Testing Correlation of Unlabeled Random Graphs

Jiaming Xu

The Fuqua School of Business
Duke University

Joint work with
Yihong Wu (Yale) and Sophie H. Yu (Duke)

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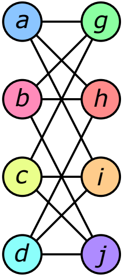
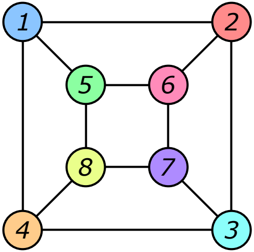
Given two graphs A and B, decide whether $A \cong B$, i.e., there exists a bijection $\pi : V(A) \to V(B)$ such that

$$(u, v) \in E(A) \Leftrightarrow (\pi(u), \pi(v)) \in E(B)$$





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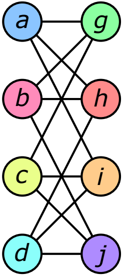

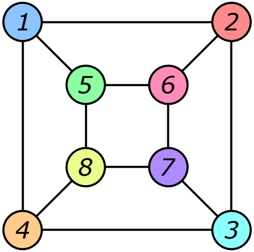

$\pi(a) = 1$

$\pi(b) = 6$

$\pi(c) = 8$

$\pi(d) = 3$

$\pi(g) = 5$

$\pi(h) = 2$

$\pi(i) = 4$

$\pi(j) = 7$

Graph isomorphism

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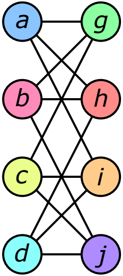

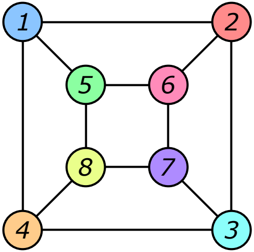

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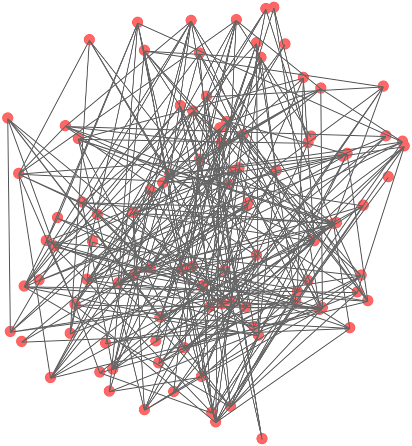
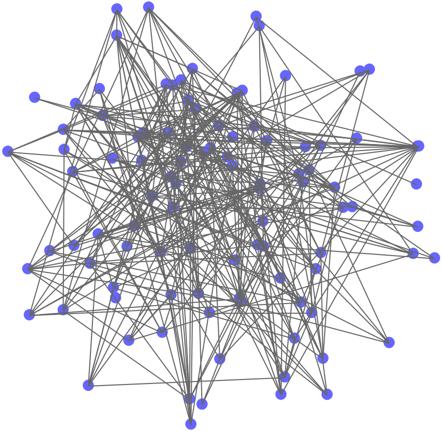
- Not known to be solvable in polynomial time in worst case
- In practice, two graphs are often not exactly isomorphic, but still want to tell whether their topologies are similar

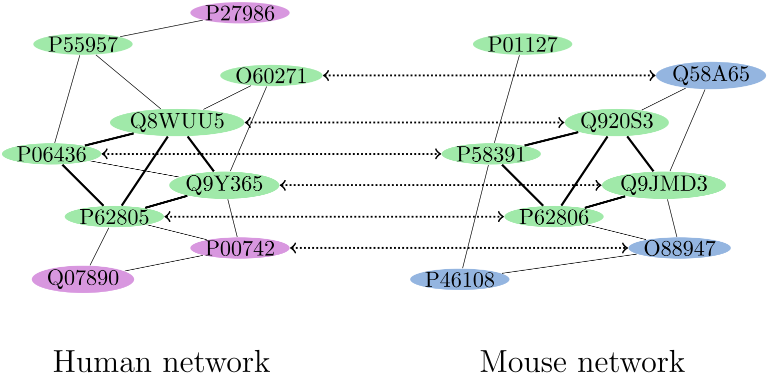
Two key challenges

- Statistical: two graphs may be correlated but not exactly isomorphic
- Computational: # of possible node mappings is $n!$ ($100! \approx 10^{158}$)





Motivation: Protein-protein interaction network

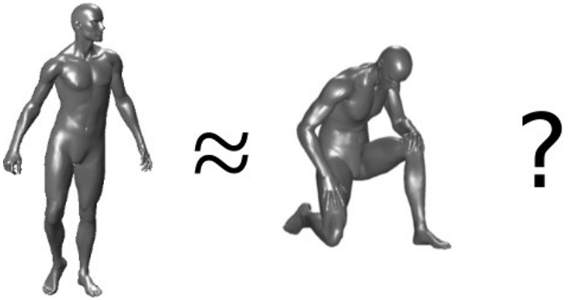


[Kazemi-Hassani-Grossglauser-Modarres '16]

Ontology: Assess the correlation of two biological networks in two different species based on network topology

Motivation: Deformable shape matching

A fundamental problem in computer vision: Detect similar objects that undergo different deformations



Shape REtrieval Contest (SHREC) dataset

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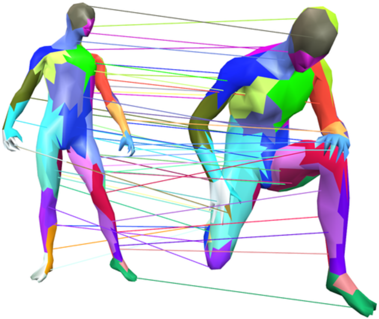


Shape REtrieval Contest (SHREC) dataset

3-D shapes → geometric graphs (features → nodes, distances → edges)
Determine whether two graphs are similar in topologies

Beyond worst-case: Noisy graph isomorphism

Definition (Hypothesis testing)

Suppose we observe two random graphs A and B:

$$\mathcal{H}_0 : A \text{ and } B \text{ are independent}$$
$$\mathcal{H}_1 : A \text{ and } B^\pi = (B_{\pi(i)\pi(j)}) \text{ are edge-correlated}$$
$$\text{conditional on a uniform permutation } \pi$$

Goal: Test $\mathcal{H}_0$ versus $\mathcal{H}_1$

- Under $\mathcal{H}_1$, the inherent edge correlation is obscured by the latent node correspondence π
- Hypothesis testing aspect of graph matching (recover π under $\mathcal{H}_1$)
- The test needs to rely on graph invariants (invariant under graph isomorphism), such as
  - Subgraph counts (e.g. # of edges or triangles)
  - Spectrum (e.g. eigenvalues of adjacency matrices or Laplacians)

Gaussian setting

A and B are two Gaussian-weighted complete graphs

Definition (Gaussian Wigner model)

$$\mathcal{H}_0 : (A_{ij}, B_{ij}) \overset{\text{i.i.d.}}{\sim} \mathcal{N}\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right)$$

$$\mathcal{H}_1 : \left(A_{ij}, B_{\pi(i)\pi(j)}\right) \overset{\text{i.i.d.}}{\sim} \mathcal{N}\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix} \right) \text{ conditional on uniform } \pi$$

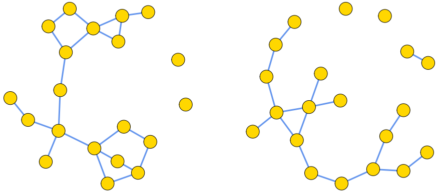
- Under both $\mathcal{H}_0$ and $\mathcal{H}_1$, A and B are Gaussian Wigner matrices marginally
- Under $\mathcal{H}_1$, ρ characterizes the correlation of edge weights

Erdős-Rényi setting

Definition (Erdős-Rényi graphs model [Barak-Chou-Lei-Schramm-Sheng'19])

$\mathcal{H}_0$: A and B are independent $\mathcal{G}(n, ps)$

$\mathcal{H}_1$: A and $B^\pi = (B_{\pi(i)\pi(j)})$ are independently edge-sampled from a common parent graph $\mathcal{G}(n, p)$ with sampling probability s



- Under both $\mathcal{H}_0$ and $\mathcal{H}_1$, A and B are $\mathcal{G}(n, ps)$ marginally
- Under $\mathcal{H}_1$, $\left(A_{ij}, B_{\pi(i)\pi(j)}\right)$ are correlated $\mathrm{Bern}(ps)$ with correlation coefficient $\rho \triangleq \frac{s(1-p)}{1-ps}$

Sharp threshold for detection: Gaussian

$\mathcal{Q}$ and $\mathcal{P}$: probability measure under $\mathcal{H}_0$ and $\mathcal{H}_1$

Theorem (Wu-X.-Yu '20)

$$\mathrm{TV}(\mathcal{P}, \mathcal{Q}) = \begin{cases} 1 - o(1) & \textit{if } \rho^2 \geq 4\,\frac{\log n}{n-1} \\ o(1) & \textit{if } \rho^2 \leq (4-\epsilon)\,\frac{\log n}{n} \end{cases}$$

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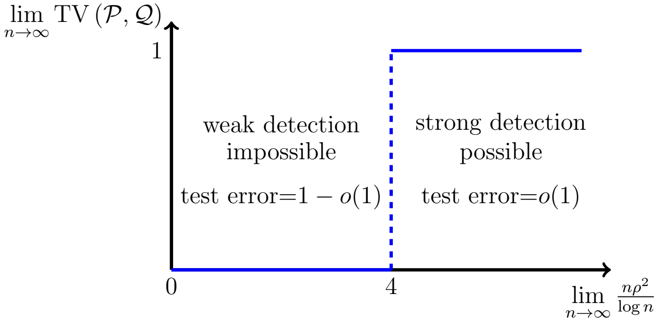

Sharp threshold for detection: dense Erdős-Rényi graphs

Theorem (Wu-X.-Yu '20)

$$\textit{If } s^2 \geq \frac{2\log n}{(n-1)p\left(\log\frac{1}{p} - 1 + p\right)}, \textit{ then } \mathrm{TV}\left(\mathcal{P}, \mathcal{Q}\right) = 1 - o\left(1\right)$$

- *(Dense regime)* $p = n^{-o(1)}$:

$$\textit{If } s^2 \leq \frac{(2-\epsilon)\log n}{np\left(\log\frac{1}{p} - 1 + p\right)}, \textit{ then } \mathrm{TV}\left(\mathcal{P}, \mathcal{Q}\right) = o(1)$$

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Remark:

- Sharp phase transition at $\frac{ns^2p\left(\log\frac{1}{p}-1+p\right)}{\log n} = 2$
- $p \mapsto p(\log\frac{1}{p} - 1 + p)$ is *not* monotone and uniquely maximized at $p_* \approx 0.203 \Rightarrow$ edge density p_* is the "easiest" for detection

Threshold for detection: sparse Erdős-Rényi graphs

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- *(Sparse regime)* $p = n^{-\Omega(1)}$*:*

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$p = n^{-\alpha}$ for a constant α:
strong detection is possible if $s^2 > \frac{2}{np\alpha}$ and impossible if $s^2 < \frac{1}{np} \wedge 0.01$

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- Counting (weighted) trees – low-degree poly. approx. of $\frac{\mathcal{P}(A,B)}{\mathcal{Q}(A,B)}$ [Mao-Wu-X.-Yu 'forthcoming]: achieve strong detection in poly-time, if

$$s^2 > 1/2.956 \quad \text{and} \quad np \geq n^{-o(1)}$$

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Polynomial-time test for $s = o(1)$ is open

Proof of positive results: QAP statistic

$$\mathcal{T}(A,B) \triangleq \max_{\pi \in \mathcal{S}_n} \sum_{i<j} A_{ij} B_{\pi(i)\pi(j)} \quad \text{(edge correlation)}$$

- This is known as Quadratic Assignment Problem
- Invariant to the node relabeling of both A and B
- Proof: First-moment calculation

Proof of negative results: second-moment method

$$\mathbb{E}_{\mathcal{Q}}\left[\left(\frac{\mathcal{P}(A,B)}{\mathcal{Q}(A,B)}\right)^2\right] = O(1) \qquad \Longrightarrow \mathrm{TV}(\mathcal{P},\mathcal{Q}) \le 1-\Omega(1)$$

Strong detection is impossible

$$\mathbb{E}_{\mathcal{Q}}\left[\left(\frac{\mathcal{P}(A,B)}{\mathcal{Q}(A,B)}\right)^2\right] = 1+o(1) \qquad \Longrightarrow \mathrm{TV}(\mathcal{P},\mathcal{Q}) = o(1)$$

Weak detection is impossible

Second-moment calculation: cycle (orbit) decomposition

- Node permutation σ on $[n]$
- Edge permutation σ^{E} on $\binom{[n]}{2}$: $\sigma^{\mathsf{E}}((i,j)) \triangleq (\sigma(i), \sigma(j))$

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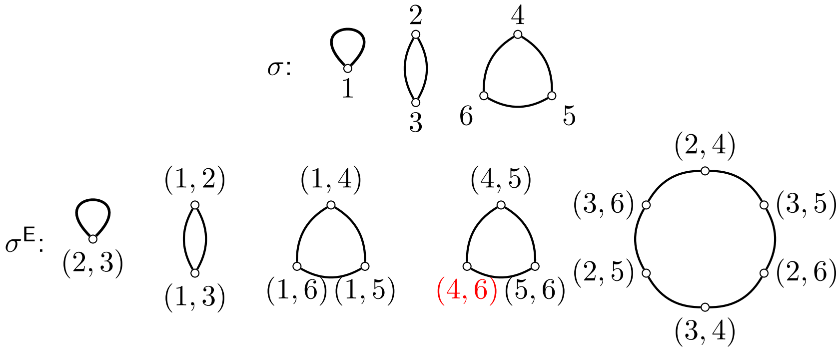
E.g. $n = 6$ and $\sigma = (1)(23)(456)$:



Cycles in node (edge) permutation are called node (edge) orbits

Second-moment calculation: cycle decomposition

$$\left(\frac{\mathcal{P}(A,B)}{\mathcal{Q}(A,B)}\right)^2 = \left(\mathbb{E}_\pi\left[\frac{\mathcal{P}(A,B|\pi)}{\mathcal{Q}(A,B)}\right]\right)^2$$

$$= \mathbb{E}_{\widetilde{\pi} \perp\!\!\!\perp \pi} \prod_{i<j} X_{ij} \quad X_{ij} \triangleq \frac{\mathcal{P}(B_{\pi(i)\pi(j)}|A_{ij})\mathcal{P}(B_{\widetilde{\pi}(i)\widetilde{\pi}(j)}|A_{ij})}{\mathcal{Q}(B_{\pi(i)\pi(j)})\mathcal{Q}(B_{\widetilde{\pi}(i)\widetilde{\pi}(j)})}$$

$$= \mathbb{E}_{\widetilde{\pi} \perp\!\!\!\perp \pi} \prod_{O \in \mathcal{O}} X_O \quad X_O \triangleq \prod_{(i,j)\in O} X_{ij}$$

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$$\boxed{\mathbb{E}_{\mathcal{Q}}[X_O] = \begin{cases} \frac{1}{1-\rho^{2|O|}} & \text{Gaussian} \\ 1+\rho^{2|O|} & \text{Erdős-Rényi} \end{cases}}$$

Failure of second-moment

We show

$$\mathbb{E}_{\mathcal{Q}}\left[\left(\frac{\mathcal{P}(A,B)}{\mathcal{Q}(A,B)}\right)^2\right] = \begin{cases} 1+o(1) & \text{if } \rho^2 \le \frac{(2-\epsilon)\log n}{n} \\ +\infty & \text{if } \rho^2 \ge \frac{(2+\epsilon)\log n}{n} \end{cases}$$

- Gaussian: suboptimal by a factor of 2
- ER graphs: suboptimal by an unbounded factor when $p = o(1)$

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Obstruction from short orbits

$$\mathbb{E}_{(A,B)\sim\mathcal{Q}}\left[\left(\frac{\mathcal{P}(A,B)}{\mathcal{Q}(A,B)}\right)^2\right] = \mathbb{E}_{\pi\perp\!\!\!\perp\widetilde{\pi}}\left[\prod_{O\in\mathcal{O}} \mathbb{E}_{\mathcal{Q}}\left[X_O\right]\right] \overset{\widetilde{\pi}=\pi}{\ge} \frac{1}{n!}\left(1+\rho^2\right)^{\binom{n}{2}}$$

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Atypically large magnitude of $\prod_{O\in\mathcal{O}:|O|=k} X_O$ for short orbits of length $k \lesssim \log n \Rightarrow$ second-moment blows up

Truncated second-moment: dense regime

It suffices to consider $k = 1$:

$$\prod_{O \in \mathcal{O}:|O|=1} X_O \approx \left(\frac{1}{p}\right)^{2e_{A\wedge B^\pi}(F)}$$

F: set of fixed points of $\sigma \triangleq \pi^{-1} \circ \widetilde{\pi}$
$A \wedge B^\pi$: Intersection graph
$e_{A\wedge B^\pi}(F)$: # of edges of subgraph of $A \wedge B^\pi$ induced by F

- Under $\mathcal{P}$: $e_{A\wedge B^\pi}(S)$ concentrates uniformly over all S when $|S|$ is large
- On this typical event $\mathcal{E}$ under $\mathcal{P}$, when $|F|$ is large,

$$\mathbb{E}_{\mathcal{Q}}\left[\prod_{O \in \mathcal{O}:|O|=1} X_O \mathbf{1}_{\{\mathcal{E}\}}\right] \lesssim \mathbb{E}_{\mathcal{Q}}\left[\left(\frac{1}{p}\right)^{2e_{A\wedge B_\pi}(F)} \mathbf{1}_{\left\{e_{A\wedge B_\pi}(F) \le \binom{|F|}{2} ps^2\right\}}\right]$$

Truncated second-moment: sparse regime

Need to consider $k = \Theta(\log n)$. It can be shown

- Long orbits:

$$\mathbb{E}_{\mathcal{Q}}\left[\prod_{|O|>k} X_O\right] \le \left(1+\rho^k\right)^{\frac{n^2}{k}} = 1 + o(1)$$

- Short incomplete orbits:

$$\mathbb{E}_{\mathcal{Q}}\left[X_O \mid O \not\subset E\left(A \wedge B^{\pi}\right)\right] \le 1$$

- Short complete orbits:

$$X_O = \left(\frac{1}{p}\right)^{2|O|}, \quad \forall O \subset E\left(A \wedge B^{\pi}\right)$$

In the subcritical regime $nps^2 < 1$, $A \wedge B^{\pi} \sim \mathcal{G}(n, ps^2)$ is a pseudoforest
$\Rightarrow H_k \triangleq \cup_{O:|O|\le k, O\subset E(A\wedge B^{\pi})} O$ is a pseudoforest

Truncated second-moment: orbit pseudoforest

On $\mathcal{E} \triangleq \{(A, B, \pi) : A \wedge B^{\pi} \text{ is a pseudoforest}\}$ under $\mathcal{P}$:

$$\mathbb{E}_{\mathcal{Q}}\left[\prod_{O \in \mathcal{O}} X_O \mathbf{1}_{\{\mathcal{E}\}}\right] \le (1+o(1))\, \mathbb{E}_{\mathcal{Q}}\left[\left(\frac{1}{p}\right)^{2e(H_k)} \mathbf{1}_{\{H_k \text{ is a pseudoforest}\}}\right]$$

$$= (1+o(1)) \sum_{H \in \mathcal{H}_k} s^{2e(H)} \quad \text{(generating function)}$$

$\mathcal{H}_k$: pseudoforests assembled from edge orbits of length at most k

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Key challenge: enumerate orbit pseudoforests $\mathcal{H}_k$ using orbit structure

Concluding remarks

- Formulate the problem of testing network correlation and characterize the statistical detection limit
- The impossibility proof applies truncated second-moment method
- The sparse setting leverages the pseudoforest structure of subcritical Erdős-Rényi graphs
- A large computational gap may exist between the statistical and computational limits

Open problem

- Sharp detection threshold in the sparse regime
- Prove the existence of or close the computational gaps
- Sharp partial recovery threshold

Reference

- Y. Wu, J. X, & S. H. Yu *Testing correlation of unlabeled random graphs*. arXiv:2008.10097.