Recent advances on random graph matching

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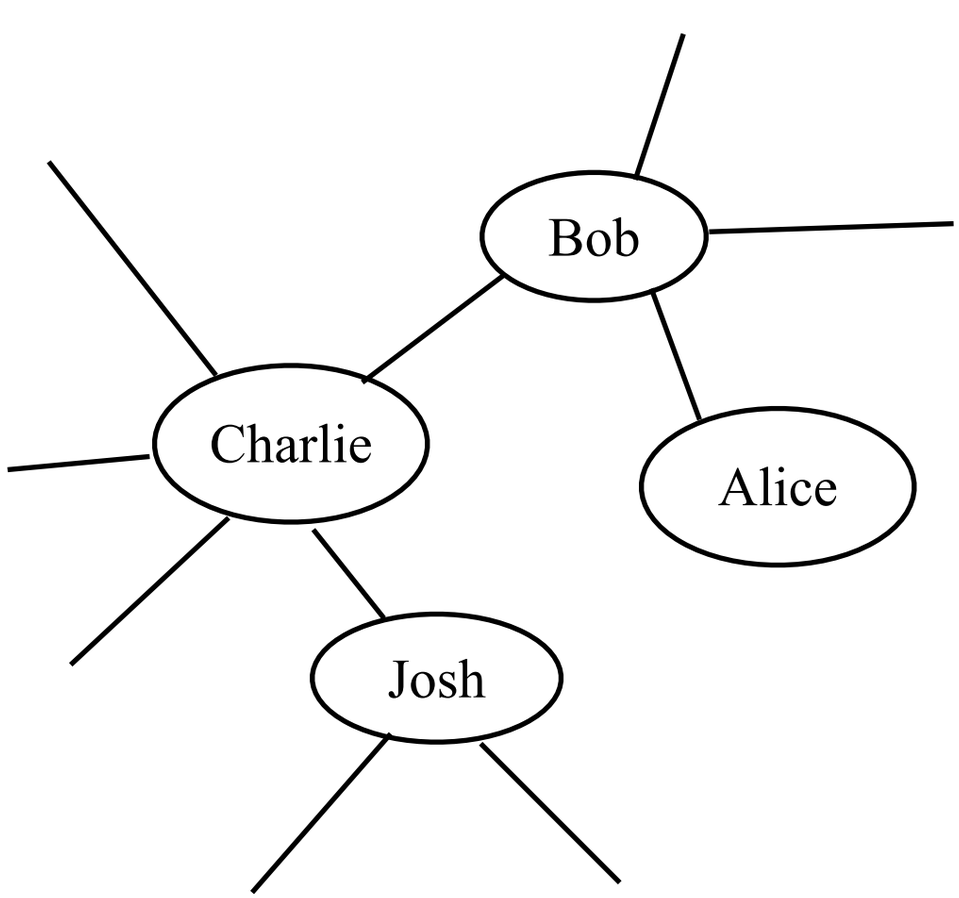
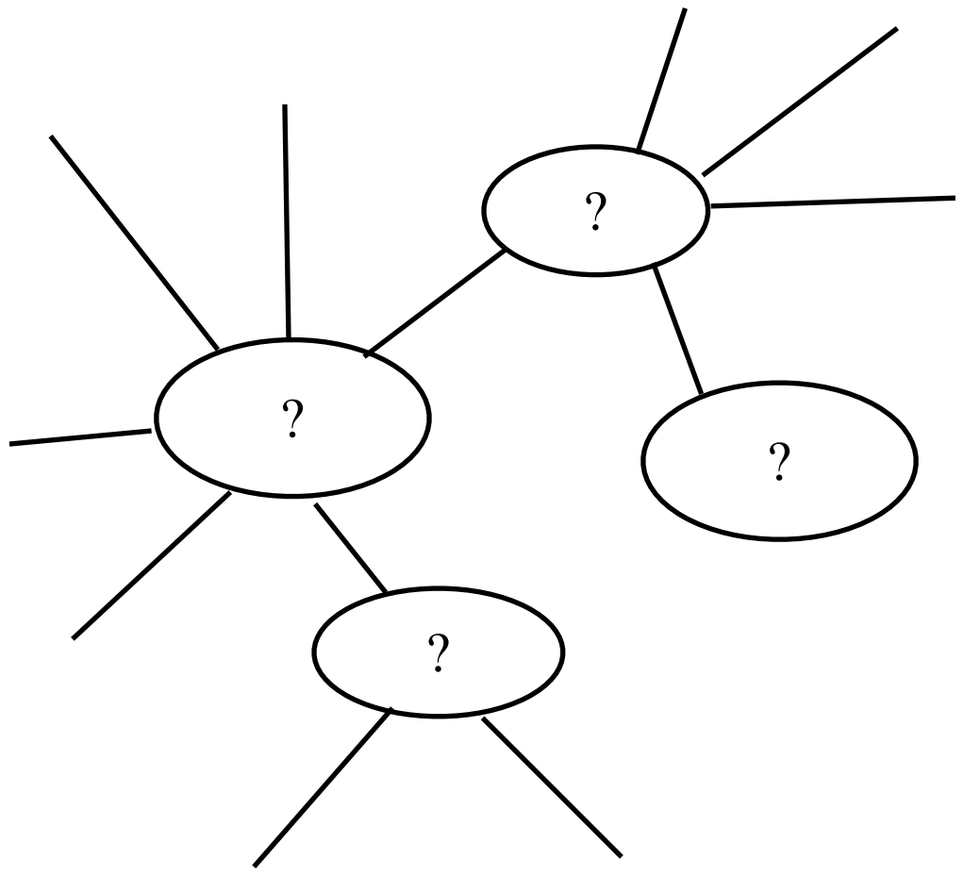
Stochastic Networks Conference 2024

Outline

- Motivation and problem setup
- Information-theoretic limits
- Efficient algorithms
- Concluding remarks

Motivating example in network de-anonymization





flickr

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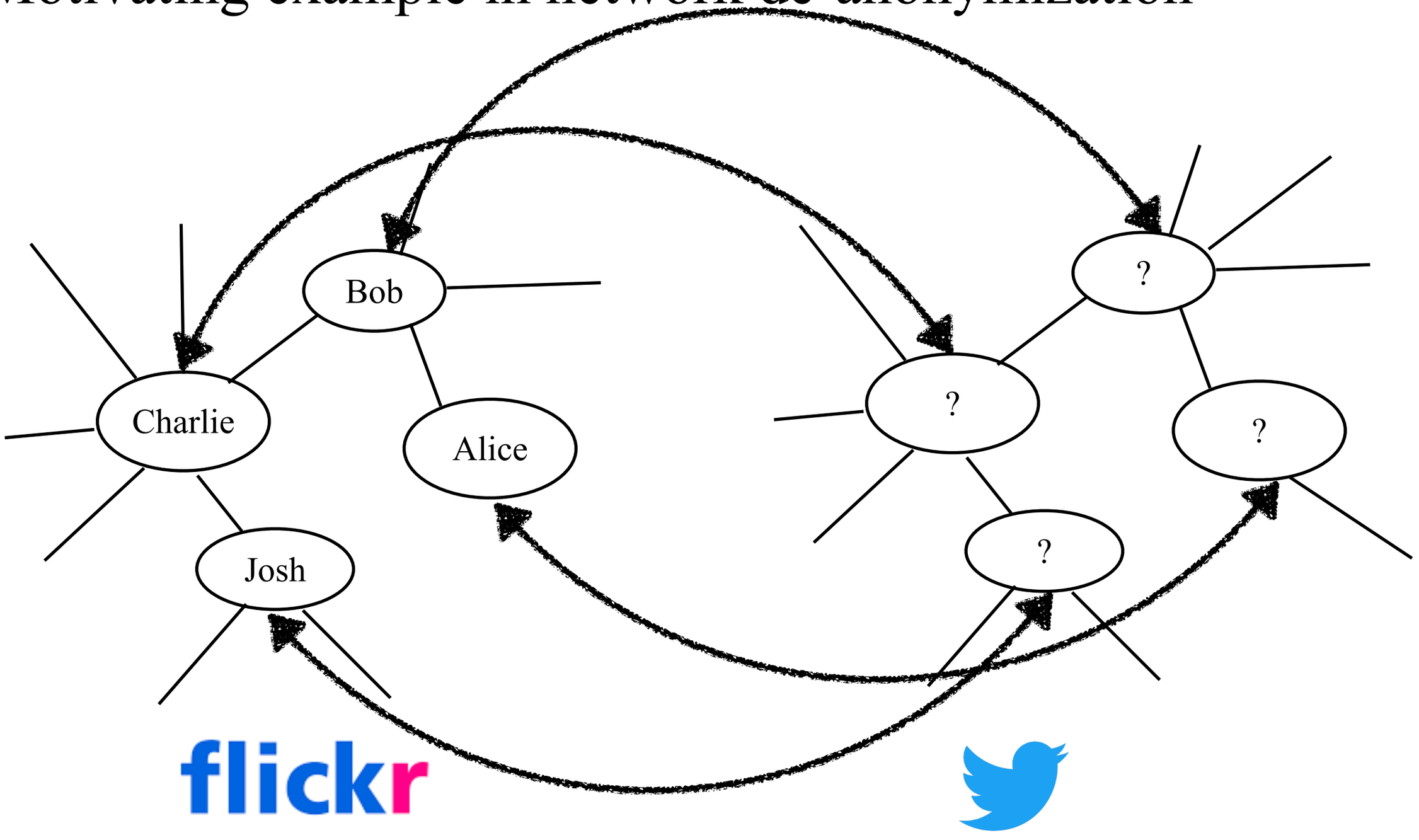

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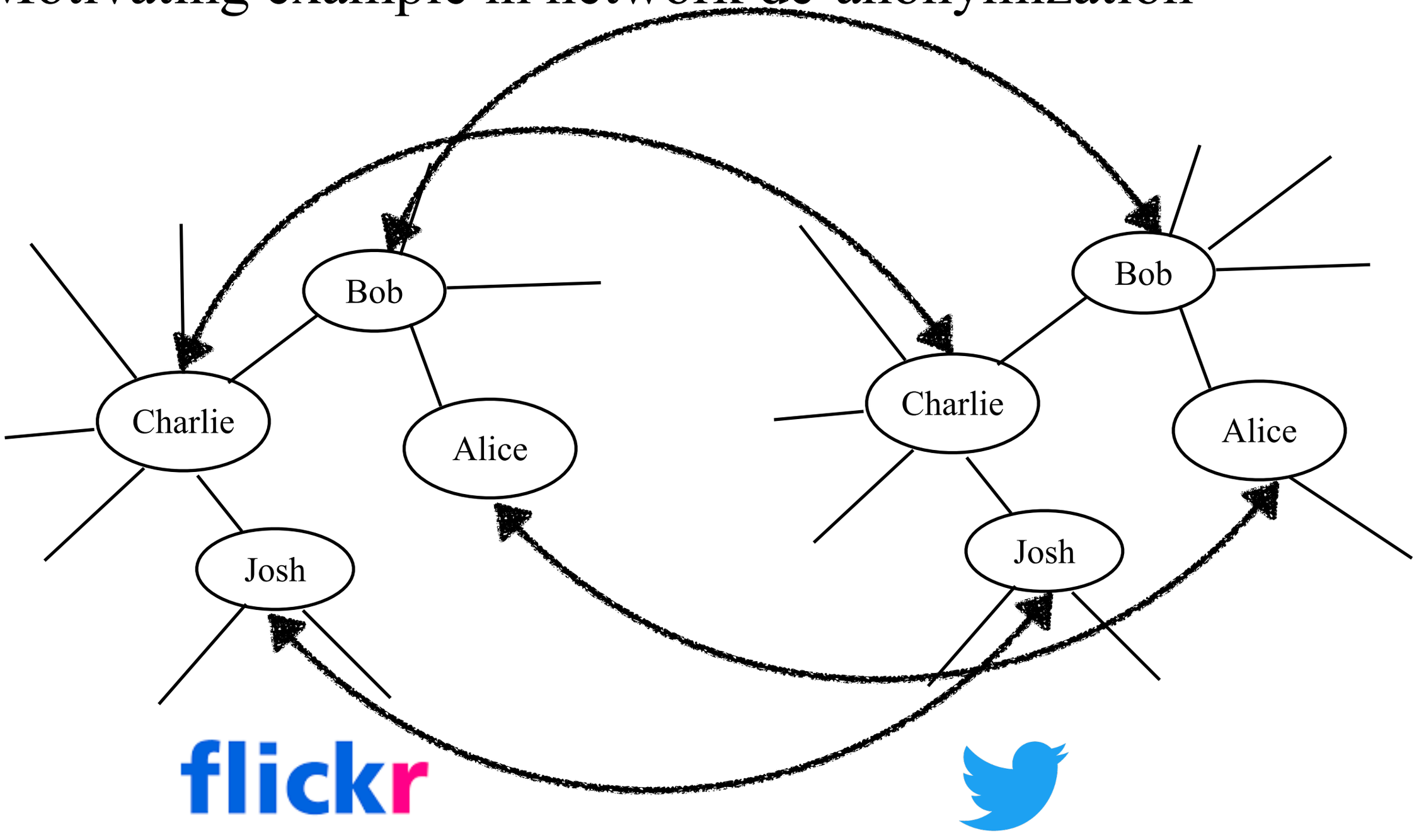

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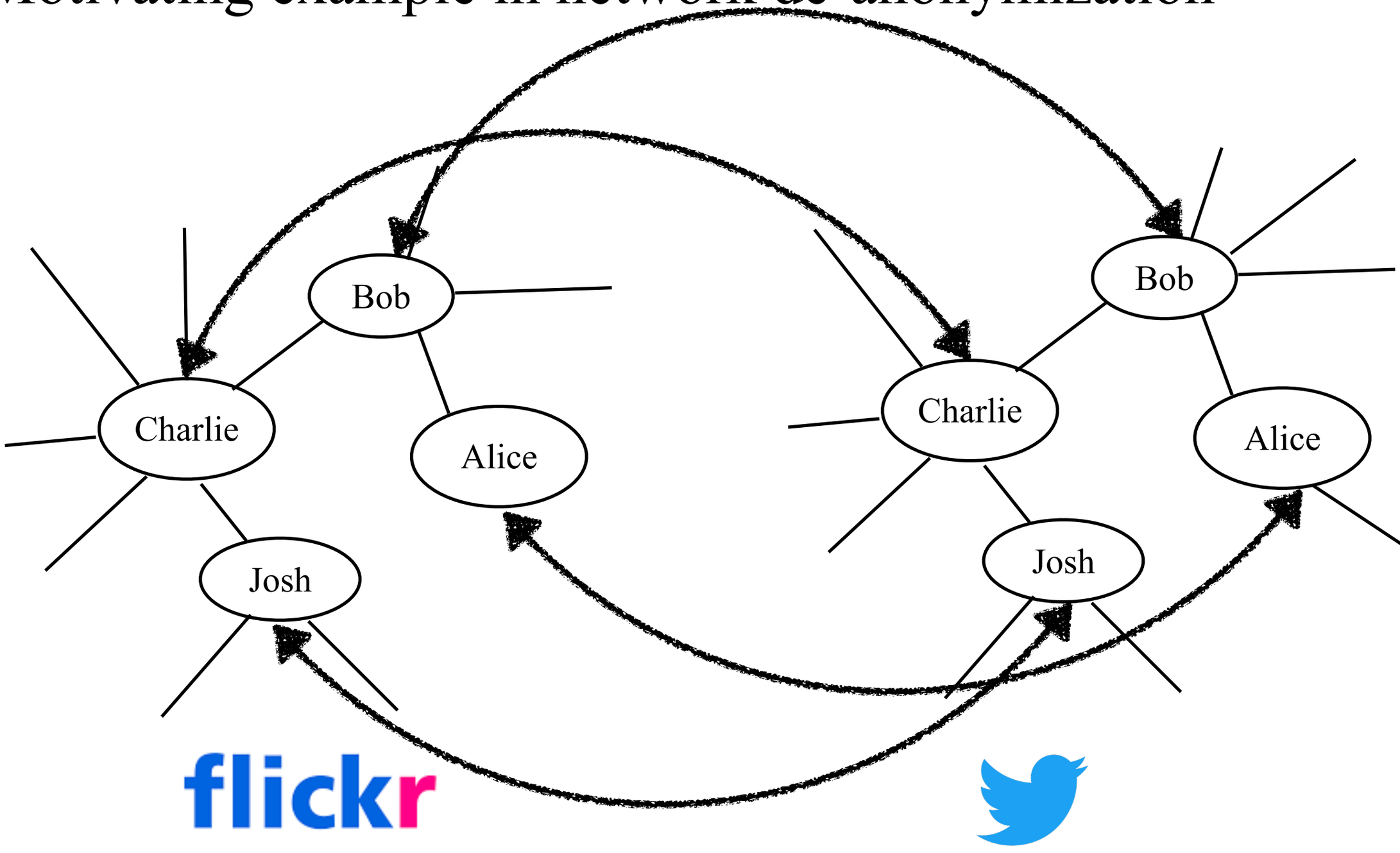
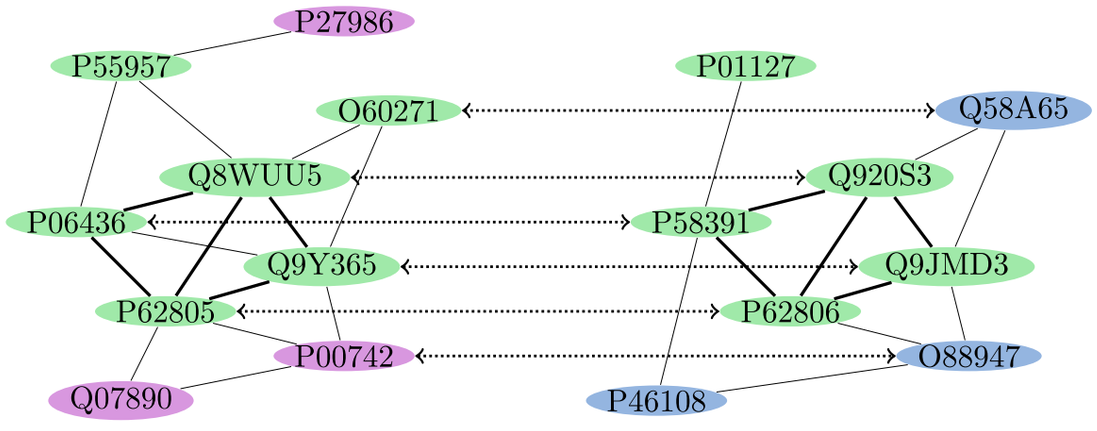


- Narayanan and Shmatikov correctly identified a fraction of users between Twitter and Flickr in 2009.

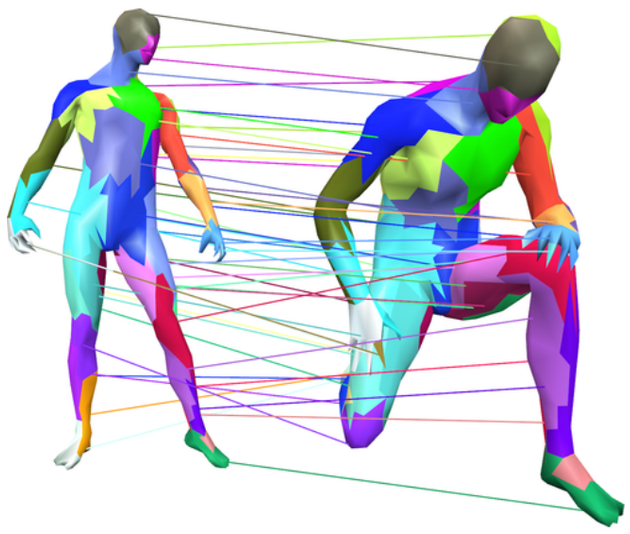
Applications

- Biology [Singh-Xu-Berger ’2008; Kazemi et al. ’2016]

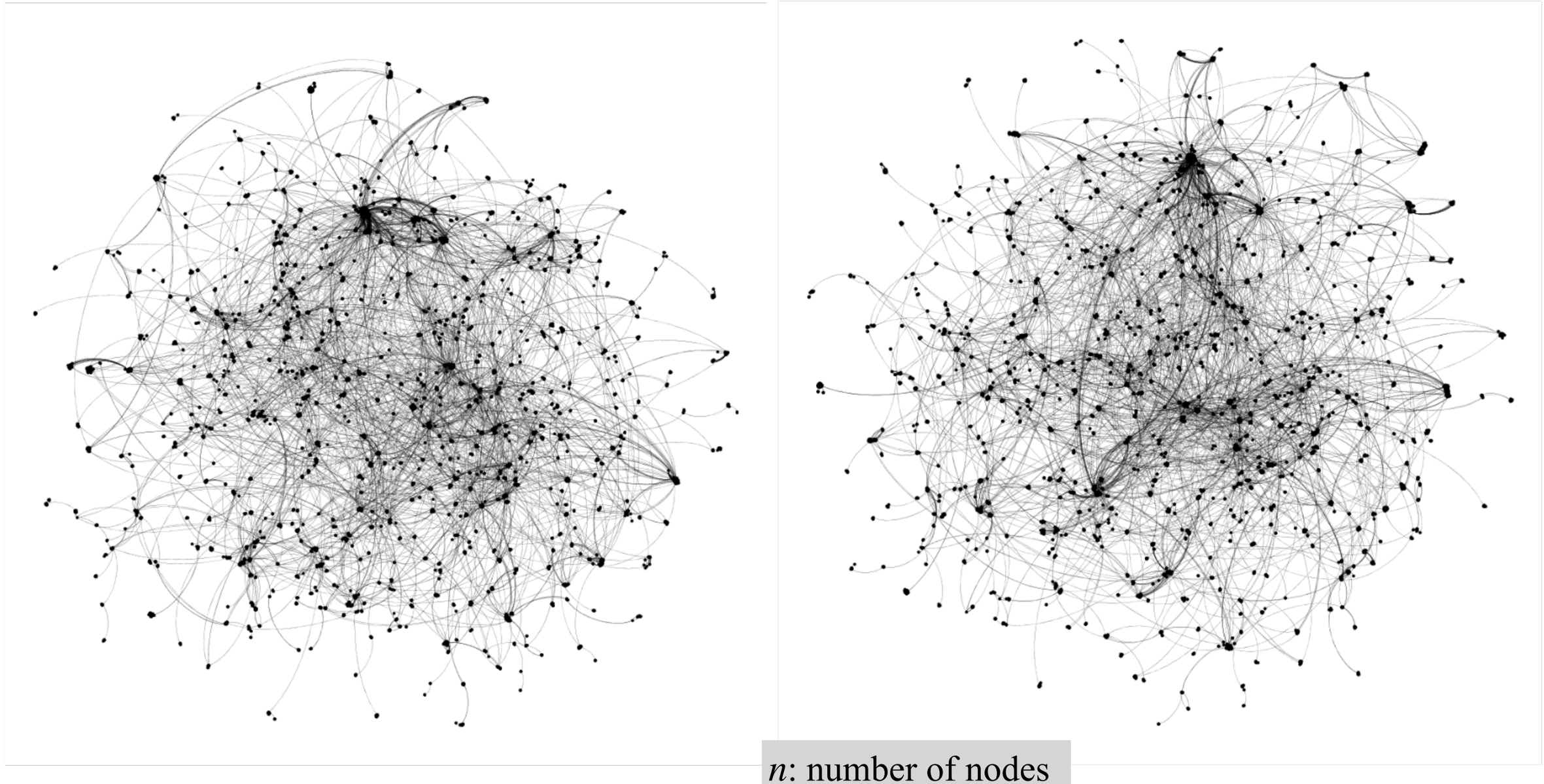


Human network Mouse network

- Computer Vision [Lähner et al. ’2016; Fan-Mao-Wu-Xu ’2020]



Real-world challenges



n: number of nodes

- **Computational:** # of possible node mappings is $n!$ $(100! \approx 10^{158})$;
- **Statistical:** two graphs may be correlated but not exactly the same;

Beyond worst-case intractability

- **NP-hard** for matching two graphs in worst case
- However, real networks are not arbitrary and have latent structures

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- **NP-hard** for matching two graphs in worst case
- However, real networks are not arbitrary and have latent structures
- Recent surge of interest on the average-case analysis
  - [Cullina-Kiyavash ’16,17]
  - [Cullina-Kiyavash-Mittal-Poor ’19, Dai-Cullina-Kiyavash-Grossglauser ’19, Ding-Ma-Wu-Xu ’18, Barak-Chou-Lei-Schramm-Sheng ’19, Fan-Mao-Wu-Xu ’19a,19b]
  - [Ganassali-Massoulié ’20, Hall-Massoulié ’20, Wu-Xu-Yu ’21, Ganassali-Lelarge-Massoulié ’21, Mao-Rudelson-Tikhomirov ’21a, 21b]
  - [Ding-Du ’21, 22, Mao-Wu-Xu-Yu ’22, Ganassali-Massoulié-Semerjian ’22]
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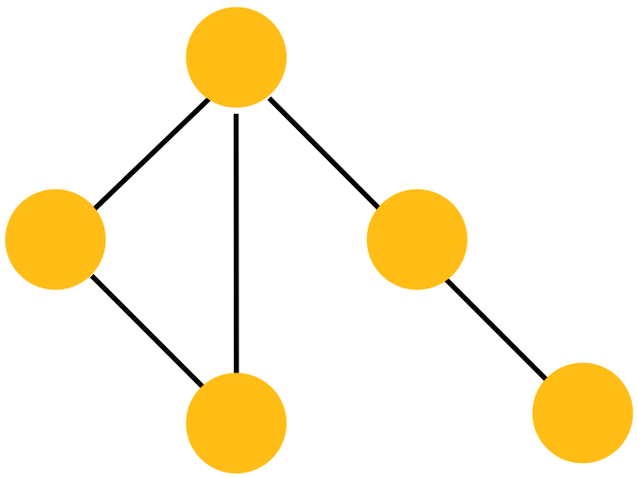
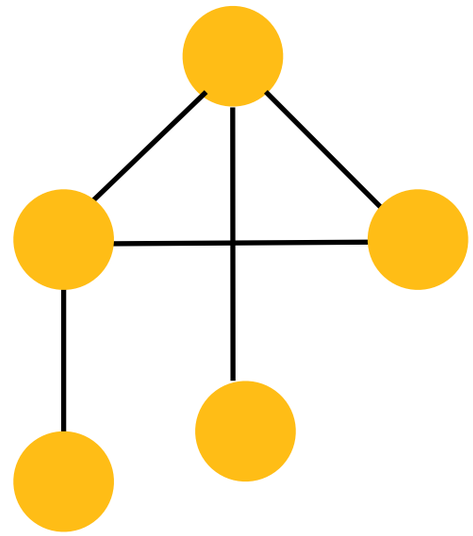


$A \sim \mathcal{G}(n, q)$



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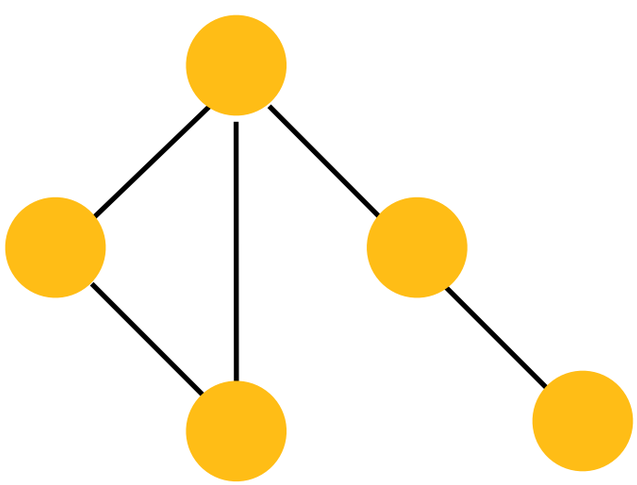

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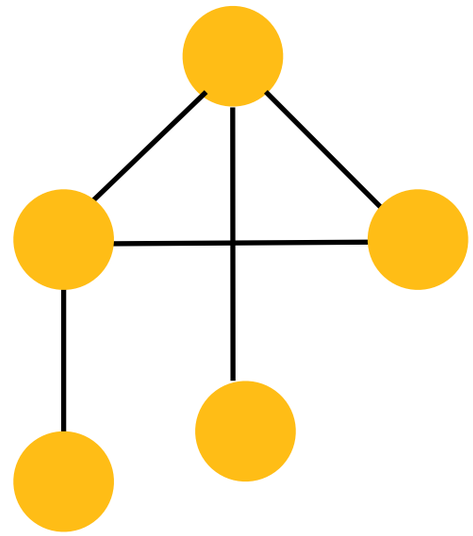

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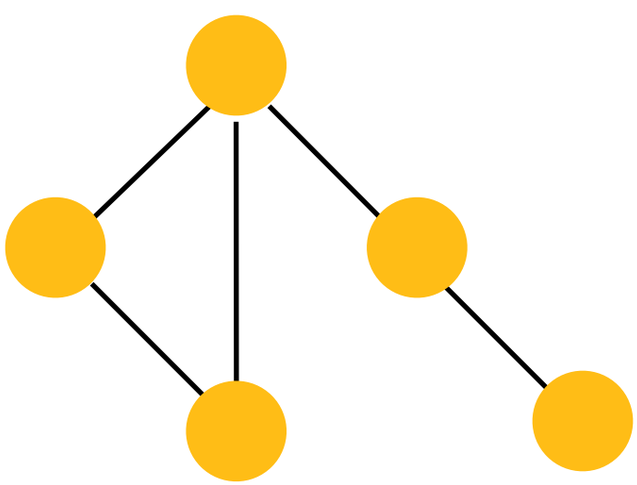

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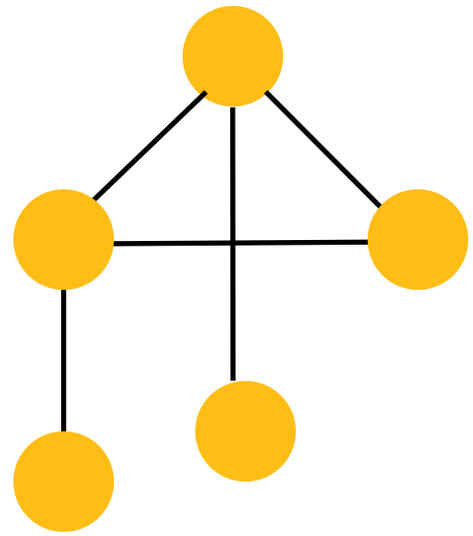
Each two nodes are connected with probability q



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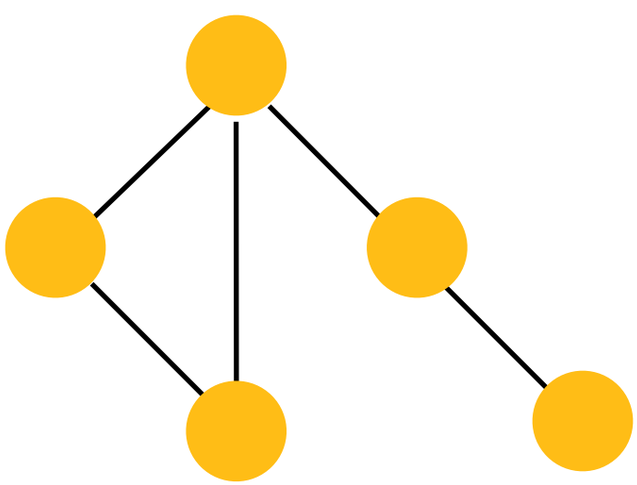

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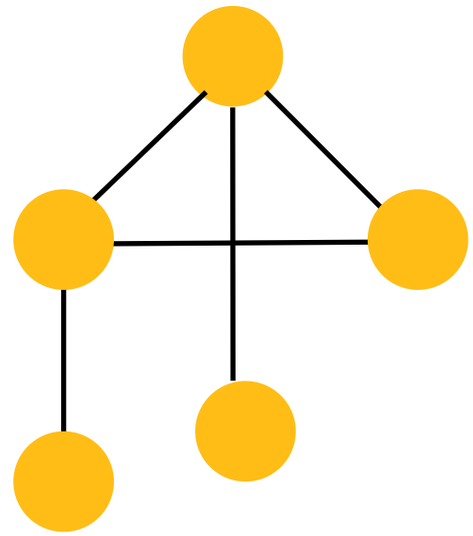

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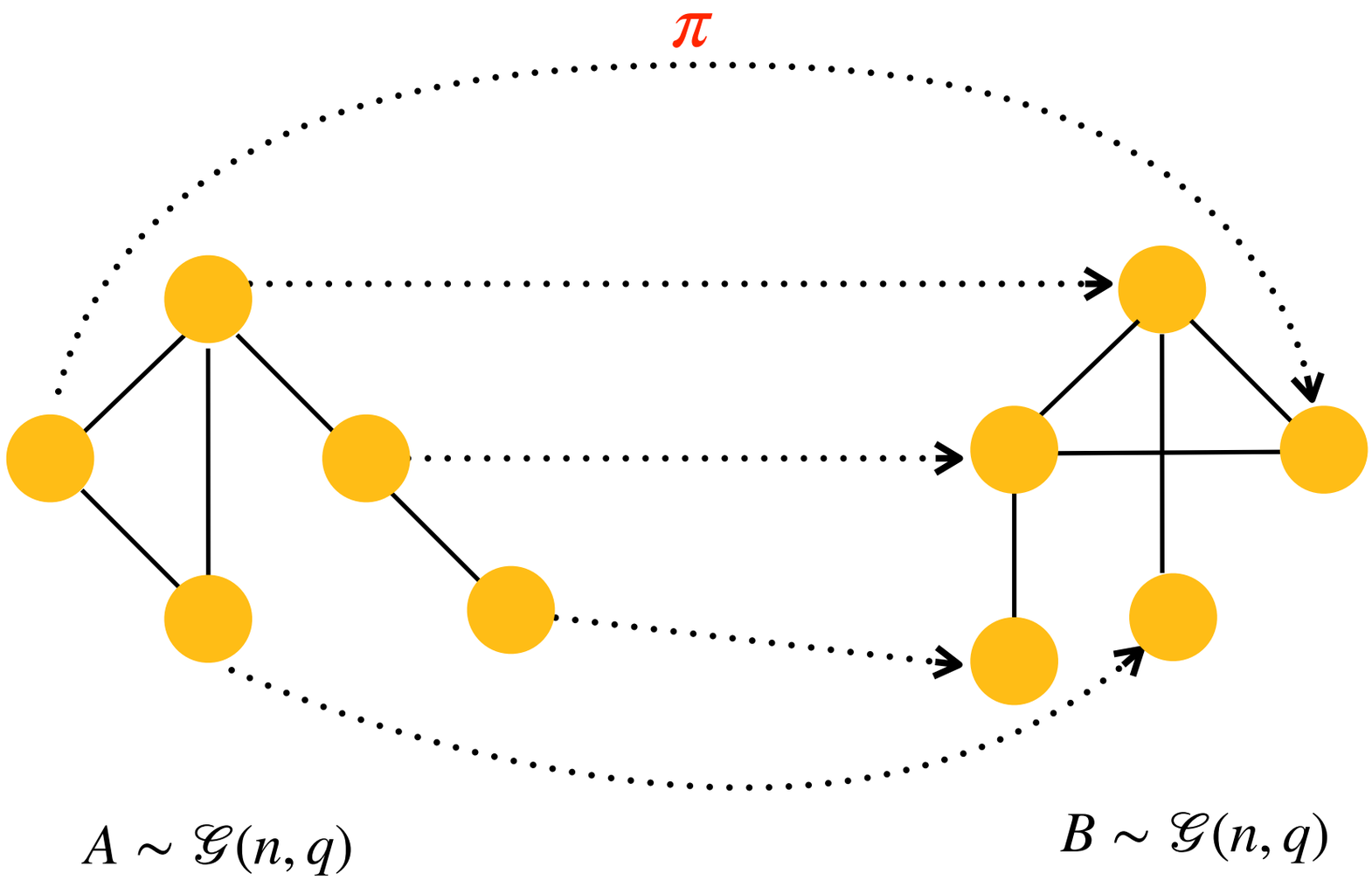

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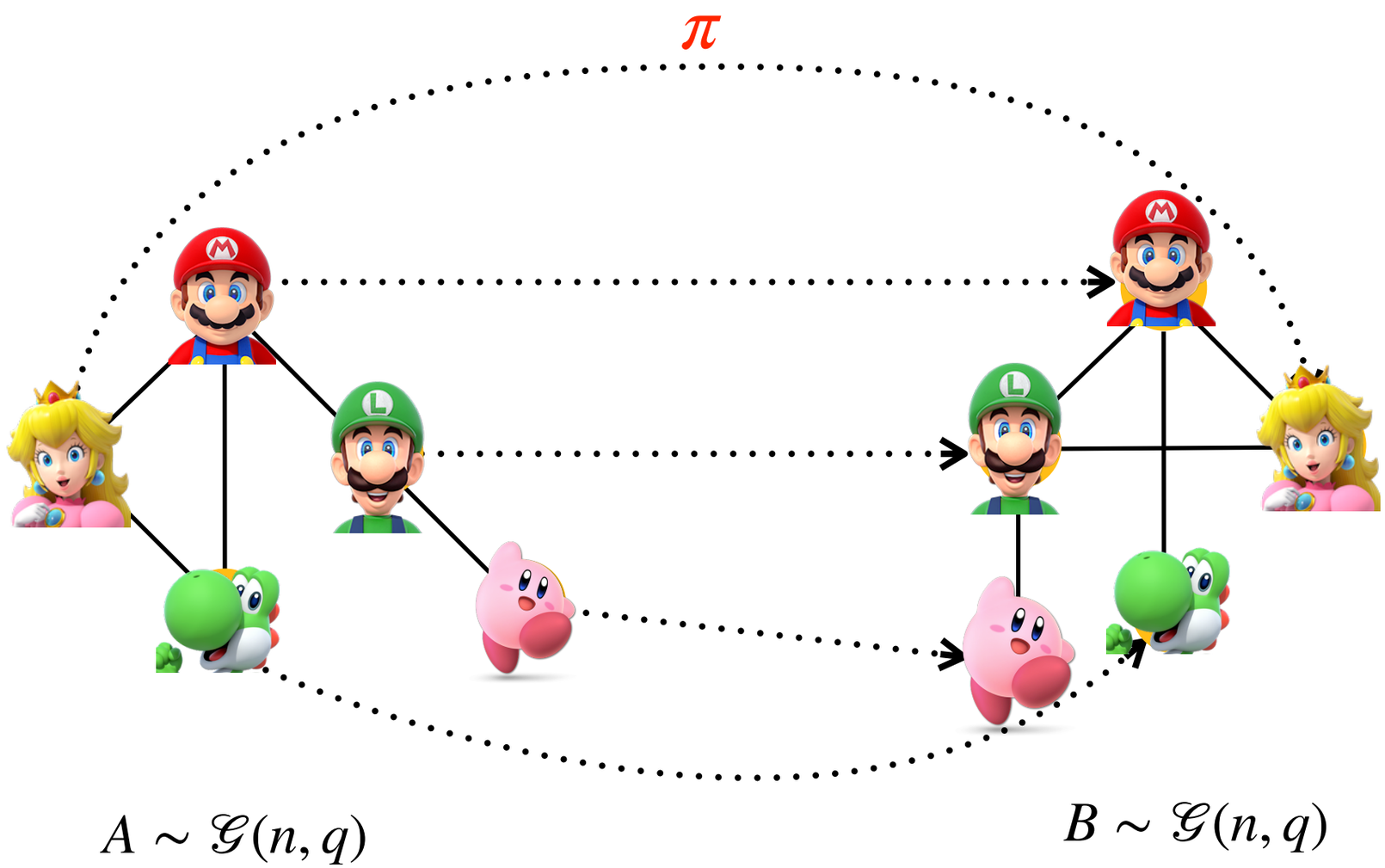

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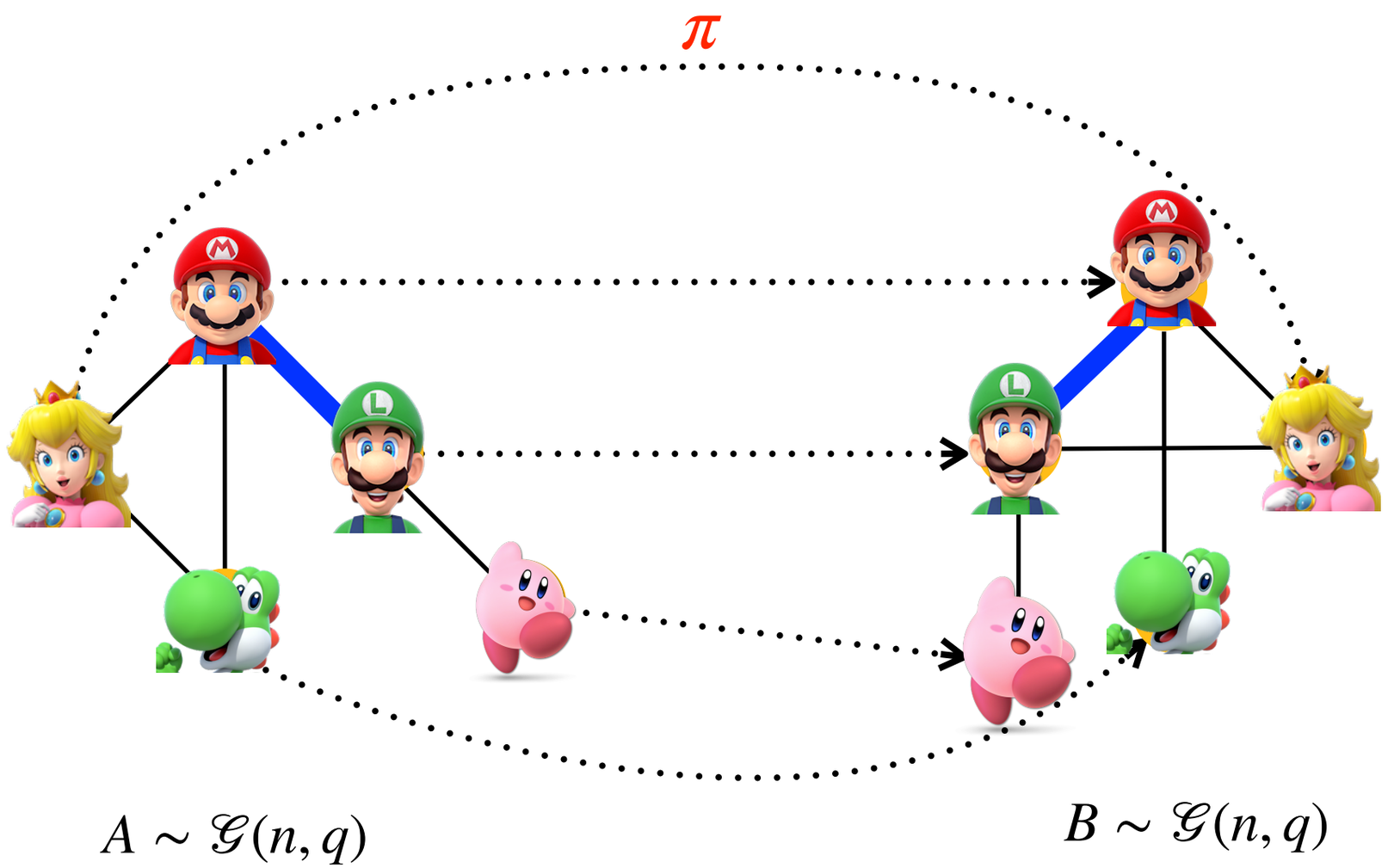

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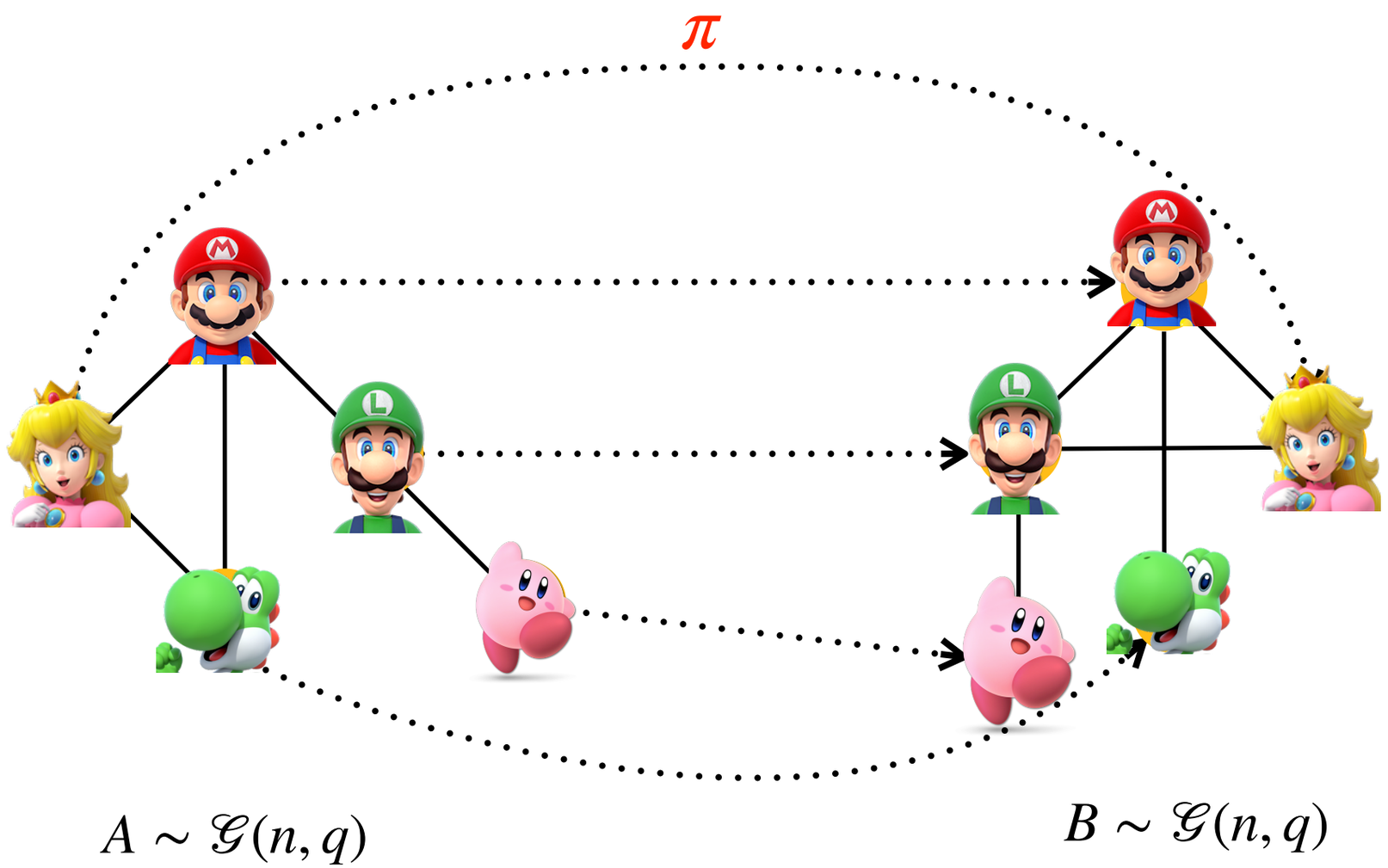

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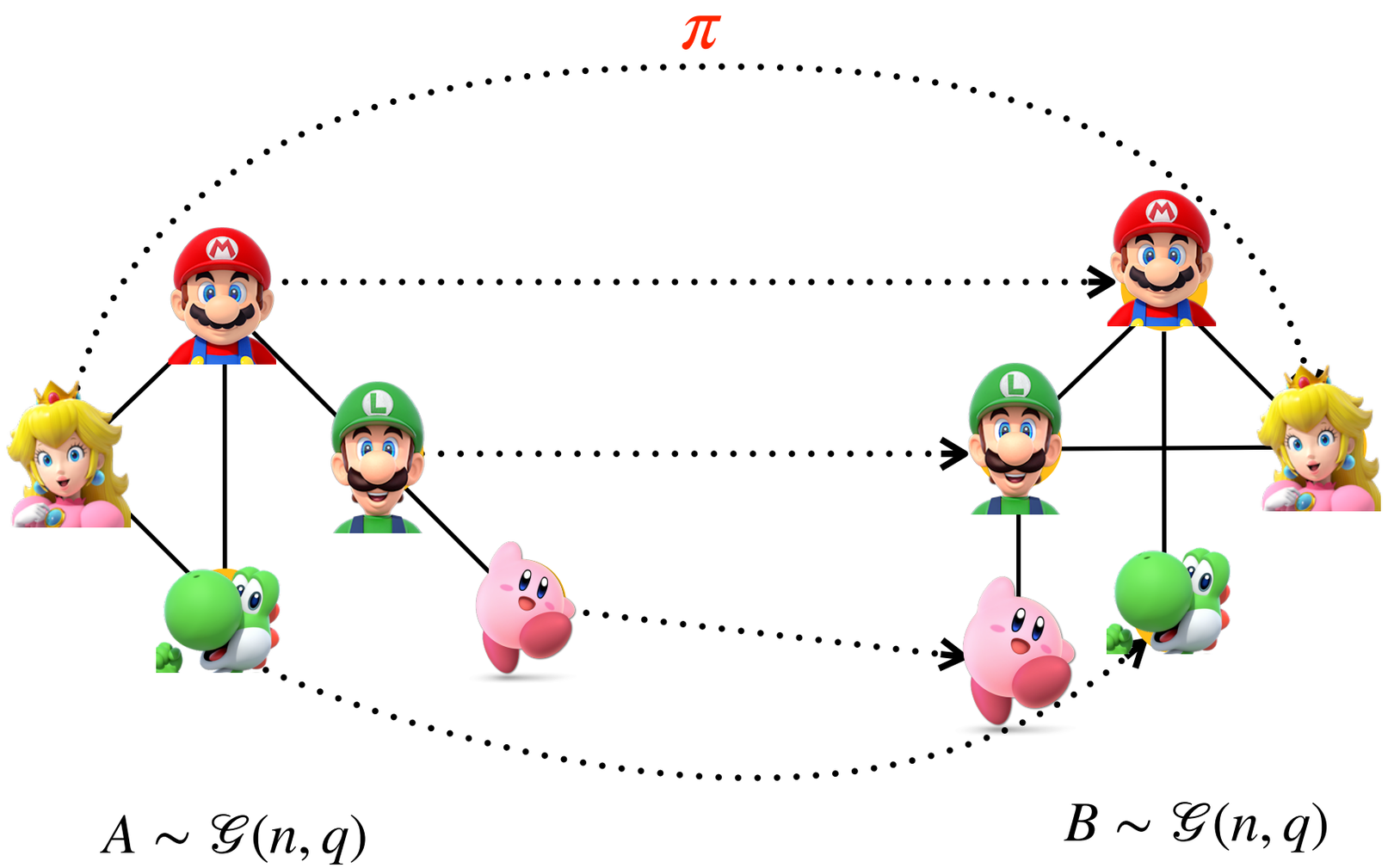

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- nq is the average degree for A and B.

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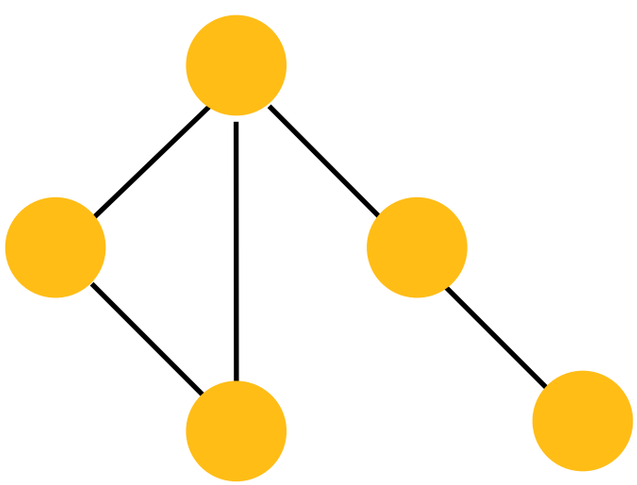

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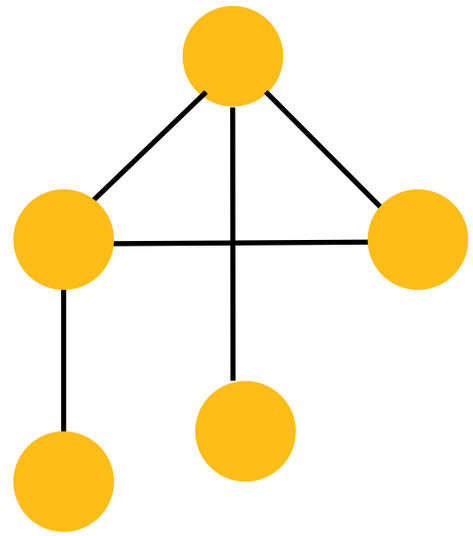

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- A and B are edge-wise correlated (ρ) under the hidden node correspondence π.
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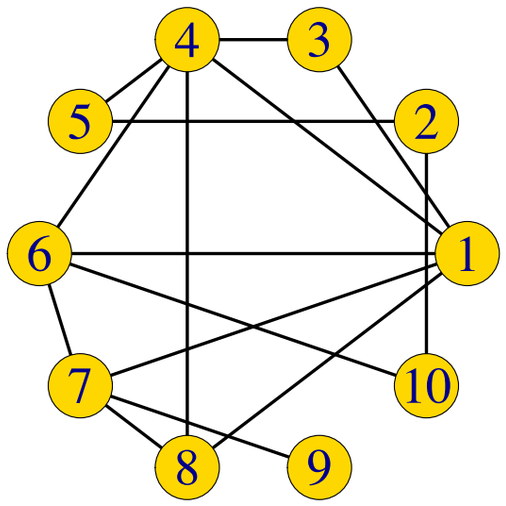
Observe A and B, we want to **recover** the hidden node correspondence π, when $n \to \infty$.

Alternative view: subsampling model

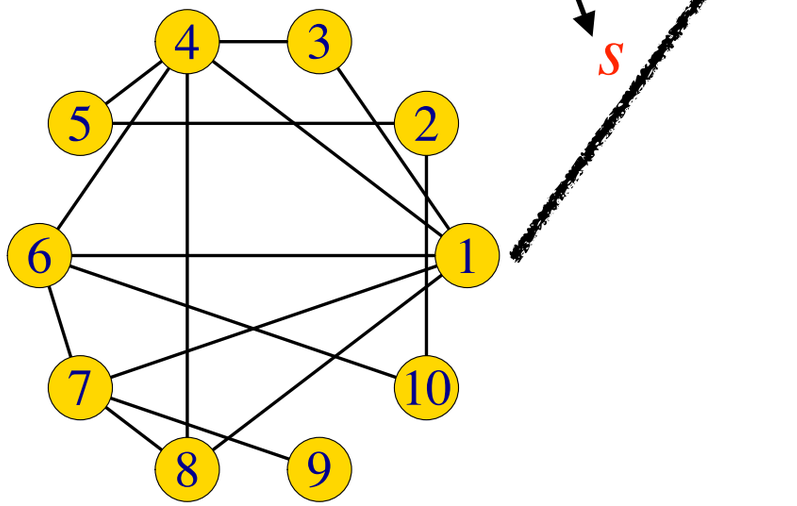


$G \sim \mathcal{G}(n,p)$

Parent graph edge probability

Alternative view: subsampling model

Edge subsampling probability



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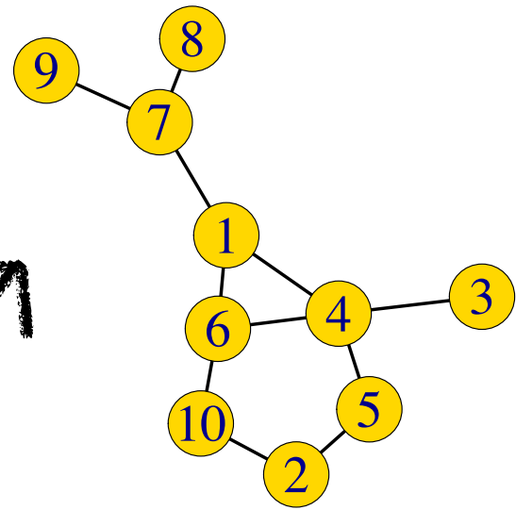

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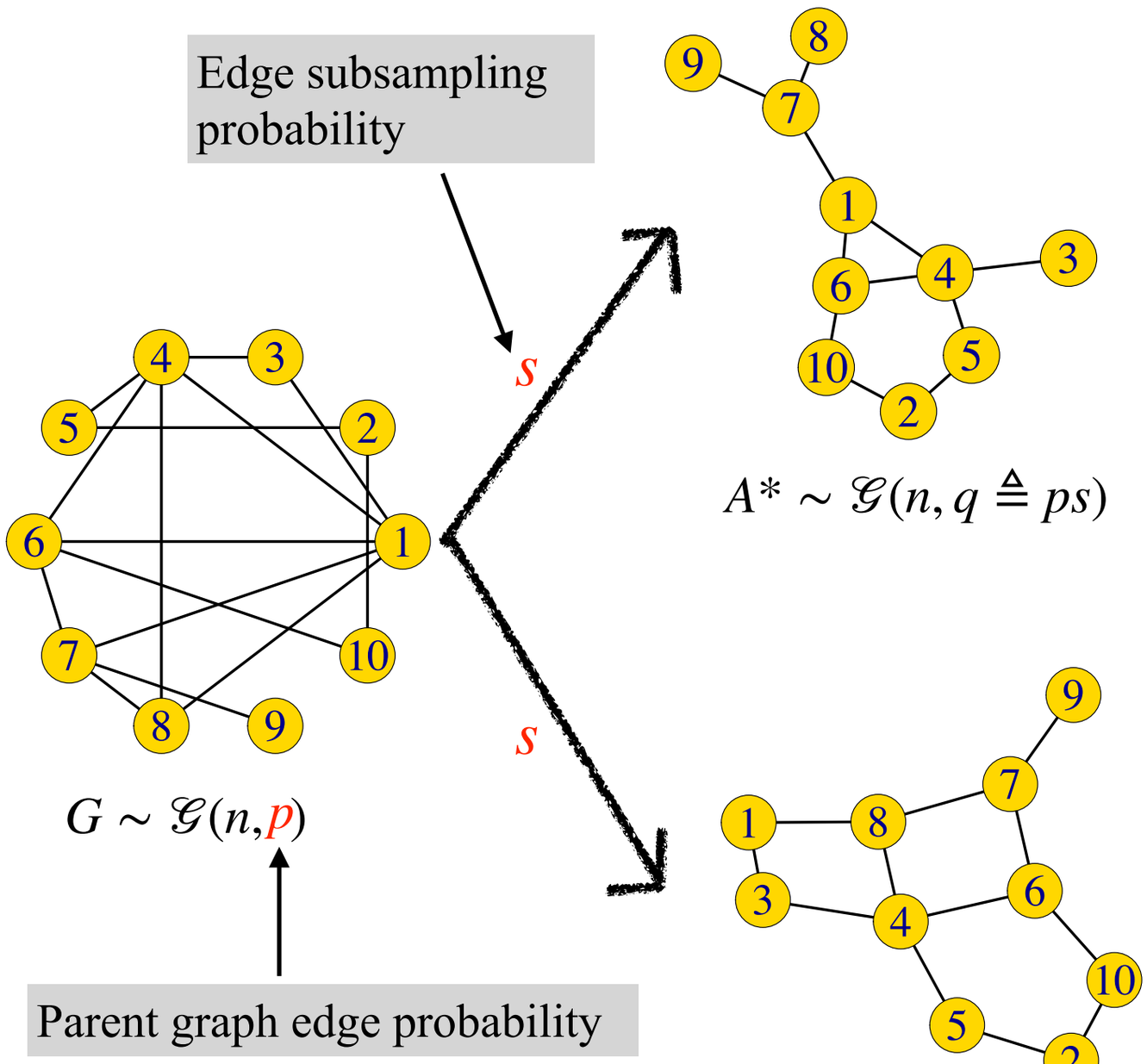

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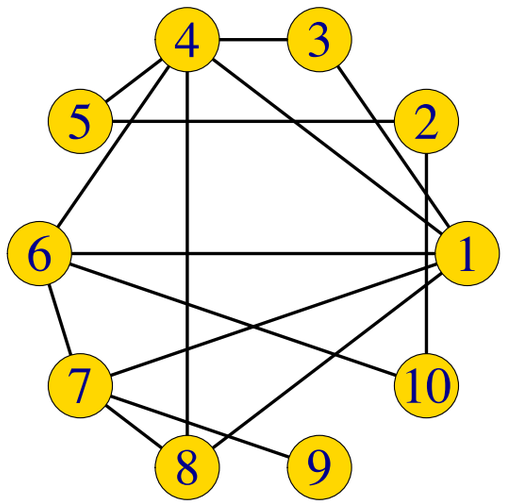

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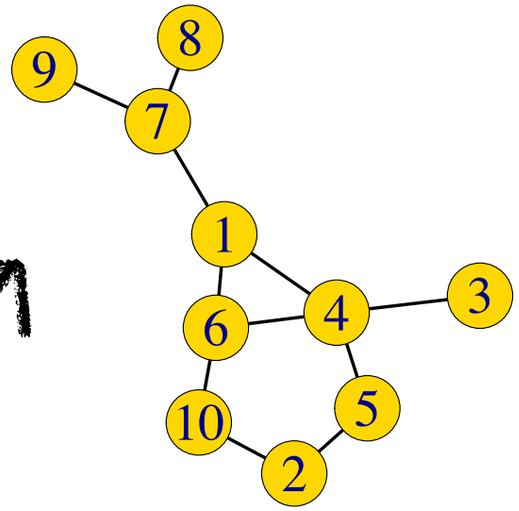

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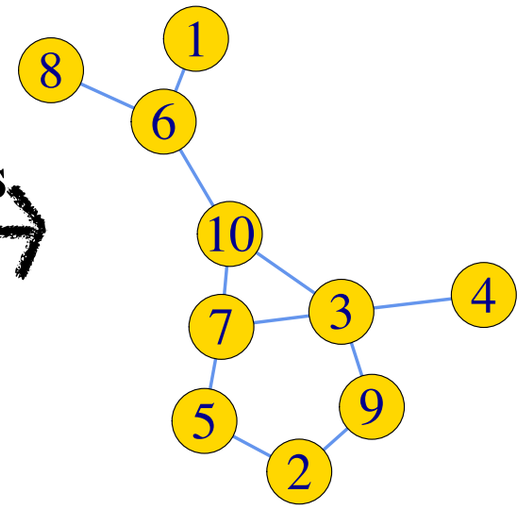
Permute the node labels

By π uniformly



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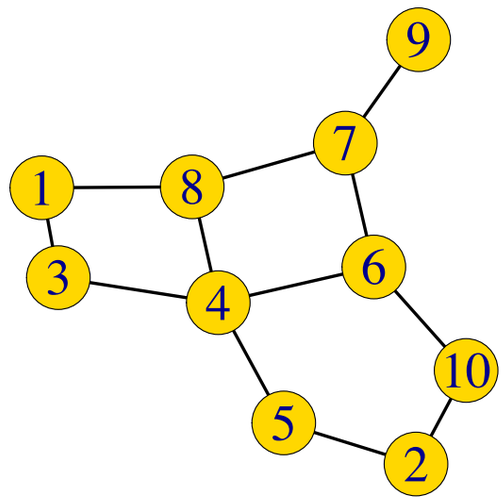

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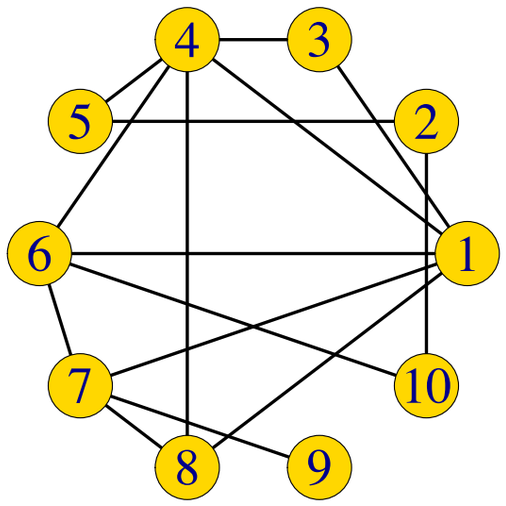

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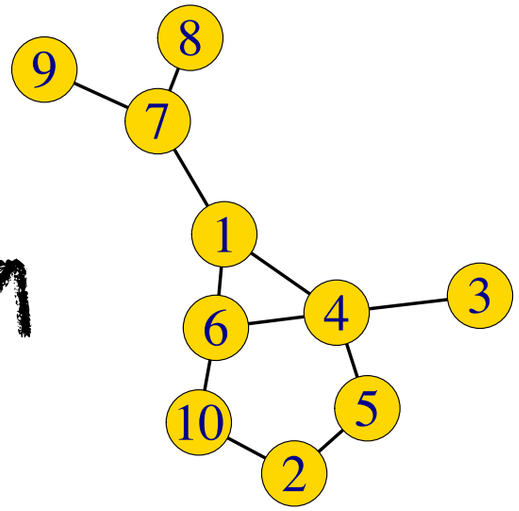

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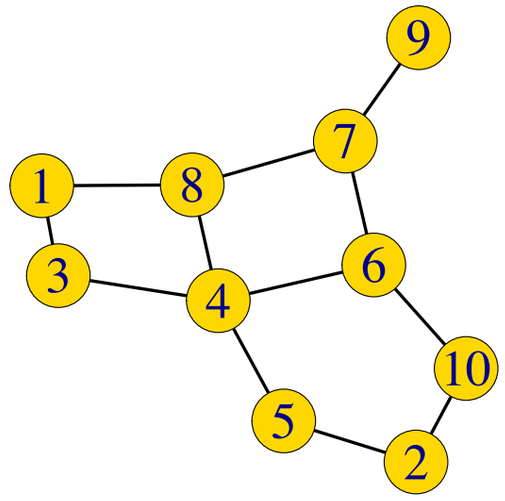

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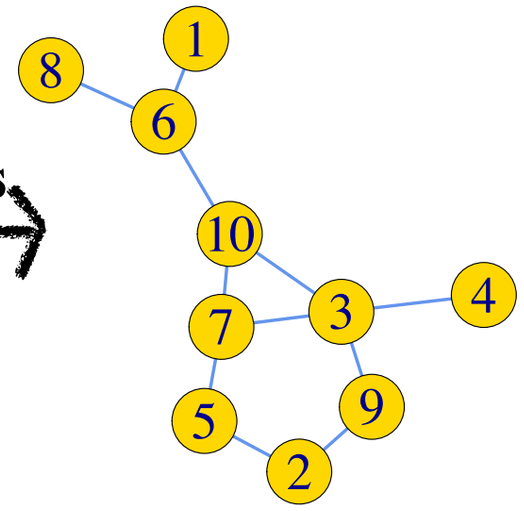

$A \sim \mathcal{G}(n, q \triangleq ps)$

- A and B are edge-wise correlated under the latent permutation π with correlation parameter $\rho = \dfrac{s-q}{1-q}$
- $A^* \wedge B \sim \mathcal{G}(n, ps^2)$

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- Maximum likelihood estimator:

$$\pi_{\mathrm{ML}} = \arg\max_{\hat{\pi}} \sum_{i<j} A_{ij} B_{\hat{\pi}(i)\hat{\pi}(j)} \quad \text{(QAP)}$$

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COWLES FOUNDATION DISCUSSION PAPER, NO. 4*

Assignment Problems and the Location of Economic Activities**

by

Tjalling C. Koopmans and Martin Beckmann

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- It is **NP-hard** to solve or even approximate.
- How much does π_{ML} have in common with π ?

$$\mathrm{overlap}\left(\pi_{\mathrm{ML}}, \pi\right) \triangleq \frac{1}{n} \left| \{ i \in [n] : \pi_{\mathrm{ML}}(i) = \pi(i) \} \right|$$

Fraction of correctly matched vertices

Sharp recovery threshold: dense Erdös-Rényi graphs

Theorem [Wu-Xu-Yu '21]

Suppose $n^{-o(1)} \le p \le 1 - \Omega(1)$. Then

$$\text{If } nps^2 \ge \frac{(2+\epsilon)\log n}{\log\frac{1}{p} - 1 + p} \Rightarrow \text{overlap}\left(\pi_{\text{ML}}, \pi\right) = 1 - o(1) \text{ w.h.p.}$$

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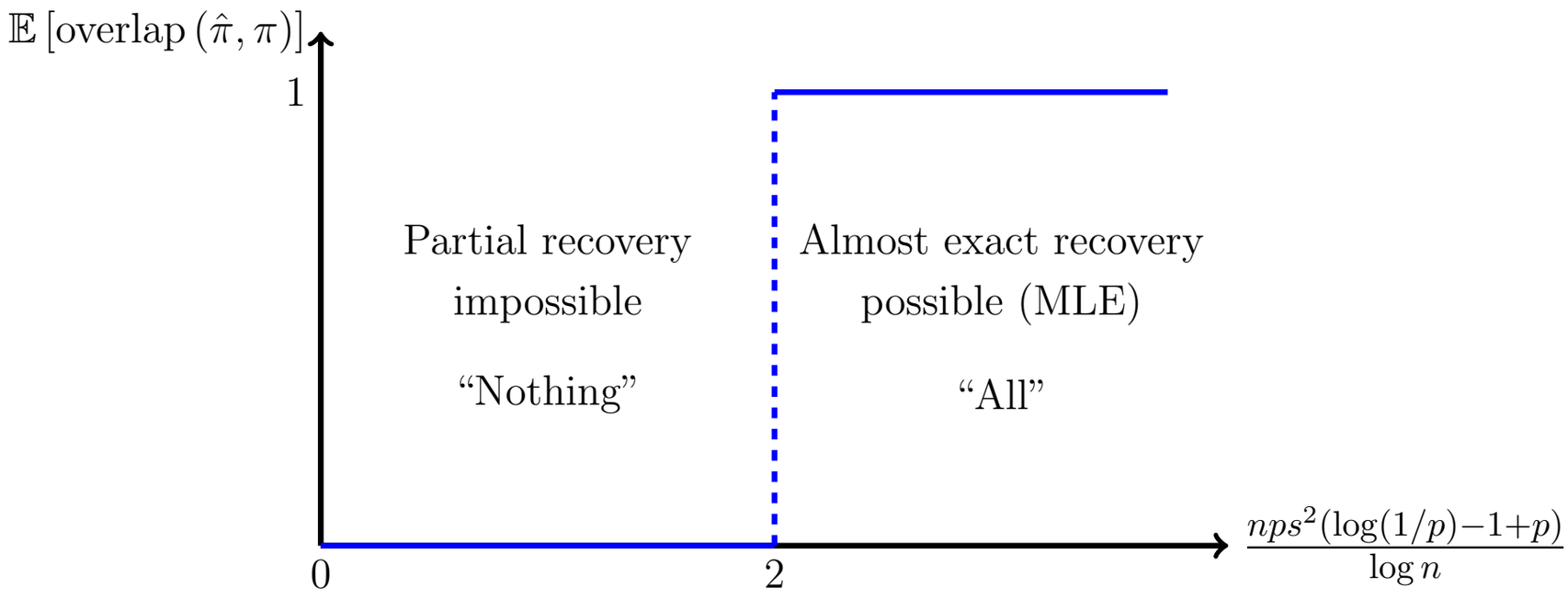

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- IT Interpretation of threshold

$$I(\pi; A, B) \approx \binom{n}{2} \times ps^2 \left(\log\frac{1}{p} - 1 + p\right)$$

Mutual info btw two correlated edges

- $H(\pi) \approx n \log n$
- Threshold is at $I(\pi; A, B) \approx H(\pi)$

Sharp recovery threshold: sparse Erdös-Rényi graphs

Theorem [Ding-Du '22]

Suppose $p = n^{-\alpha}$ for $\alpha \in (0,1]$ and $\lambda^* = \gamma^{-1}(1/\alpha)$. Then

If $nps^2 \geq \lambda^* + \epsilon \Rightarrow \exists \hat{\pi}$ s.t. overlap $(\hat{\pi}, \pi) = \Omega(1)$ w.h.p.

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Sharp recovery threshold: sparse Erdös-Rényi graphs

Theorem [Ding-Du '22]

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- "All-or-nothing" phenomenon does not exist, as almost exact recovery (overlap $= 1 - o(1)$) requires $nps^2 \to \infty$ [Cullina-Kiyavash-Mittal-Poor '19]

Exact recovery threshold

Theorem [Wu-Xu-Yu '21]

Suppose $p \le 1 - \Omega(1)$. Then

$$\text{If } nps^2 \ge \frac{(1+\epsilon)\log n}{(1-\sqrt{p})^2} \Rightarrow \text{overlap}\left(\pi_{\text{ML}}, \pi\right) = 1 \text{ w.h.p.}$$

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- $p = o(1)$: reduces to the connectivity threshold of the intersection graph $A^* \wedge B \sim \mathcal{G}(n, ps^2)$ [Cullina-Kiyavash '16 17]

- $p = \Omega(1)$: strictly higher than the connectivity threshold

Summary on information-theoretic thresholds

		Partial recovery & detection	Almost exact recovery	Exact recovery
p	$n^{-o(1)}$	$nps^2 = \frac{2\log n}{\log(1/p)-1+p}$		$\frac{nps^2(1-\sqrt{p})^2}{\log n} = 1$
	$n^{-\alpha}$	$nps^2 = \lambda^*$	$nps^2 = \omega(1)$	
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Can we develop a **scalable** algorithm to recover π with a **strong** statistical guarantee?

Outline

- Motivation and problem setup
- Information-theoretic limits
- Efficient algorithms
- Concluding remarks

State of the art (polynomial-time algorithms)

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Extremely high correlation $\rho \to 1$		
high correlation ρ close to 1		
Low correlation Constant ρ		

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$n^{\Theta(\log n)}$-time recovery when $\rho = o(1)$ [Barak-Chou-Lei-Schramm-Sheng '19]

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Extremely high correlation $\rho \to 1$	$\rho^2 > \alpha \approx 0.338$ [Mao-Wu-Xu-Yu '22]	
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The number of unlabeled trees with N edges $\approx \alpha^{-N}$ [Otter '1948]

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THE NUMBER OF TREES

RICHARD OTTER[1]

(Received June 10, 1947)

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A different (local) algorithm is shown to achieve partial recovery in the sparse regime when $\rho > \sqrt{\alpha}$ [Ganassali-Massoulié-Semerjian '22]

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When $\rho^2 > \alpha \approx 0.338$ ($\alpha \triangleq$ Otter's tree counting constant), as $n \to \infty$, our polynomial-time matching algorithm with probability $1 - o(1)$ achieves:

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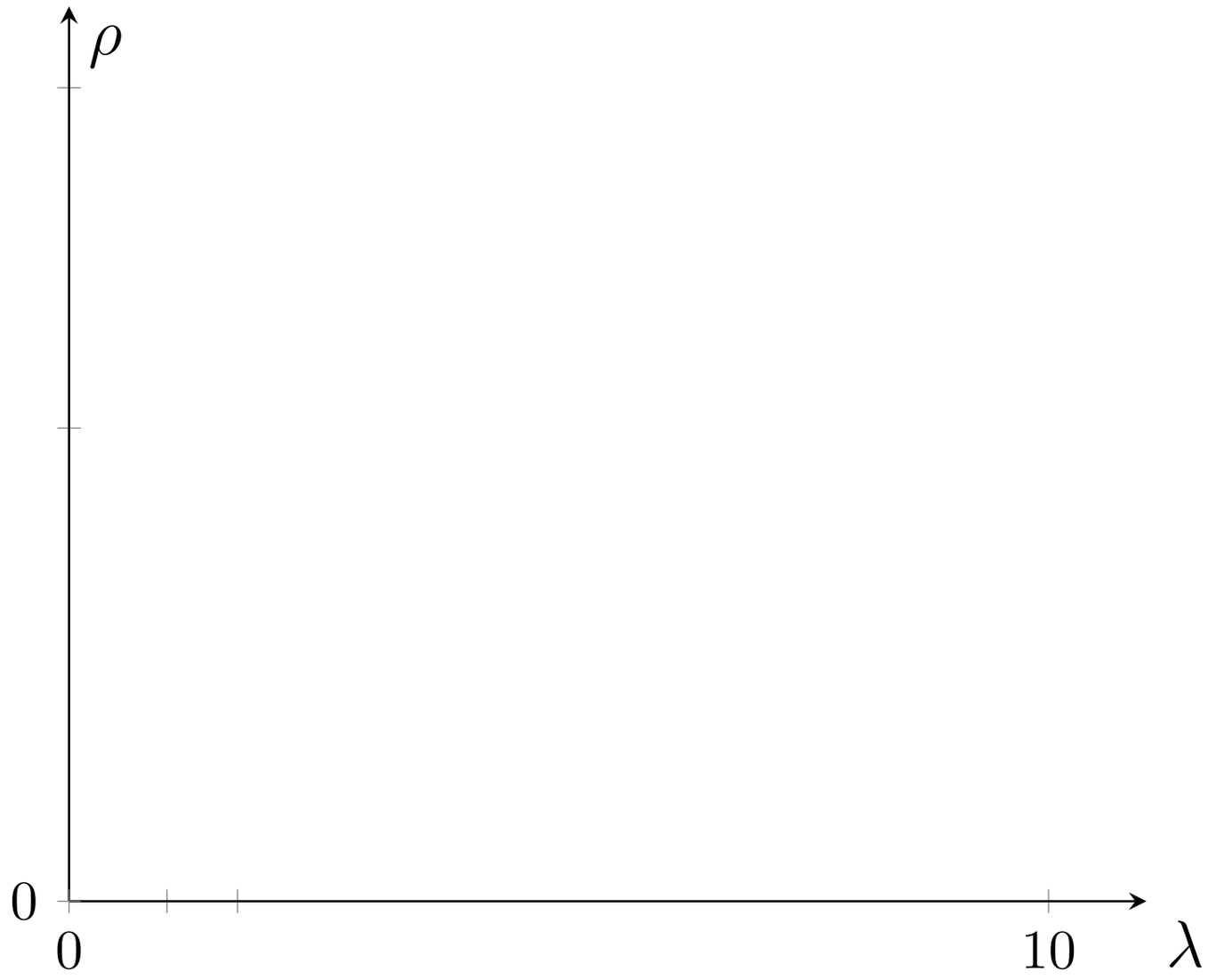
- No mismatching error;
- The intersection graph between A and B under the hidden node correspondence π $\sim \mathcal{G}\left(n, q(q+\rho(1-q))\right)$;
- $nq(q + \rho(1-q)) \geq (1+\epsilon)\log n$ is the connectivity threshold and information-theoretically necessary for exact recovery;

Phase transition diagram for exact recovery

Let's focus on the regime when $nq = \lambda \log n$, where λ is some constant.



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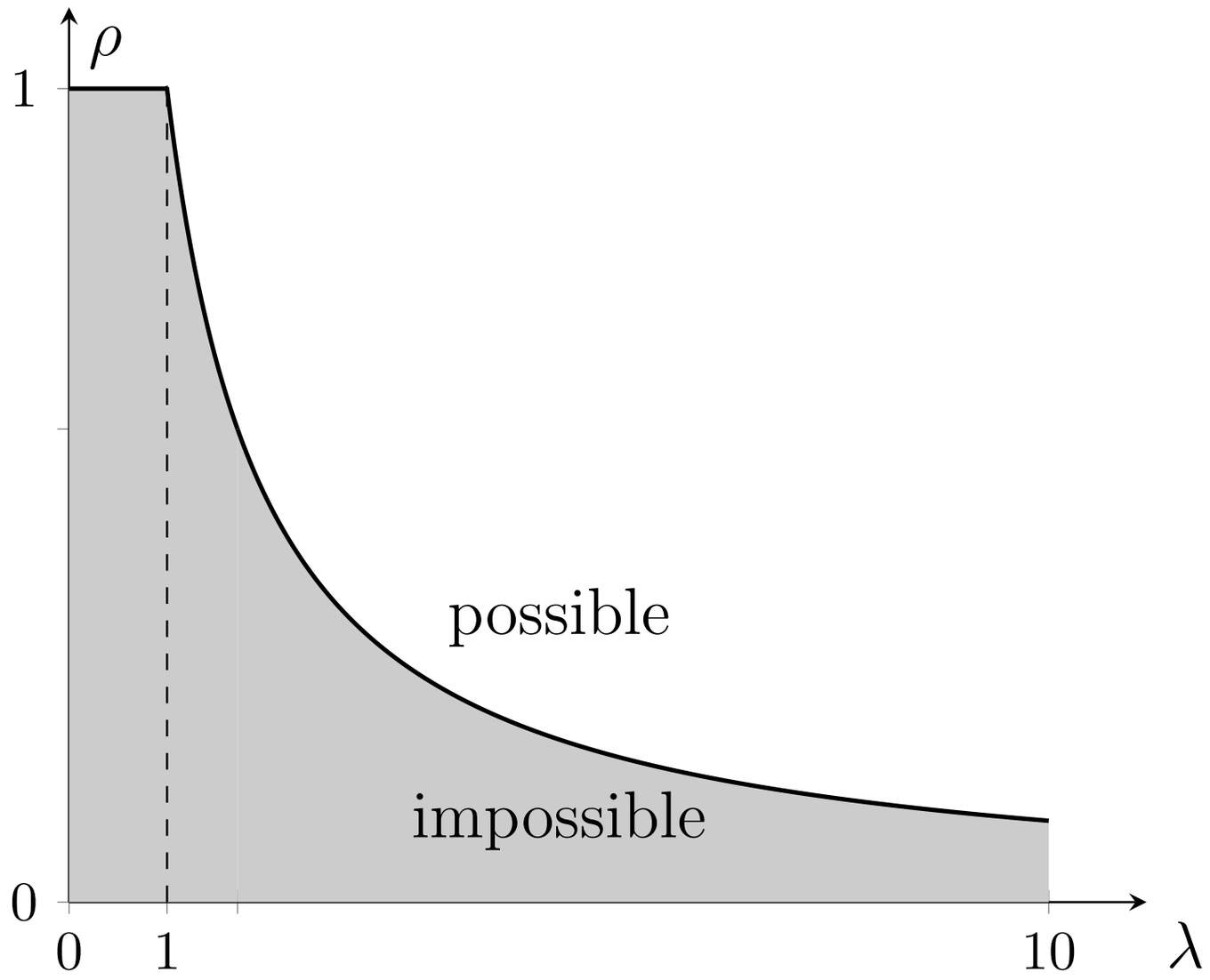

The curve is the information theoretical limit where $\rho = \min\left\{1, \frac{1}{\lambda}\right\}$ [Cullina-Kiyavash '16,'17][Wu-Xu-Yu '21].

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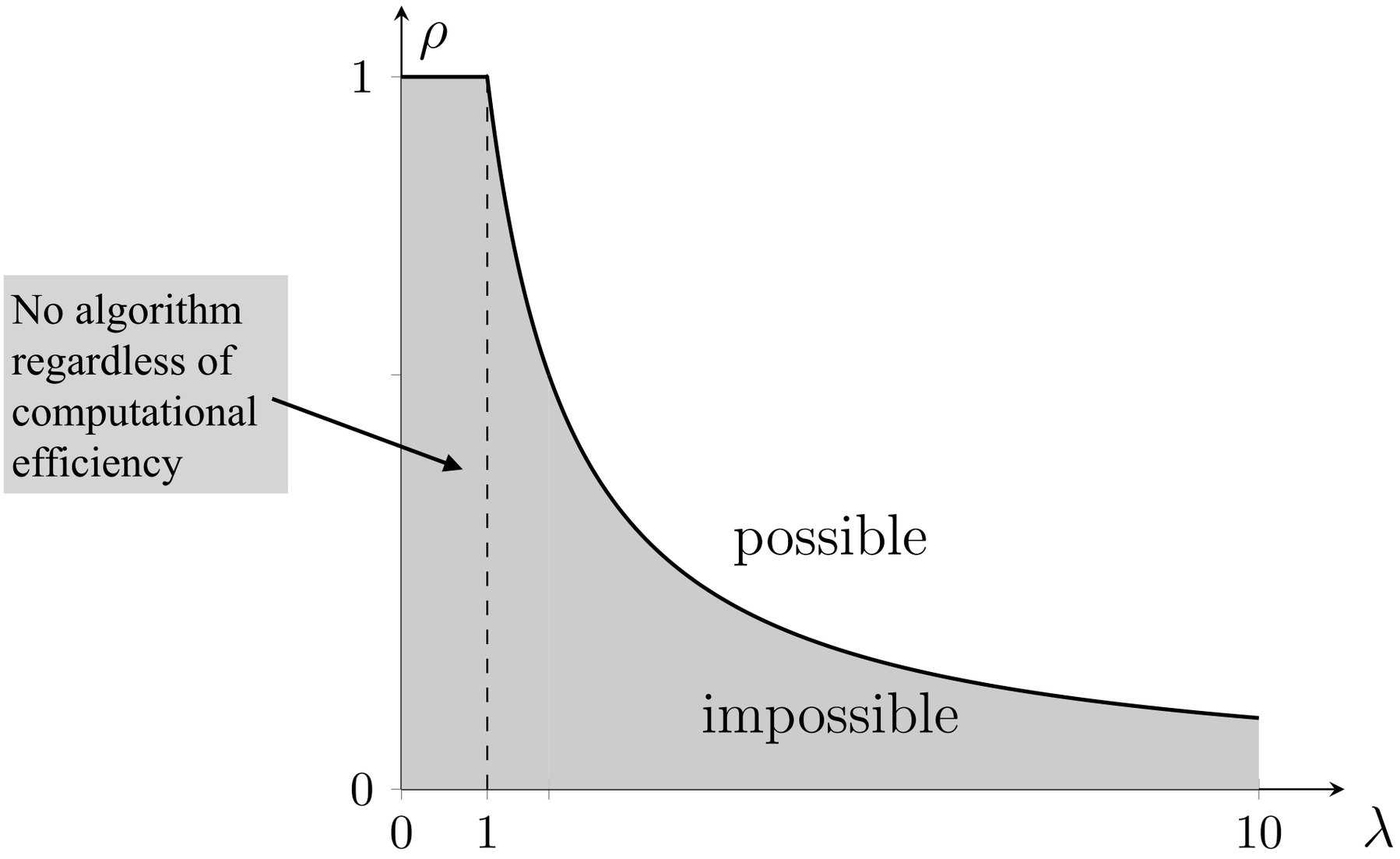

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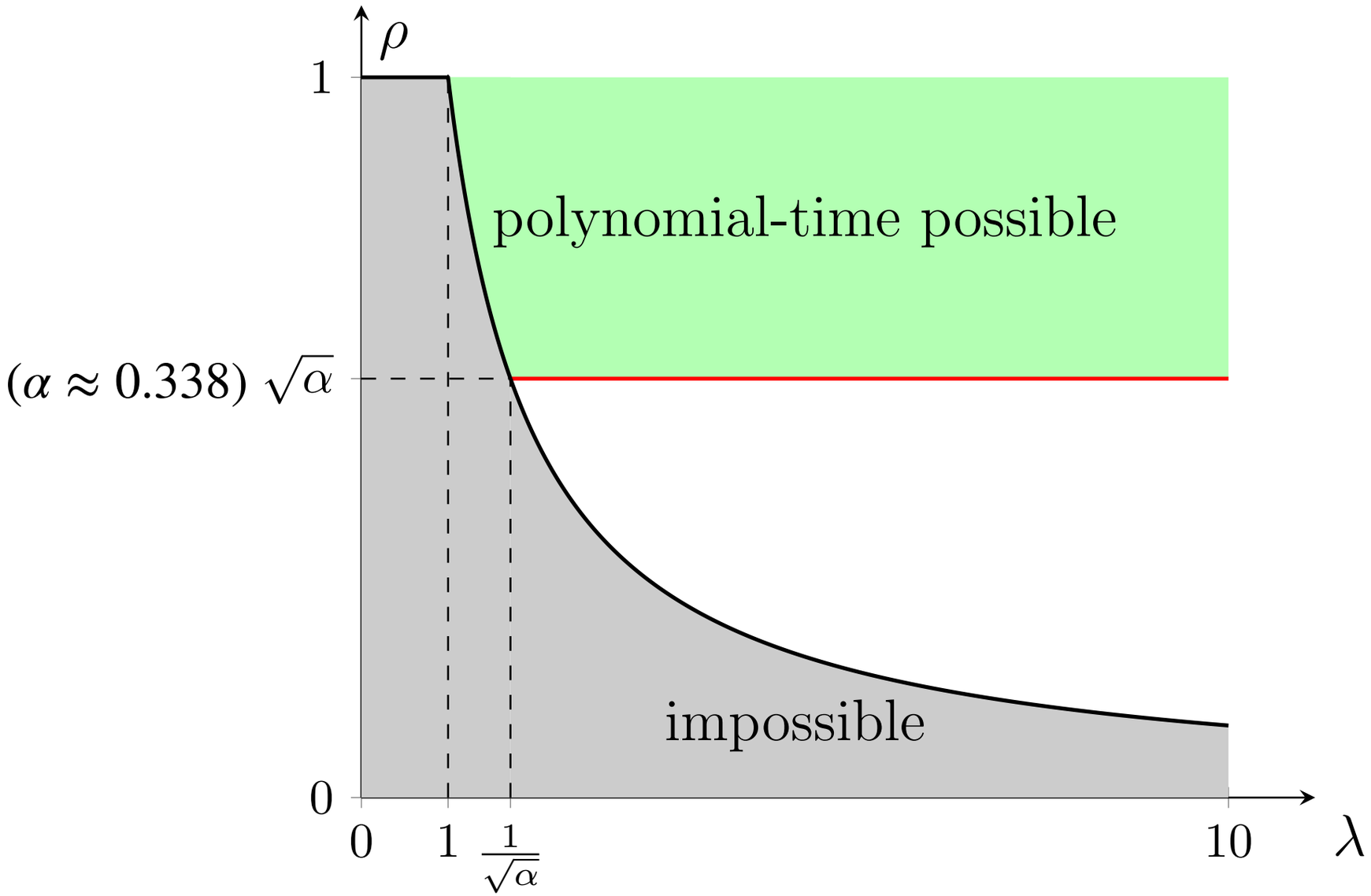

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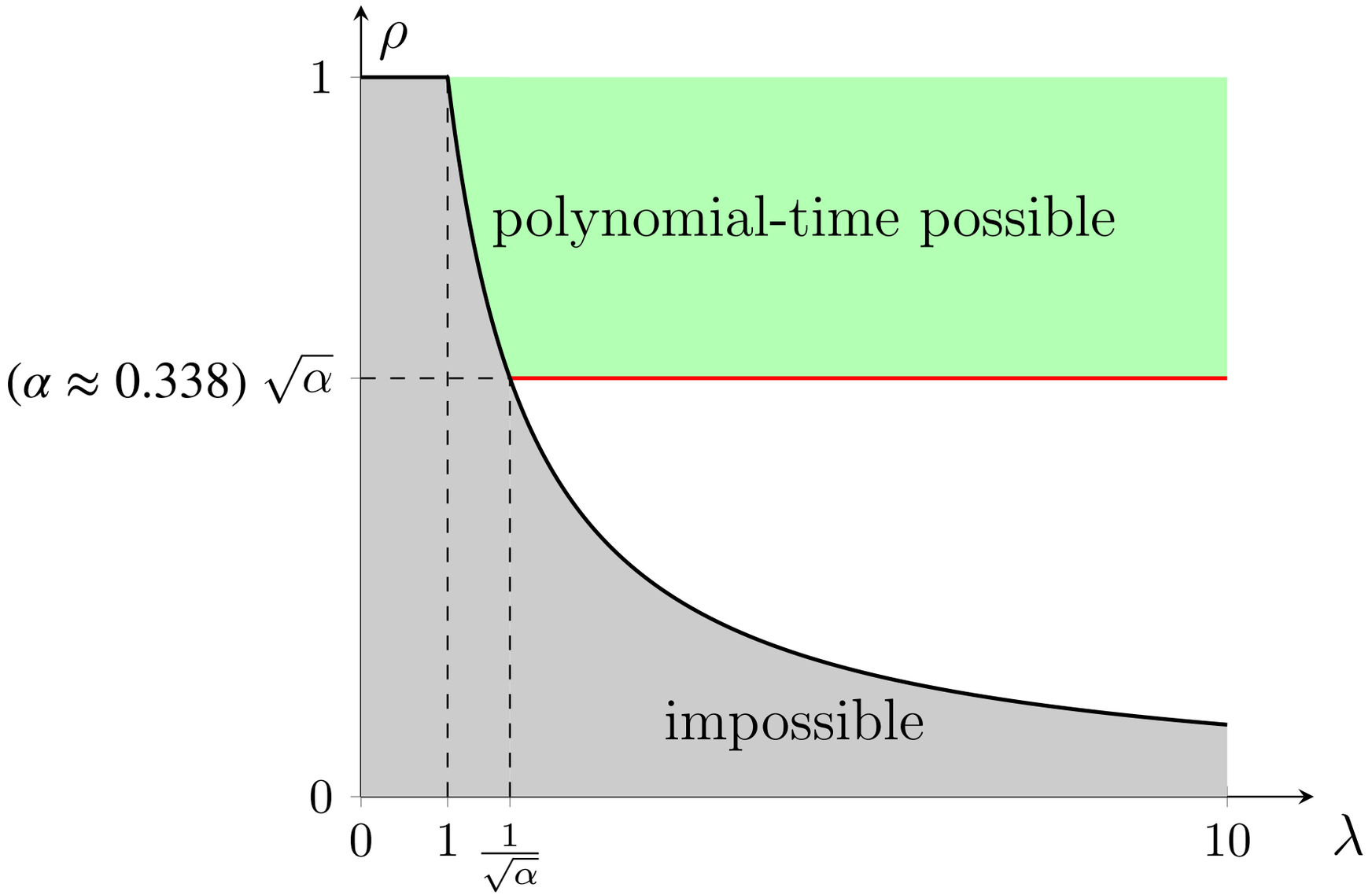

Our condition coincides with the information-theoretic limit when $1 < \lambda < \dfrac{1}{\sqrt{\alpha}}$.

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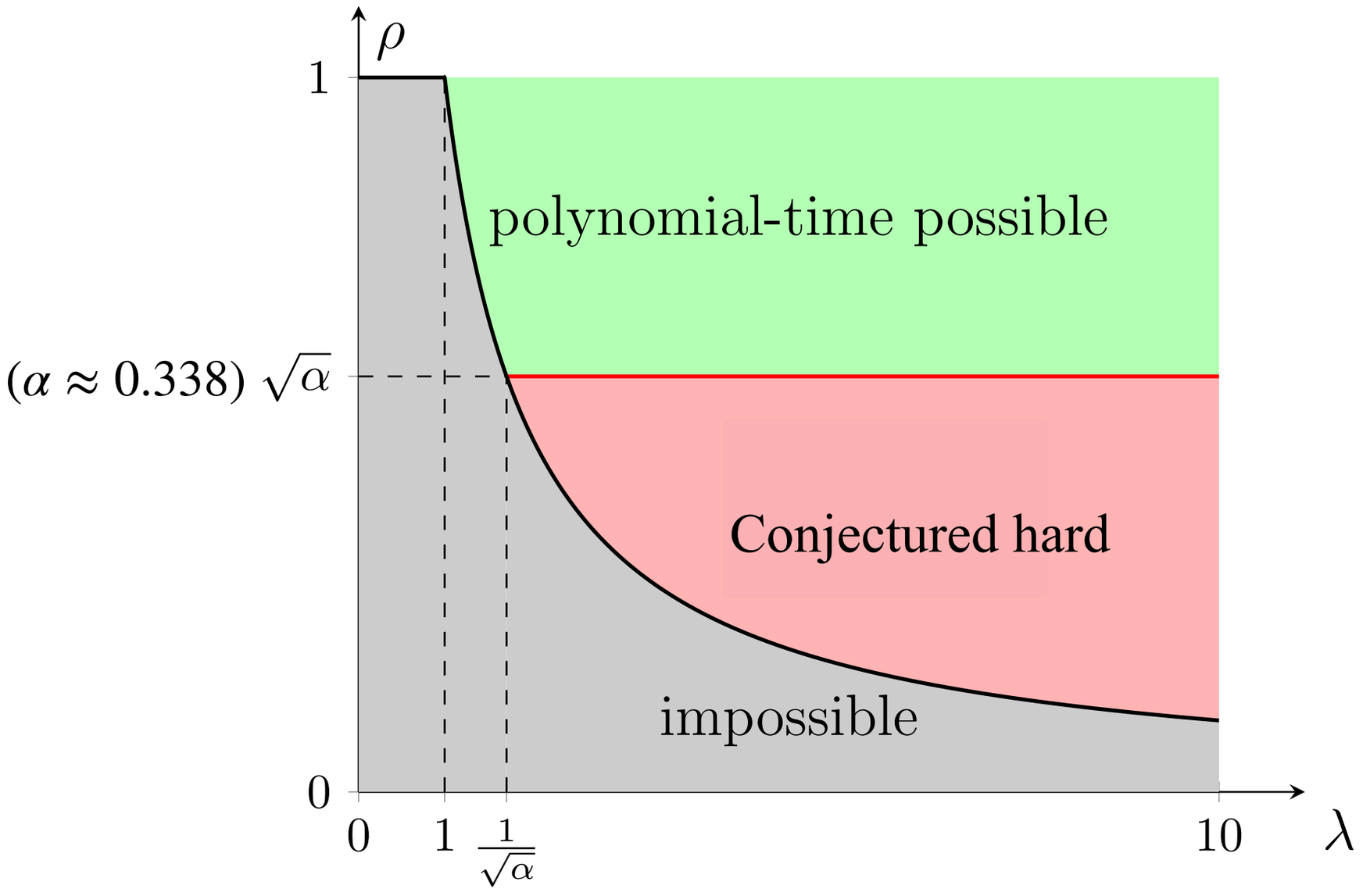
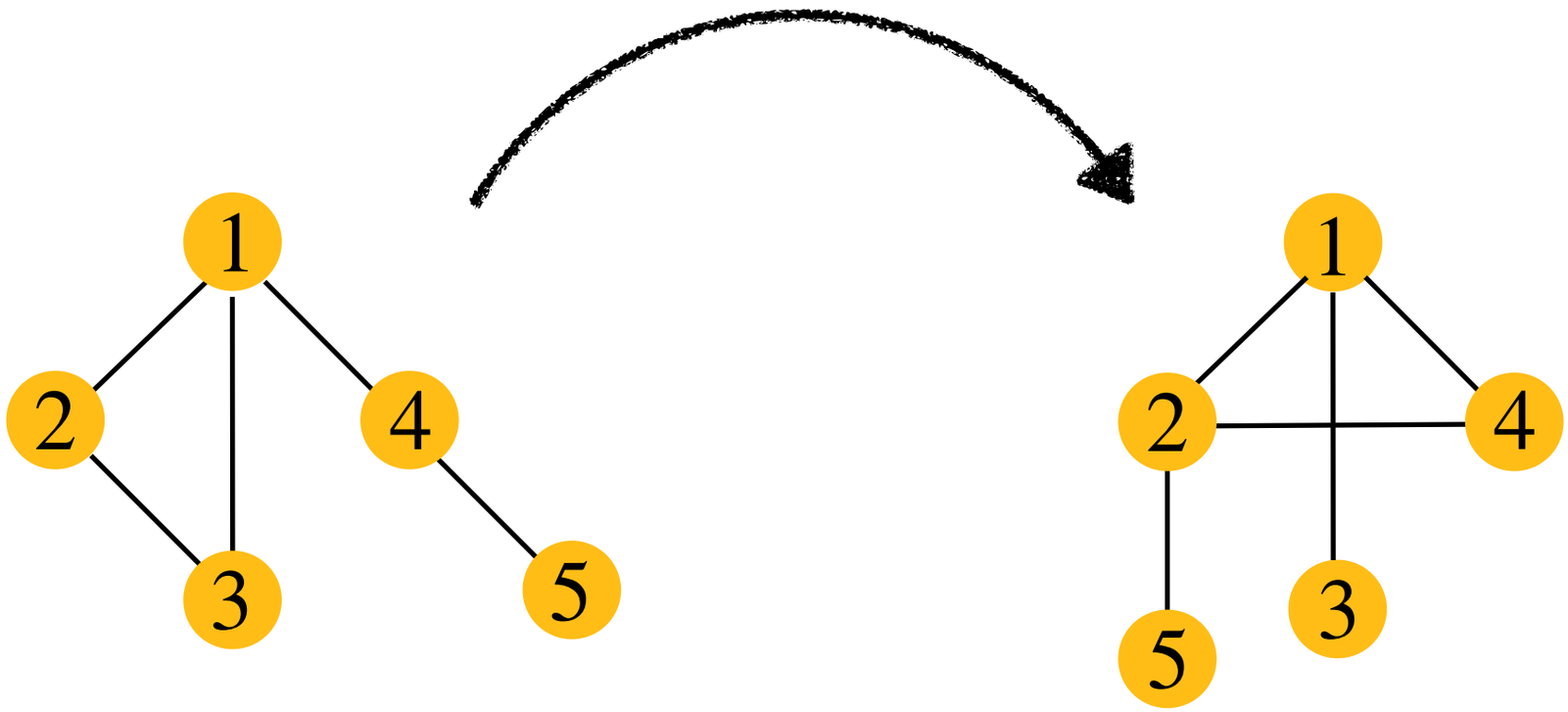


Low-degree polynomial estimators fail when $\rho < \sqrt{\alpha}$ [Ding-Du-Li '23];

Local algorithms fail when $\rho < \sqrt{\alpha}$ in sparse regime [Ganassali-Massoulié-Semerjian '22]

Meta algorithm

Algorithm



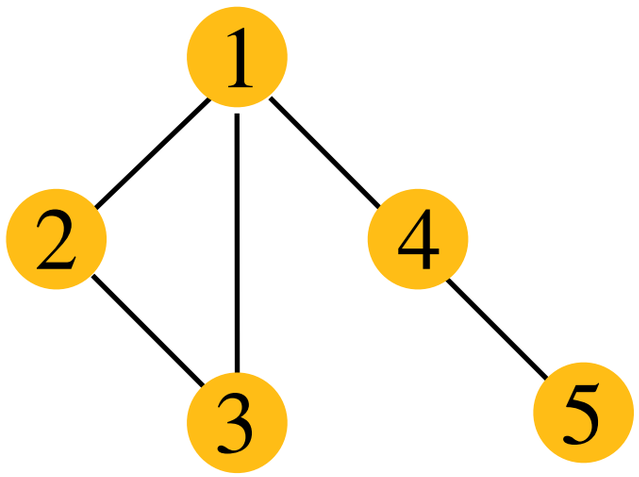
A *B*

Meta algorithm

- Step 1: signature embedding

  Based on the structure of A:

  Construct a **vertex signature** (number or vector) for each vertex in A.



A

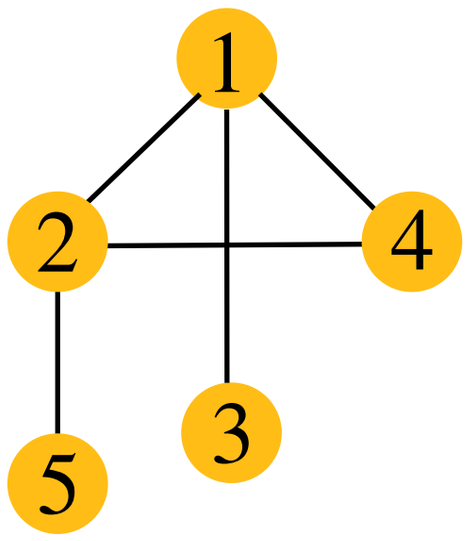
Vertex in A	Vertex signature
1	s_1
2	s_2
3	s_3
4	s_4
5	s_5

Meta algorithm

- Step 1: signature embedding

  Based on the structure of B:

  Construct a **vertex signature** (number or vector) for each vertex in B.



$B \sim \mathcal{G}(n, q)$

Vertex in B	Vertex signature
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2	t_2
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4	t_4
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Meta algorithm

Vertex in A	Vertex signature
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4	s_4
5	s_5

Vertex in B	Vertex signature
1	t_1
2	t_2
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- Step 2: Similarity score

  For any vertex pair of i in A and j in B, compute **similarity score Φ_{ij}** based on s_i and t_j.

Meta algorithm

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  We want

  - For $j = \pi(i)$ (true pair), s_i is close to $t_j \Longrightarrow \Phi_{ij}$ is relatively large;
  - For $j \neq \pi(i)$ (fake pair), s_i is far from $t_j \Longrightarrow \Phi_{ij}$ is relatively small;

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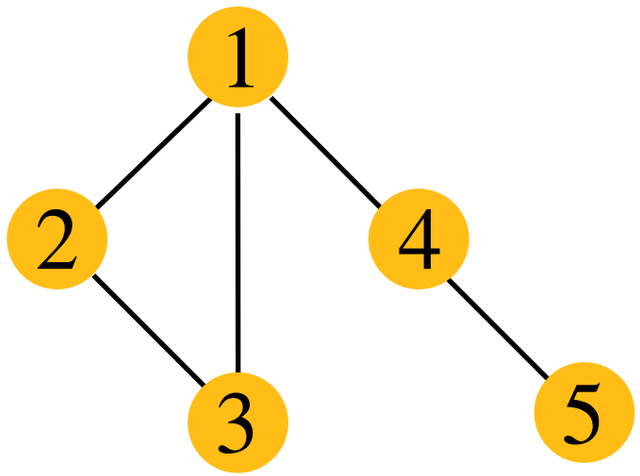

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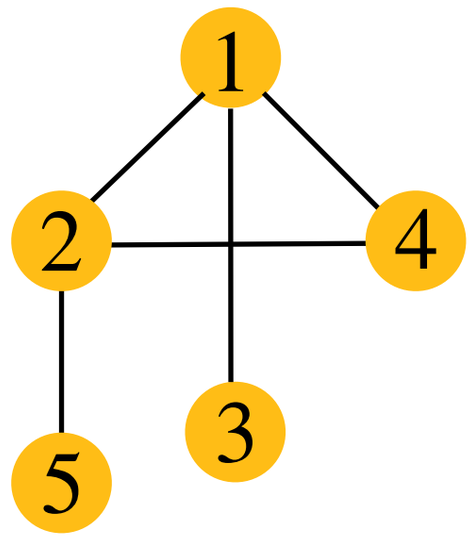
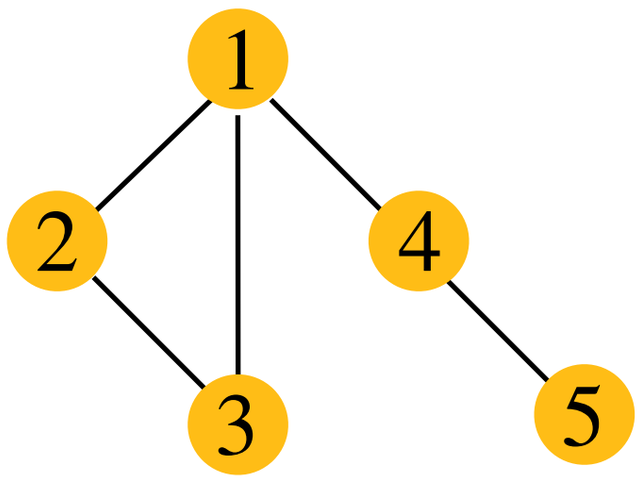
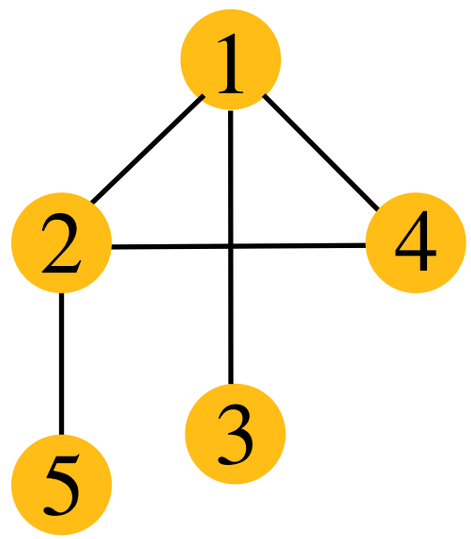


B

How to construct the vertex signature?



A

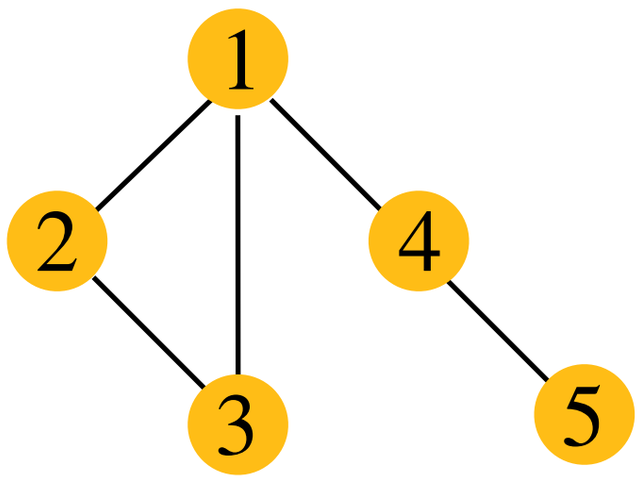


B

Examples of vertex signature:

- Degree

How to construct the vertex signature?



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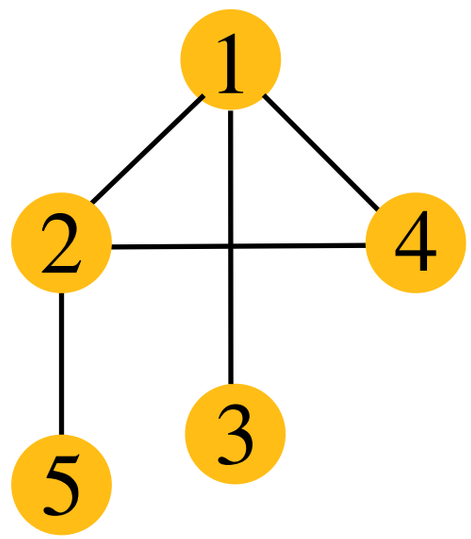

B

Examples of vertex signature:

- Degree
- Degrees of neighbors: only works for $\rho = 1 - 1/\text{polylog}(n)$ [Ding-Ma-Wu-Xu '18]

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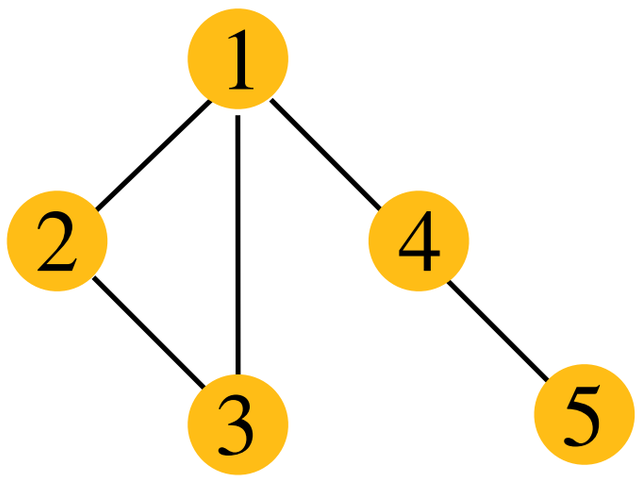

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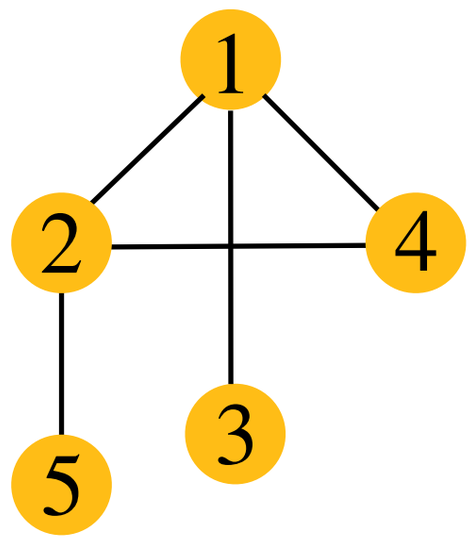

B

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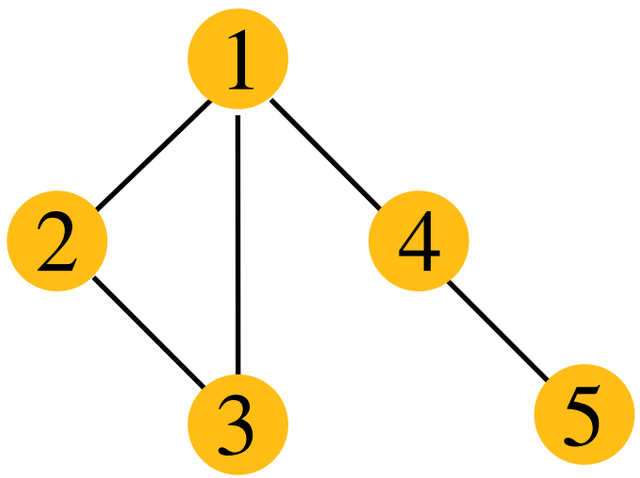

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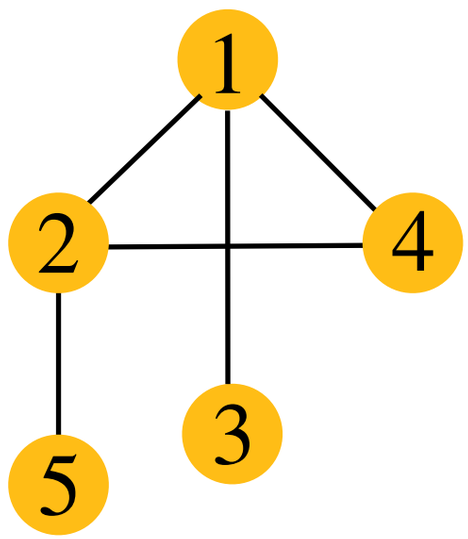

B

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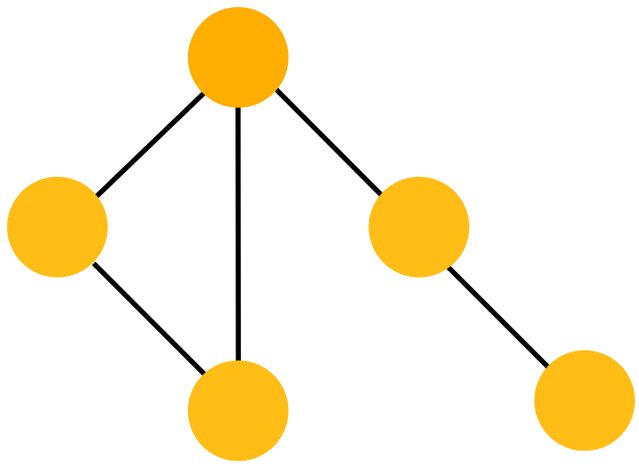
The above vertex signatures are either sensitive to noise or only work in sparse regime.

Our idea: subgraph count

Subgraph count



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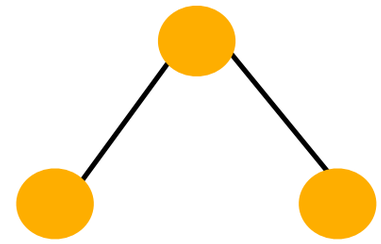
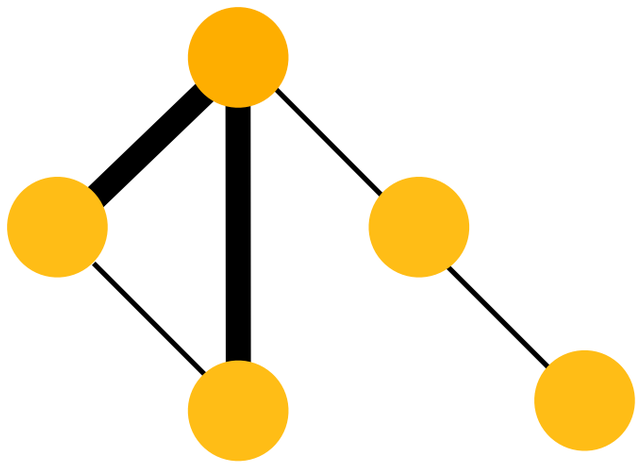


subgraph H

- # copies of H in graph A

Subgraph count



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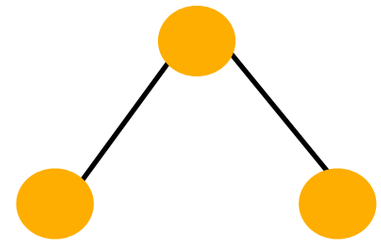

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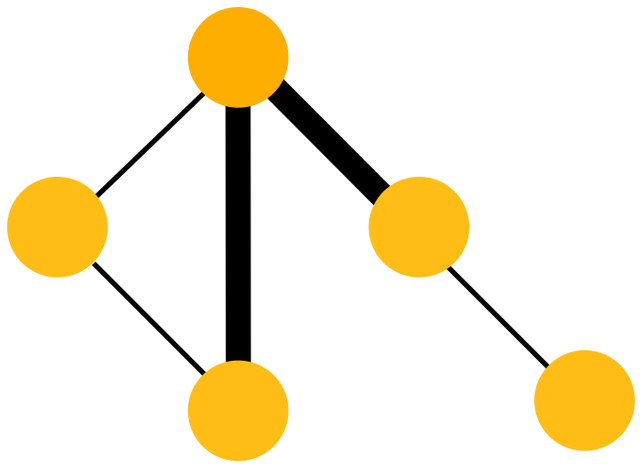

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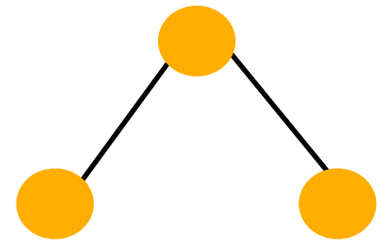

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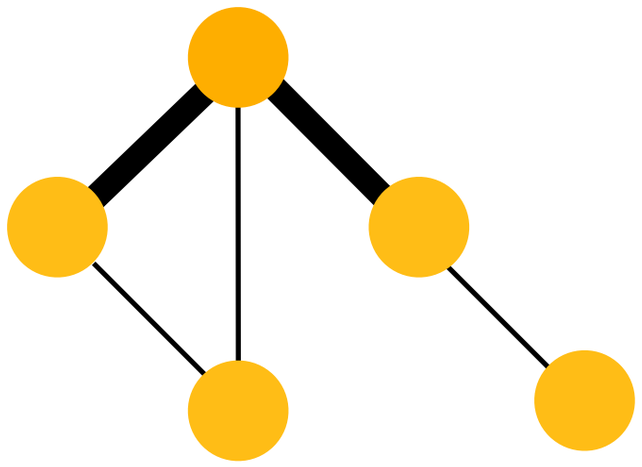

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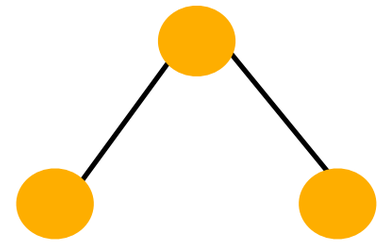

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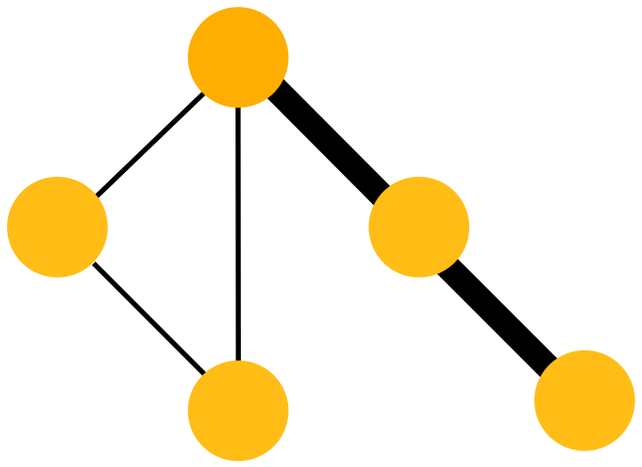

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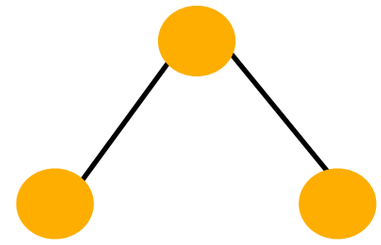

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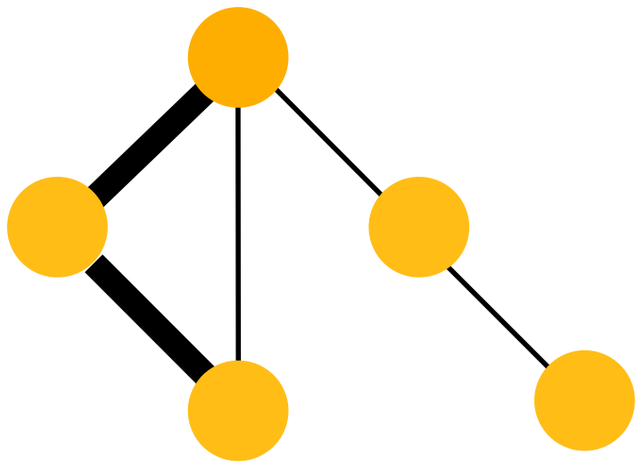

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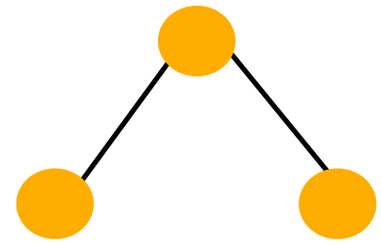

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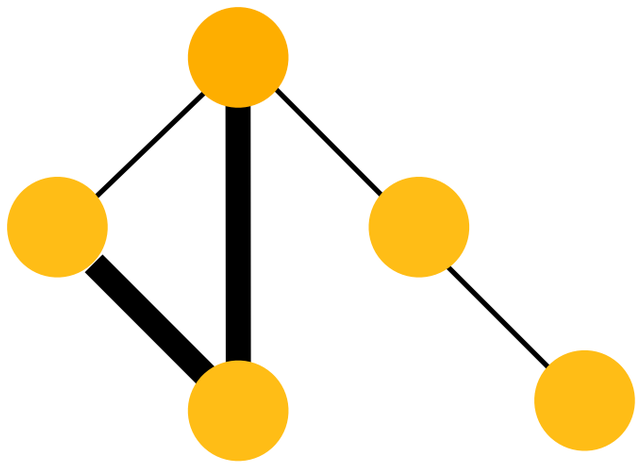

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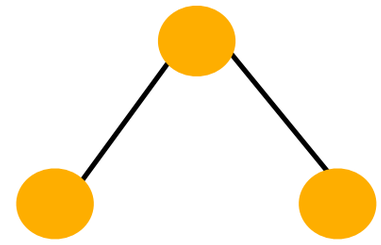

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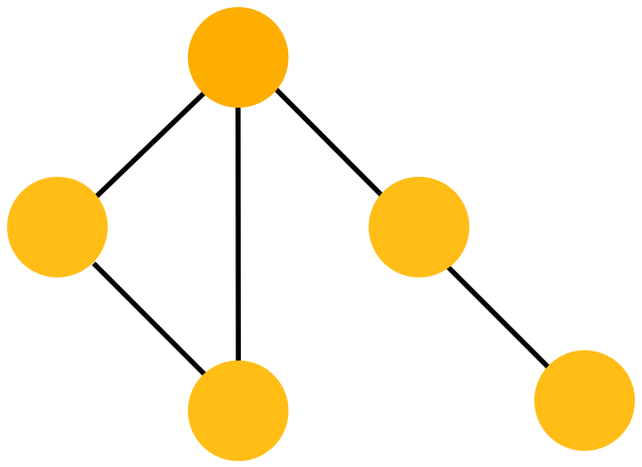

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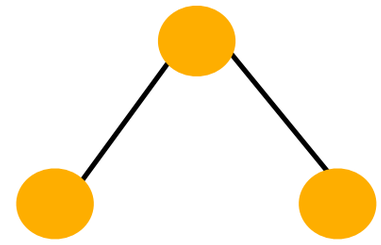
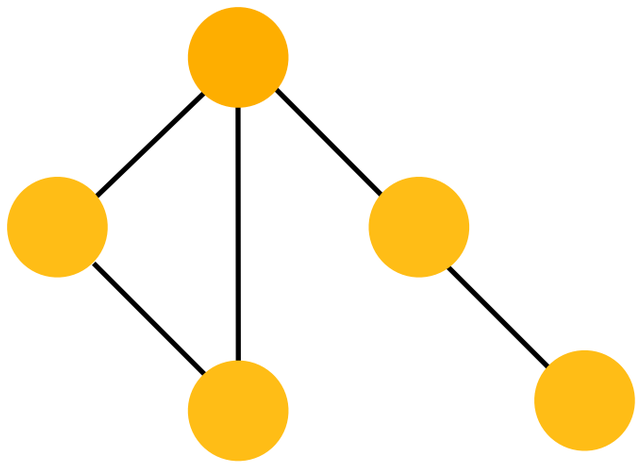


subgraph H

(6 copies of H appear in A)

- # copies of H in graph $A \Longrightarrow$ capture some graph information;

Subgraph count



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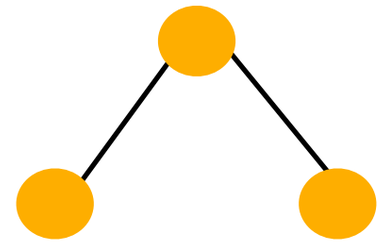

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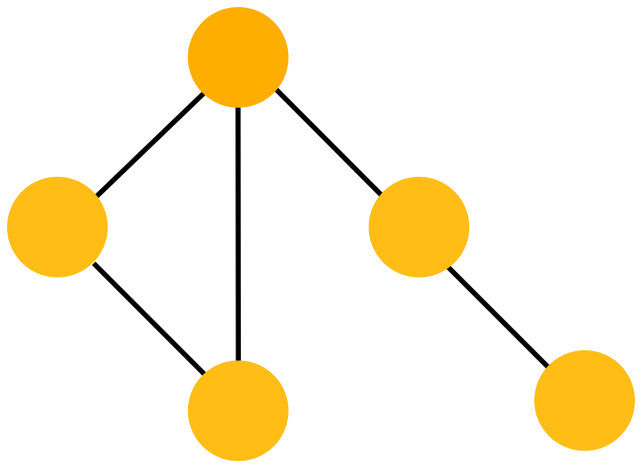

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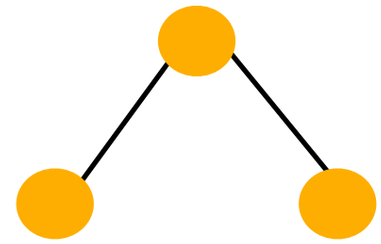

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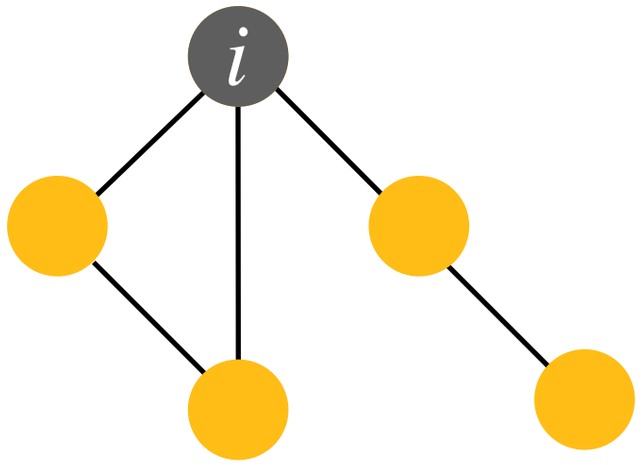
Subgraph count



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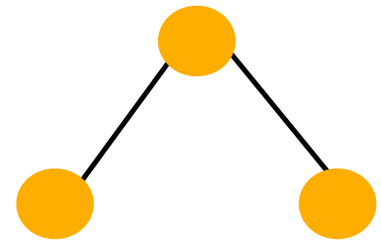

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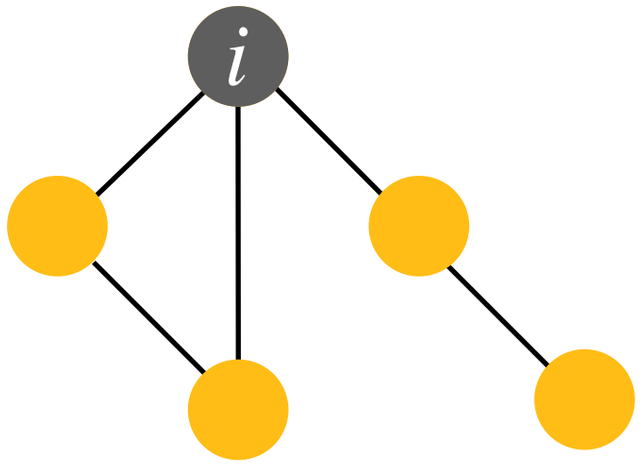
- How to capture the vertex information of i?

Rooted subgraph count



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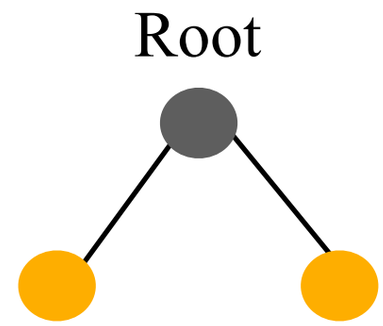
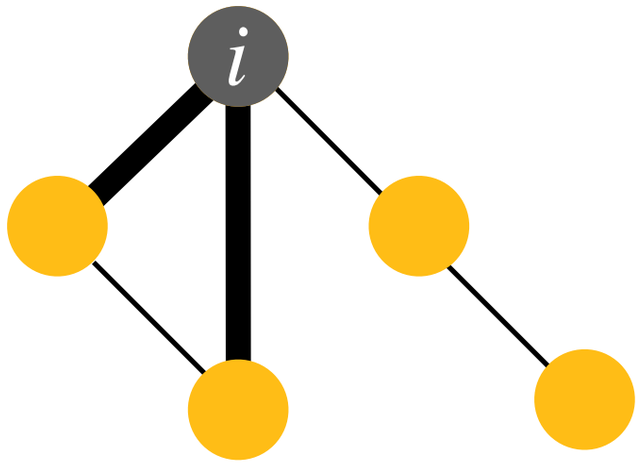


Rooted subgraph H

- # copies of H rooted at i in graph A, denoted as $W_{i,H}(A)$;

Rooted subgraph count



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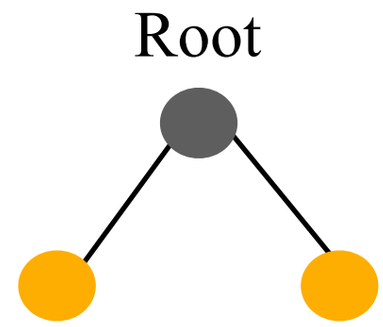

Rooted subgraph H

(3 copies of H rooted at node i appear in A)

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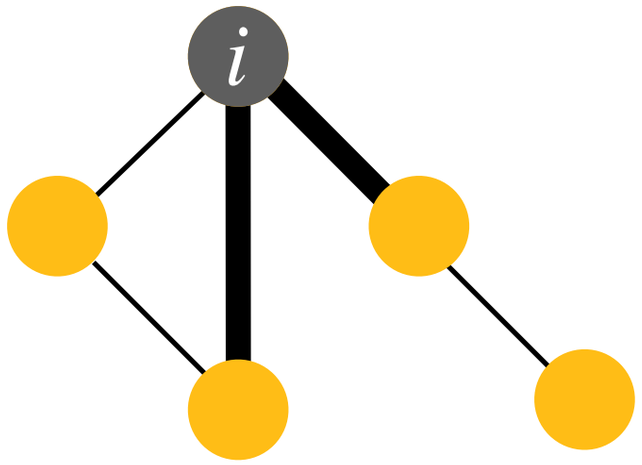

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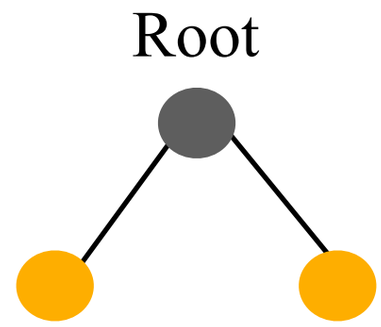

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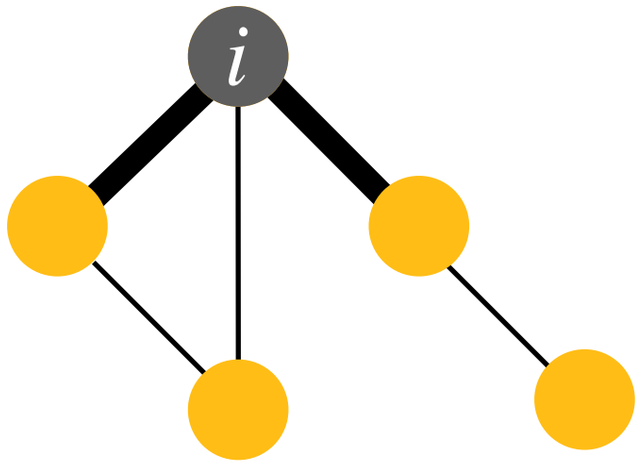

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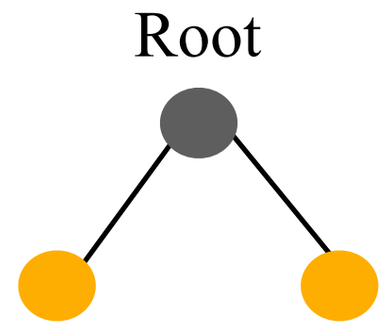

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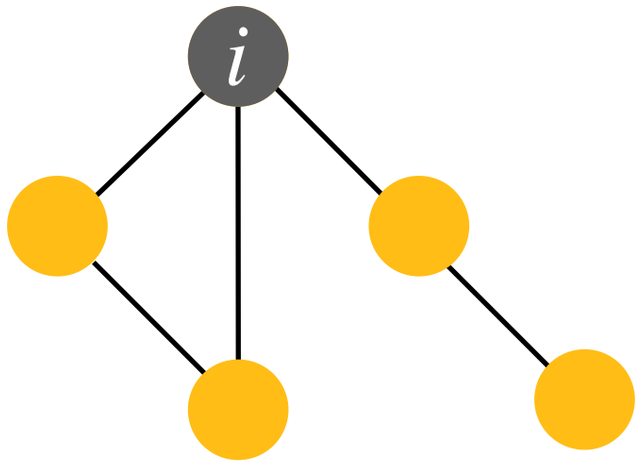

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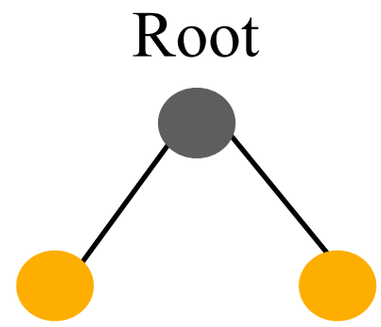


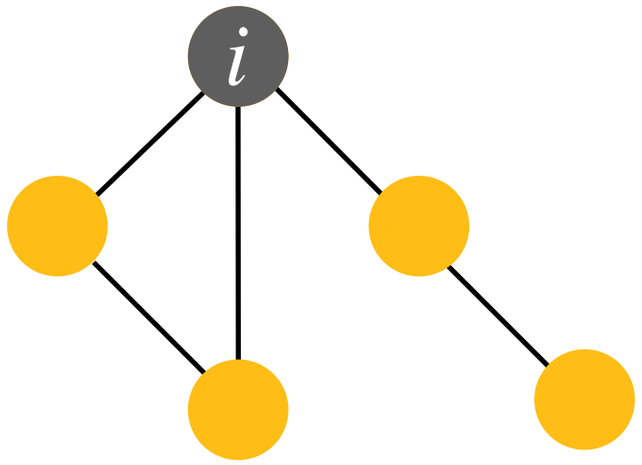
Rooted subgraph H

(3 copies of H rooted at node i appear in A)

$$W_{i,H}(A) = 3$$

- # copies of H rooted at i in graph A, denoted as $W_{i,H}(A)$;

Rooted subgraph count



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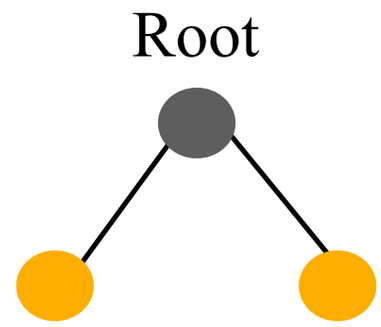


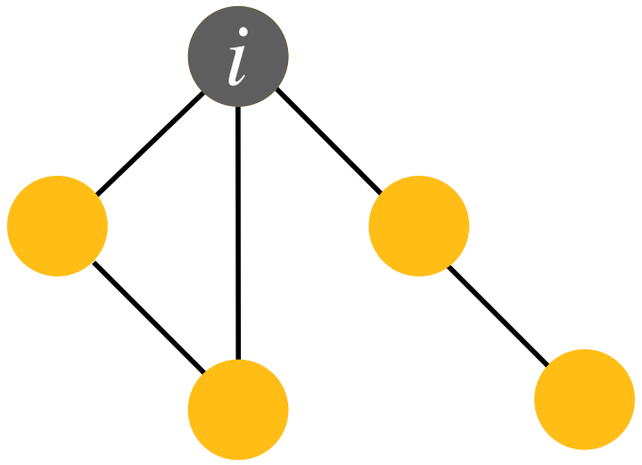
Rooted subgraph H

(3 copies of H rooted at node i appear in A)

$$W_{i,H}(A) = 3$$

- # copies of H rooted at i in graph A, denoted as $W_{i,H}(A)$;
- Capture some vertex information;

Rooted subgraph count



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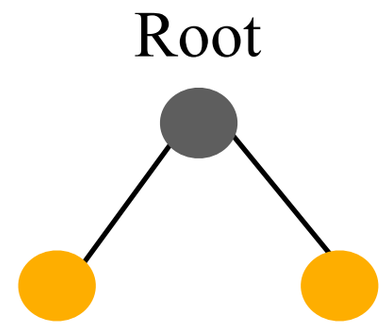

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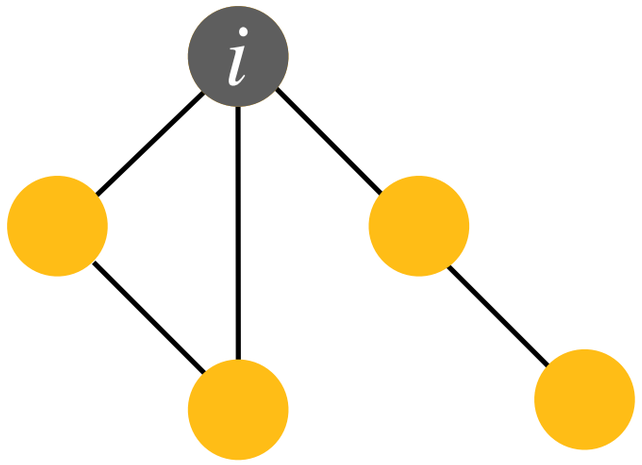
- # copies of H rooted at i in graph A, denoted as $W_{i,H}(A)$;
- Capture some vertex information;
- Idea: construct a rich family of rooted subgraphs:

Rooted subgraph count



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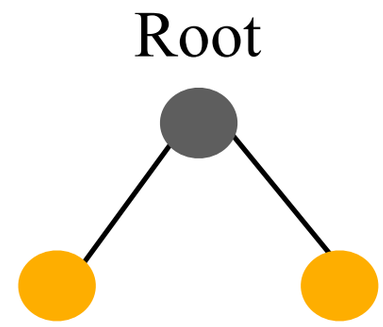

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  - Each rooted subgraph captures some information about the vertex;

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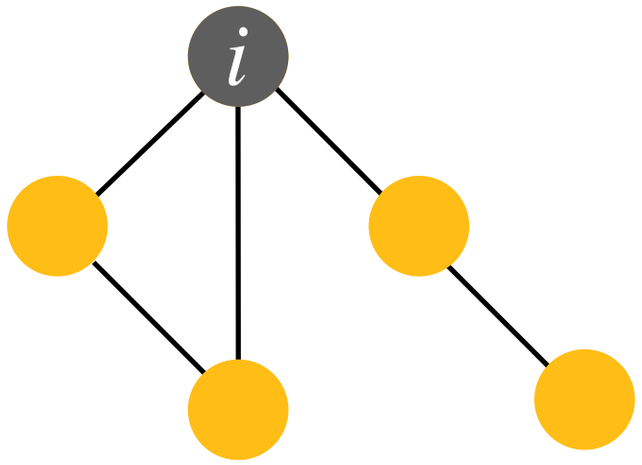

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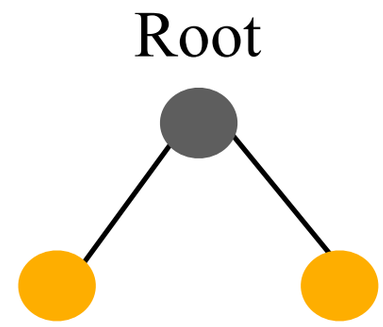

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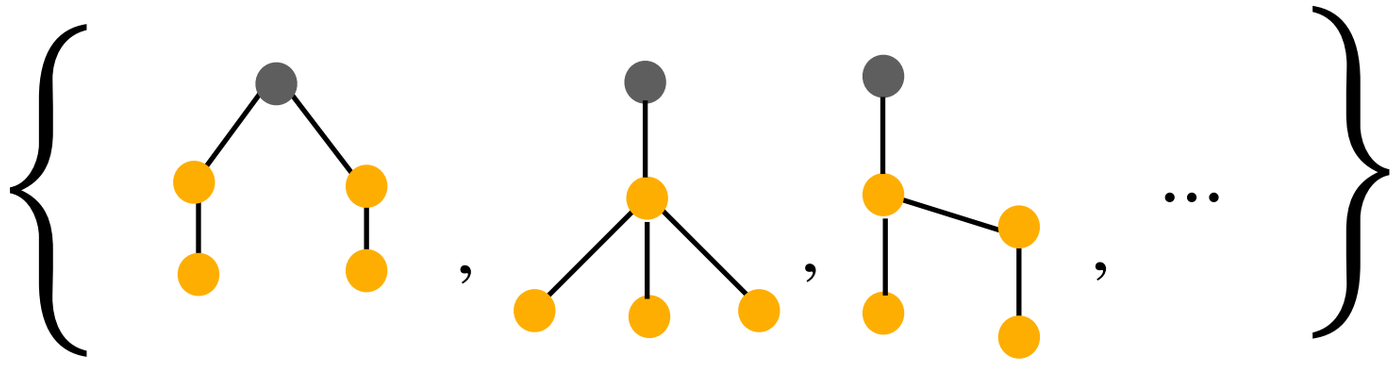
- Capture some vertex information;
- Idea: construct a rich family of rooted subgraphs:
  - Each rooted subgraph captures some information about the vertex;
  - **Combine** all the information $\Longrightarrow$ vertex signature to be more informative.

Graph matching via counting rooted subgraphs

Given **a family $\mathscr{H}$ of rooted subgraphs** with N edges:

e.g. $N = 4$, $\mathscr{H} =$ 

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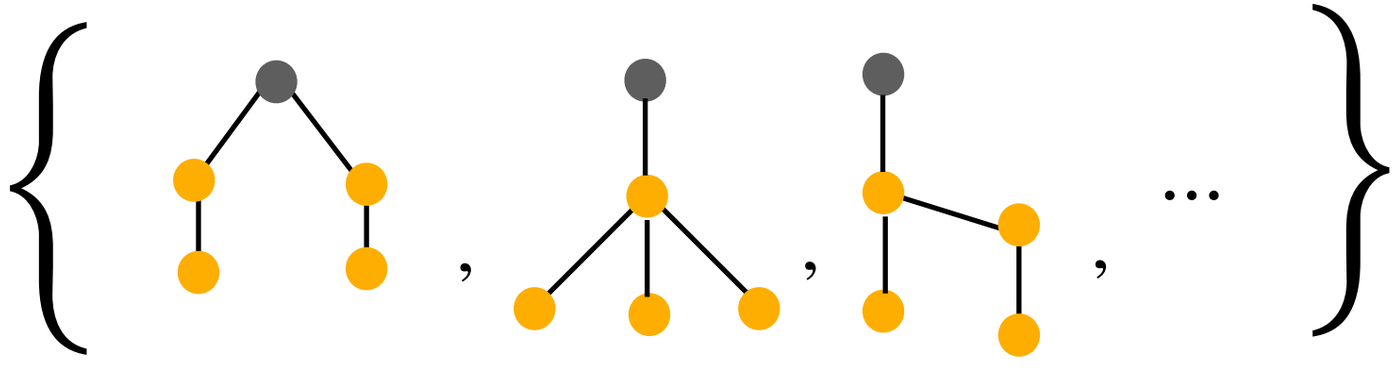
e.g. $N = 4$, $\mathscr{H} =$ 

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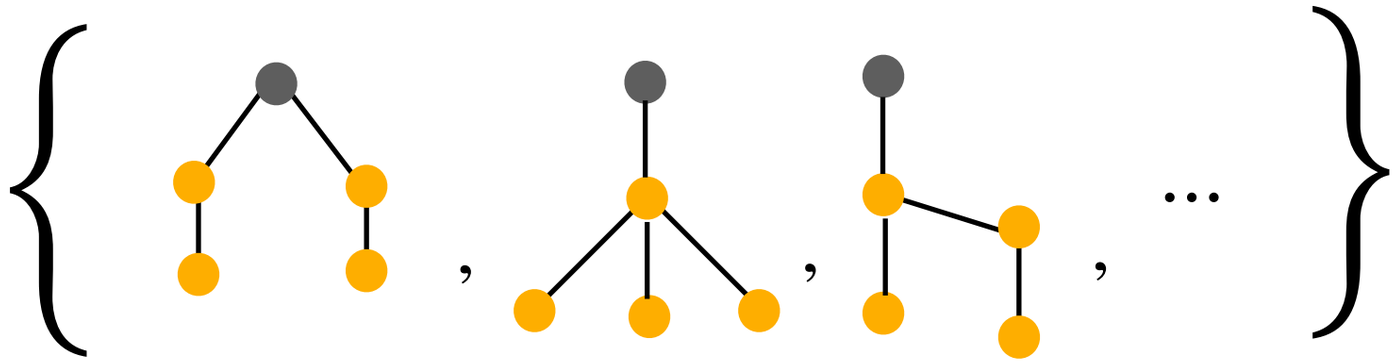
e.g. $N = 4$, $\mathscr{H} =$ 

Highly correlated across different H

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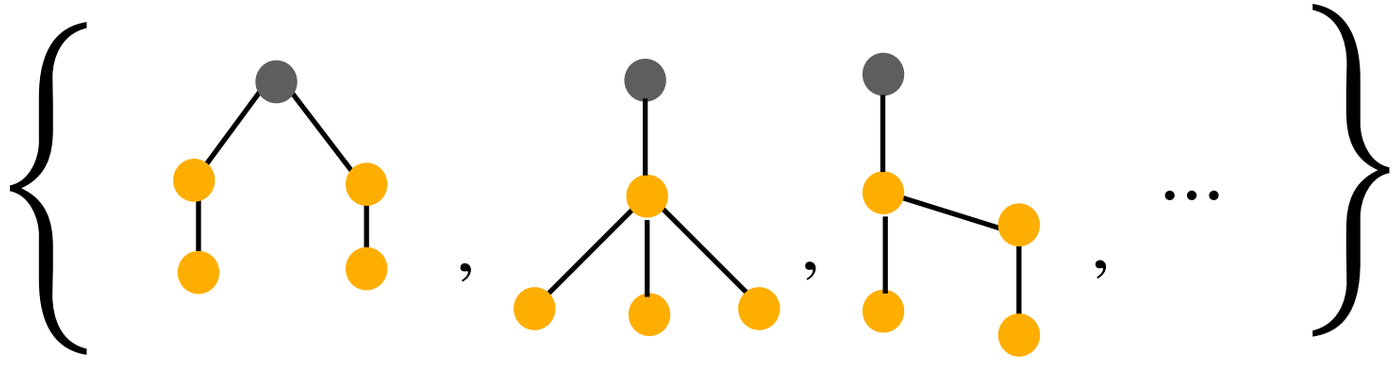
e.g. $N = 4$, $\mathscr{H} =$ 

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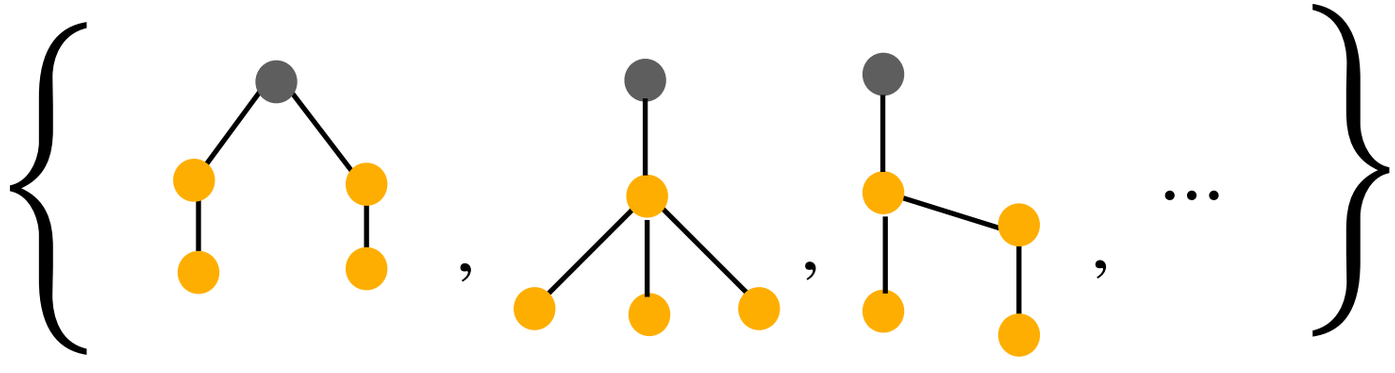
e.g. $N = 4$, $\mathscr{H} =$ 

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Count the weighted copies in $\overline{A}$
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Graph matching via counting rooted **signed** subgraphs

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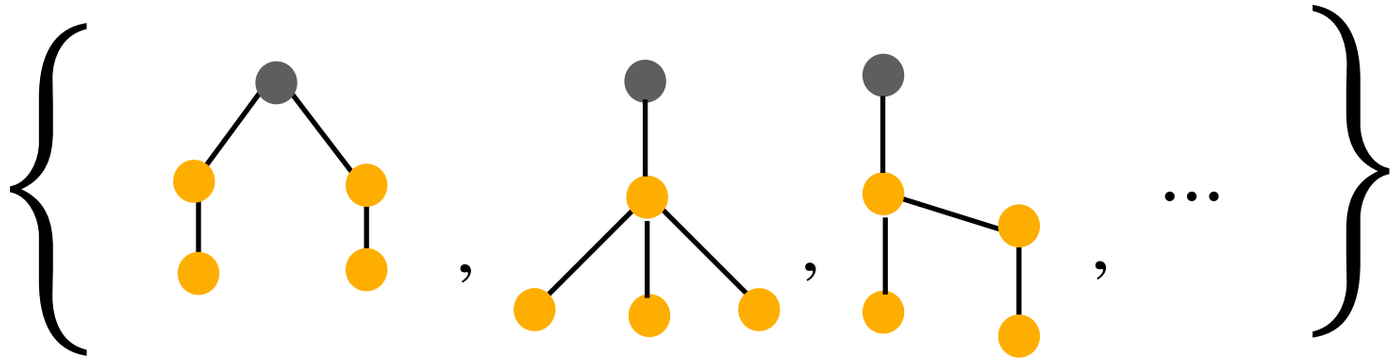
e.g. $N = 4$, $\mathscr{H} =$ 

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rooted "**signed**" subgraph count [Bubeck-Ding-Eldan-Rácz '16]

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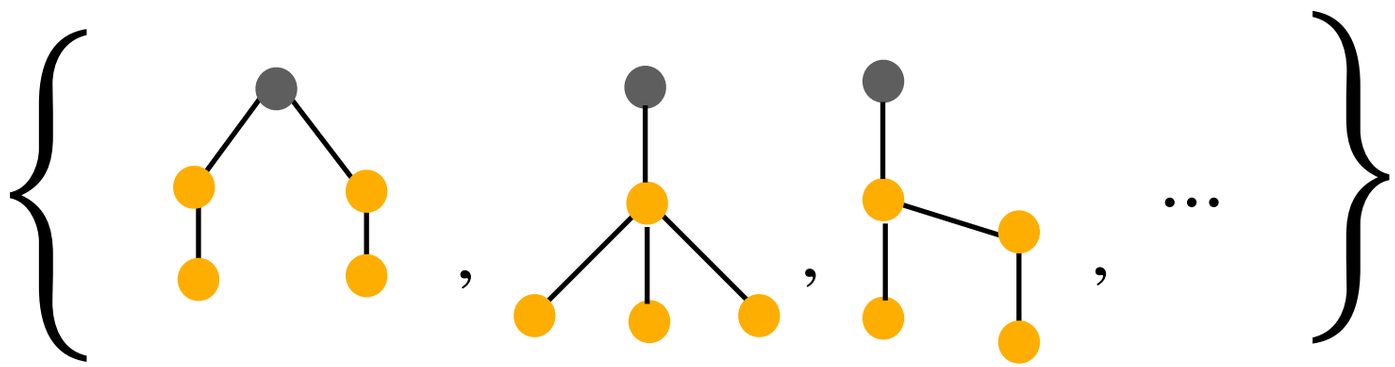
e.g. $N = 4$, $\mathscr{H} =$ 

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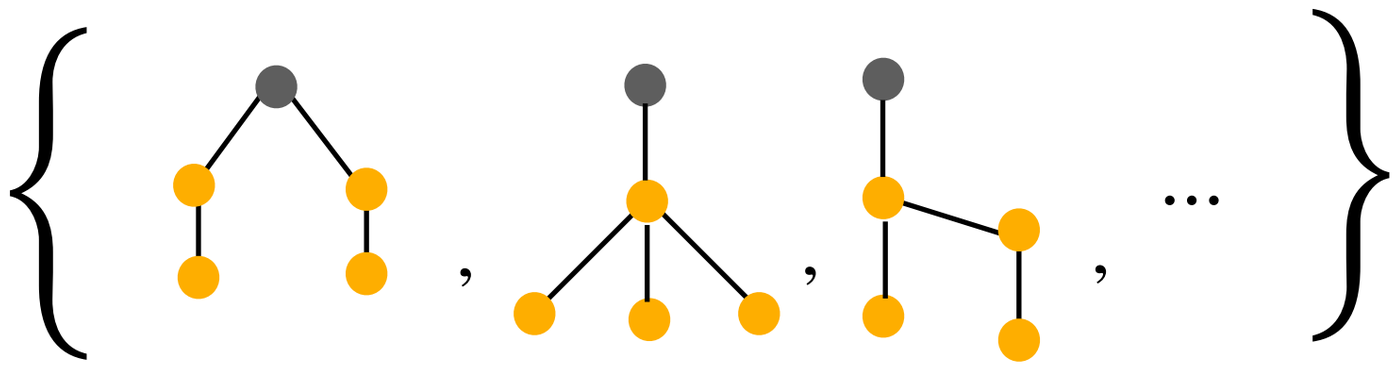
e.g. $N = 4$, $\mathscr{H} =$ 

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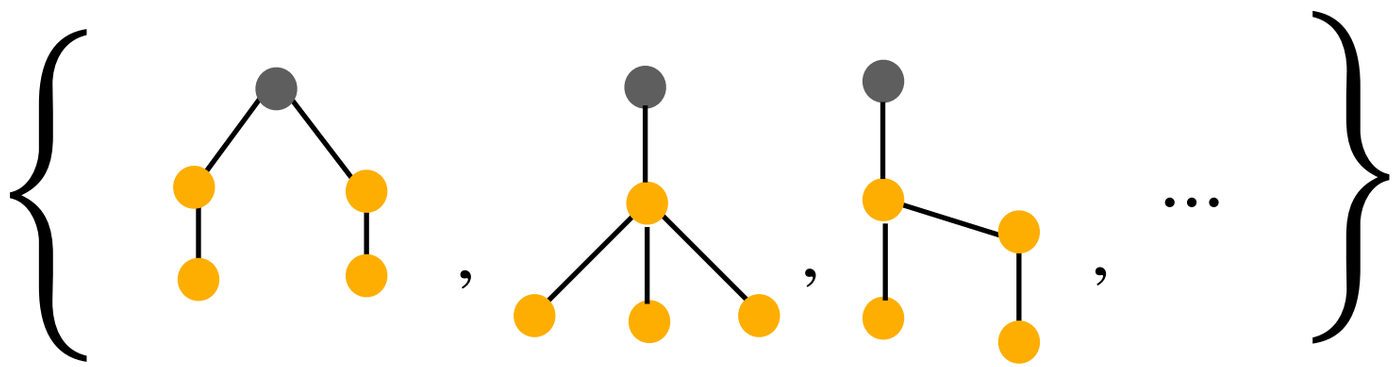
e.g. $N = 4$, $\mathscr{H} =$ 

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rooted "**signed**" subgraph count

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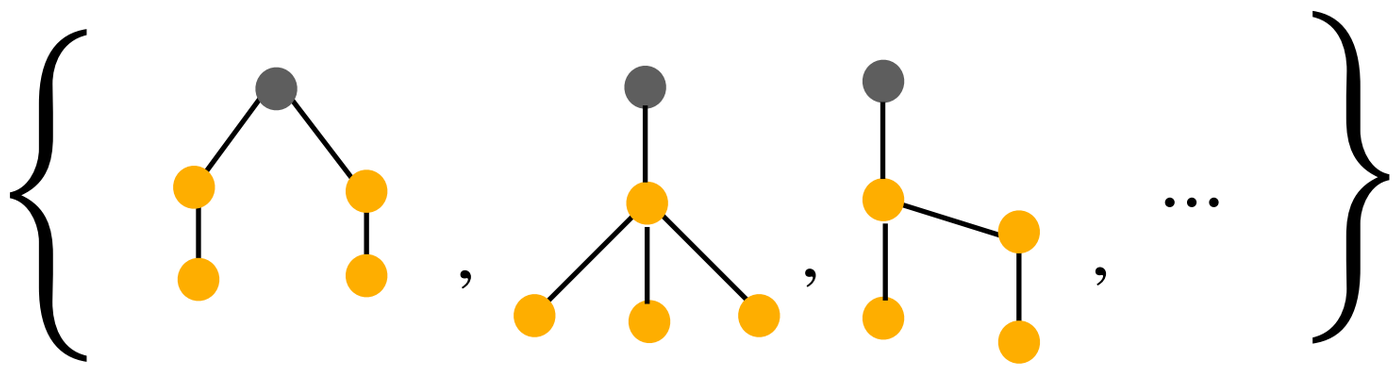
e.g. $N = 4$, $\mathscr{H} =$ 

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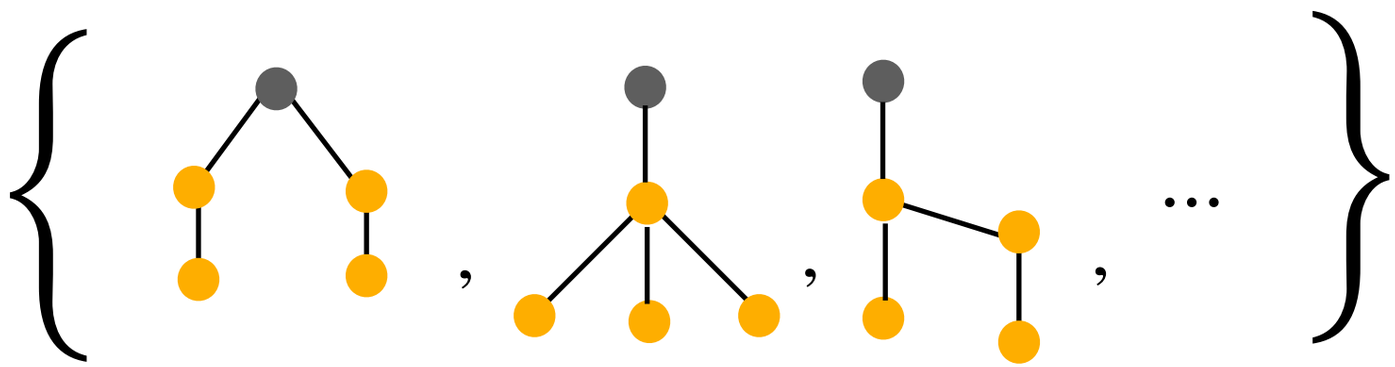
e.g. $N = 4$, $\mathscr{H} =$ 

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Polynomials of A_{ij}'s and B_{ij}'s

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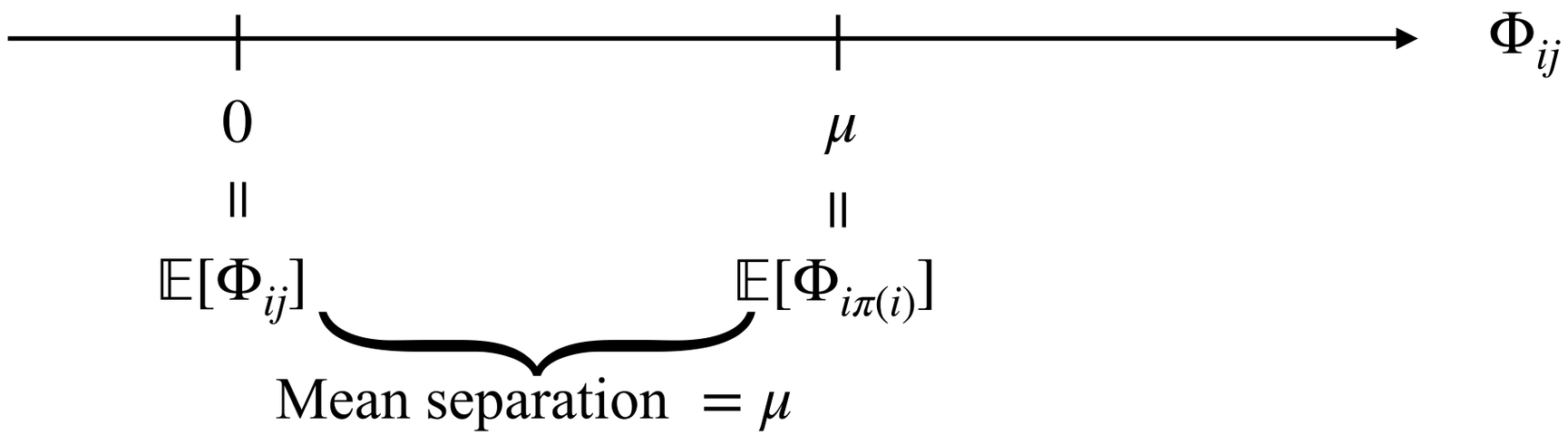
Desirable properties of similarity scores (Mean)

For each vertex i in A and j in B:

$$\Phi_{ij} = \left\langle s_i, t_j \right\rangle = \sum_{H \in \mathscr{H}} \mathrm{aut}(H) \underbrace{W_{i,H}(\overline{A}) W_{j,H}(\overline{B})}_{\text{Correlated iff } j = \pi(i) \text{ (true pair)}} .$$

Fake pair: $j \neq \pi(i)$ True pair: $j = \pi(i)$



Desirable properties of similarity scores (Variance)

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We want: **fluctuation** of Φ_{ij} to be relatively small compared to μ.

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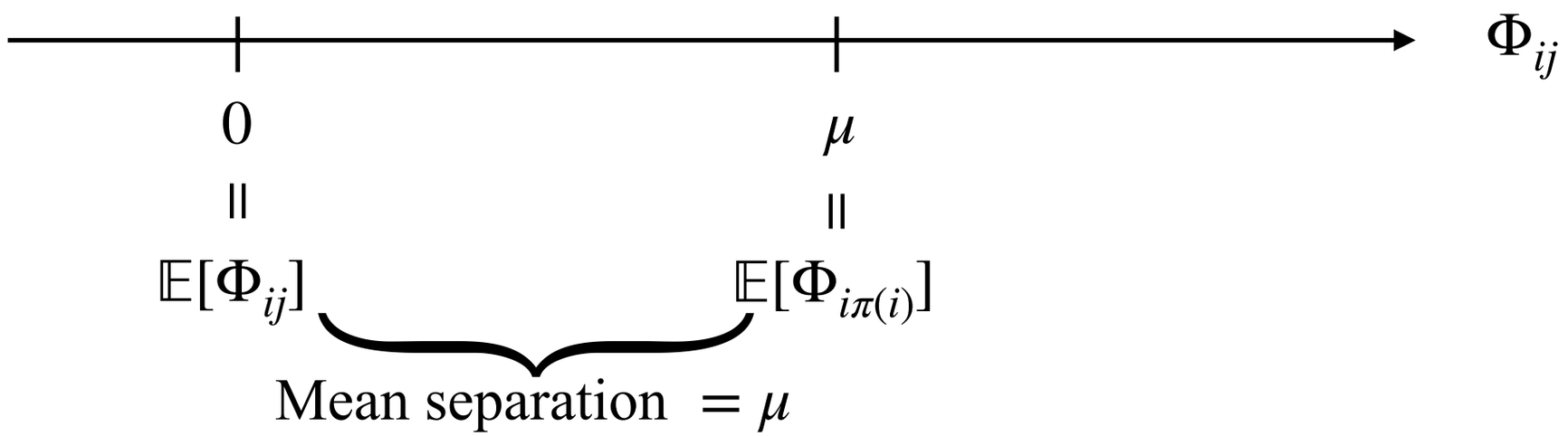

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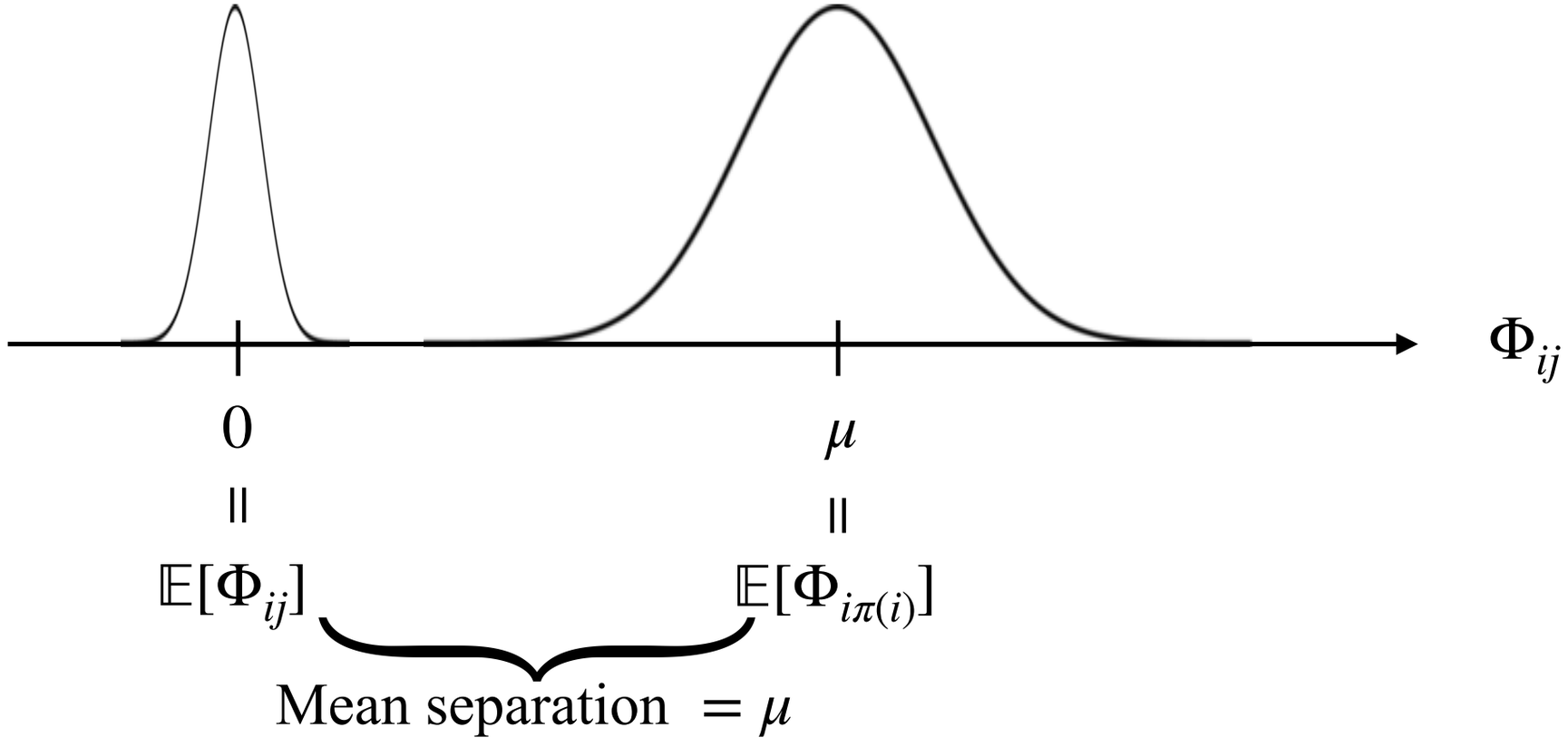

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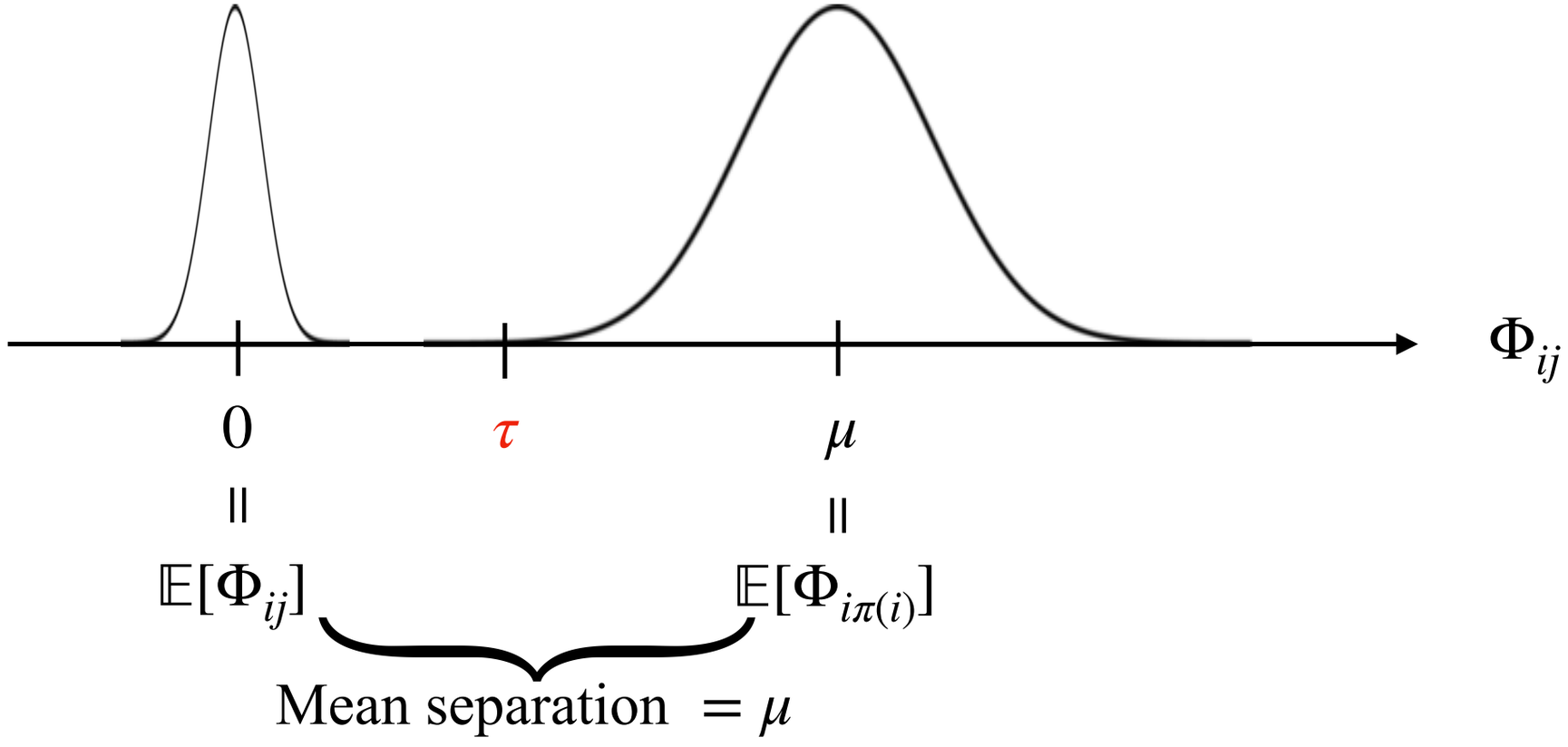

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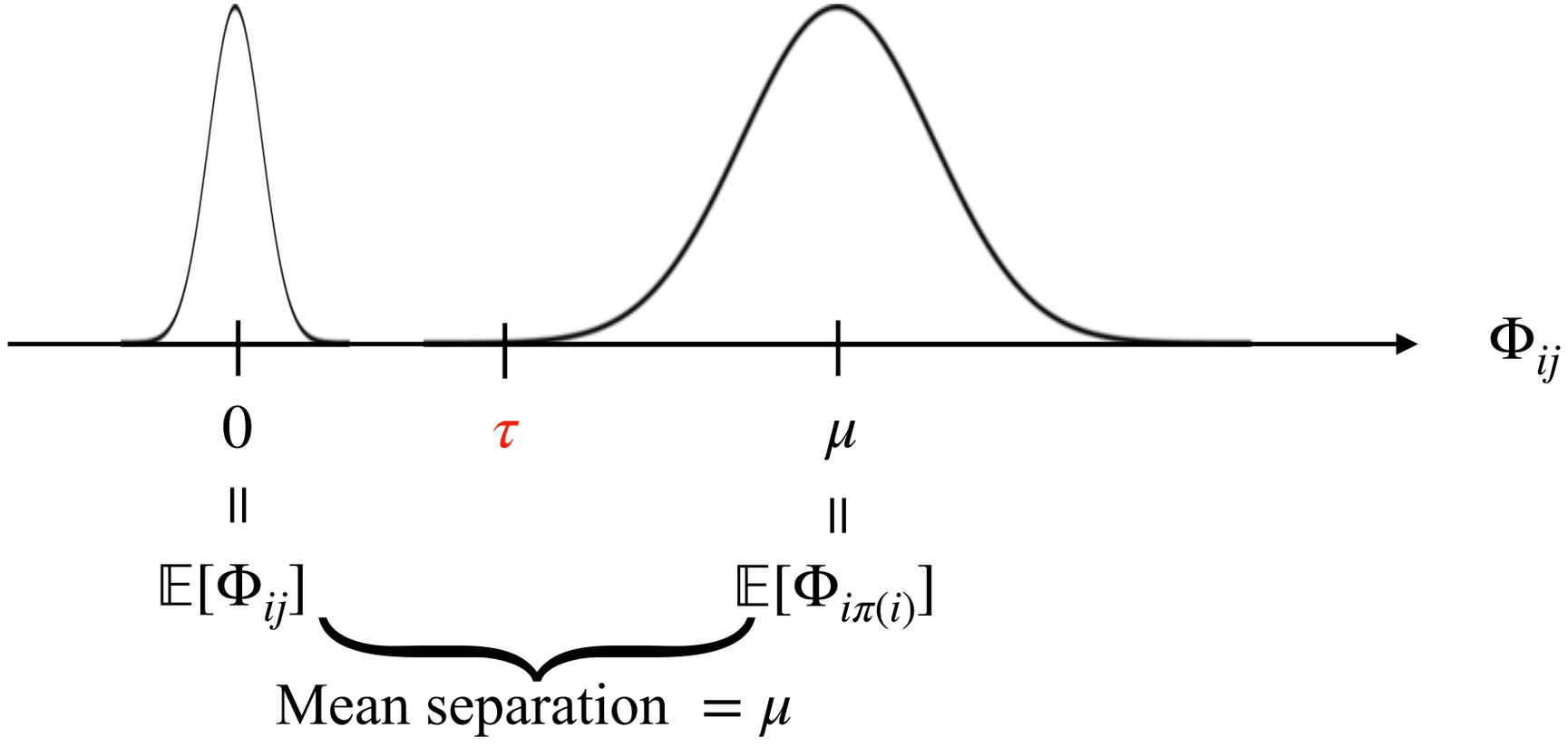

Which family of $\mathscr{H}$ (rooted subgraphs) to choose?

Which family of rooted subgraphs to count?

Suppose $\mathscr{H}$ be a family of rooted subgraphs with N edges.

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- Wishful thinking: **ignoring the cross-correlations** of $\mathrm{aut}(H) W_{i,H}(\overline{A}) W_{j,H}(\overline{B})$ and $\mathrm{aut}(I) W_{i,I}(\overline{A}) W_{j,I}(\overline{B})$ for different subgraphs H and I:

$$\frac{\mathrm{Var}[\Phi_{ij}]}{\mu^2} \approx \frac{1}{|\mathscr{H}| \rho^{2N}} \xrightarrow{\text{goal}} 0$$

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which requires

- The family $\mathscr{H}$ to be "**rich**": $|\mathscr{H}|$ grows at least exponentially in N;
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  - The family $\mathscr{H}$ to be "**rich**": $|\mathscr{H}|$ grows at least exponentially in N;
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- We also want H to be **"simple"** to count.

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} **Trees**

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A family of rooted **trees**

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$\rho^2 > \alpha$

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($\rho^2 > \alpha$; $\approx$ marked with a red cross)

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- If H is a tree, it is **"simple"** to count via color coding [Alon-Yuster-Zwick' 95].
- However, we **cannot ignore the cross-correlations.**

We need to **construct** a **special** family of trees.

A **special** family of rooted trees

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A **special** family of rooted trees - Chandeliers

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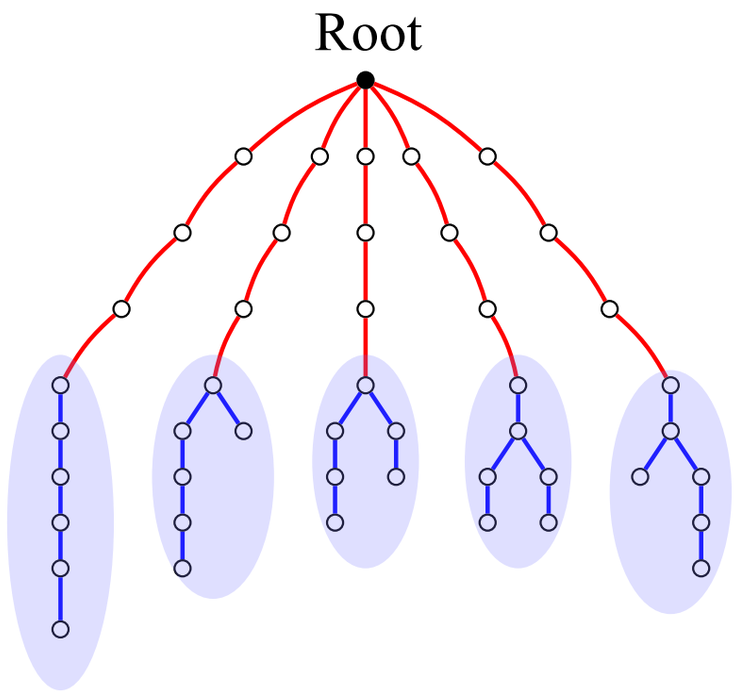

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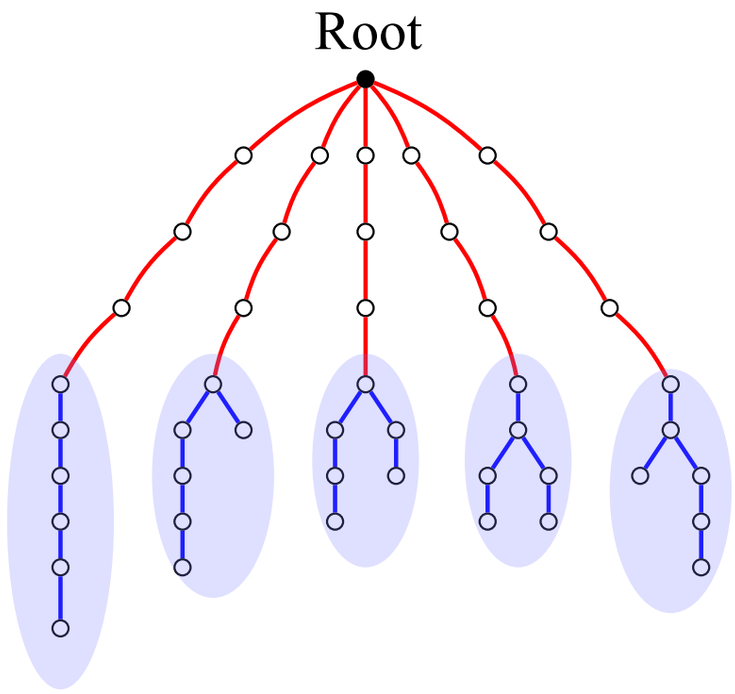

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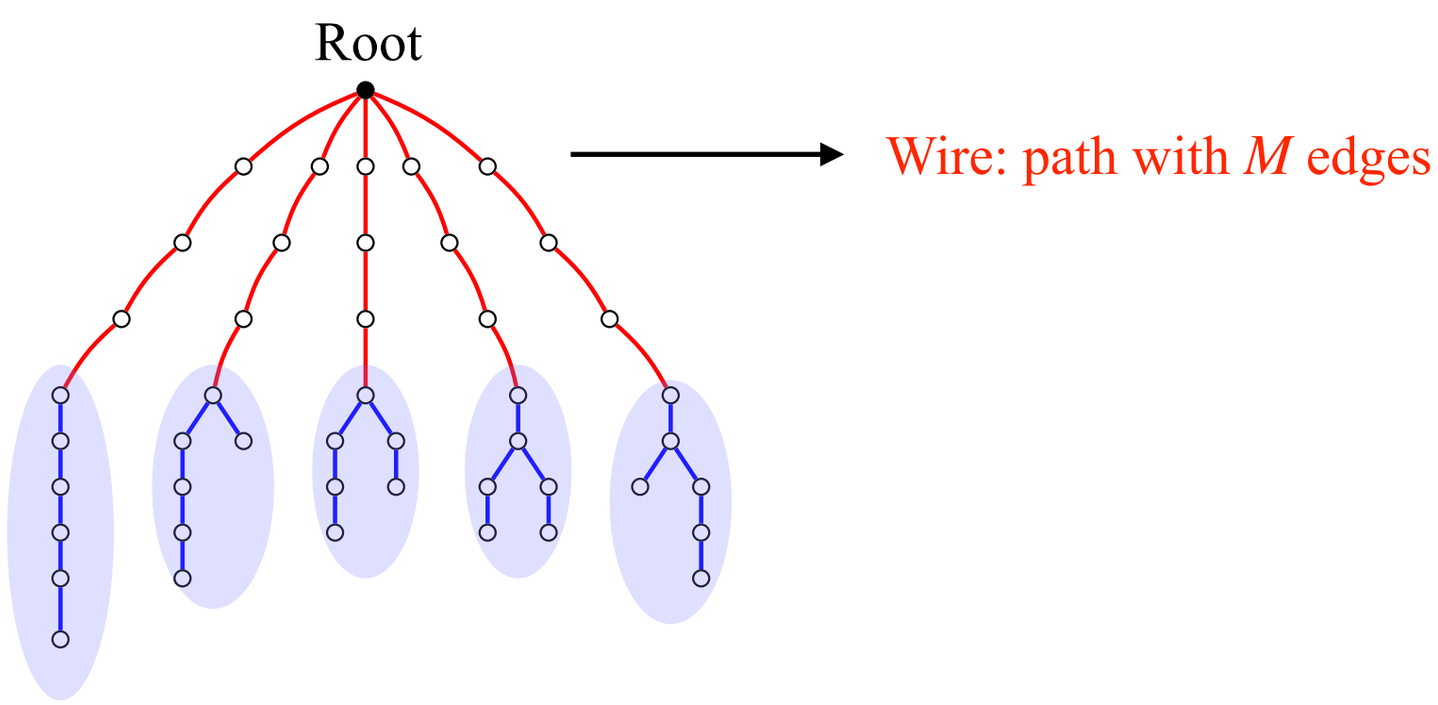
$$\Phi_{ij} = \left\langle s_i, t_j \right\rangle = \sum_{H \in \mathscr{H}} \mathrm{aut}(H) W_{i,H}(\overline{A}) W_{j,H}(\overline{B}) \,.$$

We want to construct a special family of trees:

- **Suppress** the undesirable cross-correlations to control $\mathrm{Var}[\Phi_{ij}]/\mu^2$;
- This special family of trees has to be "**rich**" enough;



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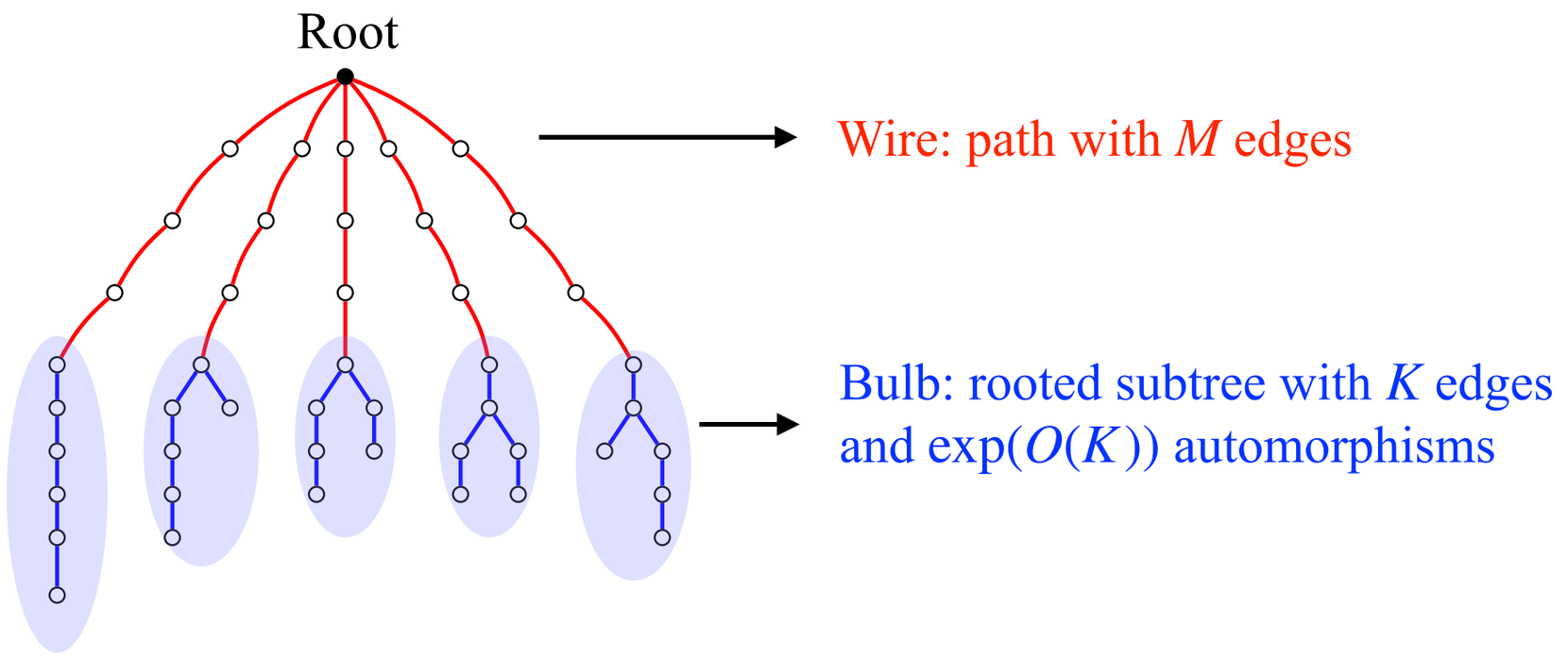

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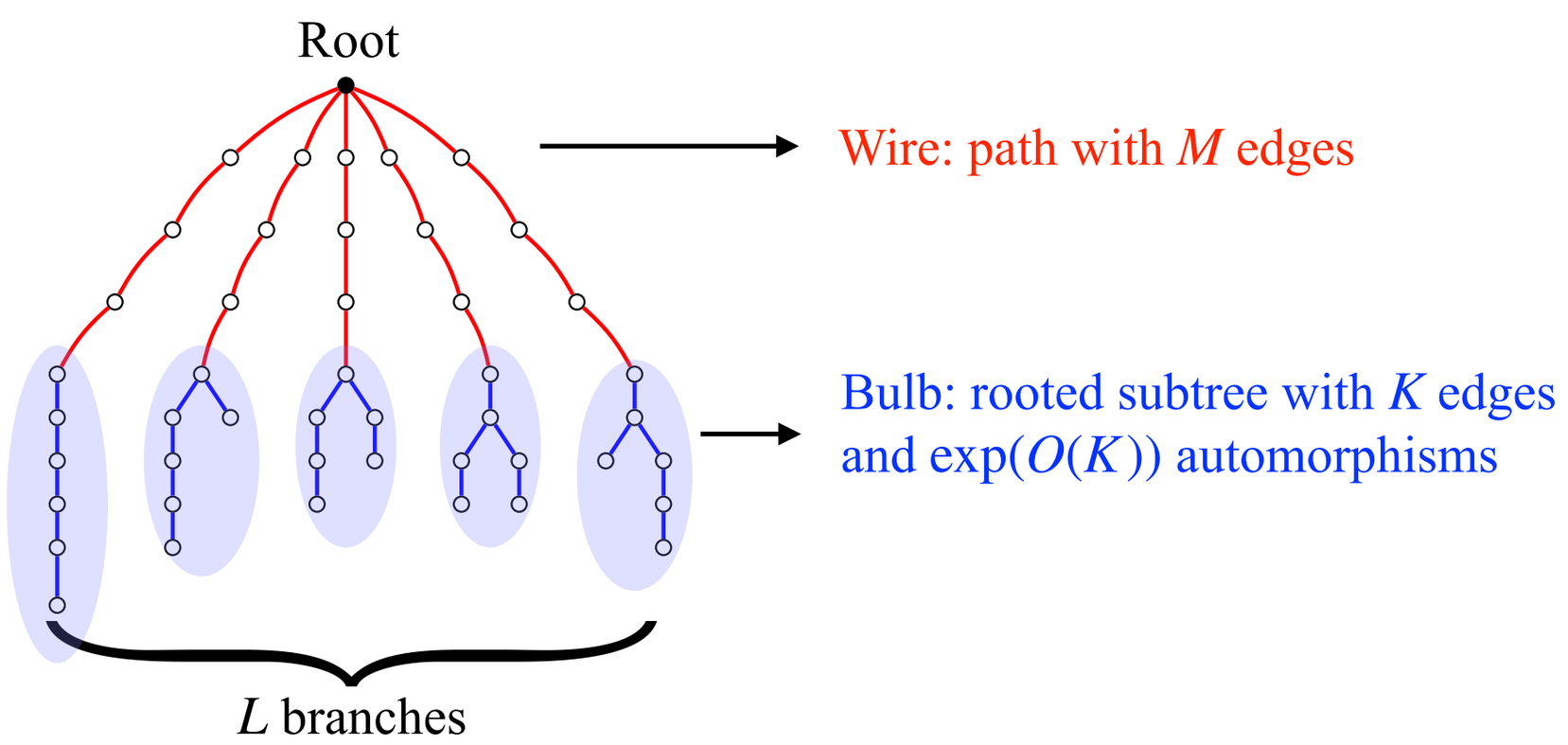

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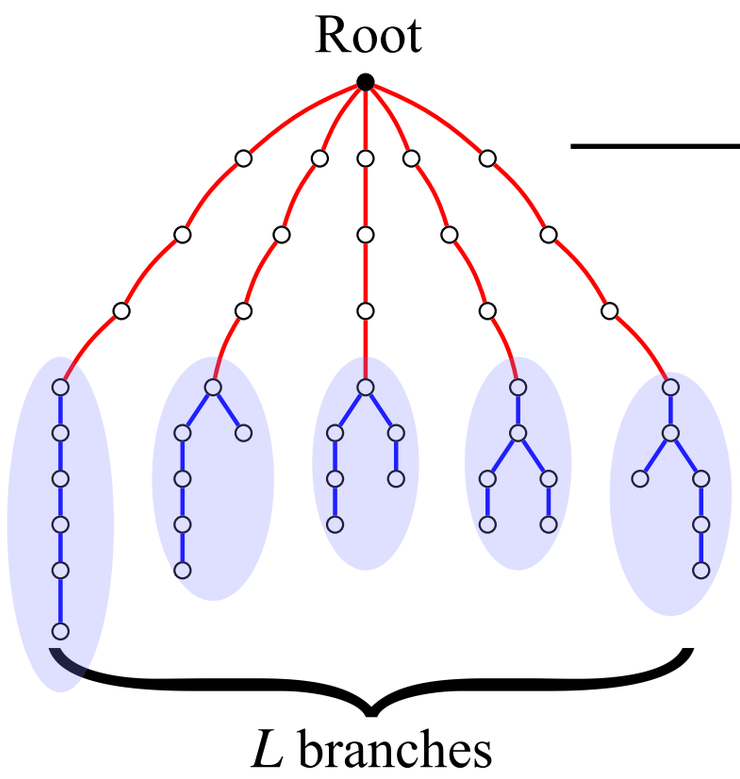

Wire: path with M edges

Bulb: rooted subtree with K edges and $\exp(O(K))$ automorphisms

Chandelier has N edges where $N = (M + K)L$

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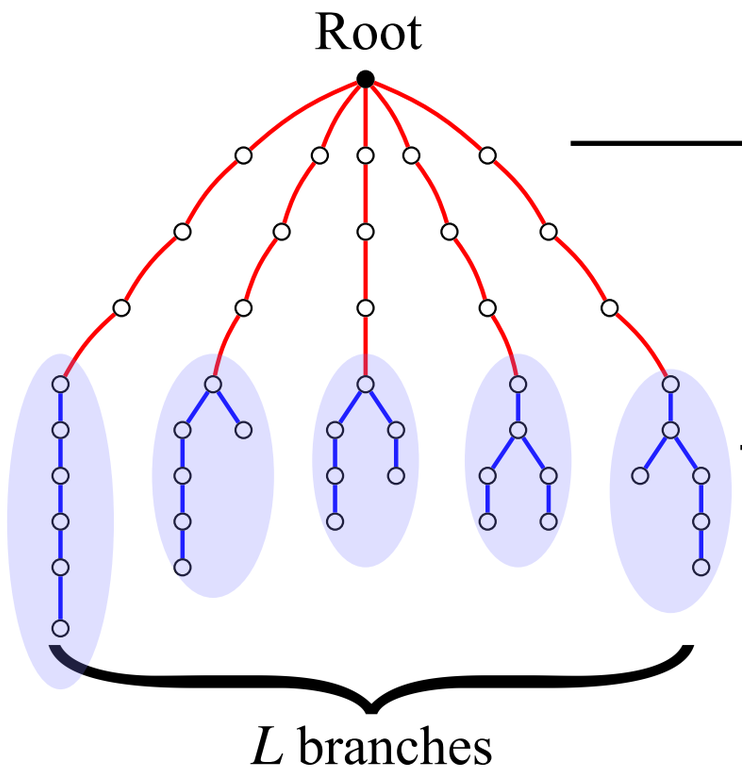

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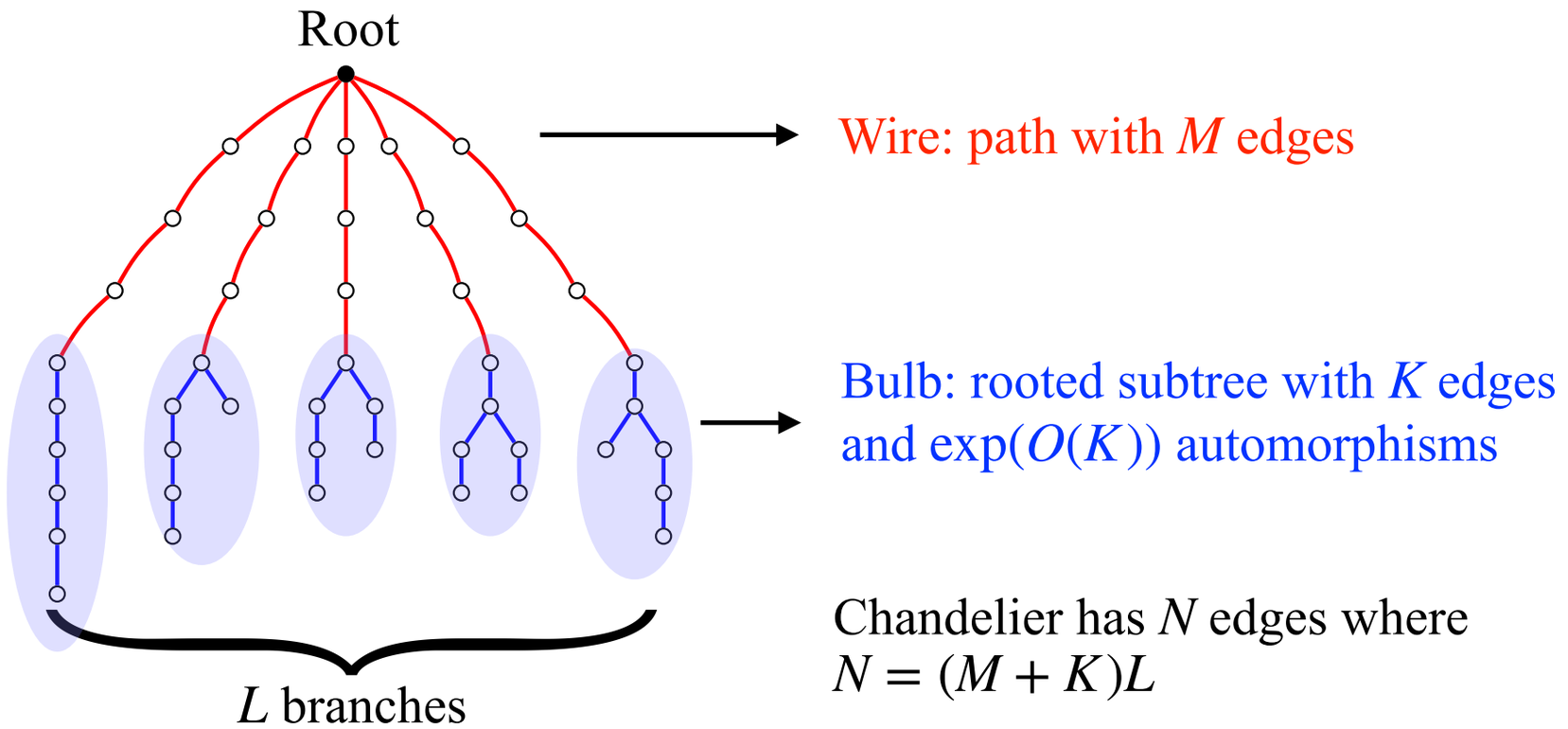

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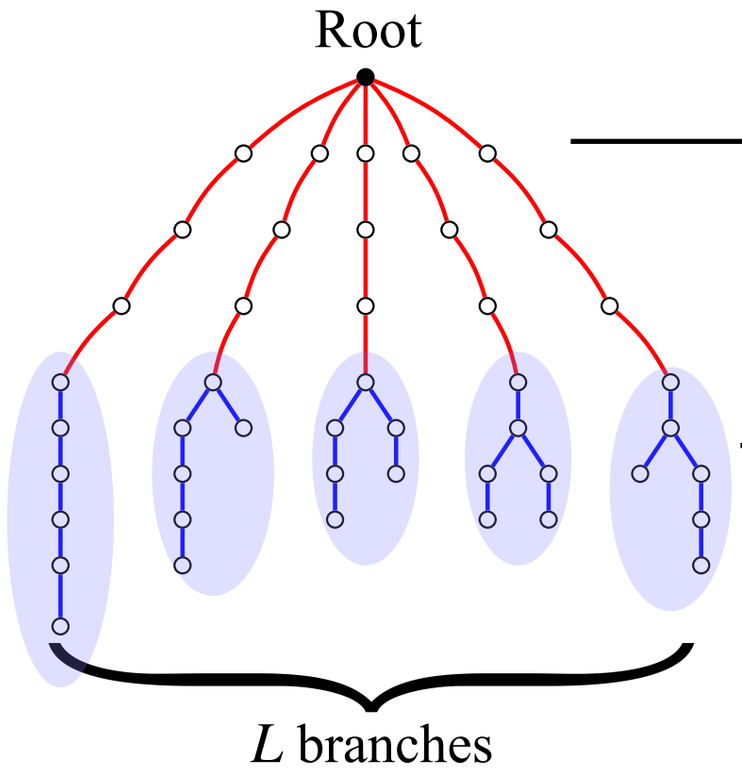
Majority of trees have $\exp(O(K))$ automorphisms [Olsson-Wagner '22]



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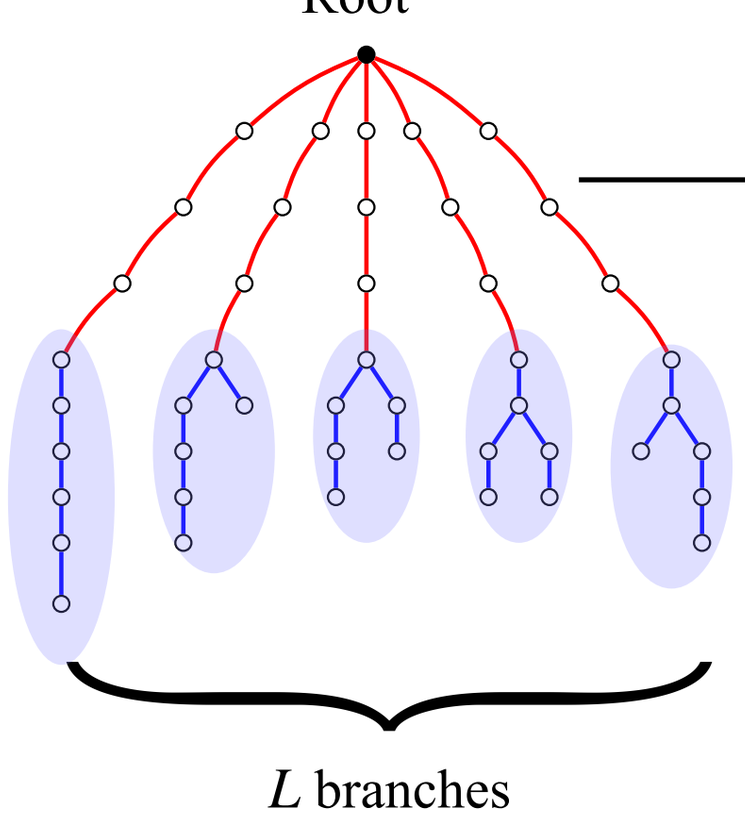
Chandelier has N edges where $N = (M + K)L$

Summary on properties of Chandeliers

Let $\mathscr{H}$ denote a special family of chandeliers:

- **Large:** The size of chandeliers $N = c \log n$, where c is some small constant.
- **Rich:** Choose $K \gg M$, to ensure $|\mathscr{H}| = (\alpha + o(1))^{-N}$.
  - Almost as rich as the whole family of all rooted trees.
- **Informative:** Choose smaller L and larger M, when graphs are sparser.



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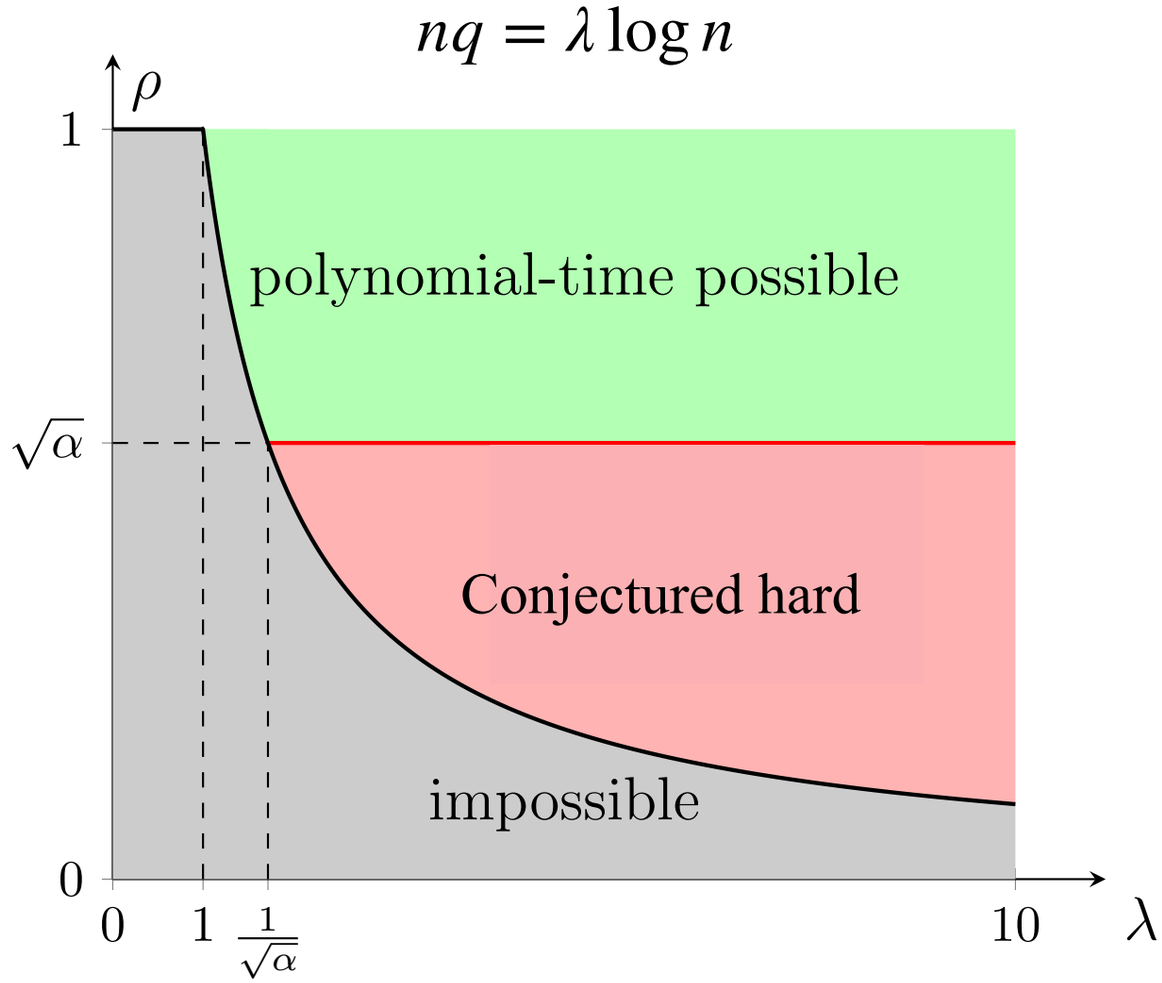
Summary on efficient algorithms



Selected references

Information-theoretic limits:

Y. Wu, J. Xu, & S. H. Yu, *Settling the sharp reconstruction thresholds of random graph matching*, IEEE Transactions on Information Theory, arXiv:2102.00082.

J. Ding & H. Du, *Matching recovery threshold for correlated random graphs,* Annals of Statistics, arXiv:2205.14650

Efficient algorithms and computational limits:

C. Mao, Y. Wu, J. Xu, & S. H. Yu, *Random graph matching at Otter's threshold via counting chandeliers*, STOC 2023, arxiv:2209.12313

L. Ganassali, L. Massoulié, & G. Semerjian, *Statistical limits of correlation detection in trees,* to appear in Annals of Applied Probability, arXiv:2209.1373

J. Ding & Z. Li, *A polynomial-time iterative algorithm for random graph matching with non-vanishing correlation*, arXiv:2306.00266.

J. Ding, H. Du & Z. Li, *Low-degree hardness of detection for correlated Erdős-Rényi graphs*, arXiv:2311.15931

Open problems and future directions

- Rigorous evidences for statistical-computational gaps
- Beyond Erdős–Rényi graphs:
  - Random geometric graph matching [Wang-Wu-Xu-Yolou' 22, …]

$$A = \Big(\kappa(x_i, x_j)\Big), \quad B = \Big(\kappa(y_i, y_j)\Big), \quad \text{where } (x_{\pi(i)}, y_i) \overset{iid}{\sim} P$$

  - Community recovery and graph matching in correlated stochastic block models [Racz-Sridhar' 21, Gaudio-Racz-Sridhar '22]
    - Matching preferential attachment graphs [Korula-Lanttanzi '14] or power-law graphs [Yu-Xu-Lin '21]
- Seeded Graph matching with initial "noisy" matched pairs [Kazemi-Hassani-Grossglausser '15, Lubars-Srikant '18, Mossel-Xu '20, Yu-Xu-Lin '20]
- Graph matching with node attribute information [Zhang-Wang-Wang-Wang '20]
- "Robust" graph matching [Ameen-Hajek ' 23]
- Random graph matching with multiple graphs [Ameen-Hajek ' 24]
- Database alignment/record linkage problem [Dai-Cullina-Kiyavash '19, …]

MLE: Quadratic Assignment Problem (QAP)

- Maximum likelihood estimator:

$$\pi_{\mathrm{MLE}} = \arg\max_{\hat{\pi}} \sum_{i<j} A_{ij} B_{\hat{\pi}(i)\hat{\pi}(j)} \quad \text{(QAP)}$$

Suppose $A, B \sim$ correlated Erdős–Rényi graph model under a hidden node mapping $\hat{\pi}$.

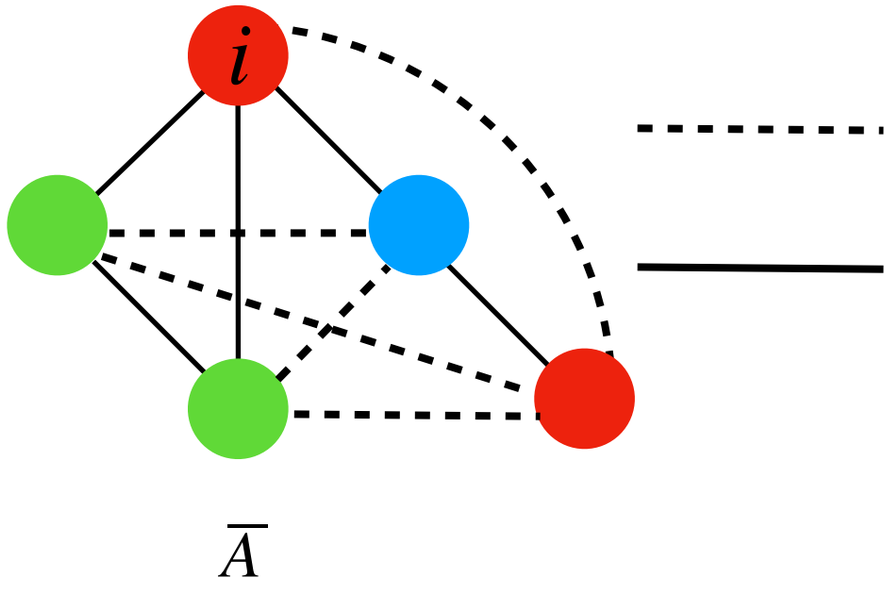
The likelihood function is

$$\begin{aligned}
\mathscr{P}(A, B \mid \hat{\pi}) = \; & (q(q+\rho-q\rho))^{\sum_{i<j} A_{ij} B_{\hat{\pi}(i)\hat{\pi}(j)}} \\
& \Big(q(1-q)(1-\rho)\Big)^{\sum_{i<j} A_{ij}(1-B_{\hat{\pi}(i)\hat{\pi}(j)}) + \sum_{i<j}(1-A_{ij}) B_{\hat{\pi}(i)\hat{\pi}(j)}} \\
& \Big(1-q(2-q-\rho+q\rho)\Big)^{\sum_{i<j}(1-A_{ij})(1-B_{\hat{\pi}(i)\hat{\pi}(j)})} \\
\propto \; & \left(\frac{(q+\rho-q\rho)(1-q(2-q-\rho+q\rho))}{q(1-q)^2(1-\rho)^2}\right)^{\sum_{i<j} A_{ij} B_{\hat{\pi}(i)\hat{\pi}(j)}} .
\end{aligned}$$

Then, we have

$$\mathrm{MLE}_{\pi} = \arg\max_{\hat{\pi}} \mathscr{P}(A, B \mid \hat{\pi}) = \arg\max \sum_{i<j} A_{ij} B_{\hat{\pi}(i)\hat{\pi}(j)} .$$

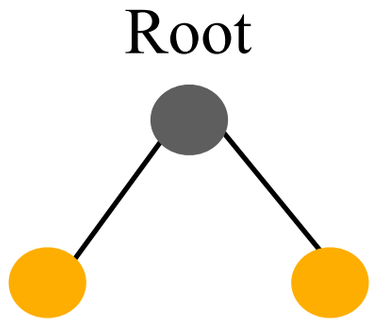
Approximately count signed trees via color coding



$\overline{A}$

$-q$

$1-q$



N edges

Rooted tree H

- Exhaustive search takes n^N times: super poly-time when $N \to \infty$
- Solution: approximate count in $n^2 e^{O(N)}$ time via color coding [Alon-Yuster-Zwick '95]
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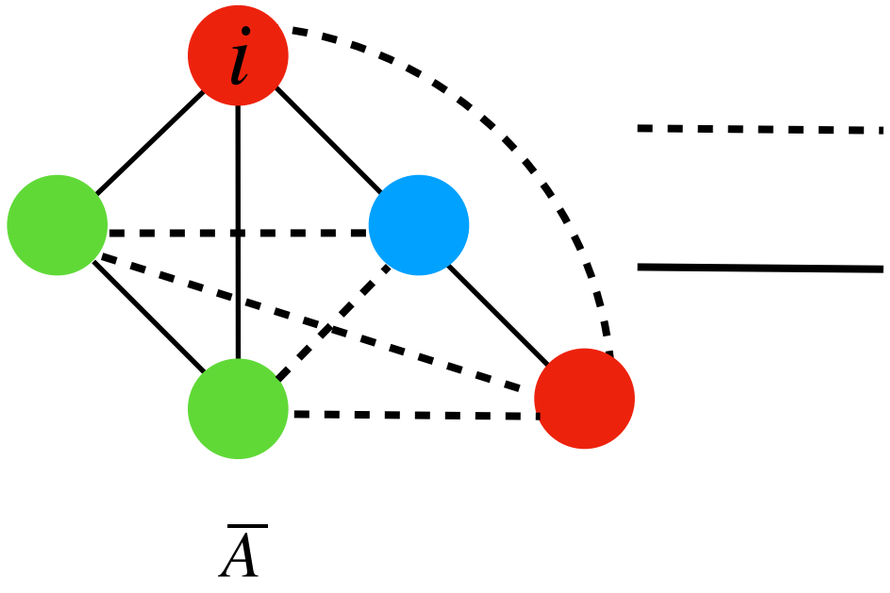

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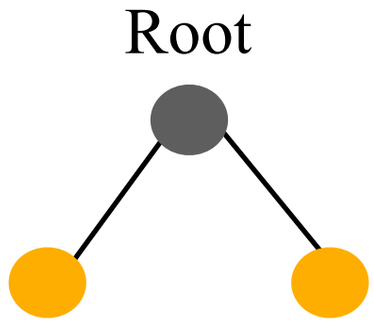

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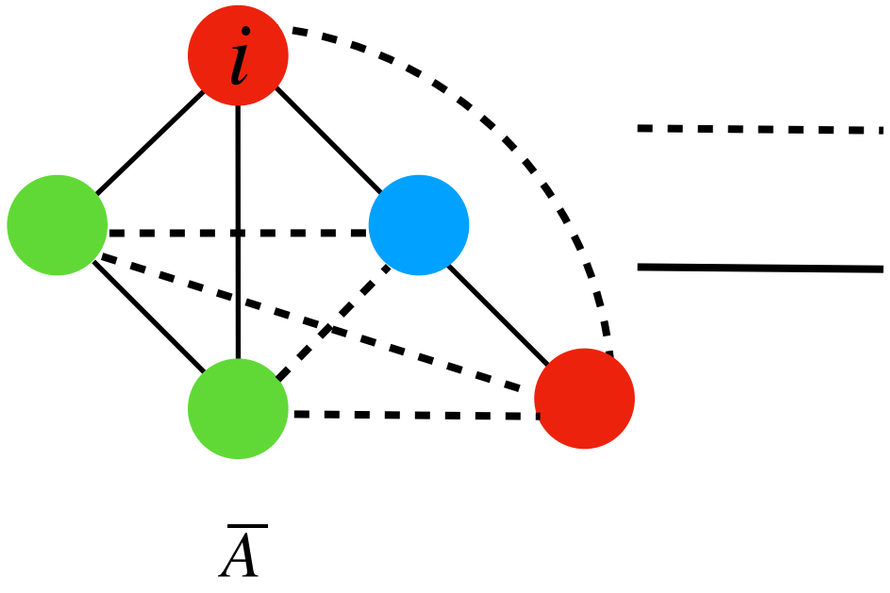

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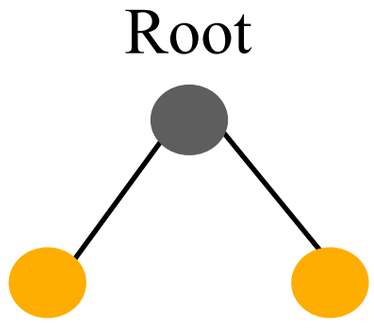

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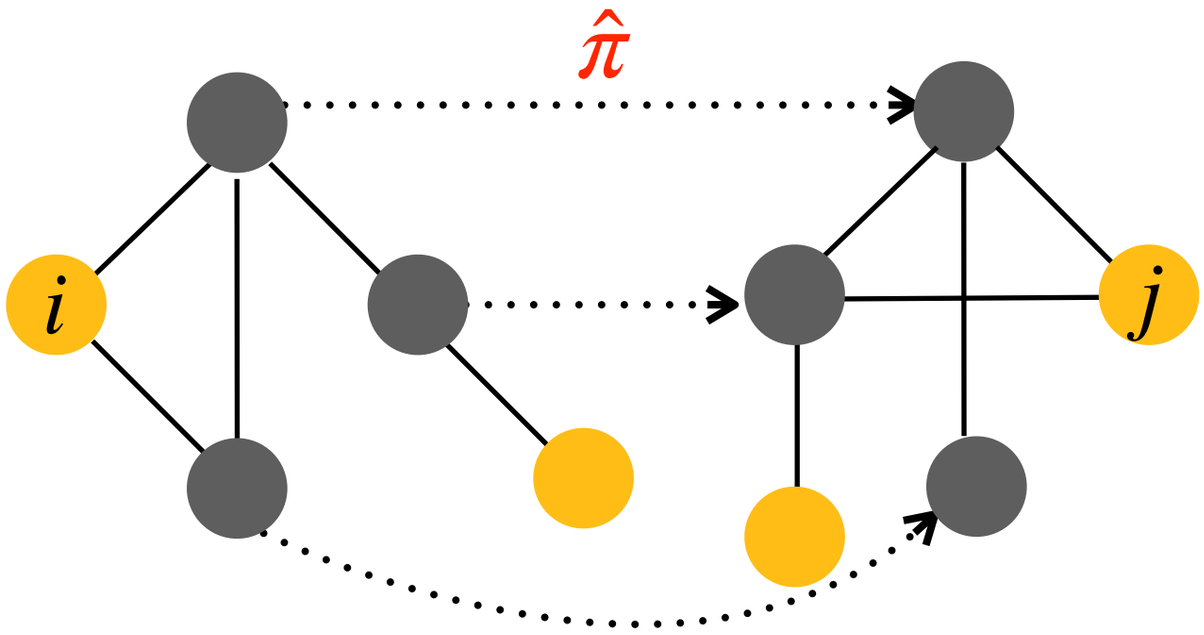

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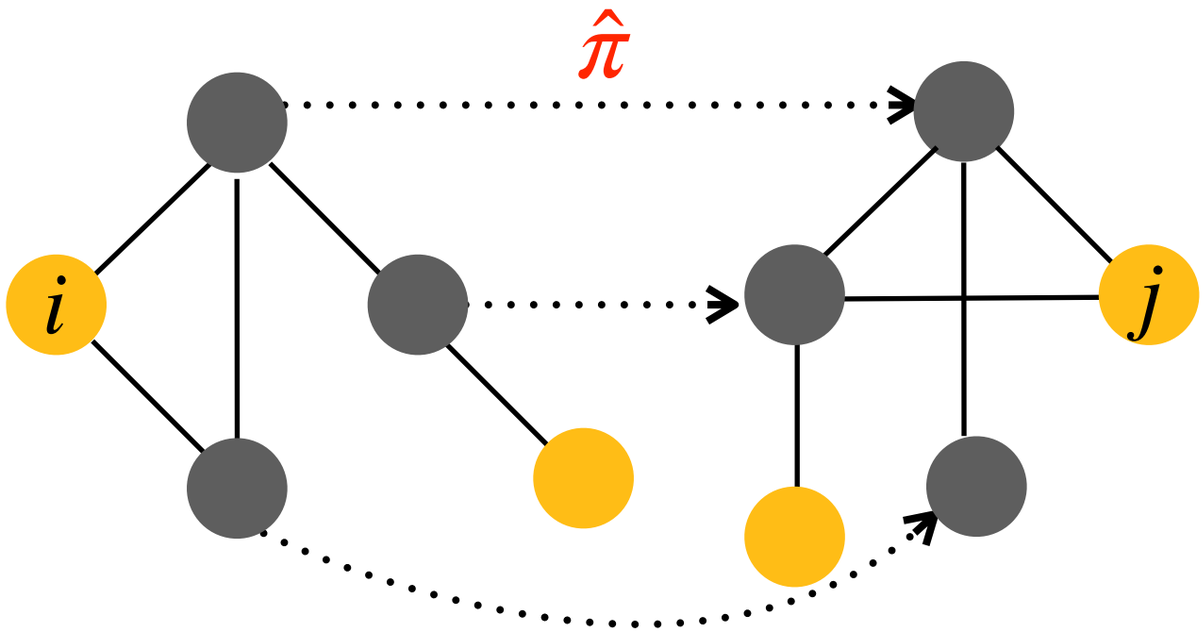

Seeded graph matching (SGM)

1. For each unmatched pair (i, j), compute "common" neighbors $N(i, j)$ under $\hat{\pi}$
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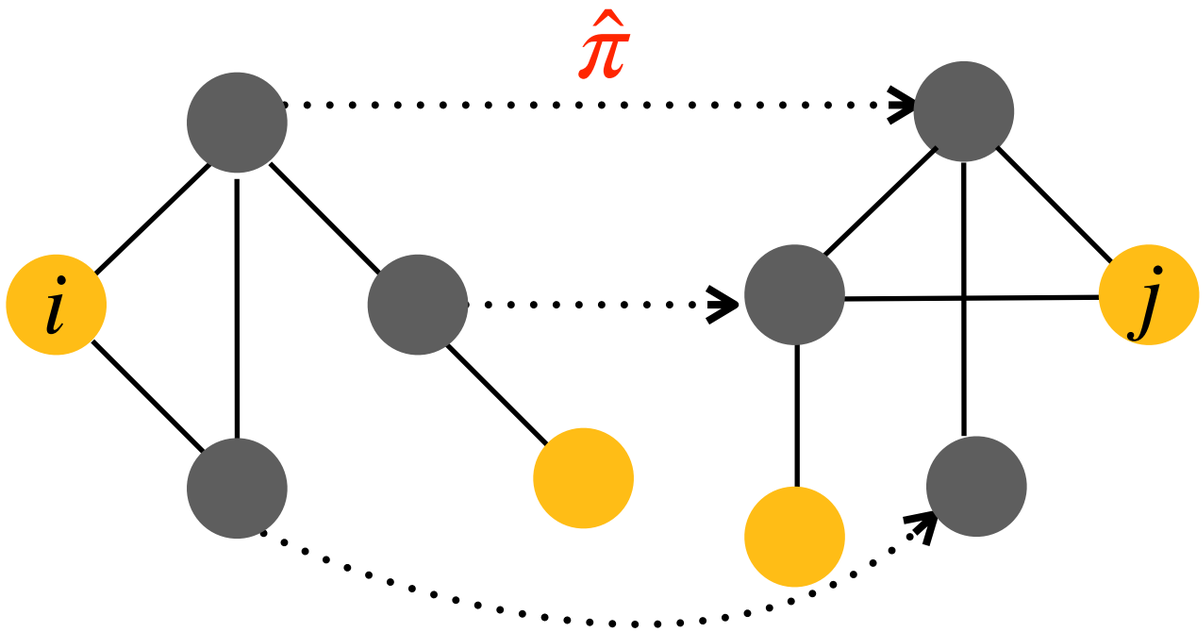

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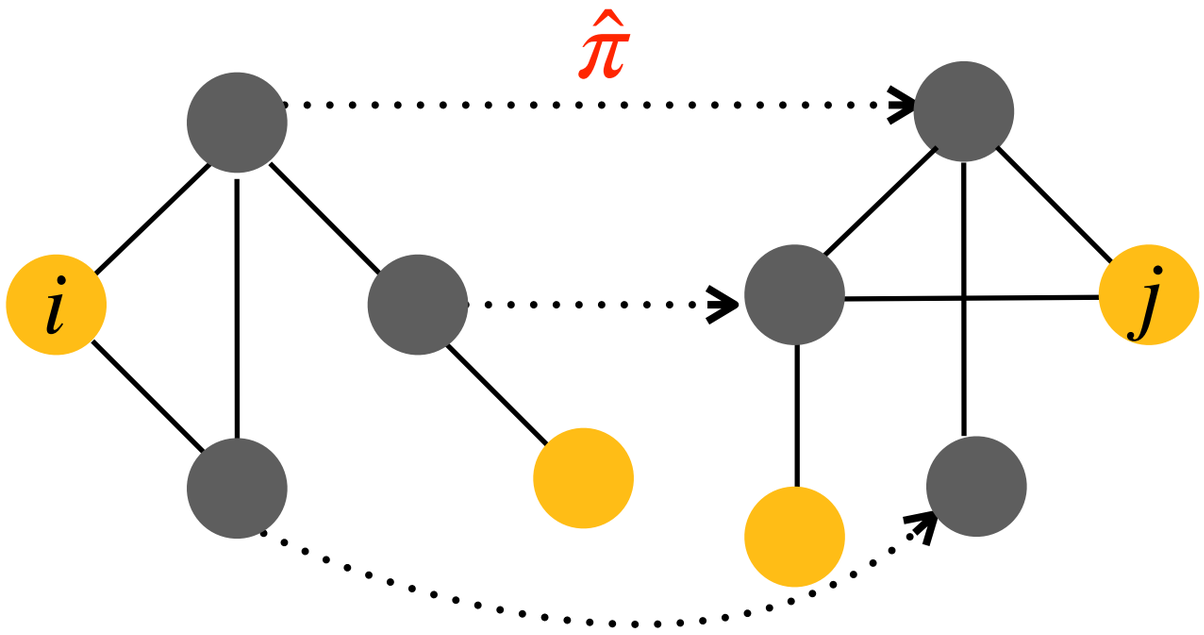

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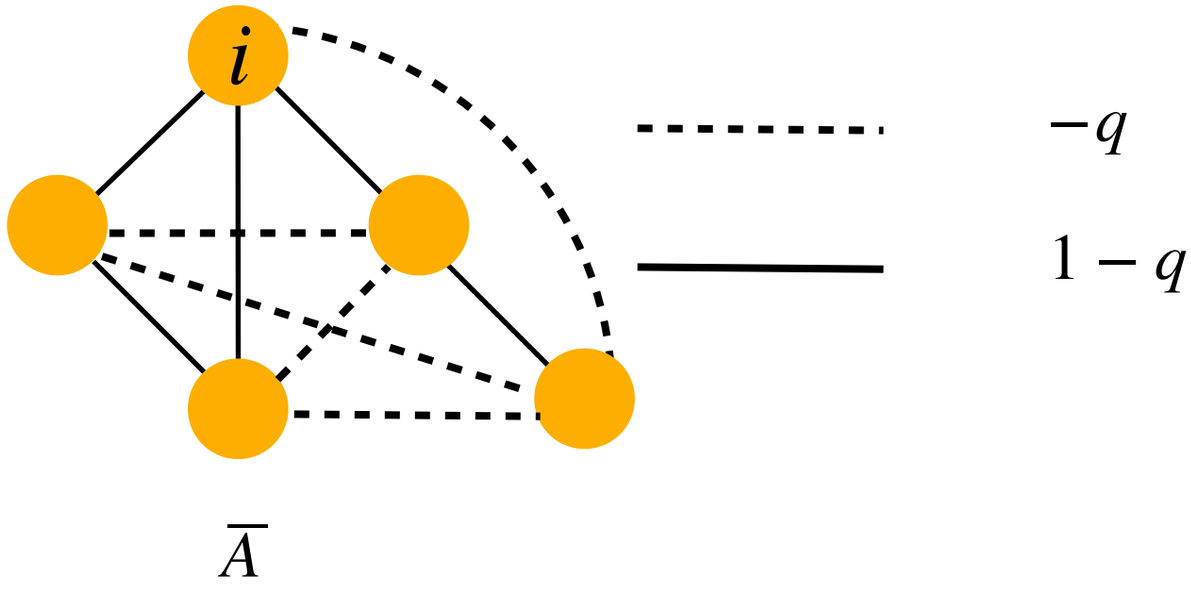
Theorem [Mao-Wu-X.-Yu '22]

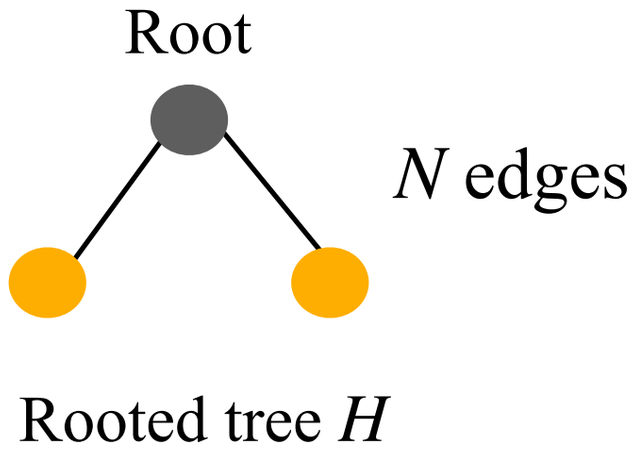
Suppose $nq(q + \rho(1 - q)) \geq (1 + \epsilon)\log n$ and $\rho \geq \epsilon$. With probability $1 - o(1)$, given any input $\hat{\pi}$ that completely coincides with π on at least $(1 - \epsilon)n$ vertices, SGM with an appropriate choice of threshold γ outputs $\tilde{\pi} = \pi$ in time $O(n^3 q^2)$.

- Proof: Intersection graph is an expander, so SGM iteratively matches all vertices

Color coding

Counting signed trees via color coding





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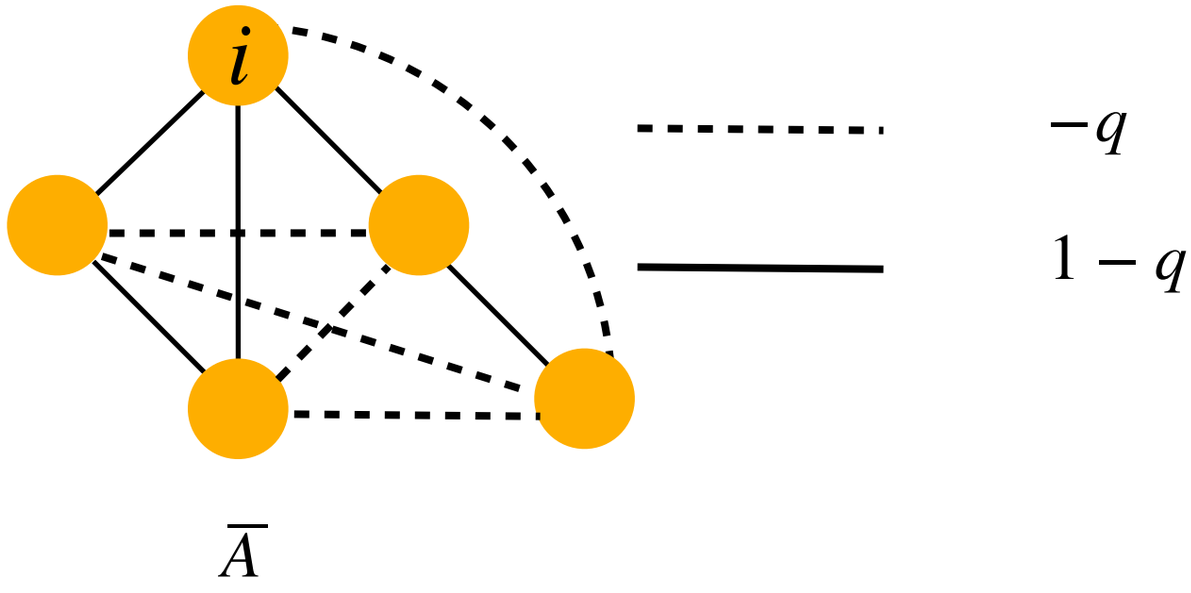

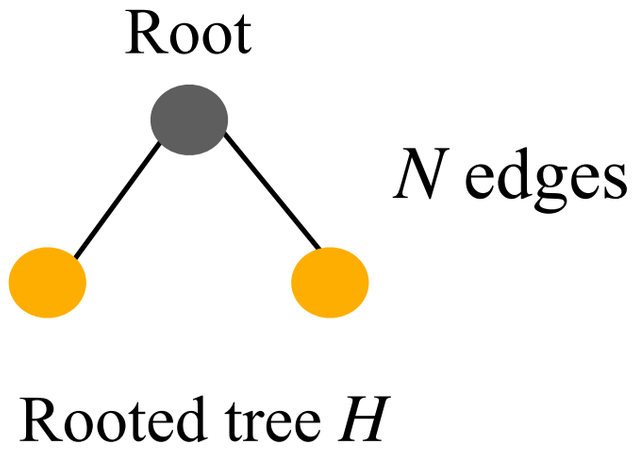

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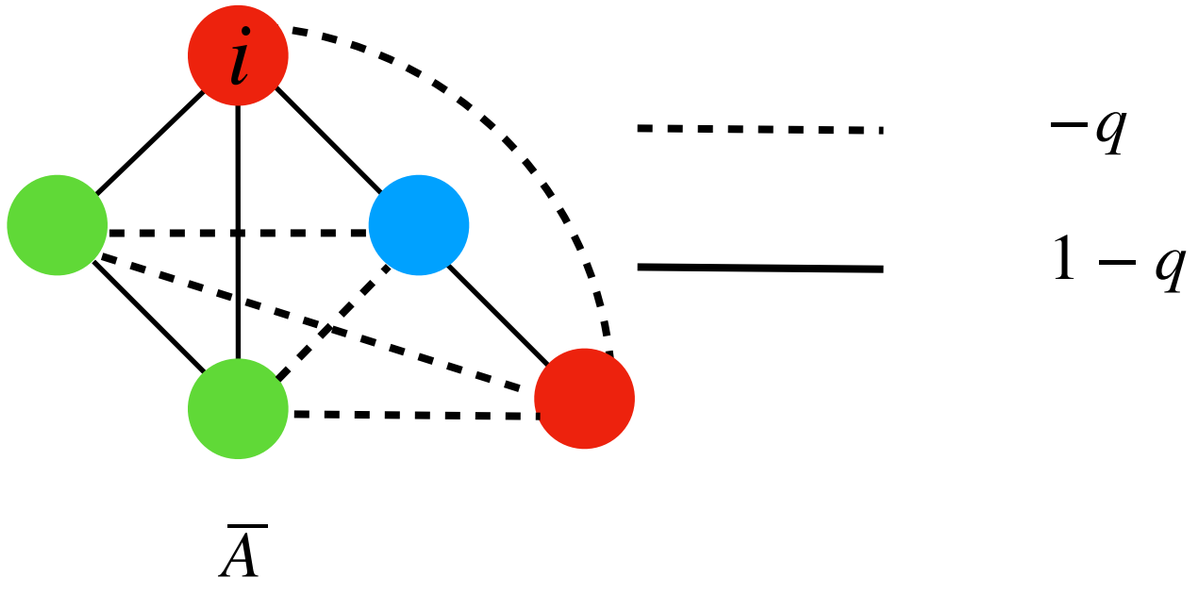

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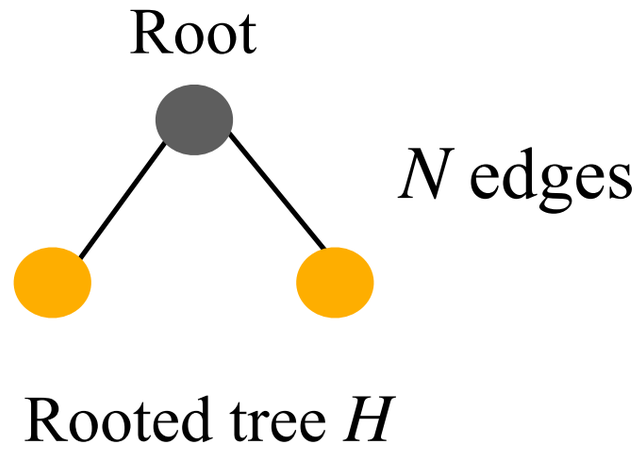

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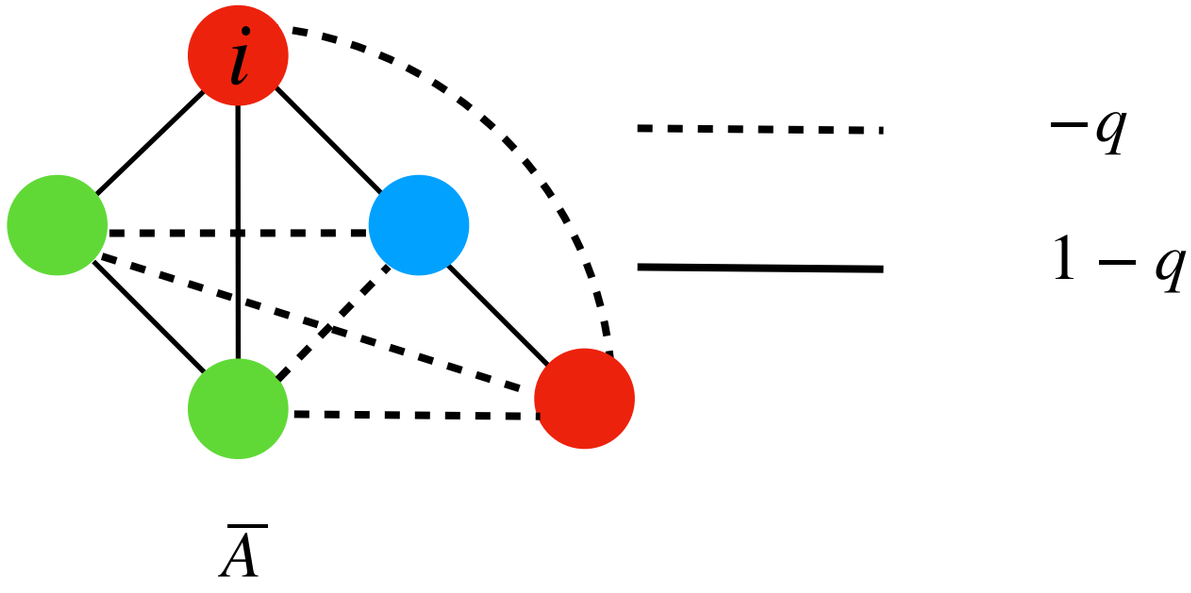

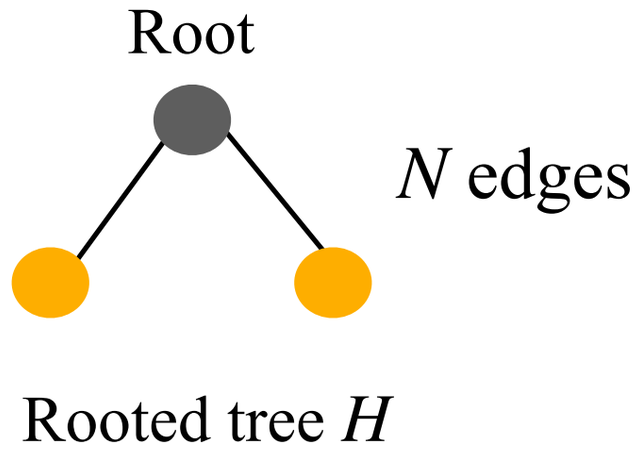

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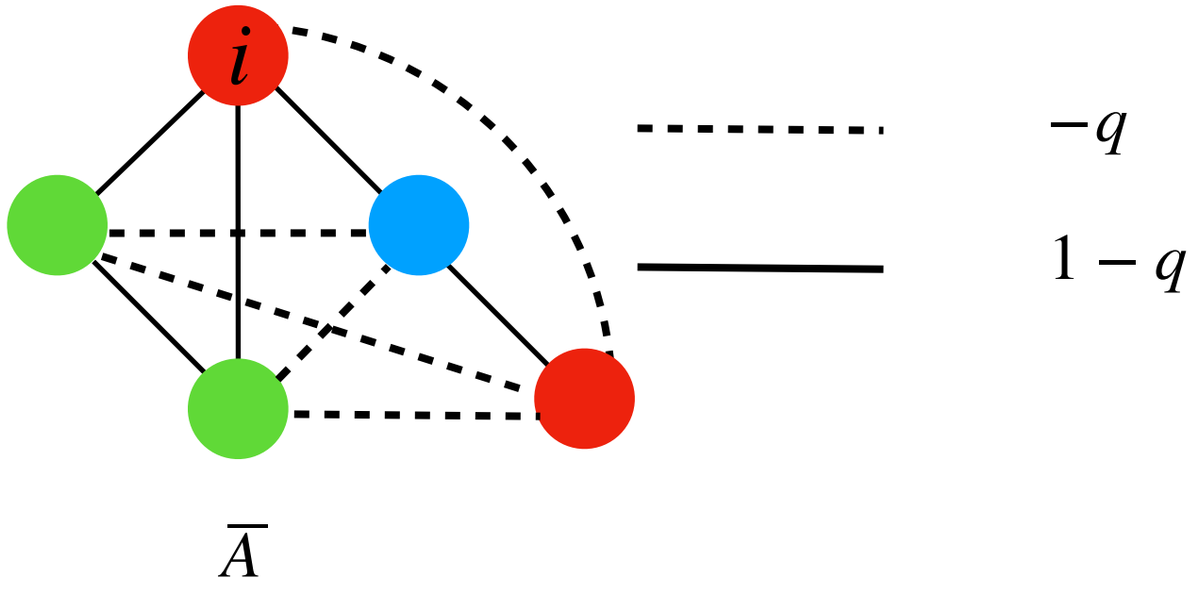

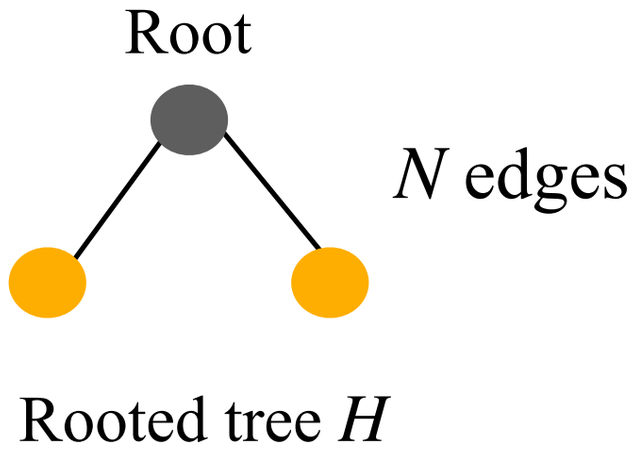

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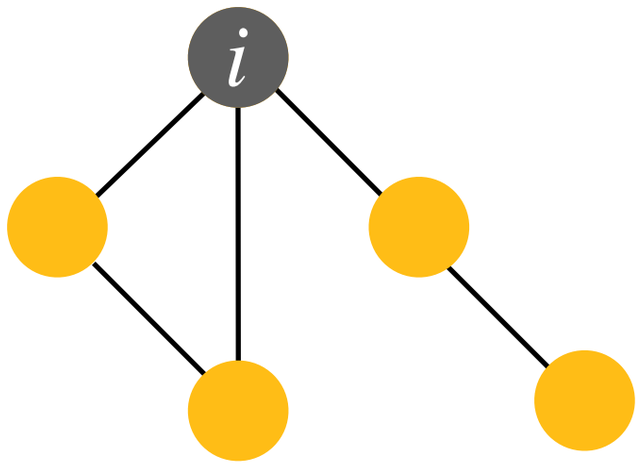
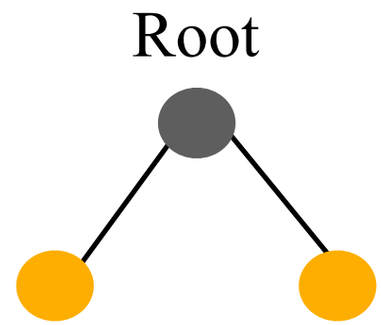
Signed rooted tree count

Rooted **signed** subgraph count

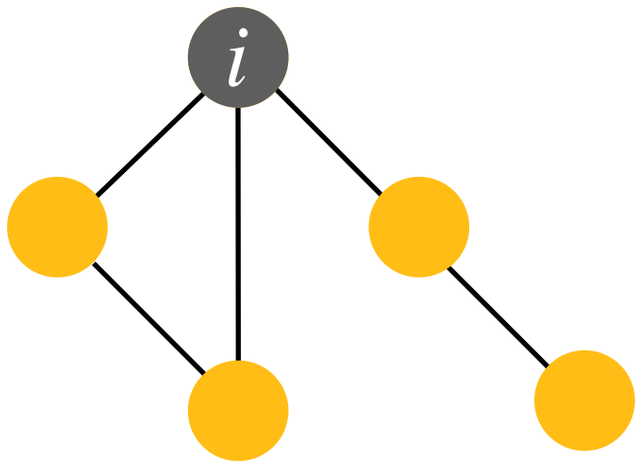


A

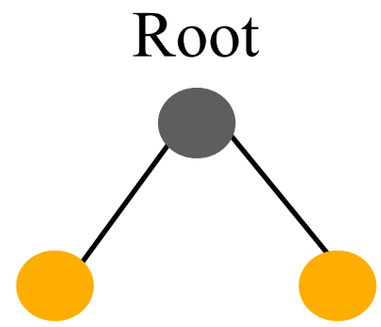


Rooted subgraph H

Rooted **signed** subgraph count



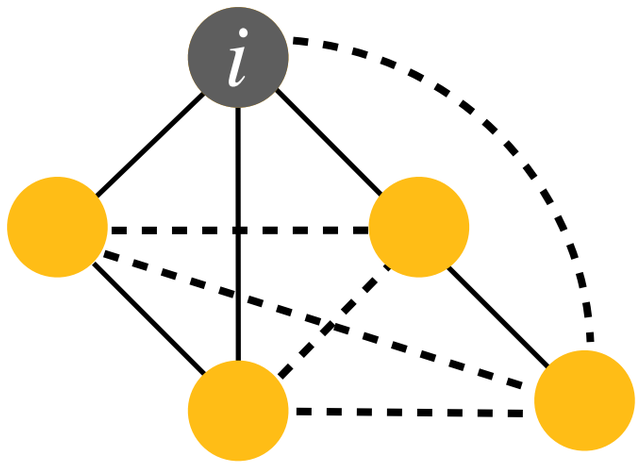
A

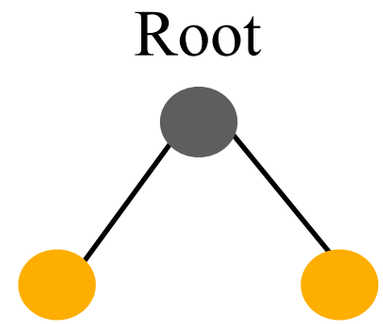


Rooted subgraph H

- Centered adjacency matrices: $\overline{A} = A - \mathbb{E}[A]$;

Rooted **signed** subgraph count





Rooted subgraph H

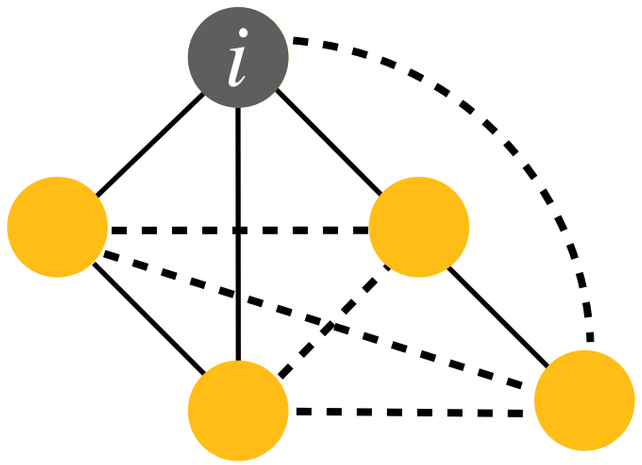
A

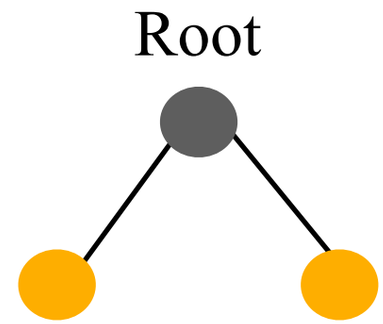
$-q$

$1-q$

- Centered adjacency matrices: $\overline{A} = A - \mathbb{E}[A]$;

Rooted **signed** subgraph count





Rooted subgraph H

A

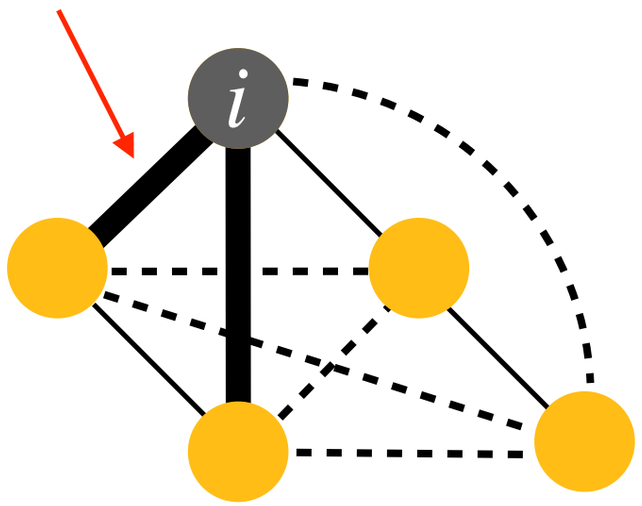
$-q$

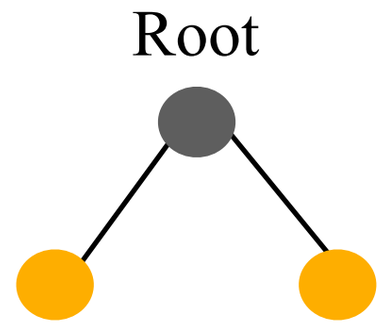
$1-q$

- Centered adjacency matrices: $\overline{A} = A - \mathbb{E}[A]$;
- Count weighted copies of rooted graph in $\overline{A} \implies W_{i,H}(\overline{A})$ (rooted **signed** subgraph count) [Bubeck-Ding-Eldan-Rácz '16];

Rooted **signed** subgraph count

This copy of H has a weight $(1-q)^2$.





Rooted subgraph H

A

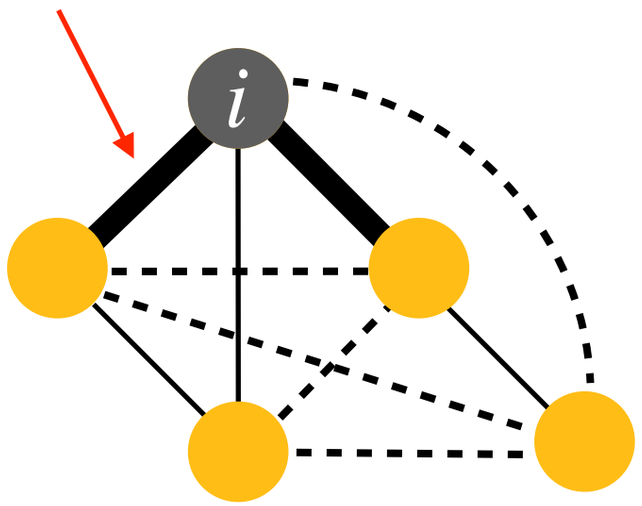
$-q$

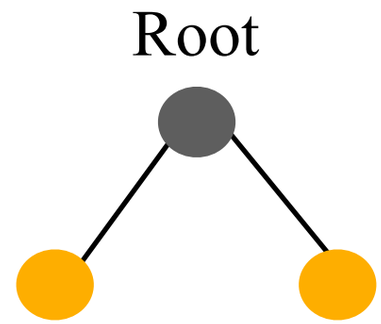
$1-q$

- Centered adjacency matrices: $\overline{A} = A - \mathbb{E}[A]$;
- Count weighted copies of rooted graph in $\overline{A} \implies W_{i,H}(\overline{A})$ (rooted **signed** subgraph count) [Bubeck-Ding-Eldan-Rácz '16];

Rooted **signed** subgraph count

This copy of H has a weight $(1-q)^2$.





Rooted subgraph H

A

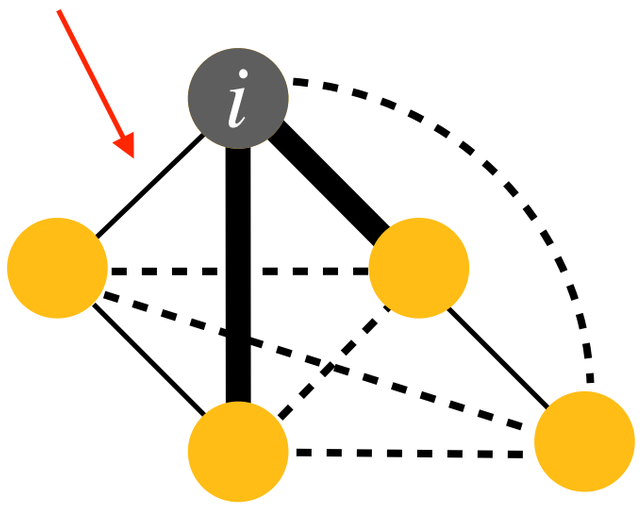
- - - - - - - - $-q$

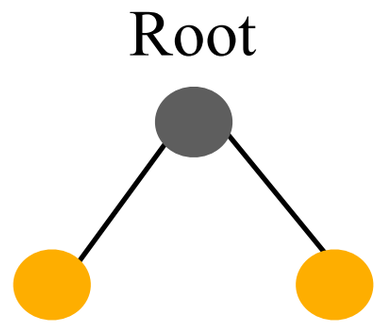
——— $1-q$

- Centered adjacency matrices: $\overline{A} = A - \mathbb{E}[A]$;
- Count weighted copies of rooted graph in $\overline{A} \implies W_{i,H}(\overline{A})$ (rooted **signed** subgraph count) [Bubeck-Ding-Eldan-Rácz '16];

Rooted **signed** subgraph count

This copy of H has a weight $(1-q)^2$.





Rooted subgraph H

A

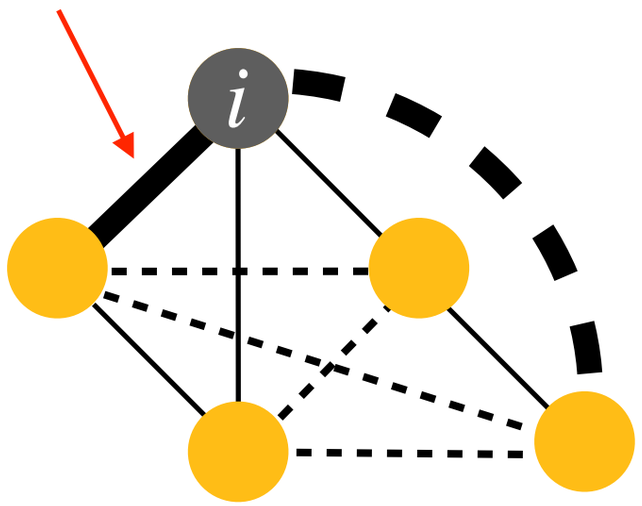
- - - - - - - - $-q$

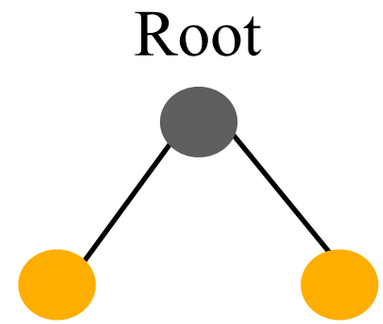
——— $1-q$

- Centered adjacency matrices: $\overline{A} = A - \mathbb{E}[A]$;
- Count weighted copies of rooted graph in $\overline{A} \implies W_{i,H}(\overline{A})$ (rooted **signed** subgraph count) [Bubeck-Ding-Eldan-Rácz '16];

Rooted **signed** subgraph count

This copy of H has a weight $(1-q)(-q)$.





Rooted subgraph H

A

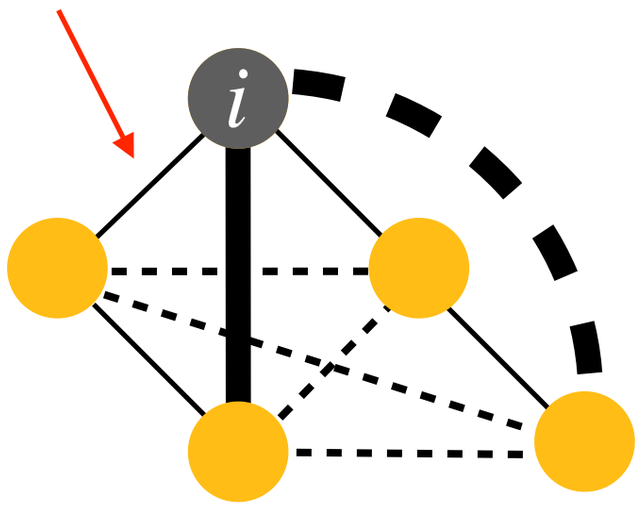
$-q$

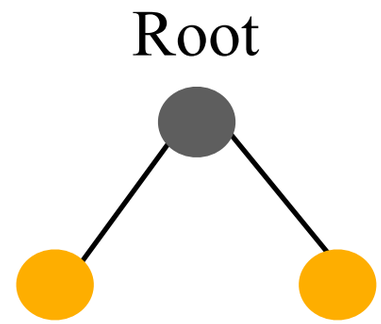
$1-q$

- Centered adjacency matrices: $\overline{A} = A - \mathbb{E}[A]$;
- Count weighted copies of rooted graph in $\overline{A} \implies W_{i,H}(\overline{A})$ (rooted **signed** subgraph count) [Bubeck-Ding-Eldan-Rácz '16];

Rooted **signed** subgraph count

This copy of H has a weight $(1-q)(-q)$.





Rooted subgraph H

A

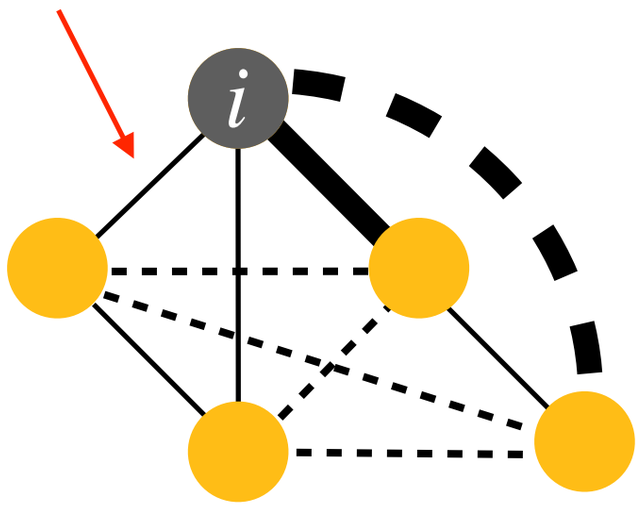
$-q$

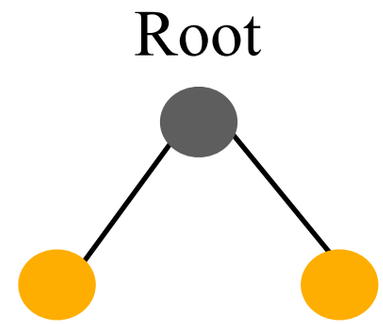
$1-q$

- Centered adjacency matrices: $\overline{A} = A - \mathbb{E}[A]$;
- Count weighted copies of rooted graph in $\overline{A} \implies W_{i,H}(\overline{A})$ (rooted **signed** subgraph count) [Bubeck-Ding-Eldan-Rácz '16];

Rooted **signed** subgraph count

This copy of H has a weight $(1-q)(-q)$.





Rooted subgraph H

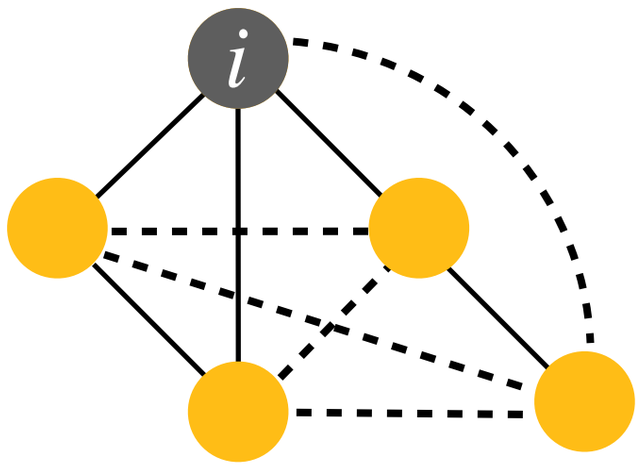
A

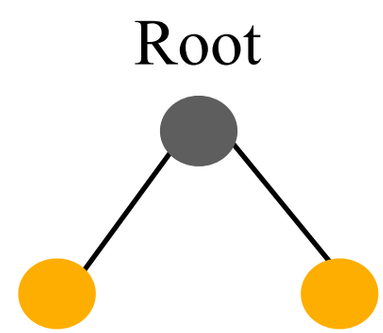
$-q$

$1-q$

- Centered adjacency matrices: $\overline{A} = A - \mathbb{E}[A]$;
- Count weighted copies of rooted graph in $\overline{A} \implies W_{i,H}(\overline{A})$ (rooted **signed** subgraph count) [Bubeck-Ding-Eldan-Rácz '16];

Rooted **signed** subgraph count





Rooted subgraph H

$W_{i,H}(\overline{A}) = 3(1-q)^2 + 3(1-q)(-q)$

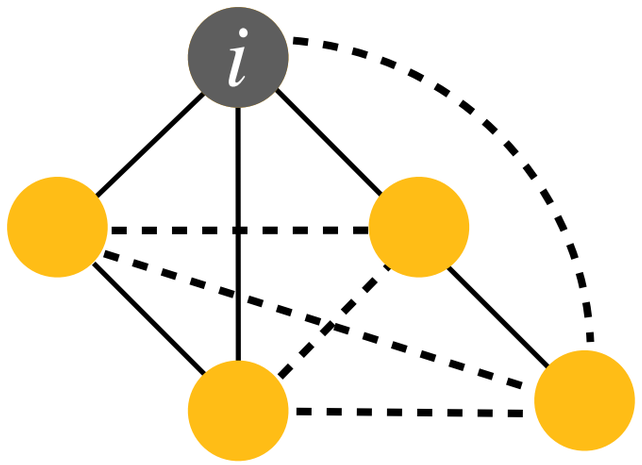
$\overline{A}$

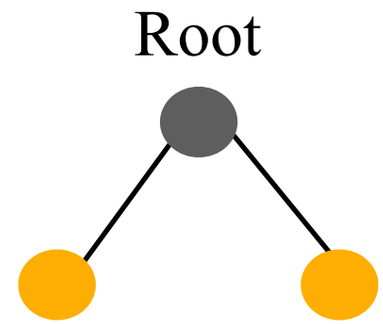
$-q$

$1-q$

- Centered adjacency matrices: $\overline{A} = A - \mathbb{E}[A]$;
- Count weighted copies of rooted graph in $\overline{A} \implies W_{i,H}(\overline{A})$ (rooted **signed** subgraph count) [Bubeck-Ding-Eldan-Rácz '16];

Rooted **signed** subgraph count





Rooted subgraph H

A

$-q$

$1-q$

- Centered adjacency matrices: $\overline{A} = A - \mathbb{E}[A]$;
- Count weighted copies of rooted graph in $\overline{A} \implies W_{i,H}(\overline{A})$ (rooted **signed** subgraph count) [Bubeck-Ding-Eldan-Rácz '16];
- Crucially, $W_{i,H}(\overline{A})$ and $W_{i,I}(\overline{A})$ are uncorrelated for distinct subgraphs H and I.