The Planted Spanning Tree Problem

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Joint work with
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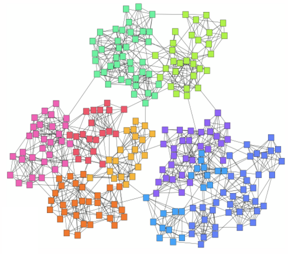
Network model with planted structure

Question: How to recover latent structure from noisy network data?



Classical examples

- Community detection under Stochastic Block Model
- Recovery of planted clique in Erdős-Rényi graphs

An expanding zoo of planted subgraph problems...

- Planted bipartite matching [Chertkov-Kroc-Krzakala-Vergassola-Zdeborová '10, Moharrami-Moore-X. '21, Ding-Wu-X.-Yang ' 23]
- Planted Hamiltonian cycle problem (TSP) [Bagaria-Ding-Tse-Wu-X. '20]
- Planted trees [Massoulié-Stephan-Towsley '18]
- Planted k-factors [Sicuro-Zdeborová '20, Gaudio-Sandon-X.-Yang '25]
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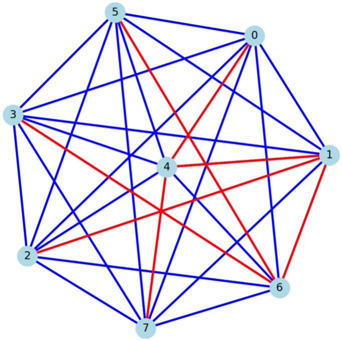
Today: Planted spanning tree model $\rightarrow$ Exact overlap characterization via local weak convergence theory

The planted spanning tree model



- A complete graph on n vertices
- A uniform spanning tree T^*
- Non-negative edge weight

$$W_e \overset{\text{ind.}}{\sim} \begin{cases} P & e \in T^* \\ Q_n & e \notin T^* \end{cases}$$

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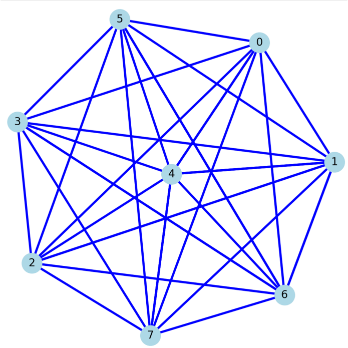

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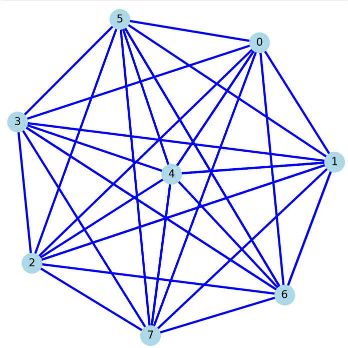

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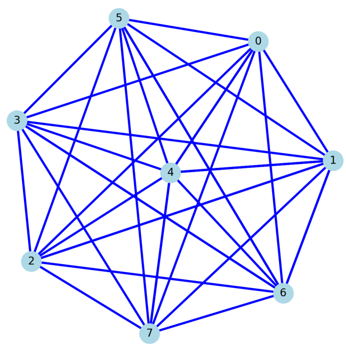

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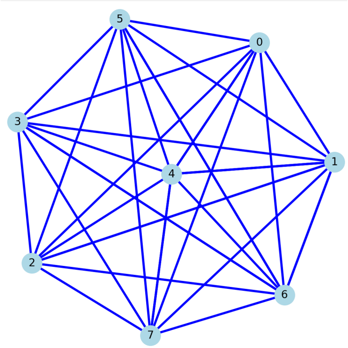

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- How much does $T_{\min}$ have in common with T^*?

Main result: Asymptotic overlap

Theorem (Moharrami-Moore-X. '25)

Let F denote the CDF of planted weight distribution P. Then

$$\lim_{n\to\infty} \frac{1}{n-1} \mathbb{E}[|T_{\min} \cap T^*|] = \int_0^\infty (1 - p_U(s) p_B(s))\, dF(s)$$

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where $p_U(s)$ and $p_B(s)$ are the largest fixed point of

$$1 - p_U(s) = \exp\left(-sp_U(s) - p_B(s)\right)\left(1 - F(s)p_U(s)\right)$$
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Remark

- $p_U(s)$ and $p_B(s)$ are probabilities of certain branching process growing to infinity
- The fixed-point equations have at most two solutions, one of which is zero

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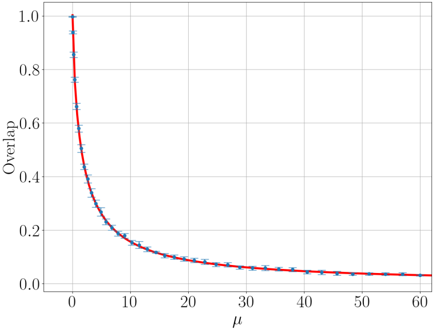

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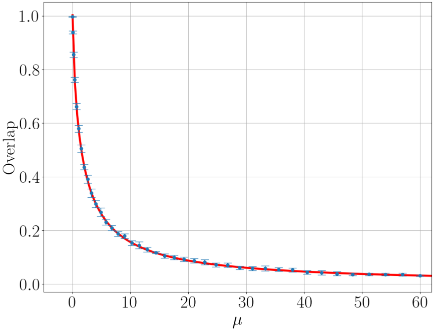

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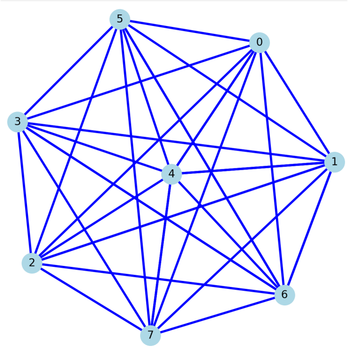
No phase transition!

Warmup: (un-planted) spanning tree problem



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- Weights are i.i.d. Q_n with $nQ_n'(0) \to 1$
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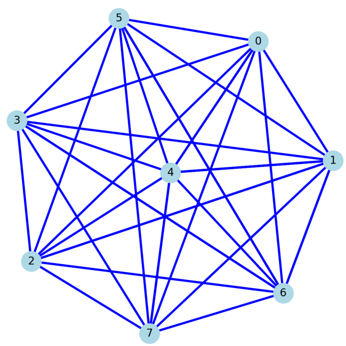

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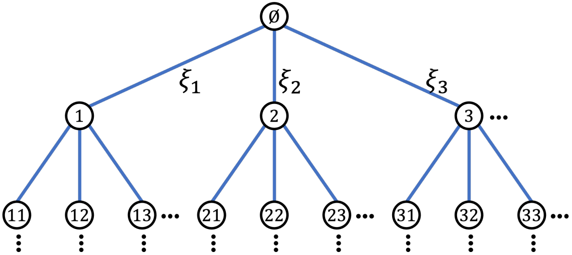
[Frieze '85, Aldous-Steele '04, Addario-Berry'15]

$$\mathbb{E}[w(T_{\min})] \to \zeta(3) \triangleq \sum_{i=1}^{\infty} i^{-3} \approx 1.202$$

Proved in several distinct and elegant ways

Poisson-weighted infinite tree approximation

Treat edge weight as distance and model "local" neighborhood as a tree
[Aldous'00, Aldous-Steele '04]



Sort iid Q_n edge weights $W_{\varnothing,1}, W_{\varnothing,2}, \ldots$ from smallest to largest:
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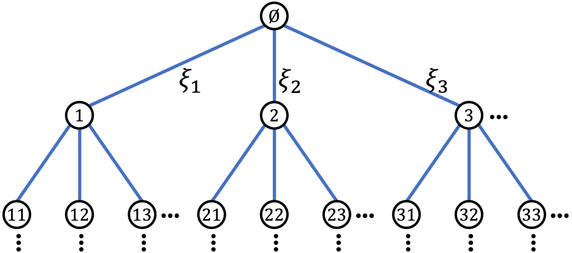

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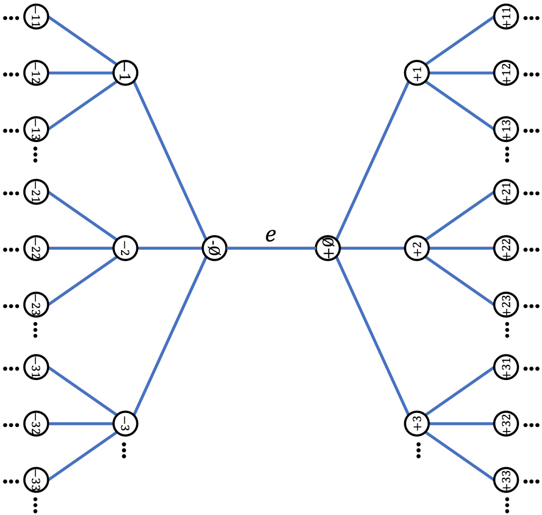
Removing all edges with weights s or greater, we get Galton-Watson tree with $\mathrm{Pois}(s)$ offspring distribution, denoted by PGW(s)

From the perspective of an edge

Pick an edge and model its local neighborhood as a two-sided infinite tree



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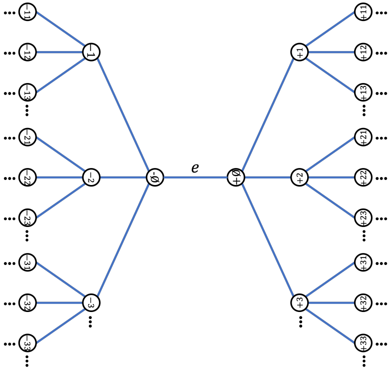

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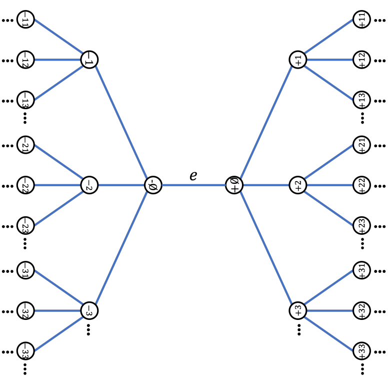

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- e in MST iff removing all edges with weight s or greater, no path exists between its two endpoints $\Leftrightarrow$ at least one side is finite

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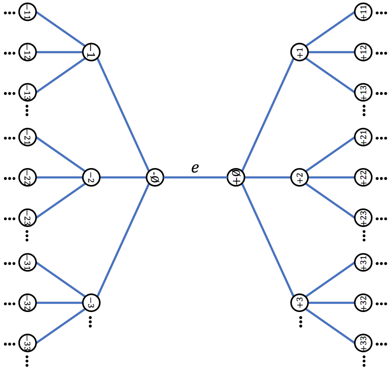

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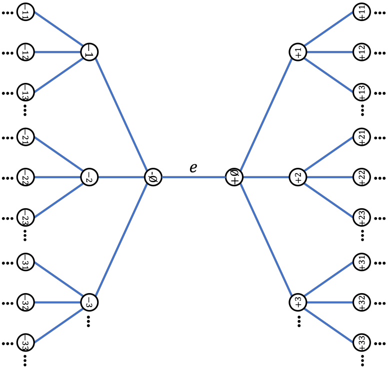

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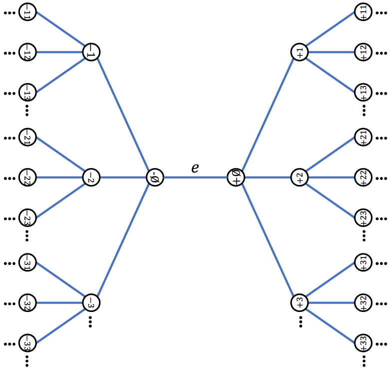

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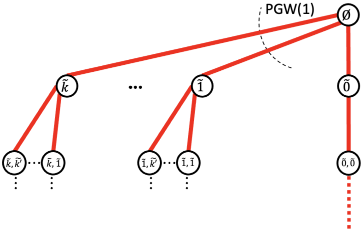

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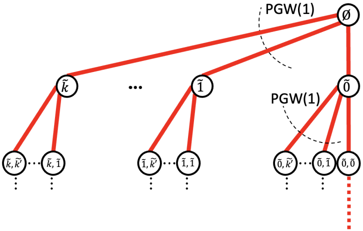

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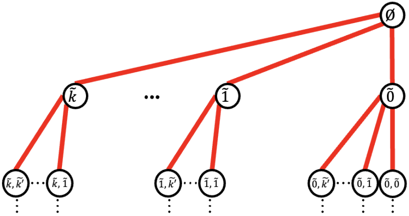
Putting together: Planted Poisson-weighted infinite tree

The local view of graph G with planted uniform spanning tree



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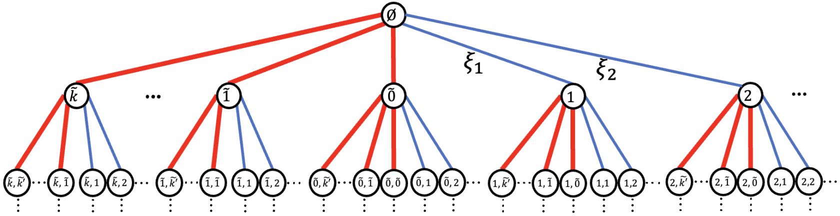

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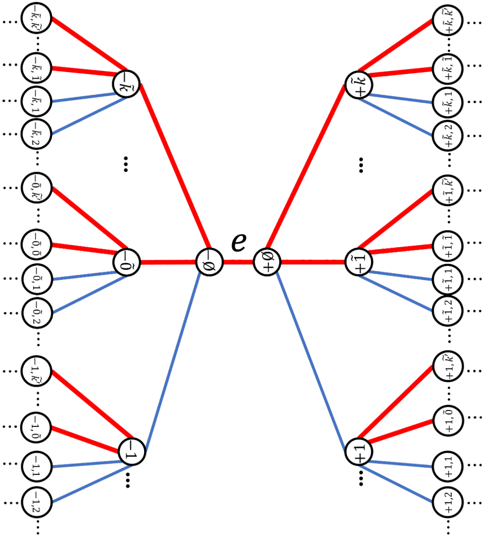

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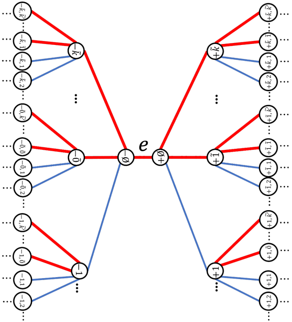

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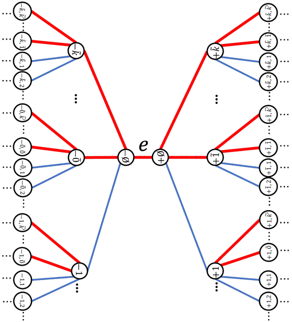

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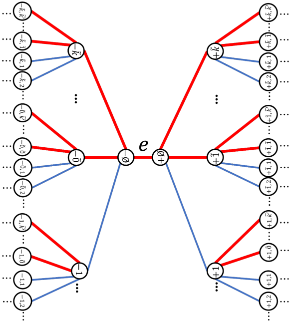

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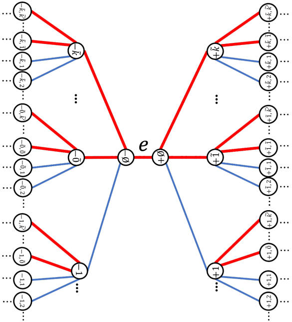

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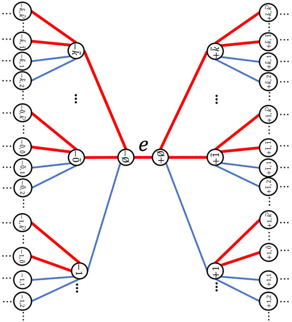

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$$\Rightarrow \frac{1}{n-1}\mathbb{E}[|T_{\min} \cap T^*|] \to \int_0^\infty (1 - p_U(s)p_B(s))\, dF(s)$$

Proving it: Local weak convergence (Aldous-Steele '04)

- Planted graph G_n converges to planted Poisson-weighted infinite trees T_∞
  - Local treelikeness of light edges
  - Uniform spanning tree converges to skeleton trees [Grimmett '80]
- MST of G_n converges to *minimum spanning forest* of T_∞ [Aldous '91]

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Open problems

1. Determine the information-theoretically optimal overlap
2. Other planted structures, such as spanning k-regular graphs [Sicuro-Zdeborová '21, Gaudio-Sandon-X.-Yang '25]

Conclusion

- Exactly characterize overlap between MST and the planted one
- Establish local weak convergence of the planted model

Extensions

- Characterize the overlap of MST for planted Hamiltonian path/cycle model [Bagaria-Ding-Tse-Wu-X. '20]
- Determine the weight of MST under the planted model, extending Frieze's $\zeta(3)$ result
- Design an efficient test for planted vs. unplanted models

Open problems

1. Determine the information-theoretically optimal overlap
2. Other planted structures, such as spanning k-regular graphs [Sicuro-Zdeborová '21, Gaudio-Sandon-X.-Yang '25]

Reference

M. Moharrami, C. Moore, & J. X., *The planted spanning tree problem*. arXiv: 2502.08790. Conference on Learning Theory 2025