Reconstruction in the Sparse Labeled Stochastic Block Model

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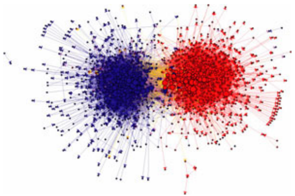
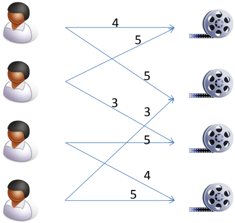
Motivation: community detection

Identify underlying communities based on the pairwise interactions represented by graph

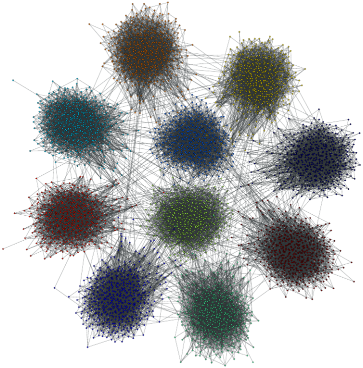


Network of political webblogs [Adamic-Glance '05]



Recommendation system

A popular model: stochastic block model



Graph generated from stochastic block model with $n = 5000$, $r = 10$, $p = 0.999$, $q = 0.001$. Ref. https://projects.skewed.de/graph-tool.

Model we use: sparse labeled stochastic block model

Two important aspects received little attention:

1. Sparse graph: Limited amount of interaction
2. Interaction can be of multiple types: ratings can be 1 to 5

We use sparse labeled stochastic block model:

1. Nodes have bounded average degree: edge probabilities $p, q = O(1/n)$.
2. Edges carry label: label represents interaction type

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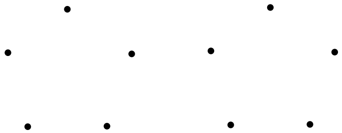
Sparse labeled stochastic block model: two symmetric communities

A random graph model on n nodes with two constants, $a, b \geq 0$ and two discrete prob. distributions, μ, ν.



$n = 10$

Sparse labeled stochastic block model: two symmetric communities

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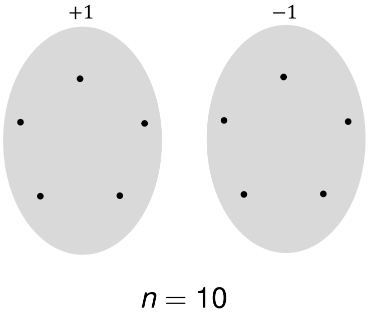
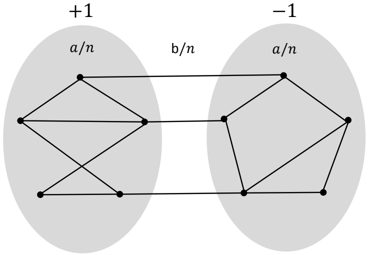
Assign each node to community $+1$ or -1 uniformly at random.



Sparse labeled stochastic block model: two symmetric communities

A random graph model on n nodes with two constants, $a, b \geq 0$ and two discrete prob. distributions, μ, ν.

Independently for each pair of nodes:
Draw an edge w.p. a/n if they are in the same community;
w.p. b/n otherwise.

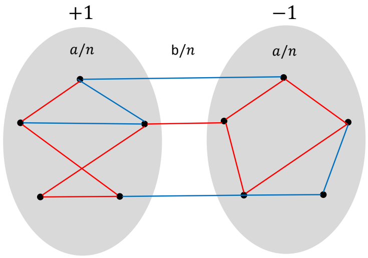


$a = 4, b = 2$

Sparse labeled stochastic block model: two symmetric communities

A random graph model on n nodes with two constants, $a, b \geq 0$ and two discrete prob. distributions, μ, ν.

Independently for each edge: Label the edge w.d. μ if the two endpoints are in the same community; w.d. ν otherwise.



$\mu(r) = 0.6, \mu(b) = 0.4$
$\nu(r) = 0.4, \nu(b) = 0.6$

Correlated reconstruction of community assignment

Isolated nodes render the exact reconstruction impossible.

Focus on correlated reconstruction, i.e., agrees with the true community assignment in more than half of all nodes.

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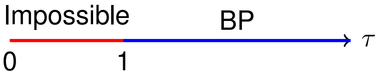
Phase transition for correlated reconstruction

[Heimlicher et al. '12] conjectures:



$$\tau = \frac{a+b}{2} \sum_{\ell \in \mathcal{L}} \frac{a\mu(\ell) + b\nu(\ell)}{a+b} \left(\frac{a\mu(\ell) - b\nu(\ell)}{a\mu(\ell) + b\nu(\ell)} \right)^2$$

It generalizes the conjecture in stochastic block model [Decelle et al. '11] [Mossel et al. '12].

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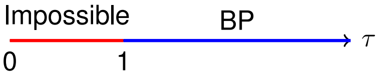

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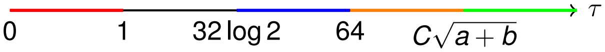
Our contribution



Impossible, MinBisection, Spectral*, Spectral

(A) Prove the impossibility result: correlated reconstruction is impossible if $\tau < 1$.

(B) Prove the achievability result: correlated reconstruction is possible if $\tau > 32 \log 2$ by MinBisection algorithm.

(C) Propose a polynomial-time Spectral algorithm and show it achieves correlated reconstruction if τ is large enough.

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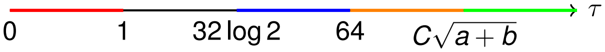

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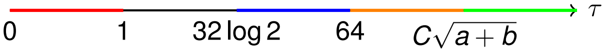

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(A) Impossibility for correlated reconstruction if $\tau < 1$

First step of proof: correlated reconstruction $\Longrightarrow$ long range correlation: The community memberships of any two nodes randomly chosen are asymptotically correlated conditional on the labeled graph [Mossel et al. '12].

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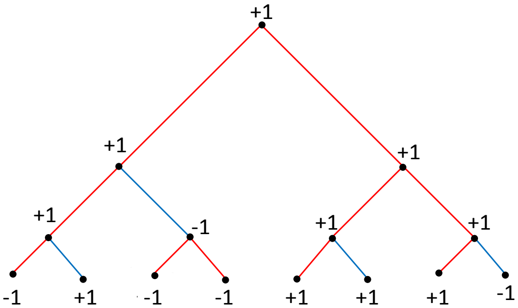
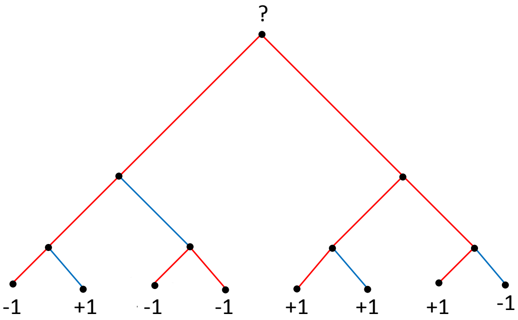
Second step of proof: The small local neighborhood of a node "looks like" a labeled Poisson tree [Mossel et al. '12].



(A) Impossibility for correlated reconstruction if $\tau < 1$

Third step: The community memberships of leaf nodes is uninformative on the community membership of root asymptotically [Heimlicher et al. '12].



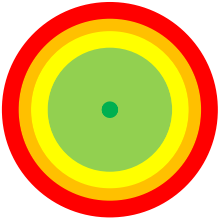
(B) Achievability of correlated reconstruction if $\tau > 32 \log 2$

Consider minimum bisection algorithm: Find two equal-sized parts with the minimum sum of weights of edges between parts with weight function $w(\ell) = \frac{a\mu(\ell) - b\nu(\ell)}{a\mu(\ell) + b\nu(\ell)}$.

Theorem

solutions of minimum bisection are correlated with the true community assignment if $\tau > 32 \log 2$ and $a \sum_\ell \mu(\ell) w^2(\ell), b \sum_\ell \nu(\ell) w^2(\ell) > 8 \ln 2$.

Proof uses the Chernoff bound and the weight function $w(\ell)$ is chosen optimally.



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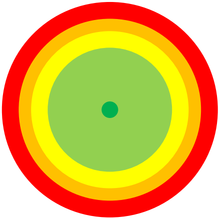

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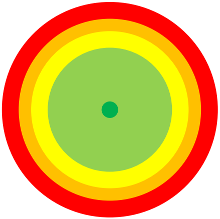

(C) Polynomial-time reconstruction for τ large enough

A polynomial-time algorithm exploiting the spectrum of weighted adjacency matrix W with weight function $w(\ell) = \frac{a\mu(\ell) - b\nu(\ell)}{a\mu(\ell) + b\nu(\ell)}$

- The expectation of W conditional on the true community assignment σ satisfies:

$$\mathbb{E}[W|\sigma] = \frac{a-b}{2n}\mathbf{1}\mathbf{1}^\top + \frac{\tau}{n}\sigma\sigma^\top.$$

- If $W \approx \mathbb{E}[W|\sigma]$, then the top left singular vector of $W - \frac{a-b}{2n}\mathbf{1}\mathbf{1}^\top$ approximates σ.
- Due to nodes with high degrees $\Omega(\frac{\log n}{\log \log n})$, W is not concentrated around its conditional mean.

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Spectral-Reconstruction algorithm: Input a, b, μ, ν, W; Output community assignment.

1. Remove nodes with degree greater than $\frac{3}{2}(a+b)$ and assign random community to these nodes.
2. Define W' by setting to zero the rows and columns of W corresponding to the nodes removed.
3. Let $\hat{x}$ denote the left singular vector associated with the largest singular values of $W' - \frac{a-b}{n}\mathbf{1}\mathbf{1}^\top$. Output $\mathrm{sign}(\hat{x})$ as the community assignment for remaining nodes.

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Theorem

Spectral-Reconstruction algorithm outputs a community assignment correlated with the true one if $\tau > C\sqrt{a+b}$.

Proof:

- The spectral norm of $W' - \mathbb{E}[W|\sigma]$ is $O(\sqrt{a+b})$. Ref. [Feige-Ofek '05], [Coja-Oghlan '10].
- The L_2 distance between $\sigma/\sqrt{n}$ and $\hat{x}$ is bounded by $O(\|W' - \mathbb{E}[W|\sigma]\|/\tau)$.

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Better performance guarantee if $\max\{a, b\} = \Omega(\log^6 n)$.

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Proof: $\|W - \mathbb{E}[W|\sigma]\| \leq 2\sqrt{2\tau}$. Ref. [Vu '07].

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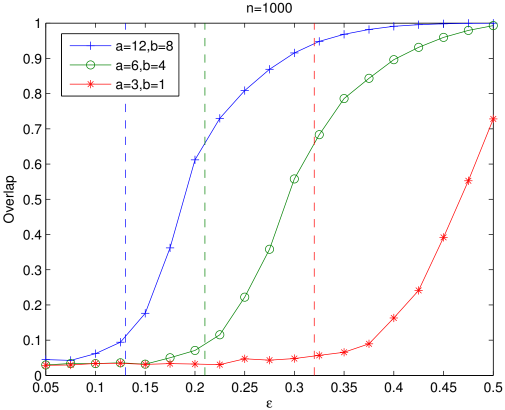
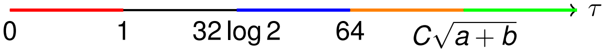
Simulation of Spectral-Reconstruction algorithm



Two labels: r and b with $\mu(\mathrm{r}) = 0.5 + \epsilon$ and $\nu(\mathrm{r}) = 0.5 - \epsilon$. The threshold $\tau = 1$ is depicted as a vertical dash line.

Summary of our results



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Future work

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