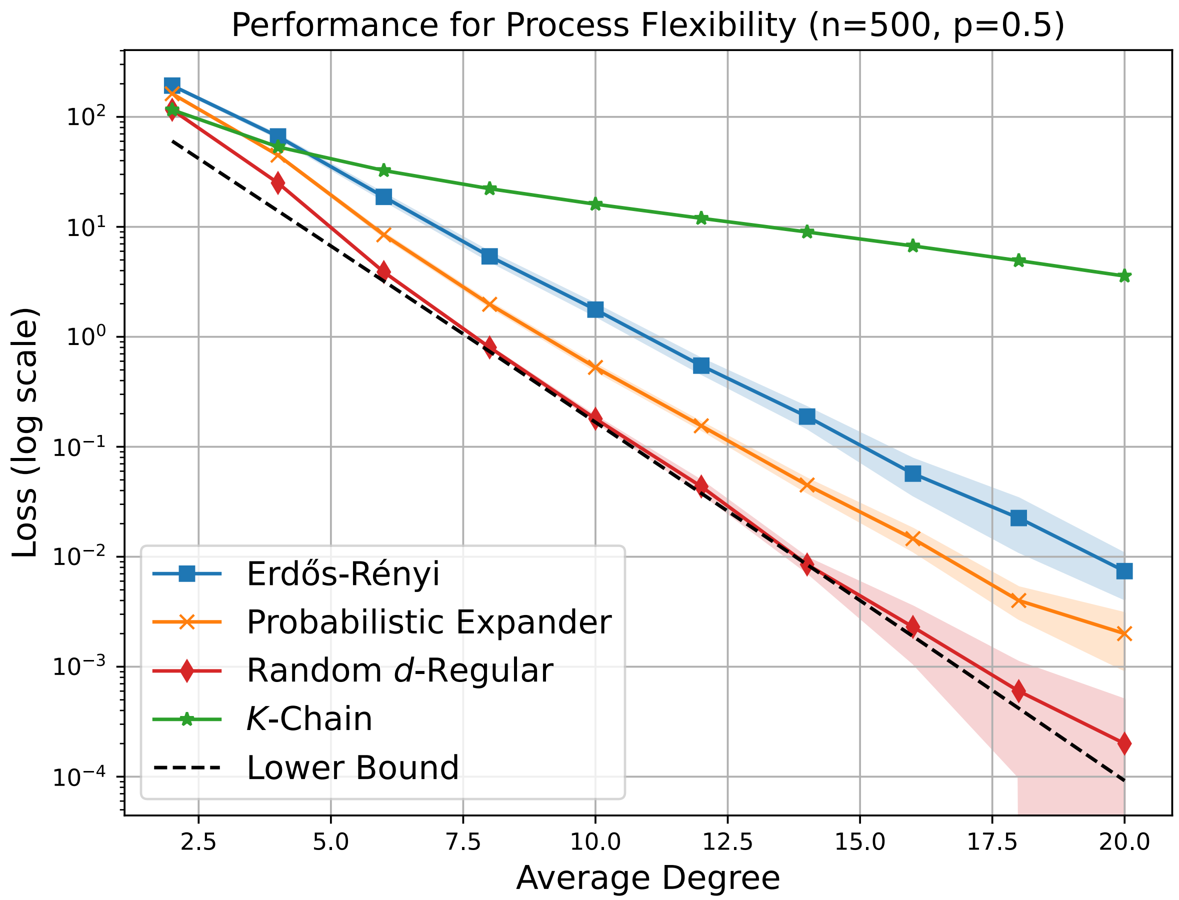
Suppose $G_n$ admits a local weak convergence to a Galton-Watson tree with degree distribution $\pi$. The fraction of unmatched vertices in a maximum matching

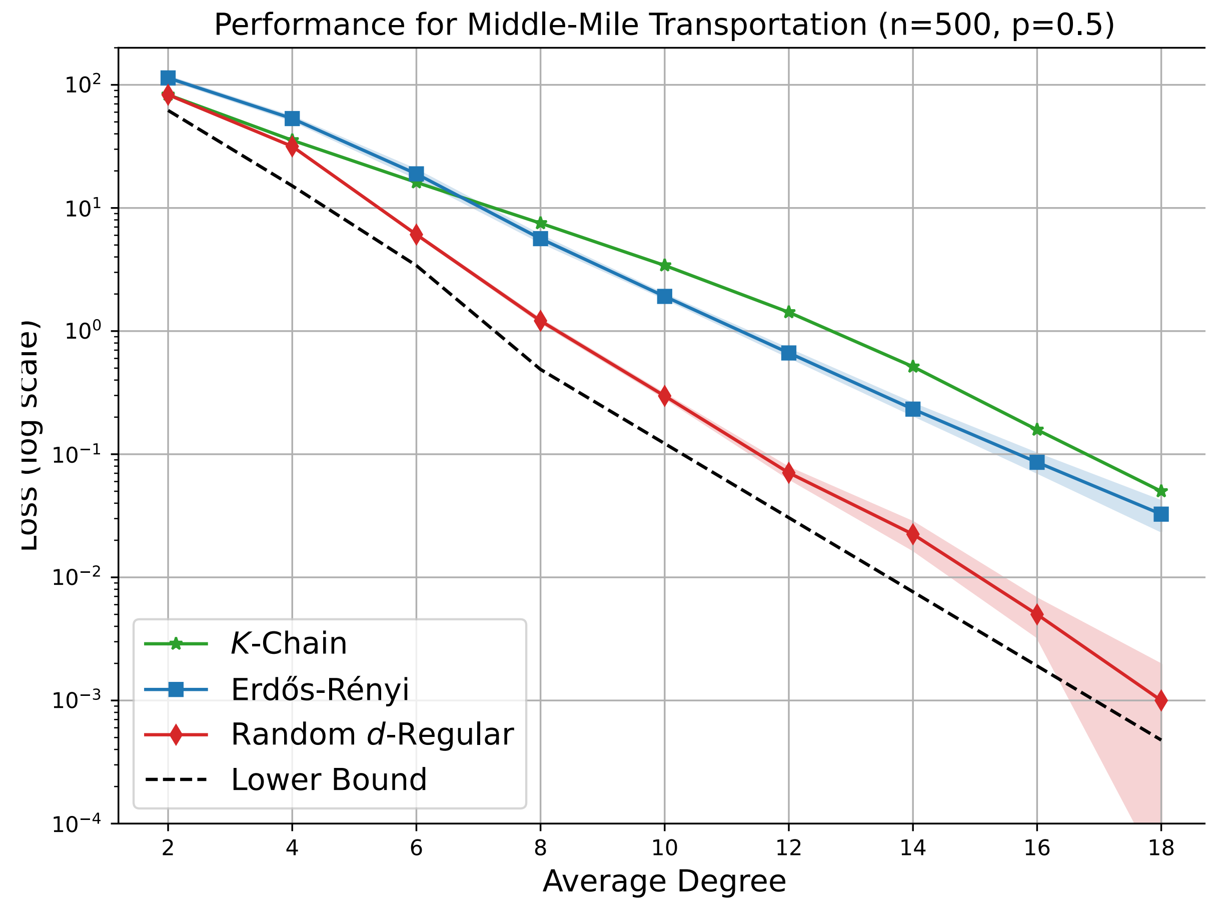
$$\alpha(G_n) \xrightarrow{\mathbb{P}} \max_{t \in [0,1]} F(t),$$

where $F(t) = t\phi'(1-t) + \phi(1-t) + \phi\left(1 - \frac{\phi'(1-t)}{\phi'(1)}\right) - 1$ and $\phi(t) = \sum_k \pi_k t^k$.

- In our setting, $G_n$ is the random $d$-regular graph with random vertex removal (with probability $q$). We show it converges locally to GW tree with degree distribution $\pi = \text{Binom}(d, p)$ with $p = 1 - q$
- Thus, $\phi(t) = (pt + 1 - p)^d$ and we show $F(t)$ has three extremum points with the first one being the maximum and $\max_{t \in [0,1]} F(t) = (1 + o_d(1))q^d$
- So, when $d \geq (1 + \delta)\log(1/\epsilon)/\log(1/q)$, $\alpha(G_n) \leq \epsilon$ and thus $\mathbb{E}_D[L(G, D)] \leq \epsilon n$

# Numerical Experiments




**Random regular graphs** can achieve performance close to the theoretical lower bounds even for moderate graph sizes and relatively small degrees

# Summary

| | Setting | Graph design | $\epsilon n$-loss | Constant loss |
|---|---|---|---|---|
| Lower bound | Both | Any | $\dfrac{\log(1/\epsilon)}{\log(1/q)}$ | $\dfrac{\log(n)}{\log(1/q)}$ |
| Chou et al. (2011) | Bipartite | Chain | $\Omega(1/\epsilon)$ | - |
| Chen et al. (2015) | Bipartite | Probabilistic Expander | $\Omega(\log(1/\epsilon))$ | - |
| Feng et al. (2024) | Unipartite | Ring; Chain; Cluster; Erdős–Rényi | $\Omega(\log(1/\epsilon))$ | $\Omega(\log(n))$ |
| Our work | Both | Random regular | $\dfrac{\log(1/\epsilon)}{\log(1/q)}^{*}$ | $\dfrac{\log(n)}{\log(1/q)}^{*}$ |

*Within a factor of $(1+\delta)$ for arbitrarily small $\delta > 0$.

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| Our work | Both | Random regular | $\frac{\log(1/\epsilon)}{\log(1/q)}$ * | $\frac{\log(n)}{\log(1/q)}$ * |

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**Future directions:**

General demand distribution? Other flexibility network design problems?

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**Reference:**

W. Li, X. Niu, Y. Wei, & J. Xu, *Optimality of Random Regular Graphs in Sparse Network Designs*, SSRN Preprint, March 2026