The Planted Matching Problem

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Joint work with
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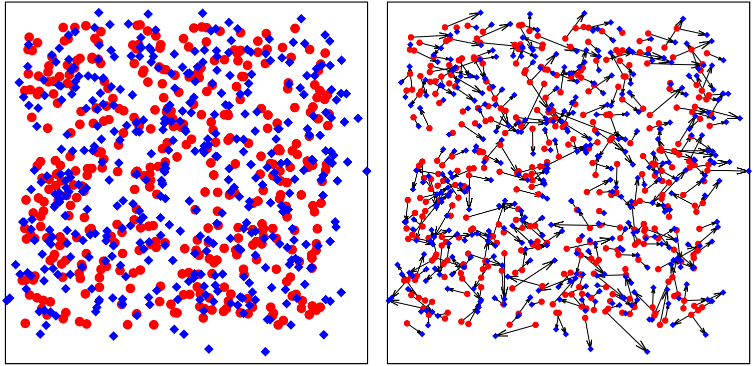
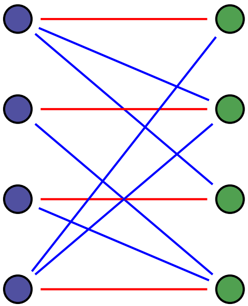
Motivating application: particle tracking



[Chertkov-Kroc-Krzakala-Vergassola-Zdeborová PNAS'10]

- Tracking particles advected by turbulent fluid flow
- Goal: recover the latent correspondence between particles

The planted matching model



- A complete bipartite graph
- A hidden perfect matching M^*
- Edge weight

$$W_e \overset{\text{ind.}}{\sim} \begin{cases} P & e \in M^* \\ Q & e \notin M^* \end{cases}$$

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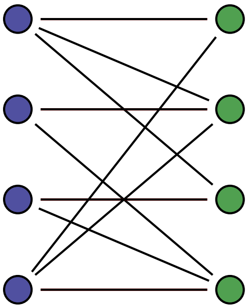

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- Goal: recover M^* from W

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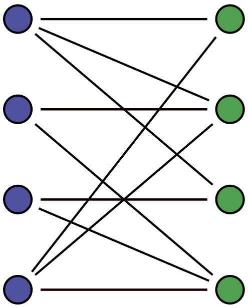

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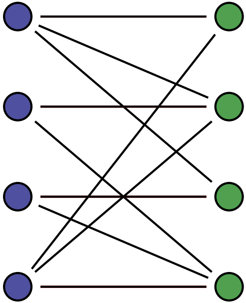

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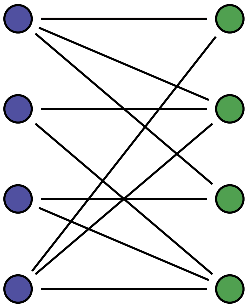

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- How much does $\widehat{M}_{\rm ML}$ have in common with M^*?

Main result: phase transition at $\lambda = 4$

Theorem (Moharrami-Moore-X. AAP'21)

$$\text{overlap: } \lim_{n\to\infty} \frac{1}{n}\mathbb{E}\left[\left|\widehat{M}_{\mathrm{ML}} \cap M^*\right|\right] = \begin{cases} 1 & \text{if } \lambda \geq 4 \\ \alpha(\lambda) & \text{if } 0 < \lambda < 4 \end{cases}$$

where $\alpha(\lambda) = 1 - 2\int_0^\infty (1 - F(x))(1 - G(x))\, V(x) W(x)\, dx,$

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and F, G, V, W is the unique solution to the ODE system

$$\dot{F} = (1 - F)(1 - G)V$$
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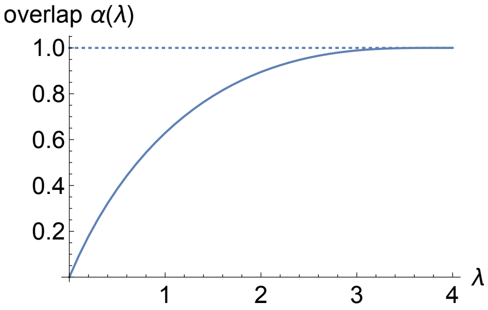

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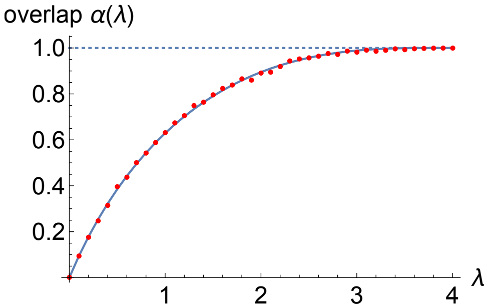
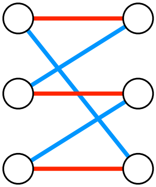


When $\lambda \geq 4$: count augmenting cycles



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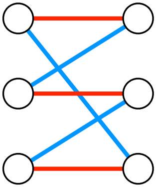

- Probability that an alternating cycle of length 2ℓ is augmenting:

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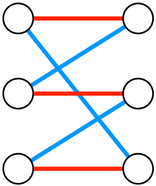

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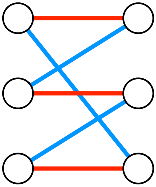

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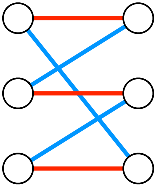

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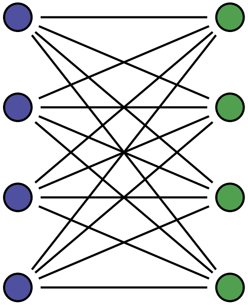
$\Rightarrow$ Sum over $\ell \Longrightarrow \mathbb{E}\left[\left|\widehat{M}_{\mathrm{ML}} \triangle M^*\right|\right] = o(n)$

Warmup: the (un-planted) random assignment problem



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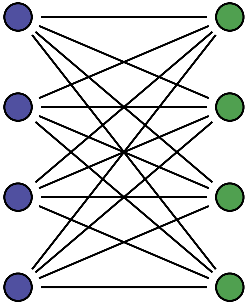

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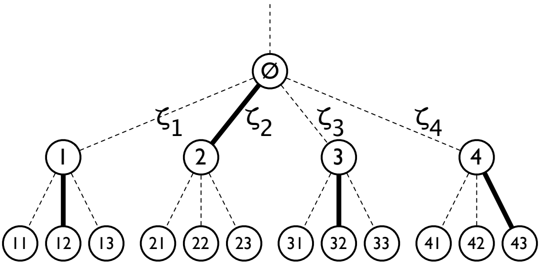
[Walkup'79, Mézard-Parisi'87, Steele'97, Aldous'01, Nair-Prabhakar-Sharma'05, Wästlund'09]

$$\mathbb{E}\left[\min_{M \in \mathcal{M}} \frac{1}{n} \sum_{e \in M} W_e\right] = 1 + \frac{1}{4} + \frac{1}{9} + \cdots + \frac{1}{n^2} \to \frac{\pi^2}{6}$$

Poisson-weighted infinite tree approximation

Cavity method: model as a tree [Mézard-Parisi '87, Aldous'00]



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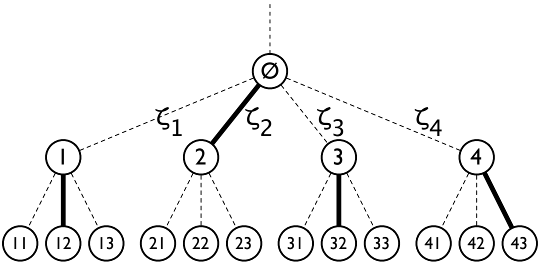

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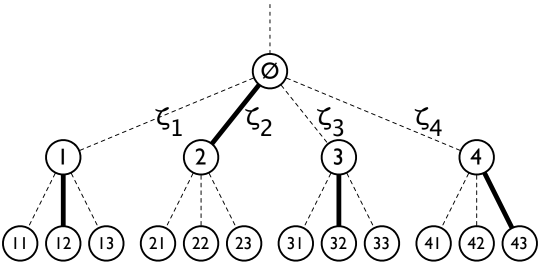

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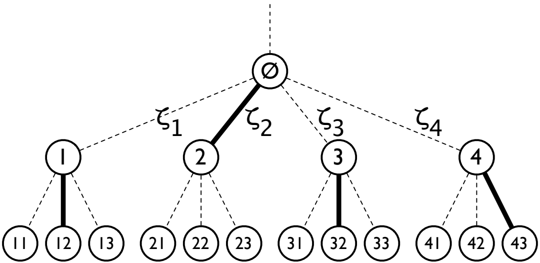

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$$X_\varnothing = \min_{i \geq 1} \{W_{\varnothing,i} - X_i\} \Longrightarrow X \overset{d}{=} \min_{i \geq 1} \{\zeta_i - X_i\}$$

From distributional to differential equations

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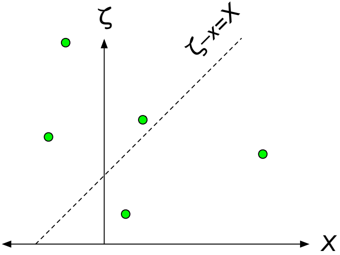
Generate pairs (ζ_i, X_i): two-dimensional Poisson process with density F'



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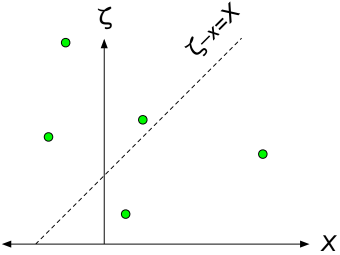

$$\bar{F}(x) = \exp\left(-\int_{-x}^{\infty} \bar{F}(t)\,dt\right) \quad \Rightarrow \quad \frac{dF(x)}{dx} = F(x)F(-x)$$

From distributional to differential equations, cont'd

$$\frac{dF(x)}{dx} = F(x)F(-x) \quad \Longrightarrow F(x) = \frac{e^x}{1+e^x}$$

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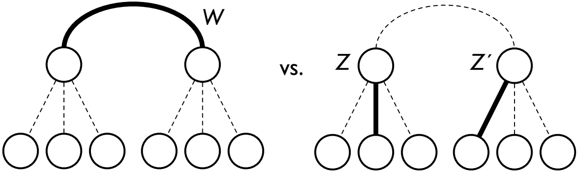
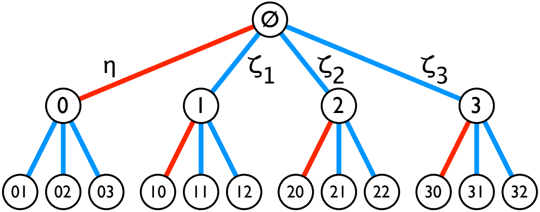


Contribution of a single edge:

$$\int_0^\infty w\mathbb{P}\left[Z + Z' \geq w\right]\mathrm{d}w = \frac{1}{4}\mathrm{Var}[Z+Z'] = \frac{1}{2}\mathrm{Var}[Z] = \frac{\pi^2}{6}$$

Planted Poisson-weighted infinite tree

Partner in planted matching is either parent or child 0,
other children sorted $1, 2, 3, \ldots$



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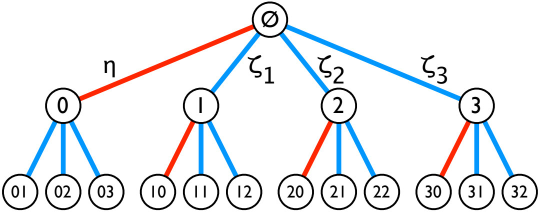

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Recursion:

$$X_\varnothing = \min \left\{ W_{\varnothing,0} - X_0, \ \min_{i \geq 1} \{W_{\varnothing,i} - X_i\} \right\}$$

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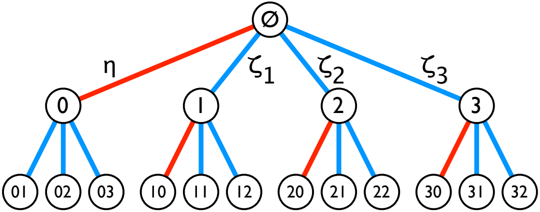

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$$Y \stackrel{d}{=} \min\{\eta - Z, Z'\}$$

$$Z \stackrel{d}{=} \min_i\{\zeta_i - Y_i\}$$

From distributional to differential equations, redux

$$\boxed{\begin{aligned} Y &\overset{d}{=} \min\left\{\eta - Z, Z'\right\} \\ Z &\overset{d}{=} \min\left\{\zeta_i - Y_i\right\}_{i=1}^{\infty} \end{aligned}}$$

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Phase transition of ODE at $\lambda = 4$

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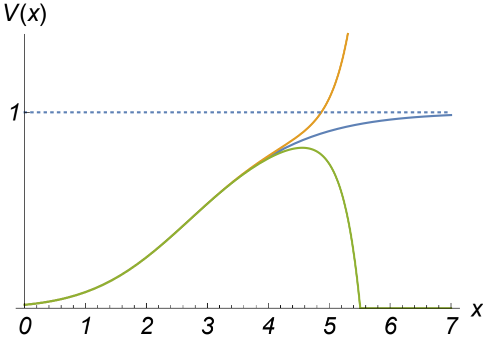
Lemma

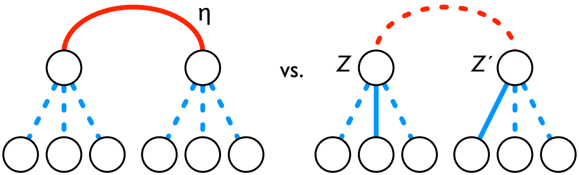
There is a unique solution if and only if $\lambda < 4$.

Geometric interpretation of uniqueness

When $\lambda < 4$, $(F = 1, G = 0, V = 1, W = 0)$ is a saddle point:
There exists a unique initial condition from which we approach the saddle along its unstable manifold



Finally, computing the overlap for $\lambda < 4$



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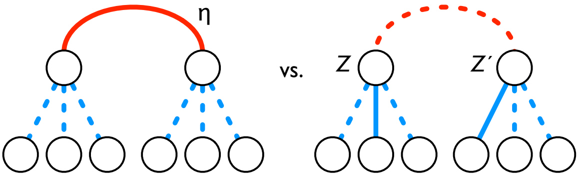

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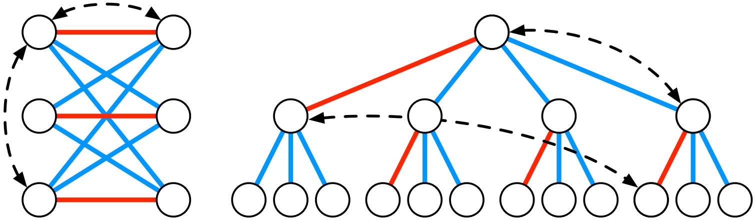
\alpha(\lambda) = \mathbb{P}\left[\eta < Z + Z'\right] &= 1 - \mathbb{E}_\eta \int_{-\infty}^{+\infty} f(x) F(\eta - x)\, dx \\
&= 1 - \int_{-\infty}^{+\infty} f(x)\, \mathbb{E}_\eta F(\eta - x)\, dx \\
&= 1 - \int_{-\infty}^{+\infty} (1 - F(x))(1 - G(x)) V(x) W(x)\, dx \\
&= 1 - 2 \int_{0}^{+\infty} (1 - F(x))(1 - G(x)) V(x) W(x)\, dx
\end{aligned}
$$

Proving it: Local weak convergence (Aldous 1992, 2001)



- Construct a *spatially invariant* M_{opt} on T_∞ using message passing
- Show $(K_{n,n}, M_{\mathsf{min}})$ converges locally to $(T_\infty, M_{\mathrm{opt}})$
  - Local treelikeness of light edges
  - Almost-doubly-stochastic matrix

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- Exact characterization of overlap of MLE by system of ODEs

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$$\lambda = 4 - \epsilon: \text{ Optimal reconstruction error is } \exp\left(-\Theta(1/\sqrt{\epsilon})\right)$$

- Sharp threshold in general (d, P, Q) model [Ding-Wu-X.-Yang '21]:

$$\sqrt{d} \int \sqrt{\mathrm{d}P \mathrm{d}Q} = 1$$

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Reference

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