A Proof of The Changepoint Detection Threshold Conjecture in Preferential Attachment Models

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Joint work with
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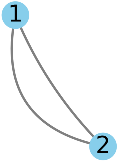
Preferential attachment models

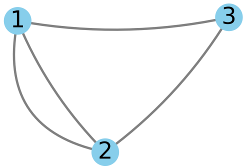
Initial graph G_2 consists of two vertices connected by m parallel edges



Preferential attachment models

At each time t, a new vertex t arrives and forms m edges, one at a time, to existing nodes $v \in [t-1]$:

$$\mathbb{P}\{t \to v\} \propto \mathsf{deg}(v) + \delta_t$$



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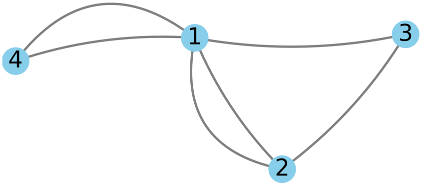

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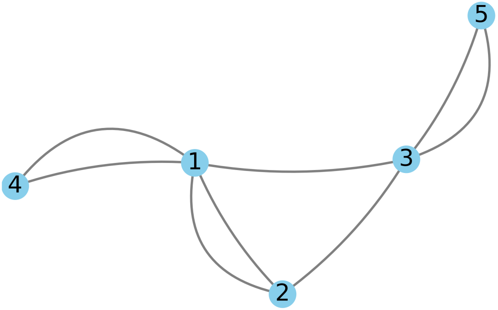

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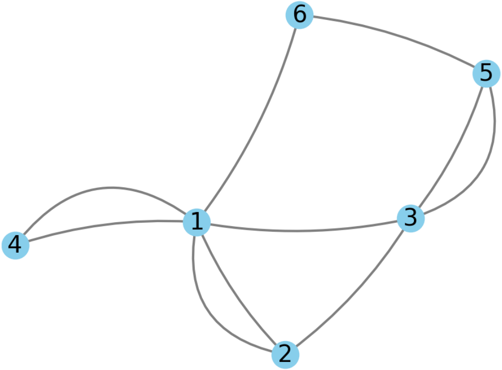

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$$\mathbb{P}\{t \to v\} \propto \deg(v) + \delta_t,$$

- $\deg(v)$ is updated after each edge is added
- $\delta_t = \infty$: uniform attachment (ignore degrees)
- $\delta_t = 0$: Barabási-Albert model [Barabási-Albert '99]
- The smaller δ_t, the stronger preference for high-degree vertices
- A most popular dynamic graph model: various properties (e.g. limiting degree distribution) are well-understood [van der Hofstad '16 '24]

Changepoint detection problem

Definition

$$\mathbb{H}_0 : \delta_t = \delta$$

$$\mathbb{H}_1 : \delta_t = \delta \mathbf{1}_{t \le \tau_n} + \delta' \mathbf{1}_{\tau_n < t \le n}$$

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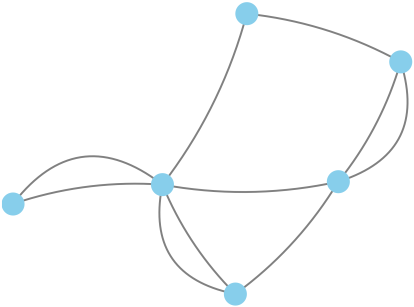

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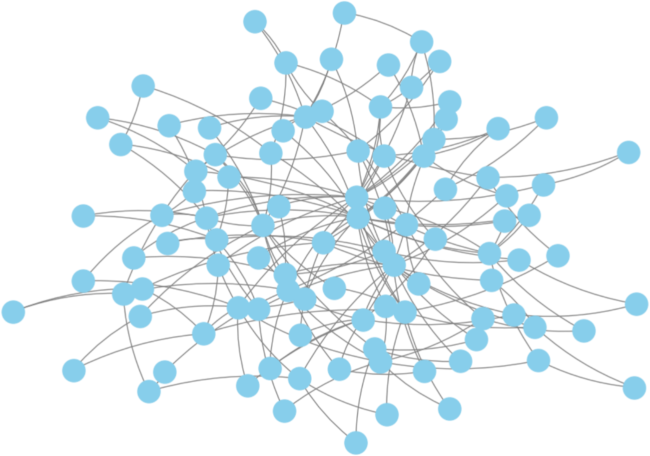
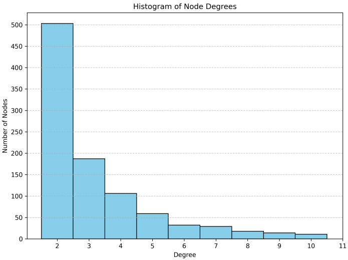
- Applications: detect structural changes in various settings, such as communication networks, social networks, financial networks, and biological networks [Cirkovic-Wang-Zhang '24].

Looks like a daunting task

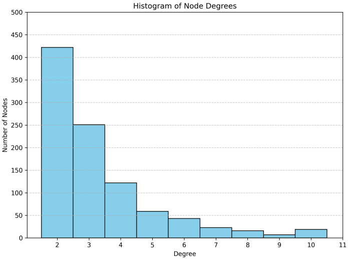


Change or no change?

A simple test based on minimum-degree



$n = 1000,\ m = 2,\ \delta(t) \equiv 0$



$n = 1000,\ m = 2,\ \delta(t) = 10 \cdot \mathbf{1}\left(t > n - n^{0.8}\right)$

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- Let $N_m(G_n)$ denote the number of degree-m vertices
- Let $p_m(\delta) = \lim_{n\to\infty} \frac{1}{n}\mathbb{E}_0\left[N_m(G_n)\right]$ under $\mathcal{H}_0$
- Consider test $T(G_n) = N_m(G_n) - np_m(\delta)$

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Theorem (Bet-Bogerd-Castro-van der Hofstad '23)

Suppose $\tau_n = n - cn^\gamma$ *for a constant* c *and* $\gamma \in (0,1)$. *If* $\gamma > 1/2$, *by choosing* $\alpha_n/\sqrt{n}$ *slowly tending to infinity,*

$$\mathbb{P}_0\left\{|T(G_n)| \geq \alpha_n\right\} + \mathbb{P}_1\left\{|T(G_n)| \leq \alpha_n\right\} \to 0$$

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- Intuition: There are $\Theta(1)$ fraction of degree-m nodes $\Rightarrow$ probability of attaching to degree-m nodes changes by $\Theta(1)$ after $\tau_n \Rightarrow$ $\mathbb{E}_1[T] = \Theta(n^\gamma)$, while $\mathrm{Std}[T] = O(\sqrt{n})$

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- If δ is unknown, can be replaced by a ML estimator
- Can establish weak detection when $\gamma = 1/2$

Changepoint detection conjecture

Conjecture (Bet-Bogerd-Castro-van der Hofstad '23)

Suppose $\tau_n = n - cn^\gamma$ *for a constant* c *and* $\gamma < 1/2$.

1. *All tests based on vertex degrees are powerless.*
2. *All tests are powerless.*

- Part 2 of the conjecture is particularly striking, because, if true, neither degree information nor any higher-level graph structure is useful for detection when $\gamma < 1/2$

Significant progress

Theorem (Kaddouri-Naulet-Gassiat '24)

Suppose $\tau_n = n - \Delta$. *If* $\Delta = o(n^{1/3})$ *for* $\delta > 0$ *or* $\Delta = o(n^{1/3}/\log n)$ *for* $\delta = 0$, *then*

$$\mathbb{P}_0(A_n) \to 0 \Longrightarrow \mathbb{P}_1(A_n) \to 0, \quad \textit{for all sequences of events } A_n$$

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- As a consequence, $\mathrm{TV}(\mathbb{P}_0, \mathbb{P}_1) \le 1 - \Omega(1) \Rightarrow$ strong detection is impossible
- Does not cover the entire regime $\Delta = o(\sqrt{n})$ and the regime $\delta < 0$
- Does not rule out the possibility of weak detection

Our resolution

Theorem (Du-Gong-X. '25)

Suppose $\tau_n = n - \Delta$. *If* $\Delta = o(n^{1/2})$, *then*

$$\mathrm{TV}(\mathbb{P}_0, \mathbb{P}_1) = o(1)$$

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- As a consequence, all tests are powerless $\Rightarrow$ resolves the changepoint detection conjecture [Bet-Bogerd-Castro-van der Hofstad '23] in positive
- We prove a stronger statement: all tests remain powerless even if, in addition to G_n, the entire network history were observed up to time $n - N$ for $\Delta^2 \ll N \ll n$
- As a corollary, we prove no estimator can locate τ_n within $o(\sqrt{n})$ with $\Omega(1)$ probability $\Rightarrow$ the estimator in [Bhamidi-Jin-Nobel'18], which achieves $|\hat{\tau}_n - \tau_n| = O_P(\sqrt{n})$, is order-optimal

Proof ideas

Challenge of directly bounding second-moment

Define the Likelihood ratio

$$L(G) \triangleq \frac{\mathbb{P}_1(G)}{\mathbb{P}_0(G)}$$

Then

$$\mathbb{E}_{G_n \sim \mathbb{P}_0}\left[L^2(G_n)\right] = 1 + o(1) \implies \mathrm{TV}(\mathbb{P}_1, \mathbb{P}_0) = o(1)$$

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- Widely used to prove impossibility of detection in high-dimensional statistics and network analysis (e.g. community detection)
- However, since only final network snapshot is observed, $L(G_n)$ involves an average over compatible network histories, making it hard to bound its second-moment directly

Consider an "easier" problem

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Theorem (Kaddouri-Naulet-Gassiat '24)

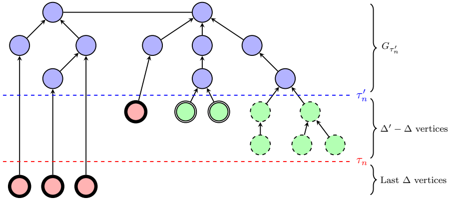
Denote $\overline{G}_n$ as the entire network history and $\overline{\mathbb{P}}_1, \overline{\mathbb{P}}_0$ as its law under $\mathcal{H}_0, \mathcal{H}_1$, respectively. Then

$$\mathrm{TV}(\overline{\mathbb{P}}_1, \overline{\mathbb{P}}_0) = 1 - o(1),$$

if and only if $\Delta \triangleq n - \tau_n \to \infty$.

Limitation of previous strategy

- Reveal arrival times of all vertices, except for a carefully chosen subset $\mathcal{S}$ of leaf vertices (**bolded red vertices** shown below):



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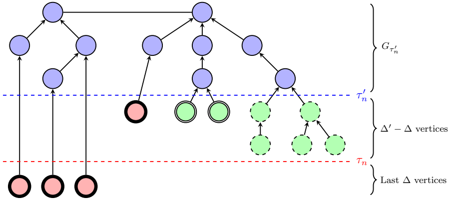

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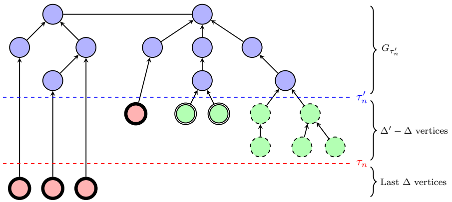

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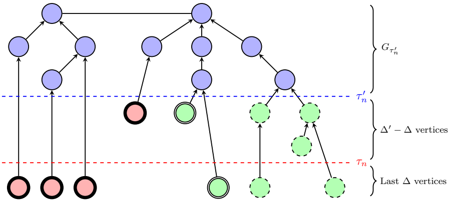
- For detection to be impossible, also need $|\mathcal{S}| \asymp \Delta' \gg \Delta^2$
$\Rightarrow \Delta \ll n^{1/3}$

Challenge in the regime $n^{1/3} \lesssim \Delta \ll \sqrt{n}$

- To prove the impossibility up to $\Delta \leq o(\sqrt{n})$, can only reveal network history up to $\tau_n' = n - \Delta'$, where $\Delta^2 \ll \Delta' \ll n$



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- Vertices arriving after τ_n may attach to vertices arrived in $[\tau_n' + 1, \tau_n]$

Our proof strategy

1. Interpolation: reduce to analyzing changepoint $\tau_n = n - 1$
2. Simplified model: reveal network history up to time $n - o(n)$
3. Bound TV by the second moment of likelihood ratio
4. Use Efron-Stein inequality and coupling

Step 1: Interpolation

- $\mathbb{P}_{n,n-k}$: distribution of G_n with changepoint at time $n-k$

$$\mathbb{P}_0 = \mathbb{P}_{n,n} \to \mathbb{P}_{n,n-1} \to \mathbb{P}_{n,n-2} \to \cdots \to \mathbb{P}_{n,n-\Delta-1} \to \mathbb{P}_{n,n-\Delta} = \mathbb{P}_1$$

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$$\mathrm{TV}(\mathbb{P}_0, \mathbb{P}_1) = \mathrm{TV}(\mathbb{P}_{n,n}, \mathbb{P}_{n,n-\Delta})$$

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- Suffices to show

$$\mathrm{TV}(\mathbb{P}_{n',n'}, \mathbb{P}_{n',n'-1}) = o\left(\frac{1}{\Delta}\right), \quad \forall n' \in [n-\Delta+1, n]$$

WLOG, focus on $n' = n$ and $\tau_n = n-1$ henceforth

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$$\begin{aligned}
\mathrm{TV}(\mathbb{P}_{n,n}, \mathbb{P}_{n,n-1}) &= \mathrm{TV}(\mathcal{P}_{G_n}, \mathcal{Q}_{G_n}) \\
&\overset{\text{DP}}{\leq} \mathrm{TV}\left(\mathcal{P}_{G_n,\overline{G}_M}, \mathcal{Q}_{G_n,\overline{G}_M}\right) \\
&\overset{\text{Jensen}}{\leq} \mathbb{E}_{\overline{G}_M \sim \mathcal{P}_{\overline{G}_M}}\left[\mathrm{TV}\left(\mathcal{P}_{G_n|\overline{G}_M}, \mathcal{Q}_{G_n|\overline{G}_M}\right)\right]
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- Reveal the network history up to time $M = n - N$, denoted by $\overline{G}_M$, where $\Delta^2 \ll N \ll n$
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$$\begin{aligned} \mathrm{TV}(\mathbb{P}_{n,n}, \mathbb{P}_{n,n-1}) &= \mathrm{TV}(\mathcal{P}_{G_n}, \mathcal{Q}_{G_n}) \\ &\overset{\text{DP}}{\leq} \mathrm{TV}\left(\mathcal{P}_{G_n,\overline{G}_M}, \mathcal{Q}_{G_n,\overline{G}_M}\right) \\ &\overset{\text{Jensen}}{\leq} \mathbb{E}_{\overline{G}_M \sim \mathcal{P}_{\overline{G}_M}}\left[\mathrm{TV}\left(\mathcal{P}_{G_n|\overline{G}_M}, \mathcal{Q}_{G_n|\overline{G}_M}\right)\right] \end{aligned}$$

- Reduce to proving

$$\mathrm{TV}\left(\mathcal{P}_{G_n|\overline{G}_M}, \mathcal{Q}_{G_n|\overline{G}_M}\right) = o\left(\frac{1}{\Delta}\right), \quad \forall \overline{G}_M$$

Step 3: Bound the second moment

- Define likelihood ratio $L \triangleq \frac{\mathcal{Q}_{G_n|\overline{G}_M}}{\mathcal{P}_{G_n|\overline{G}_M}}$. Then

$$2\mathrm{TV}\left(\mathcal{P}_{G_n|\overline{G}_M}, \mathcal{Q}_{G_n|\overline{G}_M}\right) = \mathbb{E}_{\mathcal{P}_{G_n|\overline{G}_M}}\left[|L-1|\right] \le \sqrt{\mathrm{Var}_{\mathcal{P}_{G_n|\overline{G}_M}}[L]}$$

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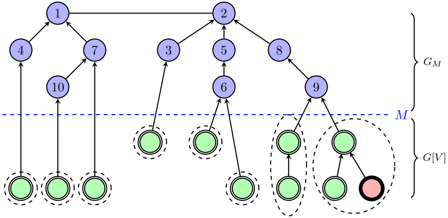
- Enough to show

$$\mathrm{Var}_{\mathcal{P}_{G_n|\overline{G}_M}}[L] = O(1/N),$$

where recall $M = n - N$ and $\Delta^2 \ll N \ll n$

Step 3: Bound the second moment

Let V denote the set of vertices arriving after time $M = n - N$. Consider the subgraph of G_n induced by V and let $\mathcal{C}(v)$ denote its connected component containing $v \in V$.



$m = 1$: connected components are denoted by dashed ellipses

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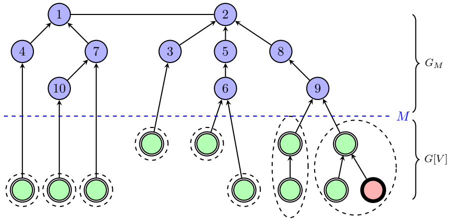

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Key: The connected components can arrive in any relative order

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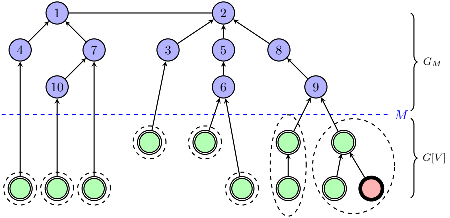

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Then

$$L \triangleq \frac{\mathcal{Q}_{G_n|\overline{G}_M}}{\mathcal{P}_{G_n|\overline{G}_M}} = \frac{C_1}{N} \sum_{v \in V} |\mathcal{C}(v)| \lambda_v X_v,$$

where C_1 is bounded constant, $\sum_{w \in \mathcal{C}(v)} \lambda_w = 1$, and $c_1 \leq X_v \leq c_2$.

Step 4: Efron-Stein inequality and coupling

- Encode $\mathcal{P}_{G_n|\overline{G}_M}$ using Nm ind. r.v.s $\{U_{t,i}\}_{M<t\leq n, 1\leq i\leq m}$

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  e.g. for $m=1$ and $\delta=0$, recall at every time t,

  $$\mathbb{P}\{t \to v\} \propto \mathsf{deg}(v)$$

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  Similar encoding scheme extends to general $m \ge 1$ and $\delta > -m$

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$$\mathrm{Var}[L] \leq \frac{1}{2} \sum_{M<t\leq n} \sum_{1\leq i\leq m} \mathbb{E}\left[\left(f(U) - f(U^{(t,i)})\right)^2\right]$$

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- Show the growth of $\mathcal{C}(t)$ is dominated by a sub-critical branching process to conclude $\mathbb{E}[|\mathcal{C}(t)|^2] = O(1)$

Concluding remarks

- We show changepoint detection threshold is $\tau_n = n - o(\sqrt{n})$, confirming a conjecture of [Bet-Bogerd-Castro-van der Hofstad '23]
- As by-product, we show changepoint localization threshold is also $\tau_n = n - o(\sqrt{n})$, matching upper bound in [Bhamidi-Jin-Nobel '18]
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Future directions

- General attachment rule: $\mathbb{P}(t \to v) \propto f(\text{deg}(v))$ [Banerjee-Bhamidi-Carmichael '22]
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References

- Hang Du, Shuyang Gong, & Jiaming Xu. *A Proof of The Changepoint Detection Threshold Conjecture in Preferential Attachment Models*, arXiv:2502.00514.