Bargaining among partners*

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Abstract

While bargaining is at the core of economics, economic theories of bargaining have produced rich, often incompatible predictions, some stipulating that bargaining be efficient and others proving that it cannot be. Using a generalized partnership model with independent private values that permits a unifying approach, we study how bargaining outcomes depend on ownership, bargaining power, and type distributions. With two agents and overlapping supports, equal bargaining power is necessary for efficiency, whereas equal ownership is not. Without overlapping supports, efficiency does not depend on ownership structure. Bargaining is efficient independent of ownership and bargaining power if and only if the gap between the supports is sufficiently large, in which case bargaining never breaks down and incomplete and complete information bargaining are equivalent. For equal (extremal) bargaining weights, variants of the k-double auction implement the optimal mechanism for uniform distributions (distributions satisfying regularity). Generalizations include decreasing marginal values and multiple agents.

Keywords: ex post efficiency, ownership, bargaining power, countervailing power
JEL Classification: D44, D82, L41

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1 Introduction

Bargaining is a central part of everyday life, business, politics and economics. Husbands and wives haggle over the household budget, parents and children negotiate over screen time, politicians bargain over pork barrels and war and peace, workers and firms over pay, and corporate leaders over business deals. The conditions under which bargaining is efficient are at the core of economic analysis and debates, from the Coase Theorem and the irrelevance of ownership that it implies to Galbraith's popular yet controversial concept of countervailing power to the impossibility results of Vickrey and Myerson and Satterthwaite to the recent upsurge of empirical studies that, among other things, document that bargaining breakdown is a systematic feature of negotiations in the real world.[1] According to theories of complete information bargaining in the tradition of Nash and Shapley, bargaining is efficient by assumption while, in a contrast that could hardly be sharper, the Myerson-Satterthwaite theorem has led some authors to conclude that, with incomplete information, the quest for efficient bargaining is "fruitless."[2] Lacking a unifying framework, this richness and the often contradictory sets of assumptions and results can easily have readers wonder what lessons to draw from the economic literature on bargaining.

In this paper, we study a general model of bargaining that encompasses as special cases the irrelevance of ownership, the relevance of countervailing power, the impossibility of ex post efficiency, and (generalized) Nash bargaining. The formal setup is a generalized partnership model that permits heterogeneous bargaining weights, type distributions with shifting supports, and arbitrary ownership structures. Information is always incomplete and bargaining is modeled as a mechanism that maximizes the weighted sum of the agents' expected payoffs, subject to incentive compatibility, interim individual rationality, and a no-deficit constraint.

Focusing first on the case with two agents, there are always bargaining weights and ownership structures such that ex post efficiency is possible. When the distributions' supports overlap, ex post efficiency requires equal bargaining weights, but is typically possible for a large, convex set of ownership structures that exclude extremal ownership. Hence, the specification with overlapping supports provides a formalization of Galbraith's hypothesis

[1]See Galbraith (1952, 1954), Coase (1960), Vickrey (1961), and Myerson and Satterthwaite (1983). For empirical studies of bargaining, see, e.g., Backus et al. (2019), Backus et al. (2020), Backus et al. (forth.), Larsen (2021), Larsen et al. (2021), Larsen and Zhang (2022) and Byrne et al. (2022). For skepticism regarding countervailing power, see, for example, Stigler (1954, p. 13), who lamented the lack of explanation for "why bilateral oligopoly should in general eliminate, and not merely redistribute, monopoly gains." Steptoe (1993) summarizes its popular appeal by noting that the notion of buyer power is sometimes embraced by courts as if it had "talismanic power."

[2]Ausubel et al. (2002, p. 1934).

that countervailing power is of “substantial, and perhaps central, importance,” (Galbraith, 1954, p. 1), and it provides a set of instances in which the Coasian irrelevance of ownership does not apply. Because ex post efficiency is not possible with overlapping supports and extremal ownership, where the agent who owns the good is the seller and the other agent is the buyer, overlapping supports and extremal ownership corresponds to the impossibility result of Myerson and Satterthwaite. If the supports do not overlap, then ownership does not matter for ex post efficiency, but if the gap between the supports is small, then bargaining weights must still be sufficiently similar for bargaining to be efficient. Thus, nonoverlapping supports provide a partial foundation for the Coasian irrelevance and the Coase Theorem, partial only because even though ownership is immaterial, bargaining power, if too skewed, can still be an impediment to efficiency. If the gap between the two supports is sufficiently large, then the Coase Theorem applies, that is, bargaining is efficient irrespective of the ownership structure and bargaining weights. If this is the case, there is no scope for countervailing power to increase social surplus and the incomplete information bargaining outcomes are the same as those under (generalized) Nash bargaining—bargaining weights only affect the distribution of surplus, not its size.

Bargaining between two agents is naturally said to break down if the two agents do not trade. Adhering to this terminology, the above results have the empirical implication that bargaining is not ex post efficient if bargaining breakdown occurs with positive probability.[3] The reasons are simple yet still slightly subtle. With nonoverlapping supports, under ex post efficiency the agent with the weak distribution always sells its share to the other agent. With overlapping supports and interior ownership, under ex post efficiency the agent with the low type should sell its share to the agent with the high type. Because two types being the same is a probability zero event, trade should occur with probability one. Last, with overlapping supports and extremal ownership, trade would not occur under ex post efficiency if the owner has the higher type, which means that, in principle, ex post efficiency is compatible with bargaining breakdown. However, as noted, overlapping supports and extremal ownership are incompatible with ex post efficiency. With nonoverlapping supports, the only reason for bargaining breakdown in our setting would be the exertion of bargaining power in the presence of extremal ownership.

To justify and rationalize the complete information approach to bargaining, which may be deemed advantageous in terms of tractability, a researcher may want to argue that, maybe, in the application at hand there is very little private information. The framework we study

[3]It may appear appropriate to add the qualification that, in the case of nonoverlapping supports, the agent with the stronger distribution is not the sole owners. But if that is the case, it is difficult to see what the two agents would bargain over in the first place.

here provides a way of formalizing and conceptualizing this otherwise vague or vacuous notion[4]—there is little private information if the gap between the supports is sufficiently large. In practice, this would correspond to a situation in which, say, a downstream firm's value for a supplier's input is always well above the supplier's cost and the downstream firm has no scope for internal production, which may be descriptive of the healthcare industry and insurers and hospitals. In contrast, in settings in which bargaining breakdown is a real-world phenomenon, such as in certain retail and media markets, a model with incomplete information that accommodates bargaining breakdown would seem more appropriate.[5]

The analysis extends directly to settings with many agents. This contrasts with complete information bargaining models, which, in the case of Nash bargaining, do not extend or, in the case of the Shapley value, become quickly computationally intractable. For extremal ownership and two agents, we also analyze a variant with decreasing marginal values, which is relevant for some applications. Among other things, we show that the buyer-optimal and the seller-optimal mechanisms can be implemented with the powerful agent posting a nonlinear tariff after its its type is realized, giving the powerless agent the discretion to determine the quantity traded.

As mentioned, bargaining is central to economics, from the Coase Theorem (Coase, 1960) to the concept of countervailing power (Galbraith, 1952, 1954; Stigler, 1954), the theory of the firm (Grossman and Hart, 1986; Hart and Moore, 1990), recent merger cases (see e.g. Lee et al., 2021), and empirical studies such as Backus et al. (2020), Larsen (2021), Larsen and Zhang (2022), Larsen et al. (2021), Backus et al. (forth.) and, Byrne et al. (2022). Our paper contributes to the literature on bargaining, including the strands of literature on complete information bargaining in the tradition of Nash (1950) and Shapley (1951) and those on incomplete information bargaining along the lines of Vickrey (1961) and Myerson and Satterthwaite (1983), by providing and analyzing a unifying framework.[6] In particular, we take the same as-if approach as Ausubel et al. (2002), Loertscher and Marx (2022), and Choné et al. (forth.), which models incomplete information bargaining as intermediated by a mechanism designer, and analyze a general partnership model (Cramton et al., 1987) that permits heterogeneous distributions and supports and unequal bargaining weights.

[4]For example, the impossibility theorem of Myerson and Satterthwaite (1983) holds for any positive densities whose supports overlap, no matter how skewed these are.

[5]For healthcare, see e.g. Ho and Lee (2017). Bargaining breakdown in retail markets is documented by Van der Maelen et al. (2017). For bargaining breakdown in media markets, see e.g. Frieden et al. (2020) or the FCC press release on "FCC Begins Proceeding to Empower Consumers During Cable & Satellite TV Blackouts," January 17, 2024, https://docs.fcc.gov/public/attachments/DOC-399876A1.pdf.

[6]There is also a strand of literature on one-to-many bargaining—such as between a developer and land owners—with complete information in which the principal lacks commitment power; see, for example, Xiao (2018) or Uyanik and Yengin (2023).

Methodologically, the paper contributes to the mechanism design literature by combining the methods that Myerson (1981), Myerson and Satterthwaite (1983), and Williams (1987) developed for settings with one-sided and two-sided private information to partnership models, which exhibit countervailing incentives, such as Lu and Robert (2001), Loertscher and Wasser (2019), and Loertscher and Marx (2023).[7] Studying informed-principal problems, Mylovanov and Tröger (2014) solve for the mechanism that is optimal for a one agent in a bilateral partnership problem. The mechanism that maximizes one agent's expected payoff is encompassed as a special case of our analysis, which sidesteps the informed-principal aspect of the problem by assuming that the mechanism is designed and run by an intermediary.[8] The partnership framework analyzed here is a generalization of the settings with one-sided and two-sided private information in Loertscher and Marx (2019, 2022).

The remainder of this paper is organized as follows. Section 2 lays out the setup with two agents. Section 3 illustrates the key forces at work while Section 4 provides the in-depth analysis of bargaining among two partners. Extensions to decreasing marginal values and more than two agents are analyzed in Section 5. Section 6 concludes the paper.

2 Setup

Let $\mathcal{N}$ denote the set of agents whose cardinality is denoted by n. Up to Section 5.2, we assume that $\mathcal{N} = \{1, 2\}$. However, we will often use the generic notation $\mathcal{N}$ as this paves the way toward the generalization in Section 5.2. There is one unit of productive resources. Agent i's ownership share is denoted r_i and satisfies $r_i \in [0, 1]$. Moreover, the productive resources are entirely owned by the agents, that is, $\sum_{i \in \mathcal{N}} r_i = 1$. Each agent i has a constant marginal value of θ_i for up to one unit of the productive resource, where θ_i is agent i's private information, which is an independent draw from its type distribution F_i with support $[\underline{\theta}_i, \overline{\theta}_i]$ and positive density f_i on its support. Type distributions are common knowledge. Agent i's bargaining (or welfare) weight is denoted $w_i \in [0, 1]$, with at least one agent having a positive weight. For the case with $n = 2$, we let $r \equiv r_1$, so that 2's ownership is $1 - r$, and, normalizing the bargaining weights by dividing by $w_1 + w_2$, we use w to denote agent 1's weight, with the implication that agent 2's weight is $1 - w$.

Incomplete information bargaining among the agents is modeled as being intermediated by a possibly fictitious designer that chooses a bargaining mechanism. Specifically, a direct mechanism is given as $\langle \mathbf{Q}, \mathbf{M} \rangle$, where $\mathbf{Q} : \times_{i \in \mathcal{N}} [\underline{\theta}_i, \overline{\theta}_i] \to [0, 1]^n$ satisfying $\sum_{i \in \mathcal{N}} Q_i(\boldsymbol{\theta}) = 1$ is

[7] Focusing on ex post efficiency, Loertscher and Marx (2024) study variants of partnership models, sometimes also referred to as "asset market models," in which agents' types can be multi-dimensional, while Liu et al. (forth.) provide an empirical application of an asset market model.

[8] Away from extremal bargaining weights, the informed-principal problem does not appear well defined.

the *allocation rule* and $\mathbf{M} : \times_{i\in\mathcal{N}}[\underline{\theta}_i, \overline{\theta}_i] \to \mathbb{R}^n$ is the *payment rule.* The mechanism is called *direct* because it asks every agent to report its type. Given $\langle \mathbf{Q}, \mathbf{M}\rangle$ and assuming truthful reporting by agents other than agent i, the interim expected allocation and payment of agent i when its report is θ_i are $q_i(\theta_i) \equiv \mathbb{E}_{\boldsymbol{\theta}_{-i}}[Q_i(\boldsymbol{\theta})]$ and $m_i(\theta_i) \equiv \mathbb{E}_{\boldsymbol{\theta}_{-i}}[M_i(\boldsymbol{\theta})]$. When its type is θ_i, its allocation is Q_i and its payment is M_i, agent i's payoff from the mechanism is $\theta_i Q_i - M_i$. Because the payoffs are linear, agent i's expected payoff given an expected allocation q_i and expected payment m_i is simply $\theta_i q_i - m_i$. Given ownership share r_i and type θ_i, the value of i's outside option is $r_i\theta_i$. A prominent allocation rule is the ex post efficient allocation rule, which we denote by $\mathbf{Q}^e(\boldsymbol{\theta})$. This allocation rule is such that $Q_i^e(\boldsymbol{\theta}) = 1$ if and only if $\theta_i > \max \boldsymbol{\theta}_{-i}$ and $Q_i^e(\boldsymbol{\theta}) = 0$ otherwise. (Ties have probability zero and can be broken arbitrarily.) We denote by $q_i^e(\theta_i) \equiv \mathbb{E}_{\boldsymbol{\theta}_{-i}}[Q_i^e(\boldsymbol{\theta})]$ agent i's interim expected allocation under the efficient allocation rule.

The mechanism $\langle \mathbf{Q}, \mathbf{M}\rangle$ satisfies *(Bayes Nash) incentive compatibility* (IC) if for $i \in \mathcal{N}$ and all $\theta_i, \hat{\theta}_i \in [\underline{\theta}_i, \overline{\theta}_i]$,

$$\theta_i q_i(\theta_i) - m_i(\theta_i) \geq \theta_i q_i(\hat{\theta}_i) - m_i(\hat{\theta}_i).$$

It satisfies interim individual rationality (IR) if for all $i \in \mathcal{N}$ and all $\theta_i \in [\underline{\theta}_i, \overline{\theta}_i]$,

$$\theta_i q_i(\theta_i) - m_i(\theta_i) \geq r_i\theta_i.$$

An immediate implication of IC is that the interim expected allocation $q_i(\cdot)$ is nondecreasing.[9]

By the revelation principle, the focus on direct mechanisms that satisfy IC and IR is without loss of generality. The problem of the designer is to choose $\langle \mathbf{Q}, \mathbf{M}\rangle$ to maximize the weighted sum of the agents' ex ante expected payoffs:

$$\max_{\mathbf{Q},\mathbf{M}} \mathbb{E}_{\boldsymbol{\theta}}\Big[\sum_{i\in\mathcal{N}} w_i\big(\theta_i Q_i(\boldsymbol{\theta}) - M_i(\boldsymbol{\theta})\big)\Big], \tag{1}$$

subject to IC and IR and a no-deficit constraint, which is to say that the designer does not pour any money into the exchange.

The description of the incomplete information bargaining mechanism is now almost complete. The cases that remain to be addressed are those in which the mechanism designer runs a budget surplus after solving the constrained maximization problem with the objective in (1) when multiple agents have the maximal bargaining weight. For these cases, the mechanism needs to determine how that budget surplus is shared among those agents. To

[9]To see this, writing IC for agent i with a true type of $\hat{\theta}_i$ that could report θ_i, we have $\hat{\theta}_i q_i(\theta_i) - m_i(\theta_i) \leq \hat{\theta}_i q_i(\hat{\theta}_i) - m_i(\hat{\theta}_i)$. Subtracting this inequality from the one in the text yields $(\theta_i - \hat{\theta}_i)(q_i(\theta_i) - q_i(\hat{\theta}_i)) \geq 0$, which implies that $q_i(\cdot)$ must be nondecreasing.

this end, let $\boldsymbol{\eta} = (\eta_i)_{i\in\mathcal{N}}$ with $\eta_i \in [0,1]$, $\sum_{i\in\mathcal{N}} \eta_i = 1$, and $\eta_i = 0$ if $w_i < \max \mathbf{w}$. Then in case the budget surplus is positive, agent i obtains the share η_i of this surplus. Notice that if $n = 2$ and $w = 1/2$, then $\eta \equiv \eta_1$ can be interpreted as agent 1's generalized Nash bargaining weight when agents 1 and 2 bargain over the budget surplus.[10]

The mechanism design setup will make use of the virtual type functions

$$\Psi_i^S(\theta) \equiv \theta + \frac{F_i(\theta)}{f_i(\theta)} \quad \text{and} \quad \Psi_i^B(\theta) \equiv \theta - \frac{1 - F_i(\theta)}{f_i(\theta)},$$

where Ψ_i^S and Ψ_i^B are referred to as agent i's *virtual cost* and *virtual value* functions, respectively. We assume that all F_i are such that Ψ_i^S and Ψ_i^B are increasing, which is the Myersonian regularity condition (Myerson, 1981).[11] To capture differential productive strengths between the agents in a simple, tractable way, we assume in the two-agent case that $[\underline{\theta}_1, \overline{\theta}_1] = [0,1]$ and $[\underline{\theta}_2, \overline{\theta}_2] = [\underline{\theta}, \underline{\theta} + 1]$ for some $\underline{\theta} \geq 0$. Further, we assume that changes in $\underline{\theta}$ merely shift the support of 2's distribution while keeping the distribution on the support fixed. That is, denoting by $F_2^P(\theta)$ with support $[0,1]$ the *primitive* distribution whose virtual type functions we denote by $\Psi_2^{S,P}(\theta)$ and $\Psi_2^{B,P}(\theta)$, we have, for $\theta \in [\underline{\theta}, \underline{\theta}+1]$, $F_2(\theta) = F_2^P(\theta - \underline{\theta})$. This implies, in particular, that for $\theta \in [\underline{\theta}, \underline{\theta} + 1]$, $\Psi_2^S(\theta) = \Psi_2^{S,P}(\theta - \underline{\theta}) + \underline{\theta}$ and $\Psi_2^B(\theta) = \Psi_2^{B,P}(\theta - \underline{\theta}) + \underline{\theta}$. Many results generalize beyond this specification. However, the comparative statics results with respect to $\underline{\theta}$ and the interpretation of changes in $\underline{\theta}$ as reflecting productivity differentials are easily and conveniently captured with this shifting-support model.

This setup generalizes prior literature by simultaneously allowing for unequal bargaining (or welfare) weights w_i, nonextremal ownership, that is, $r_i \in (0,1)$, and heterogeneous distributions. For example, Myerson and Satterthwaite (1983), Cramton et al. (1987), Che (2006), Figueroa and Skreta (2012), Makowski and Mezzetti (1993), Loertscher and Wasser (2019), Gresik and Satterthwaite (1989), Lu and Robert (2001), and Liu et al. (forth.) assume equal weights. Williams (1987) and Loertscher and Marx (2022) allow for unequal bargaining weights but assume extremal resource ownership so that each agent is either ex ante known to be a buyer or a seller.[12] Loertscher and Marx (2024) allow for nonextremal ownership

[10] To see this, letting $S > 0$ be the budget surplus and assuming that the two agents' outside options when they Nash bargain over the division of the budget surplus are 0, the generalized Nash solution maximizes $p^\eta (S-p)^{1-\eta}$ over $p \in [0, S]$, yielding $p^* = \eta S$ as the maximizer.

[11] The virtual value function Ψ_i^B captures the marginal revenue associated with agent i. To see this, consider a seller with cost c that makes a take-it-or-leave-it price offer p to agent i. The seller's problem is $\max_{p\in[\underline{\theta}_i, \overline{\theta}_i]} (1 - F_i(p))(p - c)$. The first-order condition is $-f_i(p)(\Psi_i^B(p) - c) = 0$, which by the standard "marginal revenue equals marginal cost" condition means that $\Psi_i^B(p)$ is the marginal revenue associated with i's demand. An analogous argument shows that Ψ_i^S captures the marginal cost associated with F_i.

[12] Specifically, normalizing the total resources to 1, the setup in Loertscher and Marx (2022) allows for each agent's maximum demand to be some $k_i \in (0,1]$ and its ownership to be $r_i \in \{0, k_i\}$, while Williams (1987) studies a setting with two agents in which $k_1 = k_2 = 1$ and $r_1 = 1$.

and multi-dimensional types but assume equal weights.

3 Illustration

We first provide a brief illustration of the incomplete information bargaining mechanism between two agents, focusing on the case in which both agents draw their types from $[0,1]$, i.e., $\underline{\theta}=0$, and limiting attention to bargaining weights that are either extremal or equal, i.e., $w \in \{0, \frac{1}{2}, 1\}$. We consider the possibility of resource ownership that is extremal, i.e., one agent is a buyer and the other is a seller, and the case of a "partnership model" in which each agent owns some resources. For extremal ownership, the optimal mechanisms are known from the literature, in particular Myerson (1981), Myerson and Satterthwaite (1983, MS hereafter), and Williams (1987). For partnership models, we illustrate the allocation rule of the optimal mechanisms, which will be derived in Section 4.

Extremal bargaining weights

Consider first the case in which agent 1 has all the bargaining power, that is, $w=1$, and restrict attention to take-it-or-leave-it offers, which can be shown to be optimal.[13] If $r=1$ and agent 1's type is θ_1, then agent 1's problem is to choose its take-it-or-leave-it sale price $p \in [0,1]$ to maximize $(1-F_2(p))(p-\theta_1)$. The first-order condition is $0=-f_2(p)(\Psi_2^B(p)-\theta_1)$. As stated above, we assume that Ψ_i^B is increasing, which holds, for example, for the uniform distribution on $[0,1]$, in which case $\Psi_i^B(\theta)=2\theta-1$. An increasing virtual value function implies that the first-order condition is sufficient for characterizing a maximum. The optimal sale price that agent 1 sets is thus $p={\Psi_2^B}^{-1}(\theta_1)$. For the uniform distribution, this is $\frac{1+\theta_1}{2}$, as illustrated in Figure 1(a).

Conversely, when $r=0$, agent 1 is a buyer and chooses the buyer price p to maximize $(\theta_1-p)F_2(p)$. The first-order condition is $0=f_2(p)(\theta_1-\Psi_2^S(p))$. Because Ψ_i^S is also assumed to be increasing, the first-order condition is sufficient for a maximum. For example, for the uniform distribution on $[0,1]$, $\Psi_i^S(\theta)=2\theta$. Consequently, the optimal buy price is $p={\Psi_2^S}^{-1}(\theta_1)$, which for the uniform distribution on $[0,1]$ is $\frac{\theta_1}{2}$, as illustrated in Figure 1(b).

Consider now the case with shared ownership, i.e., $r \in (0,1)$. In this case, the mechanism that maximizes agent 1's ex ante expected payoff, subject to IC and IR, is, in a sense, a combination of the optimal mechanisms for $r \in \{0,1\}$. For small values of θ_1, say for

[13]This potentially brings to mind the issue of informed-principal problems. Although this setting satisfies the conditions of Mylovanov and Tröger (2014) under which the optimal mechanism of the informed principal coincides with the optimal mechanism of a designer whose value is known, our main model sidesteps these issues by assuming that all agents participate in a mechanism. Take-it-or-leave-it offers being optimal then means the mechanism's allocation rule corresponds to that resulting from optimal take-it-or-leave-it offers.

(a) $r = 1$: optimal sell price (b) $r = 0$: optimal buy price (c) $r = 0.9$: combination

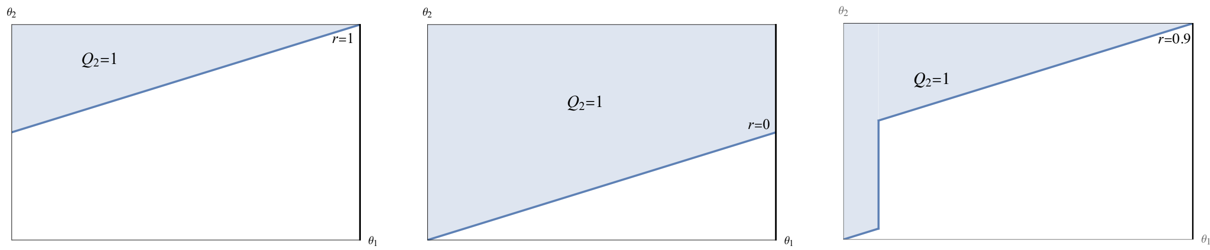


Figure 1: Agent 1 has all the bargaining power: allocation rules and prices of the mechanism that maximizes the expected payoff of agent 1 when agent 1 is a seller ($r = 1$, panel (a)), when agent 1 is a buyer ($r = 0$, panel (b)), and with shared ownership ($r = 0.9$, panel (c)). Assumes uniformly distributed types on $[0, 1]$.

$\theta_1 \leq z$ for some $z \in (0, 1)$, the allocation rule gives all of the resources to agent 2 whenever $\theta_2 \geq \Psi_2^{S^{-1}}(\theta_1)$, that is, according to the line depicted in Figure 1(b); and for $\theta_1 > z$, the allocation rule gives all of the resources to 2 whenever $\theta_2 \geq \Psi_2^{B^{-1}}(\theta_1)$, which is the line depicted in Figure 1(a). This blended allocation rule is shown Figure 1(c).

We can pin down the value of the cutoff point z using the IR constraint. The allocation rule implies that for $\theta_2 \in [\Psi_2^{S^{-1}}(z), \Psi_2^{B^{-1}}(z)]$, agent 2 is allocated the resources if and only if $\theta_1 \leq z$, and nothing otherwise. Consequently, the interim expected allocation of agent 2 for $\theta_2 \in [\Psi_2^{S^{-1}}(z), \Psi_2^{B^{-1}}(z)]$ is $F_1(z)$. For example, for F_1 and F_2 uniform on $[0, 1]$, we have $[\Psi_2^{S^{-1}}(z), \Psi_2^{B^{-1}}(z)] = [z/2, (z + 1)/2]$ and $F_1(z) = z$. As will be shown, an agent's expected gain from an incentive compatible mechanism is minimized when its interim allocation is equal to its ownership share, provided such an interim allocation exists.[14] This means that for $z = 1 - r$, all types ω_2 of agent 2 that are elements of $[z/2, (z + 1)/2]|_{z=1-r} = [(1 - r)/2, (2 - r)/2]$ are worst-off. Their expected payoffs from participating in the mechanism are $(1 - r)\omega_2 - r \int_0^{1-r} \theta_1/2 d\theta_1 + (1 - r) \int_{1-r^1} (1 + \theta_1)/2 d\theta_1 = (1 - r)\omega_2 + \frac{3}{4}r(1 - r)$ because with probability $1 - r$, these types get to consume the good and $(1 - r) \int_{1-r}^{1} (1 + \theta_1)/2 d\theta_1 - r \int_0^{1-r} \theta_1/2 d\theta_1 = \frac{3}{4}r(1-r)$ is the expected net payment they receive. Because $(1-r)\omega_2$ is the value of the outside option of an agent of type ω_2, agent 1 can *tax* the amount $T = \frac{3}{4}r(1-r)$ from every type of agent 2, for example, via an upfront fee.

Equal bargaining weights

Next, we turn to the polar opposite of extreme bargaining weights and describe the optimal mechanisms for $w = 1/2$. As shown by MS, for $r = 1$ and uniformly distributed types on $[0, 1]$, the optimal mechanism allocates the resources to agent 2 if and only if $\theta_2 \geq 1/4 + \theta_1$, which is depicted in Figure 2(a). This is the allocation rule of the *second-best* mechanism,

[14] In the context of ex post efficiency, this was first observed by Cramton et al. (1987). It extends to any incentive compatible mechanism.

that is, the mechanism that maximizes equally weighted social surplus, subject to IC, IR, and a no-deficit constraint for the designer.[15] Because ex post efficiency is not possible with $r = 1$, it differs from the ex post efficient allocation rule displayed in Figure 2(b). As shown by Cramton et al. (1987, CGK hereafter), ex post efficiency is possible, subject to the same constraints, when r is sufficiently symmetric. For example, for uniformly distributed types on $[0, 1]$, ex post efficiency is possible if and only if $r \in [0.21, 0.79]$.

(a) $r = 1$: MS second best (b) $r = 0.5$: CGK ex post efficient (c) $r = 0.9$: partnership second best

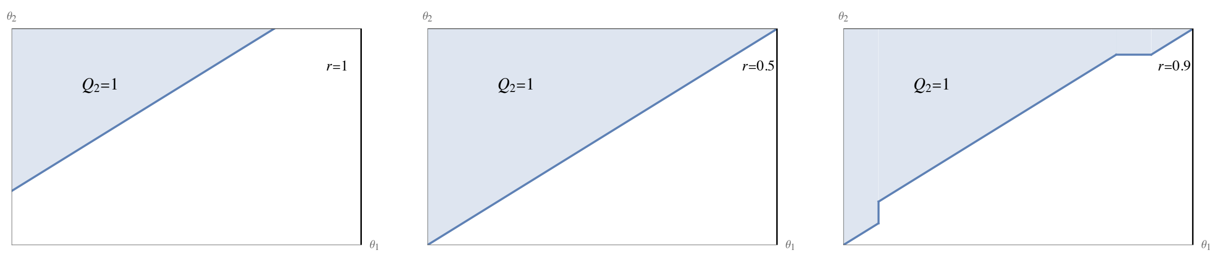


Figure 2: Equal bargaining power: panel (a) displays the allocation rule of the second-best mechanism derived for $r = 1$ by MS; panel (b) displays the ex post efficient allocation rule for $r = 0.5$, which is implementable without running a deficit as shown by CGK; and panel (c) displays the allocation rule of the second-best mechanism for the partnership model for $r = 0.9$, in which case ex post efficiency is not possible. Assumes uniformly distributed types on $[0, 1]$.

This also means that for uniformly distributed types and, say, $r = 0.9$, neither is ex post efficiency possible nor is the second-best mechanism of MS necessarily optimal. As we show, the second-best mechanism for shared ownership when ex post efficiency is not possible takes the form shown in Figure 2(c). It is, perhaps intuitively, a combination of the ex post efficient allocation rule in Figure 2(b) and a parallel inward shift of the allocation rule of second-best mechanism of MS in Figure 2(a). The bunching intervals, consisting of the vertical segment for agent 2 and the flat segment for agent 1, occur at $\theta_1 = 1 - r$ as in the mechanism that is optimal for agent 1, which ensures that agent 2's interim expected allocation in the bunching interval is equal to its ownership share $1 - r$, and at $\theta_2 = r$, which in turn ensures that agent 1's interim expected allocation is r when its type is in the bunching interval. These bunching intervals thus corresponds to the types who are worst-off, which are the types for whom the IR constraint will bind.

Both with extremal and with shared ownership, the second-best mechanism allocates based on a comparison of *weighted virtual type* functions

$$\Psi^K_{i,\alpha}(\theta) \equiv \alpha\theta + (1 - \alpha)\Psi^K_i(\theta) \tag{2}$$

[15] The case of $r = 0$ is simply the mirror image of the case with $r = 1$ and has the allocation rule that leaves the resources with agent 2 if and only if $\theta_2 \geq -1/4 + \theta_1$.

for $K \in \{B, S\}$ and $\alpha \in (0, 1)$, which are analogues to the weighted marginal revenue functions used in Ramsey pricing.[16] The larger is α, the closer is the weighted virtual type $\Psi^K_{i,\alpha}(\theta)$ to the actual type θ.

For $r = 1$, the second-best allocation rule gives the resources to agent 2 if and only if $\theta_2 \geq {\Psi^B_{2,\alpha}}^{-1}(\Psi^S_{1,\alpha}(\theta_1))$, where α is such that the no-deficit constraint is satisfied with equality. For uniformly distributed types on $[0, 1]$, as shown by MS, satisfying the no-deficit constraint requires $\alpha = 2/3$, which implies that ${\Psi^B_{2,\alpha}}^{-1}(\Psi^S_{1,\alpha}(\theta_1)) = 1/4 + \theta_1$.

With shared ownership, $r \in (0, 1)$, as in Figure 2(c), the second-best mechanism is based on a comparison of $\Psi^S_{1,\alpha}(\theta_1)$ and $\Psi^S_{2,\alpha}(\theta_2)$ for small values of θ_1 and θ_2, and a comparison of $\Psi^B_{1,\alpha}(\theta_1)$ and $\Psi^B_{2,\alpha}(\theta_2)$ for large values of θ_1 and θ_2. When the distributions are identical, these comparisons are equivalent to comparing θ_1 and θ_2, which is why the allocation rule is ex post efficient for small and large types in Figure 2(c). For intermediate types, the allocation rule gives the resources to agent 2 if and only if $\theta_2 \geq {\Psi^B_{2,\alpha}}^{-1}(\Psi^S_{1,\alpha}(\theta_1))$, just like for $r = 1$, except that the value of α is closer to 1, and hence the distortion away from efficiency is smaller—shared ownership relaxes the no-deficit constraint.

In the remainder of the paper, we provide generalizations of these insights and mechanics to heterogeneous distributions and supports, arbitrary ownership shares and bargaining weights, and to any number of agents. Before turning to that, we briefly discuss implementation of the incomplete information bargaining mechanism using variants of k-double-auctions.

k-double-auctions

So-called k-double-auctions (k-DA), first analyzed by Chatterjee and Samuelson (1983) as bargaining mechanisms between a buyer and seller, have played a prominent role in the literature to date. In a k-DA involving a buyer and a seller, both agents submit bids and, given a bid b_S by the seller and a bid b_B by the buyer, the object changes hands if and only if $b_B \geq b_S$, with the transaction price being $kb_S + (1-k)b_B$ with $k \in [0, 1]$. Thus, a take-it-or-leave-it offer by the seller (buyer) corresponds to a k-DA with $k = 1$ ($k = 0$). As noted above, for $r \in \{0, 1\}$ and $w = 1$, the optimal mechanism can be implemented with agent 1 making a take-it-or-leave-it price offer to agent 2, which is thus a form of a k-DA. For $r = 1$ and uniformly distributed types, MS observe that the 1/2-DA has an equilibrium that implements the second-best mechanism. In this equilibrium, agents 1 and 2 bid $\beta_1(\theta_1) = \frac{1}{4} + \frac{2}{3}\theta_1$ and $\beta_2(\theta_2) = \frac{1}{12} + \frac{2}{3}\theta_2$, respectively.[17]

[16] The way that the weights are defined here differs from how they were used by, say, MS who would write $\Psi^K_{i,a}(\theta) = (1-a)\theta + a\Psi^K_i(\theta)$ for a weight a. That is, α here is $1 - a$ there. The formulation here, which is the same as in Loertscher and Marx (2022), is more convenient for heterogeneous bargaining weights.

[17] For $r = 1$ and uniformly distributed types, Williams (1987) observes that this extends in the sense that for any $w \in [0, 1]$, there is a $k(w) \in [0, 1]$ such that the $k(w)$-DA implements the optimal mechanism.

For shared ownership of $r = 1/2$, CGK show that ex post efficiency is implemented by a 1/2-DA, in which the agent with the higher bid pays a per-unit price of $\frac{b_1+b_2}{2}$ to purchase the quantity of 1/2 owned by other agent, where b_i is agent i's bid. Their results also imply that for $r = 1/2$ and uniformly distributed types each agent i bids in equilibrium according to $\beta_i(\theta) = \frac{1}{6} + \frac{2}{3}\theta$. This implies that $\beta_i(1/2) = 1/2$, i.e., an agent of type $\theta = 1/2 = r = 1-r$ bids its type. This means that when of type 1/2, an agent never regrets trading ex post because the price that it is paid when selling is, with probability 1, bigger than 1/2, and the price that it pays when buying is, with probability 1, less than 1/2. For $\theta > 1/2$, an agent will never regret buying because the price that it pays will always be lower than θ, while for $\theta < 1/2$, it will never regret selling because the price that it receives is larger than θ. But there may be regret, that is, the individual rationality constraint may be violated ex post when selling (buying) for $\theta > 1/2$ ($\theta < 1/2$). However, the interim individual rationality constraint that $u_i(\theta) \geq 0$ is satisfied for all $\theta \in [0, 1]$. It is tightest when $\theta = 1/2$, but $u_i(1/2)$ remains positive.[18] In addition, an agent's interim expected allocation at the worst-off type is equal to 1/2, which is the agent's resource ownership in this example. As mentioned, the fact that agent i's interim expected net payoff is minimized at a type whose interim expected allocation is r_i is a general phenomenon.

To implement optimal mechanisms away from ex post efficiency with shared ownership $r \in (0, 1)$, one can use appropriately adjusted variants of the k-DA. These variants continue to allocate the resources to the high bidder, and at a per-unit price defined by k times agent 1's bid plus $1-k$ times agent 2's bid; however, additional payments not related to the amount traded are required to maintain IC and IR. For additional details, see Online Appendix D.

4 Bilateral bargaining

We now turn to the formal derivation of the optimal mechanisms for the setting with $n = 2$. Agent i's interim expected net payoff from participating in the mechanism when its type is θ and it reports its type truthfully, with "net" meaning net of the outside option $r_i\theta$, is

$$u_i(\theta) \equiv \theta(q_i(\theta) - r_i) - m_i(\theta). \tag{3}$$

[18] To see this, observe first that the expectation of agent j's bid conditional on agent i buying is $\mathbb{E}[\beta_j(\theta_j) \mid \theta_j \leq \theta_i] = \frac{1}{6} + \frac{1}{3}\theta_i$, implying that the expected buy price, conditional on buying, is $\frac{\beta_i(\theta_i)+\mathbb{E}[\beta_j(\theta_j)|\theta_j \leq \theta_i]}{4} = \frac{1}{12} + \frac{1}{4}\theta_i$. Likewise, the expected value of agent j's bid conditional on being higher than agent i's is $\mathbb{E}[\beta_j(\theta_j) \mid \theta_j > \theta_i] = \frac{1}{2} + \frac{1}{3}\theta_i$, implying that the expected sell price, conditional on selling, is $\frac{\beta_i(\theta_i)+\mathbb{E}[\beta_j(\theta_j)|\theta_j > \theta_i]}{4} = \frac{1}{6} + \frac{1}{4}\theta_i$. This means that in equilibrium agent i with type θ has interim expected net payoff (net of its outside option of $\theta/2$) of $u_i(\theta) = \left(\theta - \frac{\beta_i(\theta)+\mathbb{E}[\beta_j(\theta_j)|\theta_j \leq \theta]}{4}\right)\theta + \frac{\beta_i(\theta)+\mathbb{E}[\beta_j(\theta_j)|\theta_j > \theta]}{4}(1-\theta) - \frac{\theta}{2} = \frac{1}{6} + \frac{\theta^2}{2} - \frac{\theta}{2}$. This is minimized at $\theta = 1/2$, at which point it equals 1/24.

As noted in and around footnote 9, IC implies that $q_i(\cdot)$ is nondecreasing. Further, by IC, $u_i(\theta) = \max_{\hat{\theta}\in[\underline{\theta}_i,\overline{\theta}_i]} \theta(q_i(\hat{\theta})-r_i)-m_i(\hat{\theta})$, which by the envelope theorem (see e.g., Milgrom and Segal, 2002) implies that $u_i(\theta)$ is differentiable almost everywhere, satisfying $u_i'(\theta) = q_i(\theta)-r_i$ wherever u_i is differentiable, and for all $\theta, \theta' \in [\underline{\theta}_i, \overline{\theta}_i]$,

$$u_i(\theta) = u_i(\theta') + \int_{\theta'}^{\theta} (q_i(y) - r_i)dy. \tag{4}$$

The relationship in (4) is customarily referred to as *payoff equivalence theorem* because it states that, up to a constant, which in (4) is $u_i(\theta')$, agent i's interim expected (net) payoff is pinned down by the allocation rule. Equating the expression in (4) with the definition of $u_i(\theta)$ in (3) and solving for $m_i(\theta)$ yields

$$m_i(\theta) = \theta(q_i(\theta) - r_i) - u_i(\theta') - \int_{\theta'}^{\theta} (q_i(y) - r_i)dy.$$

Using $\mathbb{E}_{\boldsymbol{\theta}}[M_i(\boldsymbol{\theta})] = \mathbb{E}_{\theta_i}[m_i(\theta_i)] = \int_{\underline{\theta}_i}^{\overline{\theta}_i} m_i(\theta_i)dF_i(\theta_i)$ and changing the order of integration in the resulting double integral yields

$$\mathbb{E}_{\boldsymbol{\theta}}[M_i(\boldsymbol{\theta})] = \mathbb{E}_{\boldsymbol{\theta}}\big[\Psi_i(\theta_i, \theta')Q_i(\boldsymbol{\theta})\big] - \theta' r_i - u_i(\theta'), \tag{5}$$

where $\Psi_i(\theta_i, \theta')$ the *overall* virtual type function with *critical type* x defined as

$$\Psi_i(\theta, x) \equiv \begin{cases} \Psi_i^S(\theta) & \text{if } \theta \in [\underline{\theta}_i, x), \\ \Psi_i^B(\theta) & \text{if } \theta \in [x, \overline{\theta}_i], \end{cases} \tag{6}$$

with $x = \theta'$.[19] Observe that $\mathbb{E}_{\theta}[\Psi_i(\theta, x)] = x$.[20]

Note next that if, given $\mathbf{Q}$, there exists a $\omega \in [\underline{\theta}_i, \overline{\theta}_i]$ such that $q_i(\omega_i) = r_i$, then ω_i is a *worst-off type* of agent i, that is, $\omega_i \in \arg\min_{\theta\in[\underline{\theta}_i,\overline{\theta}_i]} u_i(\theta)$. To see this, recall that $u_i'(\theta) = q_i(\theta)-r_i$. Because q_i is nondecreasing, $u_i'(\omega_i) = 0$ characterizes the global minimum. This means that an agent's worst-off type varies nontrivially with the allocation rule. For example, for $[\underline{\theta}_i, \overline{\theta}_i] = [0, 1]$, $\omega_i = r_i^2$ if $q_i(\theta) = \theta^{1/2}$ and $\omega_i = r^{1/2}$ if $q_i(\theta) = \theta^2$. This nontrivial endogeneity of the worst-off types to the allocation rule constitutes a major complication for the designer's problem because it means that the worst-off types (for whom the IR constraints will bind) depend on the allocation rule that the designer chooses, and so the

[19]See also Lemmas B.1 and B.2 in the Online Appendix for more detailed derivations.

[20]Integrating by parts reveals that $\mathbb{E}[\Psi_i^S(\theta) \mid \theta \leq x] = x$ and $\mathbb{E}[\Psi_i^B(\theta) \mid \theta \geq x] = x$, implying that $\int_{\underline{\theta}_i}^{x} \Psi_i^S(\theta)dF_i(\theta) = F_i(x)x$ and $\int_{x}^{\overline{\theta}_i} \Psi_i^B(\theta)dF_i(\theta) = x(1-F_i(x))$, and therefore $\mathbb{E}[\Psi_i(\theta, x)] = x$ as claimed.

optimal allocation rule will, in turn, depend on the worst-off types.[21] Lemma B.2 in the Online Appendix provides a complete characterization of i's set of worst-off types given $\mathbf{Q}$, denoted $\Omega_i(\mathbf{Q})$, accounting for the possibility that the induced q_i need not intersect with r_i.

As in standard mechanism design problems, even though the worst-off type depends on the allocation rule, it is convenient to express $\mathbb{E}_{\boldsymbol{\theta}}[M_i(\boldsymbol{\theta})]$ relative to a worst-off type of agent i rather than an arbitrary type θ' because this is a type for which the IR constraint will be tightest. Thus, we can write

$$\mathbb{E}_{\boldsymbol{\theta}}[M_i(\boldsymbol{\theta})] = \mathbb{E}_{\boldsymbol{\theta}}\big[\Psi_i(\theta_i, \omega_i) Q_i(\boldsymbol{\theta})\big] - \omega_i r_i - u_i(\omega_i).$$

Given worst-off type $\omega_i \in \Omega_i(\mathbf{Q})$ for agent i, the IR constraint amounts to the requirement that $u_i(\omega_i) \geq 0$, and the no-deficit constraint requires that $\sum_{i\in\mathcal{N}} \mathbb{E}_{\boldsymbol{\theta}}[M_i(\boldsymbol{\theta})] \geq 0$. The associated Lagrangian is then

$$\mathcal{L} \;=\; \sum_{i\in\mathcal{N}} w_i \mathbb{E}_{\boldsymbol{\theta}}[\theta_i Q_i(\boldsymbol{\theta}) - M_i(\boldsymbol{\theta})] + \rho \sum_{i\in\mathcal{N}} \mathbb{E}_{\boldsymbol{\theta}}[M_i(\boldsymbol{\theta})] + \sum_{i\in\mathcal{N}} \mu_i u_i(\omega_i),$$

where ρ is the Lagrange multiplier on the no-deficit constraint and $\mu_i \geq 0$ is the multiplier on agent i's IR constraint. Defining, in analogy to (2), agent i's *weighted virtual type* with weight $\alpha \in [0,1]$ by

$$\Psi_{i,\alpha}(\theta, x) \equiv \alpha\theta_i + (1-\alpha)\Psi_i(\theta, x), \tag{7}$$

the Lagrangian can conveniently be written as

$$\mathcal{L} \;=\; \rho\mathbb{E}_{\boldsymbol{\theta}}\Big[\sum_{i\in\mathcal{N}} (Q_i(\boldsymbol{\theta}) - r_i)\Psi_{i,\frac{w_i}{\rho}}(\theta_i, \omega_i)\Big] + \sum_{i\in\mathcal{N}} (w_i - \rho + \mu_i)\, u_i(\omega_i) + \sum_{i\in\mathcal{N}} r_i w_i \mathbb{E}_{\theta_i}[\theta_i].$$

Observe that if $\rho < \max \mathbf{w}$, then the solution is unbounded because then $\mathcal{L}$ would be maximized by giving an agent with the maximum weight an infinite amount of money. Consequently, any solution satisfies $\rho \geq \mathbf{w}$ (see Online Appendix B.2).

The standard approach in mechanism design problems in which the ω_i do not depend on the allocation rule would be to maximize $\mathbb{E}_{\boldsymbol{\theta}}\big[\sum_{i\in\mathcal{N}}(Q_i(\boldsymbol{\theta}) - r_i)\Psi_{i,\frac{w_i}{\rho}}(\theta_i, \omega_i)\big]$ pointwise over $\mathbf{Q}$. Leaving temporarily aside the problem that ω_i and thus $u_i(\omega_i)$ vary with $\mathbf{Q}$, pointwise maximization would mean allocating the resources to the agent with the highest weighted virtual type $\Psi_{i,\frac{w_i}{\rho}}(\theta, \omega_i)$. However, if agent i has an interior worst-off type $\omega_i \in (\underline{\theta}_i, \overline{\theta}_i)$ and

[21] This is different for a mechanism design problem like an auction setting where by IC alone there is always one type—the lowest—that is worst-off. For example, if the seller's cost is 0 and types are uniformly distributed on $[0,1]$, then the worst-off type is 0 in an efficient auction. In an optimal auction, the set of worst-off types is $[0,1/2]$ because seller's optimal reserve is $1/2$. Evidently, type 0 is still a worst-off type.

the weight in its virtual type is less than 1, i.e., $\frac{w_i}{\rho} \in [0,1)$, then $\Psi_{i,\frac{w_i}{\rho}}(\theta, \omega_i)$ is nonmonotone with a downward discontinuity at ω_i, resulting in a violation of the monotonicity constraint of the allocation rule imposed by IC. Thus, in this case, the solution involves the ironing of agent i's weighted virtual type function as in Myerson (1981). The resources are allocated to the agent with the highest *ironed* weighted virtual type, which for agent i is denoted by $\overline{\Psi}_{i,\frac{w_i}{\rho}}(\theta, \omega_i)$.

The deeper problem, then, is that the interdependence of $\mathbf{Q}$ and $\boldsymbol{\omega} = (\omega_i)_{i \in \mathcal{N}}$ raises the question of whether the standard mechanism design methodology, whereby the objective is, first, maximized over monotone allocation rules and the payment rule is then derived based on the optimal allocation rule and the payoff equivalence theorem, is applicable. Fortunately, the answer is affirmative. As observed by Loertscher and Wasser (2019), the optimal mechanism in a partnership model is characterized by a *saddle point* $(\mathbf{Q}^*, \boldsymbol{\omega}^*)$.[22] Specifically, $\mathbf{Q}^*$ is a monotone allocation rule that maximizes $\mathbb{E}_{\boldsymbol{\theta}}\big[\sum_{i \in \mathcal{N}}(Q_i(\boldsymbol{\theta}) - r_i)\Psi_{i,\frac{w_i}{\rho}}(\theta_i, \omega_i^*)\big]$ and $\boldsymbol{\omega}^*$ is a minimizer of $\mathbb{E}_{\boldsymbol{\theta}}\big[\sum_{i \in \mathcal{N}}(Q_i^*(\boldsymbol{\theta}) - r_i)\Psi_{i,\frac{w_i}{\rho}}(\theta_i, x_i)\big]$ over $\mathbf{x} = (x_i)_{i \in \mathcal{N}}$ with $x_i \in [\underline{\theta}_i, \overline{\theta}_i]$.

To complete the characterization of the bargaining mechanism, we must also satisfy the no-deficit constraint. This is always possible because by choosing $Q_i(\boldsymbol{\theta}) = r_i$ for all i and all $\boldsymbol{\theta}$, the designer obtains revenue of 0. Moreover, in the limit as ρ goes to infinity, the allocation rule approaches that for the mechanism that maximizes the designer's expected revenue. As just observed, the designer's maximized expected revenue must be nonnegative, and it is positive whenever the problem—parameterized by r and $\underline{\theta}$—is such that there is a positive measure of types with mutually beneficial trades. By the continuity and monotonicity of the problem, this then guarantees that there is a smallest value of ρ that satisfies the no-deficit constraint.

We summarize in the following proposition:

Proposition 1. *The incomplete information bilateral bargaining allocation rule assigns the resources to the agent with the maximum ironed weighted virtual type,* $\max_{i \in \mathcal{N}} \overline{\Psi}_{i,\frac{w_i}{\rho}}(\theta_i, \omega_i)$, *with ρ equal to the smallest feasible value such that the no-deficit constraint is satisfied.*

Proof. See Online Appendix B.

To provide intuition for Proposition 1, consider the case with $w = 1$ and $r \in (0,1)$ and assume, first, that player 1 simply makes a take-it-or-leave-it offer p with the interpretation that 2 can choose whether to buy 1's share r, paying rp, or wants to sell its share $1 - r$, in

[22]They assumed equal weights and identical supports, that is, $w_i = w$ and $[\underline{\theta}_i, \overline{\theta}_i] = [0,1]$ for all $i \in \mathcal{N}$. As we show in Online Appendix B.2, these insights extend to heterogeneous weights and supports.

which case 2 receives $(1-r)p$. The optimization problem for 1 is

$$\max_{p\in[\underline{\theta},\theta+1]}(1-r)(\theta_1-p)F_2(p)+r(p-\theta_1)(1-F_2(p)),$$

whose maximizer $p^*(\theta_1,r)$ satisfies

$$\theta_1=(1-r)\Psi_2^S(p^*(\theta_1,r))+r\Psi_2^B(p^*(\theta_1,r)).$$

This is intuitive because it corresponds to the convex combination of the optimal take-it-or-leave-it offers if 1 is a buyer ($r=0$) and if 1 is a seller ($r=1$). Because Ψ_2^S and Ψ_2^B are increasing, the right-hand side is increasing and $p^*(\theta_1,r)$ is well defined. If F_2 is the uniform distribution on $[0,1]$, then $p^*(\theta_1,r)=\frac{\theta_1+r}{2}$.[23] Even though take-it-or-leave-it offers are optimal for $r\in\{0,1\}$, they are, by Proposition 1, not optimal for $r\in(0,1)$. The issue is that these leave too much money on the table.

To see this, assume that F_1 and F_2 are both uniform on $[0,1]$. The expected utility of agent 2's worst-off type, ω_2, given the pricing rule $p^*(\theta_1,r)$, is $(1-r)\omega_2-\int_0^{1-r}p^*(\theta_1,r)d\theta_1+(1-r)\int_{1-r}^1 p^*(\theta_1,r)d\theta_1=(1-r)\omega_2+\frac{1}{4}r(1-r)$ because with probability $1-r$, agent 2 gets to consume the good when its type is ω_2. Because $(1-r)\omega_2$ is the value of agent 2's outside option when its type is ω_2, it follows that, given the pricing rule $p^*(\theta_1,r)$, agent 2 can be charged the tax $\frac{1}{4}r(1-r)$ without violating its IC and IR constraints. Naturally, if the price is lower than $p^*(\theta,r)$ when $\theta_1<1-r$ and larger than that for $\theta_1>1-r$, then agent 2's utility when of type ω_2 increases, which means that the tax can be increased. Supposing that agent 1 sets the price $p^*(\theta_1,r-\Delta_S)$ for $\theta_1<1-r$ and $p^*(\theta_1,r+\Delta_B)$ for $\theta_1>1-r$, where $\Delta_S,\Delta_B\geq 0$, the tax becomes $T(\Delta_S,\Delta_B)=\frac{1}{4}r(1-r)(1+2(\Delta_S+\Delta_B))$, which is increasing in Δ_S and Δ_B. So the question is how large Δ_S and Δ_B should be. When agent 1's type is $\theta_1<1-r$, maximizing agent 1's payoff

$$\begin{aligned}&(1-r)(\theta_1-p^*(\theta_1,r-\Delta_S))F_2(p^*(\theta_1,r-\Delta_S))\\+\;&r(p^*(\theta_1,r-\Delta_S)-\theta_1)(1-F_2(p^*(\theta_1,r-\Delta_S)))+T(\Delta_S,\Delta_B)\end{aligned}\tag{8}$$

over Δ_S yields the maximizer $\Delta_S=r(1-r)$, while maximizing

$$\begin{aligned}&(1-r)(\theta_1-p^*(\theta_1,r+\Delta_B))F_2(p^*(\theta_1,r+\Delta_B))\\+\;&r(p^*(\theta_1,r+\Delta_B)-\theta_1)(1-F_2(p^*(\theta_1,r+\Delta_B)))+T(\Delta_S,\Delta_B)\end{aligned}\tag{9}$$

[23]It is interesting to note that $p^*(\theta_1,r)$ would be the optimal price set by agent 1 in a *Texas shootout*, which is a mechanism for dissolving a partnership, whereby "one owner names a price and the other owner is compelled to either purchase the first owner's shares or sell his own shares at the named price" (Brooks et al., 2010, p. 649).

over Δ_B yields the maximizer $\Delta_B = r(1-r)$. The resulting pricing rule does not correspond to the optimal allocation rule.[24] However, the maximizers of the ex ante expected utility of 1 over Δ_S and Δ_B are indeed $\Delta_S^* = r$ and $\Delta_B^* = 1-r$, resulting in the optimal allocation rule. Observe also that $T(\Delta_S^*, \Delta_B^*) = \frac{3}{4}r(1-r)$, which is the same as what was derived in Section 3.

(a) $p^*(\theta_1, r)$ (b) $p^*(\theta_1, r-\Delta_S), p^*(\theta_1, r+\Delta_B)$ (c) $\Delta_S^* = r$ and $\Delta_B^* = 1-r$

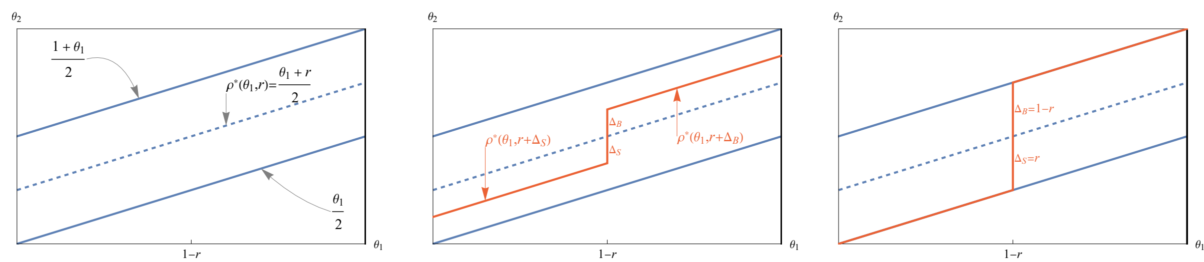


Figure 3: By choosing $\Delta_S > 0$ and $\Delta_B > 0$, the utility of the worst-off type and therefore the tax that can be extracted increases. The optimal choices are $\Delta_S^* = r$ and $\Delta_B^* = 1-r$.

With the general description of the incomplete information bargaining mechanism in hand, we can now explore how bargaining outcomes depend on w, $\underline{\theta}$, and r. We begin with the analysis of when bargaining is ex post efficient. From Myerson and Satterthwaite (1983), we know that ex post efficiency is not possible for $w = 1/2$ and $r \in \{0, 1\}$ provided that $\underline{\theta} < 1$. That is, there is no mechanism with the ex post efficient allocation that satisfies IC and IR and that does not run deficit.

4.1 Ex post efficiency

As mentioned, whether bargaining is efficient is a core question for economics and a central, sometimes controversial, issue in the literature on bargaining. There is thus ample motivation to begin the analysis of incomplete information bargaining with the question of whether or when incomplete information can be ex post efficient.

The partnership literature has, in large part, focused on the question of identifying for which ownership structures, if any, ex post efficiency is possible. For example, CGK show that with identical distributions, the set of ex post efficiency permitting ownership structures

[24]The issue is that maximizing the expected payoff for a given type θ_1 below (above) $1-r$ neglects the externality that further price reductions (increases) have on all other types. Maximizing ex ante expected utility accounts for these externalities and results in the optimal allocation rule. Put differently, if every type $\theta_1 < 1-r$ recognizes that all other types less than $1-r$ decrease their prices by Δ_S, this is as if the impact on the tax increased to $\Delta_S/(1-r)$, and analogously for types larger than $1-r$, their impact increases to Δ_B/r. Indeed, if in (8) and (9) the expression $T(\Delta_S, \Delta_B)$ is replaced by $T(\Delta_S/(1-r), \Delta_B/r)$, then the respective maximizers become $\Delta_S^* = r$ and $\Delta_B^* = 1-r$.

is nonempty, convex, and symmetric. As mentioned, for the uniform distribution on $[0,1]$ and $n = 2$, ex post efficiency is possible for all $r \in (0.21, 0.79)$. This raises the question of whether a similar property obtains for bargaining weights. Given that ours is the first paper to analyze incomplete information bargaining in a partnership model with different bargaining weights, this is an open question.[25] As we show next, the answer to the question of whether bargaining weights have the same or similar properties as property rights is quite generally no. In particular, we will see that for $\underline{\theta} \leq 1$, $w = 1/2$ is necessary for ex post efficiency, whereas typically a large set of r permits ex post efficiency.

Let $\mathcal{E}(\underline{\theta}) \equiv \{(r, w) \in [0,1]^2 \mid \text{ex post efficiency is possible}\}$ denote the set of ownership structures and bargaining weights such that ex post efficiency is possible, which in principle could be empty. Let

$$\mathcal{W}^e(\underline{\theta}) \equiv \{w \mid \exists r \in [0,1] \text{ s.t. ex post efficiency is possible}\}$$

and

$$\mathcal{R}^e(\underline{\theta}) \equiv \{r \mid \exists w \in [0,1] \text{ s.t. ex post efficiency is possible}\}$$

denote the (also possibly empty) sets of ex post permitting bargaining weights and ownership structures under which ex post efficiency is possible. By construction, $\mathcal{E}(\underline{\theta}) = \mathcal{W}^e(\underline{\theta}) \cup \mathcal{R}^e(\underline{\theta})$. We also use $\mathcal{W}^e_0(\underline{\theta}) \equiv \{w \mid \exists r \in (0,1] \text{ s.t. ex post efficiency is possible}\}$, which is the set of ex post efficiency permitting bargaining weights when agent 1 has a positive ownership share, that is, $r > 0$. This set is relevant when $\underline{\theta} > 1$ because then the initial allocation is already ex post efficient if $r = 0$.

The threshold value

$$\underline{\theta}^* \equiv \max\{\Psi_1^S(1), 1 - \Psi_2^{B,P}(0)\}$$

for $\underline{\theta}$ delineates the cases in which w has effects on the allocative efficiency of bargaining and when it does not. Notice that $\underline{\theta}^* > 1$ because $\Psi_1^S(1) > 1$ and $\Psi_2^{B,P}(0) < 0$. Observe also that $\Psi_1^S(1)$ is agent 1's highest possible virtual cost, while the lowest virtual value of agent 2 is $\Psi_2^B(\underline{\theta})$. Because for $\theta \in [\underline{\theta}, \underline{\theta}+1]$, we have $\Psi_2^B(\theta) = \Psi_2^{B,P}(\theta - \underline{\theta}) + \underline{\theta}$, it follows that $\Psi_2^B(\underline{\theta}) = \Psi_2^{B,P}(0) + \underline{\theta}$ holds. Consequently, for $\underline{\theta} \geq \Psi_1^S(1)$, agent 2 will always buy all r shares that agent 1 owns if agent 2 has all the bargaining power, and if $\underline{\theta} \geq 1 - \Psi_2^{B,P}(0)$, then agent 1 will always sell all of r if agent 1 has all the bargaining power. It is then not hard to see

[25] As mentioned, Mylovanov and Tröger (2014) derive the mechanism that is optimal for one agent in a partnership model with interior ownership, that is, in our notation for $w = 1$ and $r \in (0,1)$. But as they study informed-principal problems, they do not vary w. Rent extraction in partnership and asset market models by the designer has been studied by Loertscher and Wasser (2019), Lu and Robert (2001), and Loertscher and Marx (2023), but that problem is distinct from that of varying the bargaining weights among the agents and investigating when ex post efficiency is possible.

that for $\underline{\theta} \geq \underline{\theta}^*$, the allocation will be ex post efficient for all $r \in [0,1]$ and all $w \in [0,1]$. For example, if F_1 and F_2^P are both uniform, then we have $\Psi_1^S(1) = 2 = 1 - \Psi_2^{B,P}(0)$, implying that $\underline{\theta}^* = 2$. In general, however, there is no reason why $\Psi_1^S(1)$ should be the same as $1 - \Psi_2^{B,P}(0)$ even if $F_1 = F_2^P$.

Proposition 2. *For any $\underline{\theta} \geq 0$, the sets $\mathcal{W}^e(\underline{\theta})$ and $\mathcal{R}^e(\underline{\theta})$ are nonempty and convex. Moreover,*

(i) $\mathcal{W}^e(\underline{\theta}) = \{1/2\}$ for $\underline{\theta} \leq 1$;

(ii) $\mathcal{W}_0^e(\underline{\theta}) \subset [0,1]$ for $\underline{\theta} \in (1, \underline{\theta}^*)$ and $\mathcal{W}_0^e(\underline{\theta}) \subset \mathcal{W}_0^e(\underline{\theta}')$ for $\underline{\theta}' > \underline{\theta}$;

(iii) $\mathcal{W}^e(\underline{\theta}) = [0,1]$ for $\underline{\theta} \geq \underline{\theta}^*$;

(iv) $\mathcal{R}^e(\underline{\theta}) \subset (0,1)$ for $\underline{\theta} \in [0,1)$ and $\lim_{\underline{\theta}\uparrow 1} \mathcal{R}^e(\underline{\theta}) = \{0\}$;

(v) $\mathcal{R}^e(\underline{\theta}) = [0,1]$ for $\underline{\theta} \geq 1$.

Proof. See Appendix A.

Figure 4 illustrates Proposition 2. The left panel depicts $\mathcal{R}^e(\underline{\theta})$ and the right panel depicts $\mathcal{W}_0^e(\underline{\theta})$ as functions of $\underline{\theta}$ for the case in which F_1 and F_2^P are uniform.[26] If $\Psi_1^S(1) \neq 1 - \Psi_2^{B,P}(0)$ were the case, then 0 (or 1) would be elements of $\mathcal{W}_0^e(\underline{\theta})$ for some $\underline{\theta} < \underline{\theta}^*$, while 1 (0) would be in $\mathcal{W}_0^e(\underline{\theta})$ only for $\underline{\theta} \geq \underline{\theta}^*$.

(a) Ex post efficiency permitting set $\mathcal{R}^e(\underline{\theta})$ (b) Ex post efficiency permitting set $\mathcal{W}_0^e(\underline{\theta})$

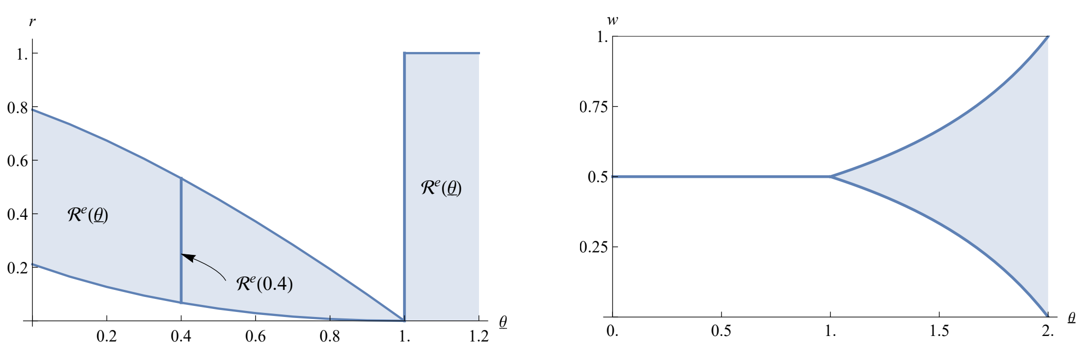


Figure 4: Ex post efficiency permitting ownership structures and bargaining weights. Assumes uniformly distributed types for agent 1 on $[0,1]$ and for agent 2 on $[\underline{\theta}, 1+\underline{\theta}]$.

Proposition 2 implies that for $\underline{\theta} \geq \underline{\theta}^*$, incomplete information bargaining is ex post efficient for all $(r,w) \in [0,1]^2$. Thus, $\underline{\theta} \geq \underline{\theta}^*$ corresponds to cases in which the predictions of incomplete information bargaining coincide with properties typically imposed in bargaining with complete information. Productivity differentials of the size $\underline{\theta}^*$ or larger, therefore,

[26] Related to panel (b), as we show in Proposition 7, for $\underline{\theta} \geq 1$ and $r > 0$, ex post efficiency is possible if and only if $1 + \left(1 - \frac{w}{\max\{w,1-w\}}\right)\frac{1}{f_1(1)} \leq \underline{\theta} - \left(1 - \frac{1-w}{\max\{w,1-w\}}\right)\frac{1}{f_2(1)}$, which, for the uniform distribution, amounts to $\frac{2-\underline{\theta}}{3-\underline{\theta}} \leq w \leq \frac{1}{3-\underline{\theta}}$ for $\underline{\theta} \in [1,2]$.

capture situations in which there is, loosely speaking, "little" private information insofar as private information is no impediment to efficient bargaining. It also corresponds to situations in which neither the assignment of property rights nor bargaining power affects whether bargaining is efficient. That is, for $\underline{\theta} \geq \underline{\theta}^*$, there are no countervailing power effects, contrasting with what Galbraith (1952) stipulated. Moreover, for all $\underline{\theta} > 1$, the assignment of property rights is irrelevant, which is in line with the Coase Theorem. A qualification regarding the equivalence of complete and incomplete information bargaining for $\underline{\theta} \geq \underline{\theta}^*$ applies when, at an ex ante stage, the agents make noncontractible investments that improve their type distributions. With incomplete information, efficient bargaining implies efficient investment whereas with complete information, hold-up from bargaining induces inefficient investments as in the theory of the firm in the tradition of Grossman and Hart (1986) and Hart and Moore (1990).[27]

In contrast, for $\underline{\theta} < \underline{\theta}^*$, sufficiently equal bargaining weights are necessary for ex post efficiency and, indeed, for $\underline{\theta} \leq 1$, ex post efficiency is not possible without equal bargaining weights. In contrast, the ownership structure is immaterial for ex post efficiency for $\underline{\theta} > 1$, and the ex post permitting set of ownership structures is multi-valued for $\underline{\theta} < 1$. Thus, specifications with $\underline{\theta} < \underline{\theta}^*$ provide a formalization of Galbraith's hypothesis that equalization of bargaining power may be a first-order issue. For $\underline{\theta} < \underline{\theta}^*$, bargaining power matters for the size as well as for the distribution of surplus.[28] The notion of *countervailing power*, introduced by Galbraith (1952), has widespread appeal but has been difficult to conceptualize in models with complete information without restricting the contracting space.[29] In a labor market context, equalization of bargaining weights between agents and workers may be achieved by allowing the workers to form unions. In healthcare, doctors may increase their bargaining power vis-à-vis insurance companies by giving up their independence and becoming employees of large hospital chains.[30]

[27]For formalizations of this point, see, for example, Milgrom (2004), Krähmer and Strausz (2007), and Liu et al. (forth.).

[28]This generalizes to a setup with interior ownership the insight from Loertscher and Marx (2022) that the incomplete information framework has the property that bargaining weights do not only affect the distribution but also the size of expected surplus.

[29]It features prominently in antitrust practice. For example, OECD (2011, pp. 50–51) and OECD (2007, pp. 58–59) raise the possibility of a role for collective negotiation and group boycotts in counterbalancing the market power of providers of payment card services. In other examples, the U.S. DOJ and FTC recognize the potential benefits from allowing physician network joint ventures in their 1996 "Statement of Antitrust Enforcement Policy in Health Care." Krueger (2018) discusses the benefits to workers of market features that boost worker bargaining power and counterbalance monopsony power. As another case in point, the Australian competition authority "has identified a range of market failures resulting from ... strong bargaining power imbalance and information asymmetry ... which ultimately cause inefficiencies" (ACCC Dairy Inquiry, 2018, p. xii).

[30]See, for example, "Doctors Say Dealing With Health Insurers Is Only Getting Worse," *Wall Street Journal*, December 12, 2024.

The intuition for why $w = 1/2$ is necessary for ex post efficiency if $\underline{\theta} < 1$ —part (i) of Proposition 2—is simple. Away from equal weights, the incomplete information bargaining mechanism discriminates against the agent with the smaller weight because, by Proposition 1, the allocation prioritizes agents on the basis of the weighted (ironed) virtual types. For $\underline{\theta} \leq 1$, with unequal bargaining weights, this prioritization differs from prioritizing agents on the basis of their true types. As the gap between the supports $[1, \underline{\theta}]$ with $\underline{\theta} > 1$ increases, unequal bargaining weights lead to less and eventually to no discrimination in the allocation rule, which is the intuition for parts (ii) and (iii) of Proposition 2. That $\mathcal{R}^e(\underline{\theta})$ is a nonempty, convex subset of $(0, 1)$ for $\underline{\theta} \in [0, 1)$ follows from the fact that if r is such that, under ex post efficiency, both agents have the same worst-off types, then revenue under ex post efficiency subject to IR is maximized and positive.[31] By continuity of this revenue function, ex post efficiency is then also possible for a convex set of ownership structures around the revenue-maximizing one. As $\underline{\theta}$ approaches 1 from below, the only way that both agents can have the same worst-off type is that their worst-off types are equal to 1, which requires $r = 0$ (in which case revenue under ex post efficiency is simply 0). This explains (iv). Part (v) follows because when $\underline{\theta} \geq 1$, ex post efficiency can easily be achieved, for example, with a posted-price mechanism with a price between 1 and $\underline{\theta}$.

4.2 Bargaining payoffs

We now look at bargaining more generally, that is, without restricting attention to ex post efficiency, by studying how the agents' expected net payoffs depend on the productivity differential, the ownership structure, and the bargaining weights. Given r, w, and $\underline{\theta}$, the allocation rules of the optimal mechanisms are as given by Proposition 1. These are nuanced variations of the allocation rules depicted in Figures 1 and 2.

Figure 5 illustrates the effects of varying $\underline{\theta}$ on the optimal allocation rules for $w = 1/2$. Increasing $\underline{\theta}$ reduces the allocative distortions arising from the exertion of bargaining power. As an illustration, assume F_2^P is the uniform distribution and $w = 1$, i.e., agent 1 has all the bargaining power. Then given $\underline{\theta} \geq 0$, we have $\Psi_2^B(\theta_2) = 2\theta_2 - (1 + \underline{\theta})$ and $\Psi_2^S(\theta_2) = 2\theta_2 - \underline{\theta}$, implying that $\Psi_2^{B^{-1}}(x) = \frac{x+\underline{\theta}+1}{2}$ for $x \in [\underline{\theta} - 1, \underline{\theta} + 1]$ and $\Psi_2^{S^{-1}}(x) = \frac{x+\underline{\theta}}{2}$ for $x \in [\underline{\theta}, 2 + \underline{\theta}]$. Because the derivative of these functions with respect to $\underline{\theta}$ is less than 1, the probability that there is trade under the mechanism that is optimal for agent 1 increases for any given θ_1 and any $r \in [0, 1]$. Conversely, and even simpler, the probability that there is trade for

[31] This insight is the driving force for why, in CGK (who assume identical distributions), the set of ex post efficiency permitting ownership structures is symmetric around equal ownership. Generalizations of this insight to asymmetric distributions with identical supports were obtained by Che (2006) and Figueroa and Skreta (2012). The proof of Proposition 1 shows that it extends to different supports.

any given θ_1 also increases in $\underline{\theta}$ when $w = 0$ because $\Psi_1^B(\theta_1)$ and $\Psi_1^S(\theta_1)$ are independent of $\underline{\theta}$ and the probability that θ_2 exceeds $\Psi_1^B(\theta_1)$ and $\Psi_1^S(\theta_1)$ increases in $\underline{\theta}$.

Away from extremal bargaining weights, increasing $\underline{\theta}$ has the additional, beneficial effect of making the budget constraint less tight. For example, for $w = 1/2$, $r = 1$, and $\underline{\theta} = 0$, the second-best mechanism has $Q_2 = 1$ if and only if $\theta_2 \geq \underline{\theta} + \theta_1 + 1/4$, while for $\underline{\theta} = 1/4$, the second-best mechanism has $Q_2 = 1$ if and only if $\theta_2 \geq \underline{\theta} + \theta_1 + 1/16$.[32] Thus, the strengthening of agent's 2's distribution increases the range of values for agent 2 such that agent 2 is allocated the resource for any given type of agent 1. Figure 5 illustrates that these comparative statics effects of $\underline{\theta}$ on the second-best allocation rule extend to $r \in (0, 1)$.

(a) $\underline{\theta} = 0$ (b) $\underline{\theta} = 0.25$

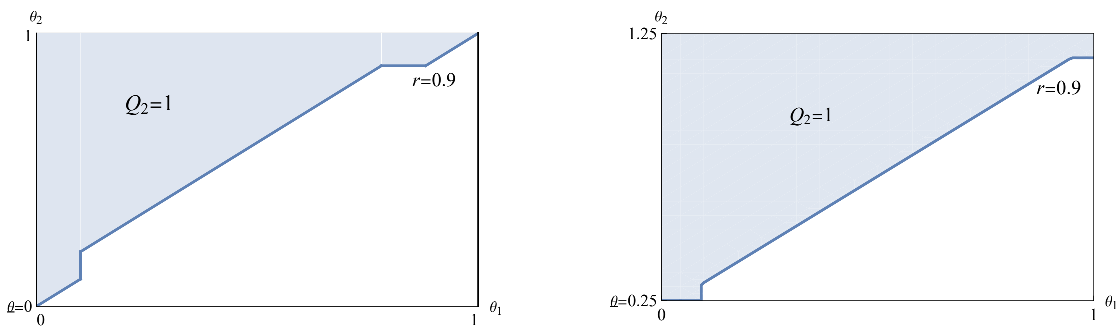


Figure 5: Allocation rule with equal bargaining weights. Assumes uniformly distributed types.

Let $U_i(r, w) \equiv \mathbb{E}_{\theta_i}[u_i(\theta_i)]$ denote agent i's expected net payoff given r and w. The expected net payoffs are the natural objects of interest because they allow us to disentangle the effects of, say, changing r on the performance of the bargaining mechanism from the direct, automatic effects that changes of r have on the agents' utilities via the value of their outside options. Denote by $U_{ir}(r, w)$ and $U_{iw}(r, w)$ the derivatives of U_i with respect to r and w, respectively.

Proposition 3. *For $i, j \in \mathcal{N}$ with $i \neq j$, we have:*

(i) $U_{ir}(r, 1/2) = U_{jr}(r, 1/2)$;

(ii) $U_{ir}(r, 1) = -U_{jr}(r, 0)$; and

(iii) $U_{iw}(r, w) = -U_{jw}(r, w) > 0$ for all w if $r \notin \mathcal{R}^e(\underline{\theta})$ and for $w \neq 1/2$ if $r \in \mathcal{R}^e(\underline{\theta})$.

Moreover, we have:

(iv) $U_{1r}(r, 1), U_{2r}(r, 0) > 0$ for r sufficiently close to 0 and $F_1 = F_2$;

(v) $U_{1r}(r, 1), U_{2r}(r, 0) < 0$ for r sufficiently close to 1 and $F_1 = F_2$;

(vi) $U_{1r}(r, 0), U_{2r}(r, 1) > 0$ for r sufficiently close to 1; and

[32] In the incomplete information bargaining mechanism in this case, $\rho = 1.5$, $\omega_1 = 1$, and $\omega_2 = 0.25$.

(vii) $U_{1r}(r,0), U_{2r}(r,1) < 0$ for r sufficiently close to 0.

Proof. See Appendix A.

In part (iii) of Proposition 3, the derivatives are 0 if r permits ex post efficiency and $w \neq 1/2$. (At $w = 1/2$, the functions $U_i(r,w)$ are not differentiable in w.) Parts (iv) and (v) reflect that an agent with all the bargaining weight prefers nonextremal ownership, assuming sufficiently symmetric distributions. As indicated in parts (vi) and (vii), the expected net payoff of the agent with no bargaining weight moves in the opposite direction.

The results of Proposition 3 can be illustrated by examining the *frontiers* for the agents' expected net payoffs. The frontier for a given r is defined by the maximum expected net payoffs that can be achieved for that r and some $w \in [0,1]$. With overlapping supports, the frontier point for a given (r,w) is uniquely defined by $(U_1(r,w), U_2(r,w))$, and each bargaining weight w is associated with a unique point on the frontier for a given r, as is illustrated in Figure 6 for the case of uniformly distributed types. As Figure 6 shows, a larger value of agent 1's bargaining weight results in a larger expected net payoff for agent 1 and a smaller expected net payoff for agent 2. But, importantly, the figure also shows the efficiency loss associated with unequal bargaining weights: the farther is w from 1/2, the smaller is $\sum_{i\in\mathcal{N}} U_i(r,w)$.

(a) $\underline{\theta} = 0$ (b) $\underline{\theta} = 0.25$

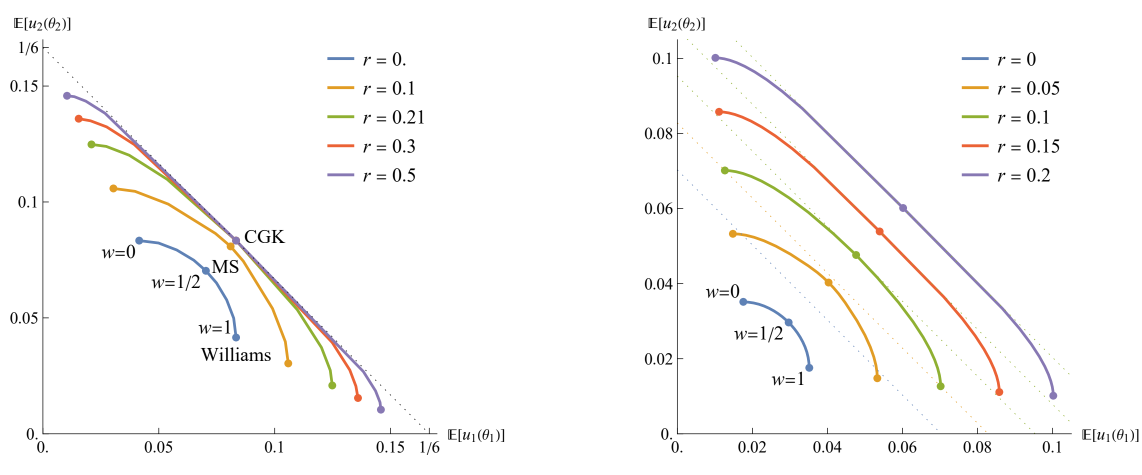


Figure 6: Frontiers for expected net payoffs. Assumes that agent 1's types are uniformly distributed on $[0,1]$ and that agent 2's types are uniformly distributed on $[\underline{\theta}, 1+\underline{\theta}]$, with $\underline{\theta}$ as indicated above each panel. Negatively sloped diagonals reflect expected net payoffs under ex post efficiency, $\mathbb{E}_{\boldsymbol{\theta}}[u_1^e(\theta_1) + u_2^e(\theta_2)]$, which depends on r in the case of heterogeneous distributions.

Figure 6(a) displays the case of identical supports. The case of $r = 0$ and $w = 1/2$, i.e., one seller and one buyer with equal bargaining weights, corresponds to the payoffs associated with the second-best mechanism derived Myerson and Satterthwaite (1983), which is labeled with “MS” in the figure. Varying w from 0 to 1 while keeping $r = 0$ maps out the frontier for extremal ownership, which was derived by Williams (1987) and is thus labeled “Williams”. Reflecting the impossibility of ex post efficiency with extremal ownership, the entire Williams frontier lies below the ex post efficient frontier, which in Figure 6(a) is given by the line with slope -1 connecting the points $(1/6, 0)$ and $(0, 1/6)$.

Once r increases to 0.21, ex post efficiency becomes possible with equal bargaining weights (labeled “CGK” in the figure). This corresponds to the range $[0.21, 0.79]$ of initial ownership shares for which efficient partnership dissolution is possible in Cramton et al. (1987) when there are two partners with uniformly distributed types on $[0, 1]$. As r increases to 0.5 (by symmetry we need only consider $r \in [0, 0.5]$), the payoff frontier continues to move closer to the ex post efficient frontier, but still only actually touches the frontier for $w = 1/2$. Without the incomplete information bargaining mechanism from Proposition 1, only the Williams frontier and the points with $w = 1/2$ and $r \in [0.21, 0.79]$ were known.

In Figure 6(b), the supports of the agents’ type distributions are only partially overlapping, with $\underline{\theta} = 0.25$. In that case, the sum of the expected net payoffs (or, equivalently, the expected gains from trade) under ex post efficiency depends on the ownership structure because the total expected net payoff varies with r when the means of the agents’ type distributions differ. This explains the presence of three different ex post efficiency dashed lines in the figure. As the figure shows, ex post efficiency is possible when bargaining weights are equal for r sufficiently close to $1/2$, but ex post efficiency is not possible, even with equal bargaining weights, for $r = 0$ and $r = 0.05$.

A difference arises with nonoverlapping supports because it is then no longer the case that each bargaining weight w is associated with a unique point on the frontier for a given r. When ex post efficiency is possible for bargaining weights other than $w = 1/2$, then the ex post efficient portion of the frontier is defined by two points corresponding to the ex post efficient expected net payoffs for $w < 1/2$ and those for $w > 1/2$, as well as the line segment in between, which represents the expected net payoffs that can be achieved when $w = 1/2$. A range of possible expected net payoffs is possible when $w = 1/2$, corresponding to the different possible allocations between the two agents of the expected budget surplus under ex post efficiency when IR binds for the agents’ worst-off types, denoted by $\Pi^e(\underline{\theta}, r)$. Recall that we denote by $\eta \in [0, 1]$ the share of $\Pi^e(\underline{\theta}, r)$ obtained by agent 1, or, alternatively, one can view η as the probability that $\Pi^e(\underline{\theta}, r)$ is allocated to agent 1 and $1-\eta$ as the probability that it is allocated to agent 2. Frontiers for nonoverlapping supports are illustrated in Figure 7 for uniformly distributed types. As shown in Figure 7(a), for $\underline{\theta} = 1.5$, ex post efficiency is possible for $w \in [1/3, 2/3]$, but not for more extreme values of w, and as shown in Figure 7(b), for $\underline{\theta} = 2$, ex post efficiency is achieved for all $w \in [0, 1]$, in line with Proposition 2.

(a) $\underline{\theta} = 1.5$ (b) $\underline{\theta} = 2$

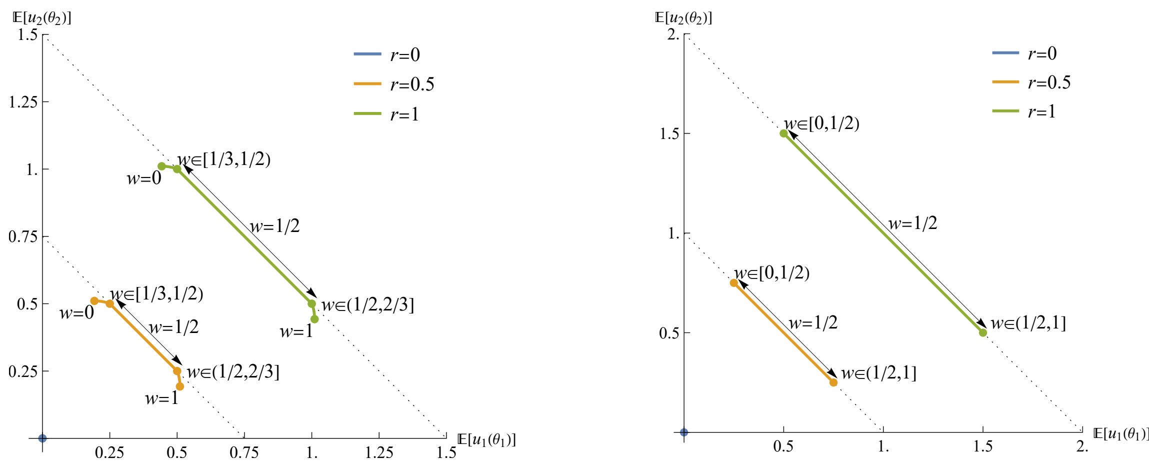


Figure 7: Frontiers for expected net payoffs. Assumes that agent 1's types are uniformly distributed on $[0, 1]$ and that agent 2's types are uniformly distributed on $[\underline{\theta}, 1 + \underline{\theta}]$, with $\underline{\theta}$ as indicated above each panel. Negatively sloped diagonals reflect expected net payoffs under ex post efficiency, $\mathbb{E}[u_1^e(\theta_1) + u_2^e(\theta_2)]$. When $w = 1/2$, a range of outcomes are possible, as parameterized by $\eta \in [0, 1]$.

While our figures assume uniformly distributed types, the result that the expected net payoff frontiers are concave holds generally, as shown in the following proposition:

Proposition 4. *For any ownership r, the frontier of expected net payoffs as w varies over $[0, 1]$ is concave to the origin; away from ex post efficiency, the slope of the frontier is $-w/(1-w)$; at ex post efficiency, it is -1.*

Proof. See Appendix A.

Proposition 4 generalizes Loertscher and Marx (2022, Prop. 4) to a partnership setup. It follows from Proposition 4 that movement toward the equalization of bargaining weights along the expected net payoff frontier weakly increases social surplus. And, from Proposition 2, for $\underline{\theta} \in [0, 1)$, ex post efficiency is only achieved for full equalization of the bargaining weights.

Nonlinear effects of the outside option

In Nash bargaining, an agent's payoff is linear in its outside option—with outside options o_i for agents $i = \{1, 2\}$ and total surplus to divide of S, agent i's Nash bargaining payoff, net of its outside option, is $\frac{1}{2}(S - o_1 - o_2)$. In contrast, with incomplete information bargaining, this relation is no longer linear even though the expected value of agent i's outside option, $r_i \mathbb{E}_{\theta_i}[\theta_i]$, is linear in r_i. An agent's outside option affects the agent's worst-off type, which enters both into the IR constraint and into the determination of the second-best allocation rule. These effects render the relation nonlinear.

As an illustration, Figure 8(a) shows the frontiers for agents' expected net payoffs for different resource ownership, where the frontiers are traced out by varying agent 1's bargaining weight from zero to one. As shown in that figure, agents prefer to have higher bargaining power rather than lower bargaining power, all else equal. We can also trace out an agents' expected net payoff for a given bargaining weight varying the resource ownership, as shown in Figure 8, which highlights that the effects of changes in an agent's outside option vary with the bargaining weights. Figure 8(a) shows the frontiers of expected net payoffs for given bargaining weights as ownership varies, and Figure 8(b) shows agent 1's expected net payoff for given bargaining weights as its ownership varies.

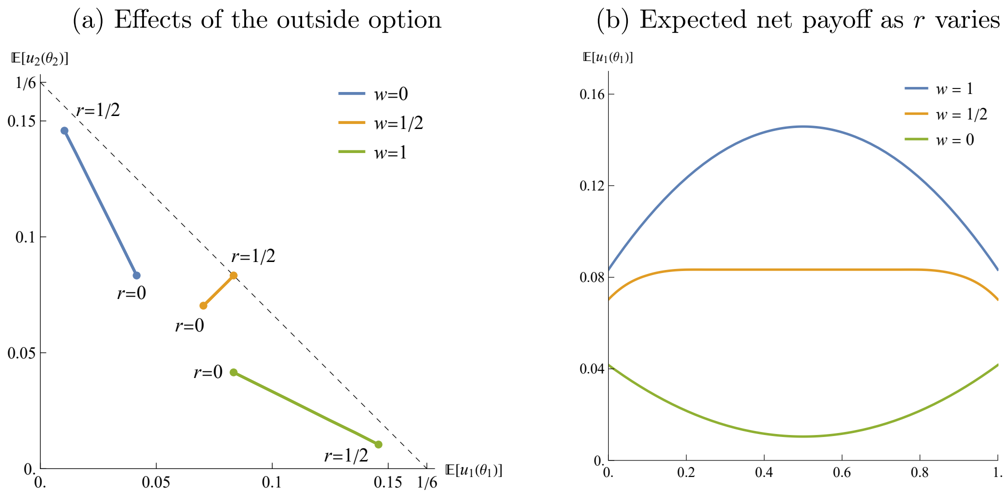


Figure 8: Frontiers for expected net payoffs (panel a) and agent 1's expected net payoff (panel b) as r varies for given w. The negatively sloped diagonal in panel (a) is the ex post efficient frontier. For $w = 1/2$ and $r \in [0.21, 0.79]$, ex post efficiency is achieved. Assumes that agents' types are uniformly distributed on $[0, 1]$.

4.3 Agents' preferences

For $\theta \geq \underline{\theta}^*$, neither bargaining weights nor ownership shares affect the efficiency of bargaining because, as with complete information, bargaining is always efficient. This raises the question of what effects bargaining weights and ownership have on agents' payoffs. For example, if $\underline{\theta} \geq \underline{\theta}^*$, are the two instruments perfect substitutes? To address these and related questions, we now investigate what preferences the agents have over r and w. To this end, define the agents' expected payoffs (including their outside options) as $V_1(r, w) \equiv U_1(r, w) + r\mathbb{E}_{\theta_1}[\theta_1]$ and $V_2(r, w) \equiv U_2(r, w) + (1 - r)\mathbb{E}_{\theta_2}[\theta_2]$.

Given (r, w), in some cases the agents would, prior to types being realized, benefit if they could commit to adjustments to r and/or w. Specifically, we can define the "better than set" for agent i, $\mathcal{B}_i(r, w) \equiv \{(r', w') \mid V_i(r', w') > V_i(r, w)\}$. Then, given (r, w), there is scope for mutually beneficial adjustments to ownership and bargaining weights if $\mathcal{B}_1(r, w)$ and $\mathcal{B}_2(r, w)$ have a nonempty intersection.

As illustrated in Figure 9(a), For $\underline{\theta} \geq \underline{\theta}^*$, there is no scope for mutually beneficial adjustments because the outcome is always ex post efficient and so any changes are zero sum, simply transferring payoffs between the agents—as shown in the figure, agent 1 generally benefits from higher r and higher w (agent 1's better-than set extends to the upper-right corner), while the opposite is true for agent 2 (agent 2's better-than set extends to the lower-left corner), and there is a jump in an agent's payoff as its bargaining weight exceeds that of the other agent. However, for cases in which ex post efficiency is not achieved, mutually beneficial adjustments are possible. For example, for $\underline{\theta} = 1.5$, if we take as the initial point $(r, w) = (0.5, 0)$, then there is overlap in the better-than sets, as shown in Figure 9(b). But, given the inefficiency of the outcome associated with the initial point, there are social surplus benefits to having more equal bargaining weights. Thus, there are bargaining weights in $(0, 1)$ and associated ownership close to $r = 0.5$ such that both agents are better off.

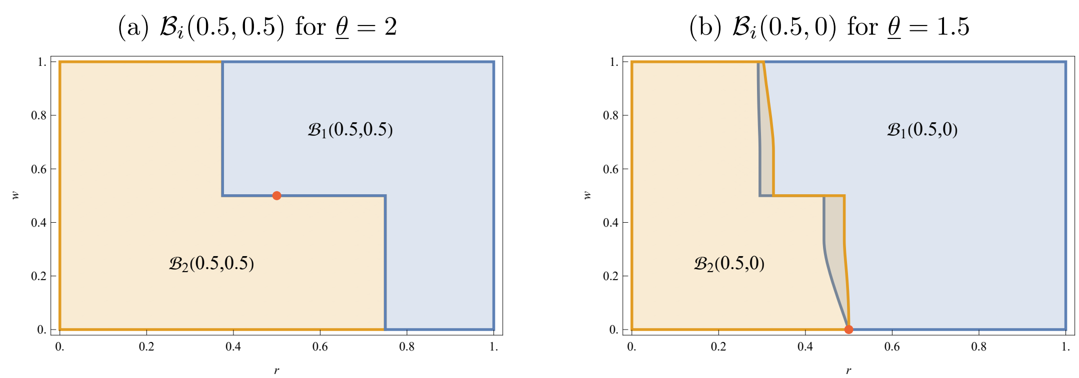


Figure 9: Better than sets $\mathcal{B}_i(r, w)$ for expected payoffs. Assumes uniformly distributed types. The initial point is indicated with a red dot.

For analogous reasons, there is no scope for mutually beneficial adjustments if $w = 0.5$ and r is such that ex post efficiency is possible with equal bargaining weights, but otherwise, there can be scope for mutually beneficial adjustments.

5 Extensions

We now analyze two extensions. We first replace the assumption of constant marginal values by that of decreasing marginal values in the form of quadratic utility, which allows us to capture cases in which trade is not necessarily zero-one and tariffs are nonlinear. Then we extend the model to multiple agents and analyze ownership structures and bargaining weights that permit ex post efficiency while assuming, again, constant marginal values.

5.1 Decreasing marginal values

The model with constant marginal values has the bang-bang property that, with probability 1, the optimal allocation to agent i is either 0 or 1. To enrich the model, we now consider a setting with two agents and quadratic utility in which each agent's consumption utility when of type θ and when allocated $q \in [0,1]$ units is $U(\theta, q) = \theta q - \frac{1}{2}q^2$.[33] For example, as noted by Liu et al. (forth.), emission permit usage does typically not take a bang-bang form, suggesting that for some applications the constant marginal values model is not the empirically most compelling one.

For simplicity, we assume extremal ownership and set $r = 1$ here. We also assume that the two distributions have identical supports , which we set equal to $[0, 1]$. With constant marginal values $\underline{\theta} = 0$ is without loss of generality within the domain of problems with identical supports. However, with decreasing marginal values, this is not the case. The maximizer of $U(\theta_i, Q)$ being $Q = \theta_i$ means that the problem is not always subject to scarcity, which drives some of the results that follow. Moreover, with a free disposal constraint, no agent i can be induced to consume more than θ_i, which will affect, in particular, the results for the buyer-optimal mechanism.

As we will show, given weights $w_1 = w$ and $w_2 = 1 - w$ and $\rho \geq \max\{w, 1 - w\}$, the allocation rule of the optimal mechanism maximizes:[34]

$$\sum_{i \in \mathcal{N}} U(\overline{\Psi}_{i,\frac{w_i}{\rho}}(\theta_i), Q_i) = \sum_{i \in \mathcal{N}} \left(Q_i \left(\overline{\Psi}_{i,\frac{w_i}{\rho}}(\theta_i) - \frac{1}{2} Q_i \right) \right), \tag{10}$$

subject to the feasibility constraint $\sum_{i \in \mathcal{N}} Q_i \leq 1$ and the free-disposal constraint $Q_i \leq \theta_i$.

The ex post efficient allocation rule $\mathbf{Q}^e(\boldsymbol{\theta})$ maximizes $\sum_{i \in \mathcal{N}} U(\theta_i, Q_i)$. Thus, for $\sum_{i \in \mathcal{N}} \theta_i \leq 1$, $\mathbf{Q}^e(\boldsymbol{\theta}) = (\theta_1, \theta_2)$, and otherwise, $\mathbf{Q}^e(\boldsymbol{\theta}) = (Q, 1 - Q)$, where

$$Q = \arg \max_{Q \in [0,1]} U(\theta_1, Q) + U(\theta_2, 1 - Q) = \frac{1 + \theta_1 - \theta_2}{2}.$$

[33] See also Choné et al. (forth.), who in an extension of their procurement problem analyze a setting in which the suppliers' consumption utility is quadratic.

[34] With quadratic utility, the agents are no longer risk neutral, so in principle nondegenerate lotteries could be used to alleviate the incentive constraints. However, along with our assumption of increasing virtual values and virtual costs and with the assumption of continuous densities, quadratic utility satisfies the sufficient conditions derived by Maskin and Riley (1989, Prop. 5) for the optimal mechanism to be deterministic in the sense that, conditional on a type profile $\boldsymbol{\theta}$, the allocation is deterministic.

Because the problem is not necessarily subject to scarcity, i.e., the constraint $\sum_{i\in\mathcal{N}} Q_i^e \le 1$ can be slack, in which case revenue ex post is positive—the buyer can be charged without requiring payments to the seller—it is not clear whether in general ex post efficiency is impossible. However, for example for the case in which both distributions are uniform on $[0,1]$, ex post efficiency is not possible.[35] If ex post efficiency is not possible, then the second-best allocation rule maximizes the objective in (10) with $w_1 = w_2 = 1/2 < \rho$, where the inequality follows from the impossibility of ex post efficiency. In contrast to the case of constant marginal values, where even in the second-best mechanism, the allocation is equal to the ex post efficient allocation whenever there is trade with $r = 1$, the second-best allocation rule can differ from ex post efficiency for almost all type profiles with decreasing marginal values as shown in the following proposition and corollary.

Proposition 5. *With decreasing marginal values and $r = 1$, if ex post efficiency is not possible, then the second-best allocation rule differs from $\mathbf{Q}^e(\boldsymbol{\theta})$ for all but a zero-measure set of types if $\frac{F_1(x)}{f_1(x)} \le \frac{1-F_2(x)}{f_2(x)}$ for all $x \in [0, 1/2]$.*

Proof. See Appendix A.

Because for $r = 1$ and uniformly distributed types, the ex post efficient mechanism runs a deficit, Proposition 5 has the following corollary:

Corollary 1. *With decreasing marginal values, $r = 1$, and uniform distributions, the second-best allocation rule differs from $\mathbf{Q}^e(\boldsymbol{\theta})$ for all but a zero-measure set of types.*

Extremal bargaining weights

If the seller has all the bargaining weight ($w = 1$), then the allocation rule is the pointwise optimizer of $Q_1\left(\theta_1 - \frac{1}{2}Q_1\right) + Q_2\left(\Psi_2^B(\theta_2) - \frac{1}{2}Q_2\right)$, accounting for the possibility of virtual types being outside $[0,1]$ and for free disposal. And if the buyer has all the bargaining weight ($w = 0$), then the allocation rule is the pointwise optimizer of $Q_1\left(\Psi_1^S(\theta_1) - \frac{1}{2}Q_1\right) + Q_2\left(\theta_2 - \frac{1}{2}Q_2\right)$, accounting for constraints. Consequently, when the seller has all the bargaining weight, i.e., $w = 1$, if $\theta_1 + \Psi_2^B(\theta_2) \le 1$, then $Q_1(\boldsymbol{\theta}) = \theta_1$ and $Q_2(\boldsymbol{\theta}) = \max\{0, \Psi_2^B(\theta_2)\}$, while if $\theta_1 + \Psi_2^B(\theta_2) > 1$, then agent 1 is allocated $Q^S(\boldsymbol{\theta})$ defined as

$$Q^S(\boldsymbol{\theta}) \quad = \quad \arg \max_{Q\in[0,\min\{\theta_1,1\}]} U(\theta_1, Q) + U(\Psi_2^B(\theta_2), 1-Q) = \min\{\theta_1, 1, \frac{1}{2}(1 + \theta_1 - \Psi_2^B(\theta_2))\},$$

[35] In this case, the expected budget surplus under binding IR for agents' worst-off types, i.e., $u_1(\omega_1) = 0$ and $u_2(\omega_2) = 0$, is $\Pi^e = \mathbb{E}_{\boldsymbol{\theta}}\left[\sum_{i\in\mathcal{N}} Q_i(\boldsymbol{\theta})(\Psi_i(\theta_i, \omega_i) - \frac{1}{2}Q_i(\boldsymbol{\theta}))\right] - \sum_{i\in\mathcal{N}}\left(r_i\omega_i - \frac{1}{2}r_i^2\right)$. Computing this, we get $\Pi^e = \frac{7}{16} - \frac{1}{2} = -\frac{1}{16}$.

and agent 2 is allocated $\max\{0, \min\{\Psi_2^B(\theta_2), 1 - Q^S(\boldsymbol{\theta})\}\} = 1 - Q^S(\boldsymbol{\theta})$. In contrast, when the buyer has all the bargaining weight, i.e., $w = 0$, if $\Psi_1^S(\theta_1) + \theta_2 \leq 1$, then $Q_1(\boldsymbol{\theta}) = \min\{\theta_1, \Psi_1^S(\theta_1)\} = \theta_1$ and $Q_2(\boldsymbol{\theta}) = \theta_2$, while if $\Psi_1^S(\theta_1) + \theta_2 > 1$, then agent 1 is allocated $Q^B(\boldsymbol{\theta})$ defined as

$$Q^B(\boldsymbol{\theta}) = \arg \max_{Q \in [0, \min\{\theta_1, 1\}]} U(\Psi_1^S(\theta_1), Q) + U(\theta_2, 1 - Q) = \min\{\theta_1, 1, \frac{1}{2}(1 - \theta_2 + \Psi_1^S(\theta_1))\},$$

while agent 2 is allocated $1 - Q^B(\boldsymbol{\theta})$, disposing of any excess beyond θ_2. Note that because $\Psi_1^S(\theta_1) > \theta_1$ for all $\theta_1 > 0$, the free disposal constraint that requires $Q_1 \leq \theta_1$ will be binding.

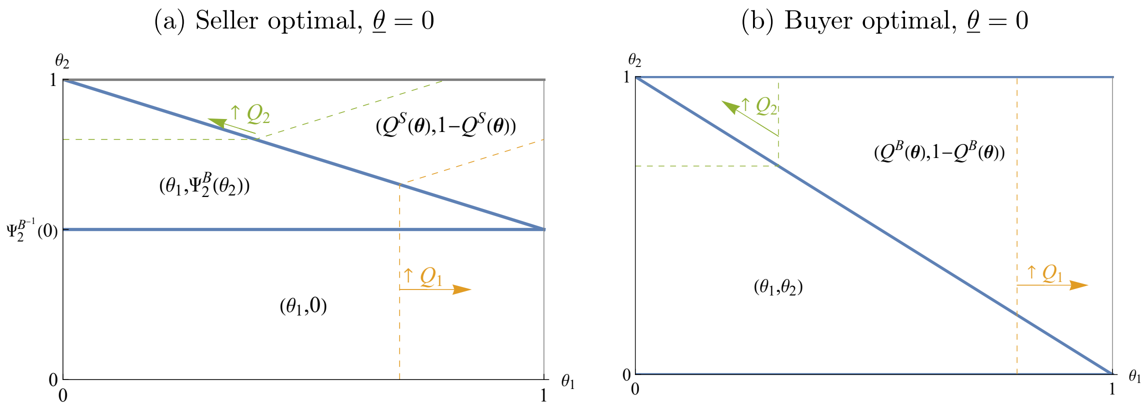


Figure 10: Seller-optimal and buyer-optimal allocation rules (Q_1, Q_2) for quadratic utility, with representative contour lines indicated. Assumes uniformly distributed types on $[0, 1]$.

Figure 10 illustrates these allocation rules for the case of uniformly distributed types. Relative to ex post efficiency, the allocation rule of the buyer-optimal mechanism is less distorted than that of the seller-optimal mechanism in the following sense. Under the seller-optimal mechanism, the allocation is never ex post efficient, whereas under the buyer-optimal mechanism, it is ex post efficient as along as $\theta_1 + \theta_2 \leq 1$. This arises because the free disposal constraint prevents the buyer from over-allocating to the seller, relative to efficiency when $\theta_1 + \theta_2 \leq 1$, whereas the seller never wants to over-allocate to the buyer relative to efficiency, so the free disposal constraint plays no role. Straightforward calculations also show that social surplus is larger under the buyer-optimal mechanism than under the seller-optimal mechanism for any $\boldsymbol{\theta}$ such that $\theta_2 \in (1 - \theta_1, 1 - \theta_1/2)$. Only for $\theta_2 > 1 - \theta_1$ is social surplus under the buyer-optimal mechanism smaller than under the seller-optimal mechanism because $Q^e(\boldsymbol{\theta}) < Q^S(\boldsymbol{\theta}) < \theta_1$. As this suggests, with quadratic utility and uniformly distributed types on $[0, 1]$, buyer power is less harmful than seller power,[36] i.e.,

[36]With quadratic utility and uniformly distributed types on $[0, 1]$, expected social surplus under the seller-optimal mechanism is 0.2760 and under the buyer-optimal mechanism it is 0.2917. For comparison, under

monopoly power is worse for social surplus than monopsony power.[37] This contrasts with the model with constant marginal values and uniform distributions, where social surplus is the same with buyer power as it is with seller power.

It should, however, be noted that results above depend on the assumptions about supports, even keeping the distributions fixed as uniform. For example, if the support of both agents' distributions is $[0,1]$, then every seller type other than the one with $\theta_1 = 1$ is overendowed with the good, and so the seller's free-disposal constraint typically binds. In contrast, if both distributions are uniform on $[1,2]$, then the allocation rule of the seller-optimal mechanism would be $Q_1(\boldsymbol{\theta}) = 1$ and $Q_2(\boldsymbol{\theta}) = 0$ if $\theta_2 < \frac{1+\theta_1}{2}$ and $Q_1(\boldsymbol{\theta}) = Q^S(\boldsymbol{\theta})$ and $Q_2(\boldsymbol{\theta}) = 1 - Q^S(\boldsymbol{\theta})$ otherwise. The allocation for the buyer-optimal mechanism would be $Q_1(\boldsymbol{\theta}) = 1$ and $Q_2(\boldsymbol{\theta}) = 0$ if $\theta_2 < 2\theta_1 - 2$ and otherwise $Q_1(\boldsymbol{\theta}) = Q^B(\boldsymbol{\theta})$ and $Q_2(\boldsymbol{\theta}) = 1 - Q^B(\boldsymbol{\theta})$. In this case, expected social surplus is once again the same under the seller-optimal and buyer-optimal mechanisms.

Optimal nonlinear tariffs

As noted, with constant marginal values and extremal ownership, the seller-optimal and buyer-optimal mechanisms can be implemented with a take-it-or-leave-it price offer from the agent with all the bargaining weight. We now show that this property extends to the model with decreasing marginal values, where the agent with all the bargaining weight can offer any nonlinear tariff it wishes.

We begin with the case with $w = 1$, in which case the seller of type θ_1 offers the seller-optimal tariff $t^S(q;\theta_1)$ to agent 2. Agent 2 then chooses the quantity q and pays $t^S(q;\theta_1)$ to agent 1, where $t^S(q;\theta_1)$ is given by:

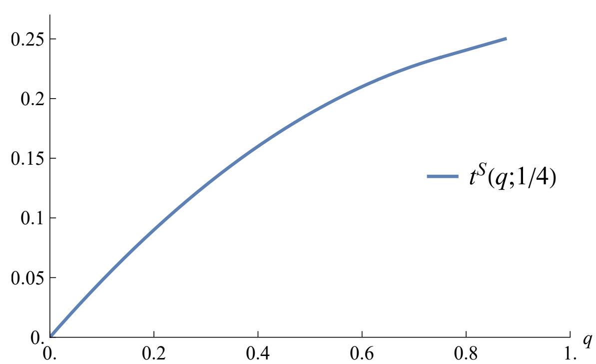
$$t^S(q;\theta_1) \equiv \begin{cases} \int_0^q (\Psi_2^{B^{-1}}(x) - x)dx & \text{if } q \in [0, 1-\theta_1], \\ \int_0^{1-\theta_1} (\Psi_2^{B^{-1}}(x) - x)dx + \int_{1-\theta_1}^q (\Psi_2^{B^{-1}}(2x + \theta_1 - 1) - x)dx & \text{if } q \in (1-\theta_1, 1 - Q^S(\theta_1, 1)], \\ \infty & \text{otherwise.} \end{cases}$$

It is easy to see that given $t^S(q;\theta_1)$ agent 2 optimally chooses $q = \Psi_2^B(\theta_2)$ when $\theta_2 \in$

ex post efficiency, expected social surplus is 0.3125.

[37] One can show that these result extends to standard monopoly and monopsony settings the following sense. Suppose that there is a continuum of agents with mass 1 and types that are uniformly distributed on $[0,1]$. In the monopoly case, the agents are buyers, and in the monopsony case, they are sellers endowed with 1 unit each. Assume that the monopoly's weakly increasing marginal cost for the Qth unit with $Q \in [0,1]$ is $MC(Q)$ satisfying $MC(0) = 0$ and the monopsony's weakly decreasing willingness to pay for the Qth unit is $1 - MC(Q)$. Then, with constant marginal values, monopoly and monopsony are equivalent insofar as they induce the same quantity traded and the same deadweight loss. In contrast, with quadratic utility, the deadweight loss under monopsony is smaller than that under monopoly. In particular, the monopsony may optimally allocate efficiently whereas the monopoly never does.

$[0, \Psi_2^{B^{-1}}(1-\theta_1)]$ and $q = 1 - Q^S(\theta_1, \theta_2)$ when $\theta_2 > \Psi_2^{B^{-1}}(1-\theta_1)$.[38] Thus, the buyer's optimal choice of q given t^S implements the seller-optimal allocation. Figure 11(a) illustrates this payment schedule. For F_2 uniform, $t^S(q;\theta_1) = \frac{1}{4}q(2-q)$ for $q \leq 1-\theta_1$ and $t^S(q;\theta_1) = \frac{1}{4}(1 + 2\theta_1 q + \theta_1^2)$ for $q \in (1-\theta_1, 1-Q^S(\theta_1, 1)]$. This nonlinear tariff is concave in q, that is, it involves quantity discounts.



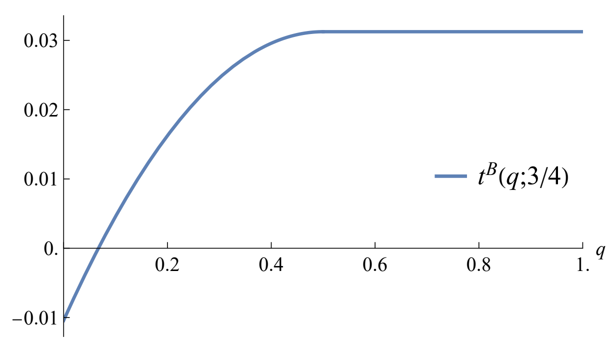


Figure 11: The seller-optimal payment schedule in panel (a) assumes $r = 1$, $w = 1$, and uniformly distributed types on $[0, 1]$. The schedule is offered by agent 1 to agent 2, who chooses the quantity q to buy at price $t^S(q;\theta_1)$. The buyer-optimal payment schedule in panel (b) assumes $r = 1$, $w = 0$, and $F_1(\theta) = \theta^2$ with both agents having type support $[0, 1]$. The schedule is offered by agent 2 to agent 1, who chooses the quantity q to sell at price $t^B(q;\theta_2)$.

Analogously, when $w = 0$, the buyer-optimal mechanism can be implemented through a nonlinear tariff $t^B(q;\theta_2)$ that agent 2 with type θ_2 offers to agent 1, with the understanding that agent 1 chooses the quantity q to sell to agent 2 in exchange for the payment $t^B(q;\theta_2)$. If, given θ_2, the quantity $Q^B(\theta_1, \theta_2)$ is equal to θ_1 for all values of θ_1 (as is the case when F_1 is the uniform distribution), then the buyer-optimal allocation always has $Q_1(\boldsymbol{\theta}) = \theta_1$, which can be implemented by agent 2 offering to pay zero and agent 1 optimally choosing to "sell" $1 - \theta_1$ units to agent 2. Agent 1 is willing to hand over those units to agent 2 without payment because it would simply dispose of them otherwise. Agent 2 then consumes the minimum of what it receives and θ_2.

In contrast, if for a given θ_2 we have $Q^B(\theta_1, \theta_2) < \theta_1$ for some θ_1 such that $\Psi_1^S(\theta_1) + \theta_2 > 1$, then agent 1's quantity will sometimes be less than θ_1. For example, this arises with $F_1(\theta) = \theta^2$, $\theta_2 > 1/2$, and $\theta_1 > 2(1-\theta_2)$. Then the buyer-optimal allocation is

$$\left(\frac{1}{2}(1 - \theta_2 + \Psi_1^S(\theta_1)),\ \frac{1}{2}(1 + \theta_2 - \Psi_1^S(\theta_1))\right),$$

[38] Agent 2 maximizes $q\theta_2 - \frac{1}{2}q^2 - t(q;\theta_1)$ over q. For $q \in [0, 1-\theta_1]$, the first-order condition is $q = \Psi_2^B(\theta_2)$, and for larger q, the first-order condition is $q = \frac{1}{2}(1 - \theta_1 + \Psi_2^B(\theta_2)) = 1 - Q^S(\theta_1, \theta_2)$. Note that for $\theta_2 > \Psi_2^{B^{-1}}(1-\theta_1)$, $Q^S(\theta_1, \theta_2) = 1 + \theta_1/2 - \theta_2$.

which allocates less than θ_1 to agent 1 and less than θ_2 to agent 2. In this case, a nontrivial payment schedule is required. Defining the threshold type $\tau(\theta_2)$ to be the value of θ_1 that satisfies $\frac{1}{2}(1-\theta_2+\Psi_1^S(\theta_1)) = \theta_1$ for a given θ_2, then for $\theta_1 \leq \tau(\theta_2)$, the buyer-optimal allocation for agent 1 is θ_1, but for $\theta_1 > \tau(\theta_2)$, the buyer-optimal allocation for agent 1 is $Q^B(\theta_1,\theta_2) < \theta_1$. The buyer-optimal allocation is implemented with the following payment schedule, which is the amount paid by agent 2 to agent 1 as a function of the quantity q that agent 1 chooses to sell to agent 2:

$$t^B(q;\theta_2) \equiv \begin{cases} K(\theta_2) - \int_{\tau(\theta_2)}^{1-q} \left(\Psi_1^{S^{-1}}(2x-1+\theta_2) - x\right)dx & \text{if } q \in [0, 1-\tau(\theta_2)), \\ K(\theta_2) & \text{if } q \in [1-\tau(\theta_2), 1], \end{cases}$$

where $K(\theta_2)$ is defined below. Given this schedule, agent 1 chooses the quantity q to sell (and the quantity $1-q$ to retain) to maximize $\theta_1 \min\{\theta_1, 1-q\} - 1/2(\min\{\theta_1, 1-q\})^2 + t^B(q;\theta_2)$, whose first-order condition for $1-q < \theta_1$ is $q = \frac{1}{2}\left(1+\theta_2-\Psi_1^S(\theta_1)\right) = 1 - Q^B(\theta_1,\theta_2)$, ensuring the buyer-optimal allocation. Defining $K(\theta_2)$ so that agent 1's worst-off type, i.e., $\theta_1 = \overline{\theta}_1$ has payoff $\overline{\theta}_1 - \frac{1}{2}\overline{\theta}_1^2$, which is agent 1's outside option, completes the specification of the payment schedule.[39] The payment schedule is illustrated in Figure 11 for the case of $F_1(\theta) = \theta^2$ on $[0,1]$. As shown there, agent 1 is paid by agent 2 when it sells a sufficiently large quantity to agent 2, but would have to pay agent 2 if it chose not to sell anything.

5.2 Multilateral bargaining

We now show how incomplete information bargaining extends to more than two agents. In the interest of space, our focus here is on conditions that permit ex post efficiency, thereby mirroring the analysis in Section 4.1, even though the analysis away from ex post efficiency extends as well.

We let $\mathcal{N} = \mathcal{N}_U \cup \mathcal{N}_D$ consist of agents $i \in \mathcal{N}_U$ and agents $j \in \mathcal{N}_D$. For all $i, j \in \mathcal{N}$ we allow for $F_i \neq F_j$. For all $i \in \mathcal{N}_U$, the support of F_i is $[0,1]$, while for $j \in \mathcal{N}_D$, F_j has the support $[\underline{\theta}, \underline{\theta}+1]$, with $F_j(\theta) = F_j^P(\theta - \underline{\theta})$ for $\theta \in [\underline{\theta}, \underline{\theta}+1]$, where F_j^P is j's primitive distribution with support $[0,1]$. That is, we continue to adhere to the shifting-support model. All distributions exhibit increasing virtual values and virtual costs. Let $n_U \equiv |\mathcal{N}_U|$, $n_D \equiv |\mathcal{N}_D|$, and $n = n_U + n_D$. As in Section 4, we assume constant marginal values and that the agents own the entire resources, that is, $\sum_{i\in\mathcal{N}} r_i = 1$ and $r_i \geq 0$ for all $i \in \mathcal{N}$. As in the bilateral setting, we denote by $\mathcal{R}^e(\underline{\theta})$ the set of ownership structures that permit ex

[39] The payment $K(\theta_2)$ is defined by $Q^B(\overline{\theta}_1,\theta_2) - 1/2Q^B(\overline{\theta}_1,\theta_2)^2 + K(\theta_2) - \int_{\hat{\tau}(\theta_2)}^{Q^B(\overline{\theta}_1,\theta_2)} (\Psi_1^{S^{-1}}(2x-1+\theta_2) - x)dx = \overline{\theta}_1 - \frac{1}{2}\overline{\theta}_1^2$.

post efficiency.

As shown in Appendix B, the result of Proposition 1 extends to the case of $n > 2$. That is, the incomplete information bargaining allocation rule assigns the resources to an agent with the maximum ironed weighted virtual type, $\max_{i\in\mathcal{N}} \overline{\Psi}_{i,\frac{w_i}{\rho}}(\theta_i, \omega_i)$, with ρ equal to the smallest feasible value such that the no-deficit constraint is satisfied. However, in contrast to the case of $n = 2$, in some cases with $n > 2$, there is positive probability of having more than one agent with the maximum ironed weighted virtual type, so one must address the possibility of ties. In the event of such a tie, the mechanism randomizes over the tied agents with randomization probabilities that satisfy the condition that the type associated with agent i's ironing parameter is worst-off for agent i.[40]

We begin by considering the case of $n_D = 1$ and $n_U \geq 2$. In a setting with $n_D = 1$, we let Δ^U denote the set of n_U-dimensional vectors $\mathbf{x}$ such that $(\mathbf{x}, 1 - \sum_{i=1}^{n_U} x_i) \in \Delta$, i.e., $\Delta^U \equiv \{\mathbf{x} \in [0,1]^{n_U} \mid \sum_{i=1}^{n_U} x_i \leq 1\}$. Further, we let

$$\mathcal{R}_U^e(\underline{\theta}) \equiv \{\mathbf{r}_U \in \Delta^U \mid (\mathbf{r}_U, 1 - \textstyle\sum_{i=1}^{n_U} r_{U,i}) \in \mathcal{R}^e(\underline{\theta})\}.$$

With these definitions, the result of Proposition 2 that the set of ex post efficiency permitting ownership structures converges as $\underline{\theta}$ approaches 1 from below to the singleton set in which the agent with support $[\underline{\theta}, 1+\underline{\theta}]$ owns all the resources generalizes to a setting with $n_U \geq 2$ agents with support $[0,1]$ and one agent with support $[\underline{\theta}, 1+\underline{\theta}]$ by essentially the same logic. And for $\underline{\theta} \geq 1$, again, any ownership structure permits ex post efficiency.

Proposition 6. *Assume equal bargaining weights, $n_U \geq 2$, and $n_D = 1$. The set $\mathcal{R}_U^e(\underline{\theta})$ satisfies: for $\underline{\theta} \in [0,1)$, $\mathcal{R}_U^e(\underline{\theta})$ is nonempty with $\mathcal{R}_U^e \subset [0,1)^{n_U} \backslash \{\mathbf{0}\}$ and $\lim_{\underline{\theta}\uparrow 1} \mathcal{R}_U^e(\underline{\theta}) = \{\mathbf{0}\}$; and for $\underline{\theta} \geq 1$, $\mathcal{R}_U^e(\underline{\theta}) = \Delta^U$.*

Further, we can characterize bargaining weights that permit ex post efficiency. With $n_D \geq 2$, for ex post efficiency to be possible, all agents in $\mathcal{N}_D$ must have the same bargaining weight, and we provide conditions under with the bargaining weights of agents in $\mathcal{N}_U$ are constrained to be equal to or close to those of the agents in $\mathcal{N}_D$. For the purposes of stating Proposition 7, we define $\overline{w}_U \equiv \max_{j\in\mathcal{N}_U \text{ s.t. } r_j>0} w_j$, which is the maximum bargaining weight among the agents in $\mathcal{N}_U$ that have positive ownership.

Proposition 7. *Ex post efficiency requires that: (i) all agents in $\mathcal{N}_D$ have the same bargaining weight w_D; (ii) for $\underline{\theta} \in [0,1)$, any agent $i \in \mathcal{N}_U$ has $w_i = w_D$; (iii) for $\underline{\theta} \geq 1$, any agent*

[40] See Appendix B, for a proof of existence of the incomplete information bargaining mechanism and details of the requirements for tie-breaking rules.

$i \in \mathcal{N}_U$ with $r_i > 0$ has

$$\frac{\max\{\overline{w}_U, w_D\} - w_i}{\max\{\overline{w}_U, w_D\}} \leq (\underline{\theta} - 1)\, f_i(1).$$

Further, for $\underline{\theta} \geq 1$, if $n_D = 1$, then ex post efficiency is possible if and only if any agent $i \in \mathcal{N}_U$ with $r_i > 0$ has

$$1 + \left(1 - \frac{w_i}{\max\{w_i, w_D\}}\right)\frac{1}{f_i(1)} \leq \underline{\theta} - \left(1 - \frac{w_D}{\max\{w_i, w_D\}}\right)\frac{1}{f_D(\underline{\theta})},$$

where we use D as the index for the agent in $\mathcal{N}_D$.

Proof. See Online Appendix C.2.

As Proposition 7 shows, for overlapping supports, ex post efficiency requires that all agents have the same bargaining weight, consistent with Proposition 2 for the case of only two agents. For nonoverlapping supports, it is still the case that for ex post efficiency all agents with support $[\underline{\theta}, \underline{\theta} + 1]$ must have the same bargaining weight. But, in that case, the bargaining weights of the agents with support $[0, 1]$ can differ, as long as they do not differ too much from each other and from the common bargaining weight of the agents with the higher support.

Turning to the case of $n_D = 2$ and $n_U = 1$, Figure 12 illustrates the result of Makowski and Mezzetti (1993) that for $\underline{\theta} \in (0, 1)$ sufficiently large, ex post efficiency is possible even if the agent in $\mathcal{N}_U$ owns all the resources. This is illustrated by the top vertex in the triangle being included in the ex post efficiency permitting set for $\underline{\theta} = 0.8$ in panel (a) and for $\underline{\theta}$ approaching 1 in panel (b).

Figure 12(a) shows that a shift of resources from an agent in $\mathcal{N}_U$ to an agent in $\mathcal{N}_D$ can cause ex post efficiency to no longer be possible. For example, starting from the boundary at the tip of the right corner white triangle in Figure 12(a), shifting resources to agent 1 would cause ex post efficiency to no longer be possible (moves the ownership structure into the white corner triangle). Thus, it may be advantageous to have some inefficient sellers to balance the market power of the agents in $\mathcal{N}_D$.

Related to Figure 12(b), note that when $\underline{\theta} \geq 1$ and $r_3 = 0$, the setup is essentially that of Cramton et al. (1987) with two symmetric partners and we have the usual CGK problem that ex post efficiency is not possible. This is reflected in the figure by the fact that the intersection of the orange $\underline{\theta} \to 1$ region with the bottom edge of the triangle, where the agent 3 has zero resources, spans $(0.21, 0.79, 0)$ to $(0.79, 0.21, 0)$, but does not include the bottom corners.

(a) Agents 1 and 2 in $\mathcal{N}_D$; agent 3 in $\mathcal{N}_U$ (b) Agents 1 and 2 in $\mathcal{N}_D$; agent 3 in $\mathcal{N}_U$

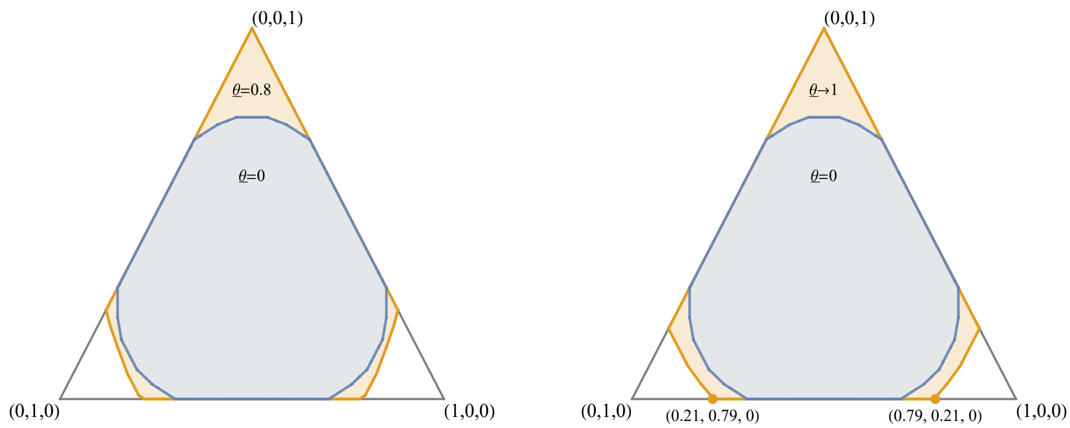


Figure 12: Ex post efficiency permitting set with equal bargaining weights. Assumes that agents 1 and 2 are in $\mathcal{N}_D$ and agent 3 is in $\mathcal{N}_U$ and that types are uniformly distributed on the respective supports.

6 Conclusions

We analyze a unifying model of bargaining using an independent private values setting in which information is always incomplete. Our framework provides conditions under which complete and incomplete information bargaining are equivalent. If these conditions are met, then bargaining is always efficient, that is, neither bargaining power nor ownership affects whether the outcome is ex post efficient. This is a formalization or conceptualization of the notion of "little private information" sometimes invoked to justify the complete information approach to bargaining, which may have advantages in terms of tractability. (As discussed, a caveat regarding the equivalence applies for problems involving investment, where incomplete and complete information bargaining continue to diverge because there is hold-up with complete information and not with incomplete information.) While ownership structures can affect whether bargaining is efficient, they are, loosely speaking, less important than bargaining power insofar as, typically, there are many ownership structures that permit ex post efficiency, whereas with overlapping supports, ex post efficiency is only possible with equal bargaining power.

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A Appendix: Proofs

Proof of Proposition 2. We begin with some preliminary discussion and results. Given $\underline{\theta}$ and r, let $\Pi^e(\underline{\theta}, r)$ denote that the expected budget surplus under ex post efficiency when agents' IR constraints are satisfied with equality for their worst-off types. It then follows that ex post efficiency is possible if and only if $\Pi^e(\underline{\theta}; r) \geq 0$, where Π^e can be characterized as follows:

Lemma A.1. *With two agents, if $\hat{\theta}_i^e(r)$ is agent i's worst-off type under ex post efficiency, then (dropping the argument on $\hat{\theta}_i$ for readability)*

$$\Pi^e(\underline{\theta}, r) = \mathbb{E}_{\theta_1}\left[\Psi_1(\theta_1, \hat{\theta}_1^e)q_1^e(\theta_1)\right] + \mathbb{E}_{\theta_2}\left[\Psi_2(\theta_2, \hat{\theta}_2^e)q_2^e(\theta_2)\right] - r\hat{\theta}_1^e - (1-r)\hat{\theta}_2^e.$$

Thus, we can define the set of *ex post efficiency permitting* ownership structures $\mathcal{R}^e(\underline{\theta})$ as $\mathcal{R}^e(\underline{\theta}) \equiv \{r \in [0,1] \mid \Pi^e(\underline{\theta}, r) \geq 0\}$. Focusing on the existence of ex post efficiency permitting ownership structures, Lemma A.2 shows that for $\underline{\theta} < 1$, there exists ownership $r^* \in [0,1]$ that equalizes the agents' worst-off types; moreover, r^* maximizes $\Pi^e(\underline{\theta}, r)$ with respect to r and ensures that $\Pi^e(\underline{\theta}, r)$ is positive.This implies that $r^* \in \mathcal{R}^e(\underline{\theta})$, which guarantees that

$\mathcal{R}^e(\underline{\theta})$ is nonempty.

It is convenient here to prove the result for the more general case of $n \geq 2$ agents, where we have an ownership vector $(r_1, \ldots, r_n) \in \Delta \equiv \{\mathbf{x} \in [0,1]^n \mid \sum_i^n x_i = 1\}$. To establish that $\Pi^e(\mathbf{r}^*) > 0$, the proof of Lemma A.2 shows that when all agents have equal worst-off types, the VCG mechanism (satisfying interim individual rationality) has a budget surplus in expectation.

Lemma A.2. *Given* $\underline{\theta} \in [0,1)$*, there exists ownership* $\mathbf{r}^* \in \Delta$ *that equalizes agents' worst-off types; moreover,* $\mathbf{r}^*$ *maximizes* $\Pi^e(\mathbf{r})$ *and* $\Pi^e(\mathbf{r}^*) > 0$*.*

Proof. See Online Appendix C.1.

Lemma A.2 extends existing results to the case of agents with differing supports. As noted above, a corollary of Lemma A.2 is that $\mathcal{R}^e(\underline{\theta})$ is nonempty.

Ownership: For $\underline{\theta} \in [0,1)$, Lemma A.2 shows that ownership that equalizes the agents' worst-off types is always possible and maximizes the designer's revenue under ex post efficiency; moreover, that maximized revenue is positive, implying that $\mathcal{R}^e(\underline{\theta})$ is nonempty. But, for $\underline{\theta} \in [0,1)$, Myerson and Satterthwaite (1983) show that ex post efficiency is impossible with extremal ownership, implying that neither 0 nor 1 are elements of $\mathcal{R}^e(\underline{\theta})$, i.e., $\mathcal{R}^e(\underline{\theta}) \subset (0,1)$.

Turning to the limit result, because an agent's worst-off type must be in its type support, in the limit at $\underline{\theta}$ goes to 1 from below, equalized worst-off types must approach 1. Using a result first established by Cramton et al. (1987) (see Lemma B.2 in Appendix B), agent 2's worst-off type $\hat{\theta}_2$ must satisfy either (i) $q_2^e(\hat{\theta}_2) = 1 - r$ or (ii) $\hat{\theta}_2 = \underline{\theta}$ and $q_2(\underline{\theta}) > 1 - r$. Thus, noting that $q_2^e(1) = F_1(1) = 1$, a worst-off type of 1 requires that $r = 0$. Further, when r approaches zero and $\underline{\theta}$ approaches 1, the expected net payoff, and hence maximized revenue, approaches zero, with the implication that the set of ownerships inducing positive revenue under ex post efficiency approaches the singleton set $\{0\}$.

For $\underline{\theta} \geq 1$, ex post efficiency is straightforward to achieve for any ownership because resources owned by the agent with support $[0,1]$ can be sold to the agent with support $[\underline{\theta}, 1 + \underline{\theta}]$ at a posted price $p \in [1, \underline{\theta}]$.

Bargaining weights: **Case (i)**: For $\underline{\theta} \leq 1$, ex post efficiency is possible for $w = 1/2$ and $r \in \mathcal{R}^e(\underline{\theta})$, as just seen, so we are left to show that for $w \neq 1/2$, it is not possible. To this, it suffices to recall the allocation rule of the incomplete information bargaining mechanism in Proposition 1. This rule is not ex post efficient if $w \neq 1/2$ because then the ironed virtual type function of the two agents differ. **Case (ii)**: For $\underline{\theta} \in (1, \underline{\theta}^*)$, ex post efficiency is possible for $w = 1/2$ for any r. With nonoverlapping supports, i.e., $\underline{\theta} > 1$, small enough departures from equal bargaining weights will only affect the weighted ironed virtual type functions in such a way that agent 2's weighted ironed virtual type is still larger than agent 1's for all possible θ_2 and θ_1. The larger is $\underline{\theta}$, the larger can these departures from equality be, which proves the result $\mathcal{W}^e(\underline{\theta})$ increases in the set inclusion sense. Convexity follows from the monotonicity of the virtual type functions in w_i. For $\underline{\theta} < \underline{\theta}^*$, either $\Psi_1^S(1) > \underline{\theta}$, implying that trade will not be ex post efficient for $w = 0$ or sufficiently close to 0 or $\underline{\theta} < 1 - \Psi_1^B(0)$, implying that trade will not be ex post efficient for $w = 1$ or sufficiently close to 1 (or both). **Case (iii)**: For $\underline{\theta} \geq \underline{\theta}^*$, agent 2 is always a buyer and agent 1 always a seller of r units. Because the weighted virtual type functions are monotone in w and ex post efficiency obtains for $w \in \{0, 1\}$, ex post efficiency obtains for any $w \in [0, 1]$. ■

Proof of Proposition 3. The proof for parts (i) and (ii) follows from the envelope theorem applied to the Lagrangian associated with the designer's problem. Part (iii) reflects the property shown in Proposition 4 that the expected net payoff frontier has slope $-w/(1-w)$ if $r \notin \mathcal{R}^e(\underline{\theta})$ and the convex hull of the frontier has slope -1 if $r \in \mathcal{R}^e(\underline{\theta})$, where for $r \in \mathcal{R}^e(\underline{\theta})$ and $w = 1/2$, the expected net payoff depends on the parameter η.

It remains to prove parts (iv)–(vii) of the proposition. We begin by focusing on the case with $w = 1$ and proving that $U_{1r}(r, 1) > 0$ for r sufficiently close to 0, and that $U_{1r}(r, 1) < 0$ for r sufficiently close to 1, when $F_1 = F_2$. For $w = 1$, the interim expected allocations are:

$$q_1(\theta_1) = \begin{cases} F_2(\Psi_2^{S^{-1}}(\theta_1)) & \text{if } 0 \leq \theta_1 \leq F_1^{-1}(1-r), \\ F_2(\Psi_2^{B^{-1}}(\theta_1)) & \text{if } F_1^{-1}(1-r) < \theta_1 \leq 1, \end{cases}$$

and

$$q_2(\theta_2) = \begin{cases} F_1(\Psi_2^S(\theta_2)) & \text{if } 0 \leq \theta_2 < \Psi_2^{S^{-1}}(F_1^{-1}(1-r)), \\ 1 - r & \text{if } \Psi_2^{S^{-1}}(F_1^{-1}(1-r)) \leq \theta_2 \leq \Psi_2^{B^{-1}}(F_1^{-1}(1-r)), \\ F_1(\Psi_2^B(\theta_2)) & \text{if } \Psi_2^{B^{-1}}(F_1^{-1}(1-r)) < \theta_2 \leq 1. \end{cases}$$

Using the characterization of worst-off types from Cramton et al. (1987) (see also Lemma B.2 in the Online Appendix), this gives us a worst-off type for agent 1 of

$$\omega_1 = \begin{cases} \Psi_2^B(F_2^{-1}(r)) & \text{if } F_2(\Psi_2^{B^{-1}}(F_1^{-1}(1-r))) \leq r, \\ F_1^{-1}(1-r) & \text{if } F_2(\Psi_2^{S^{-1}}(F_1^{-1}(1-r))) < r < F_2(\Psi_2^{B^{-1}}(F_1^{-1}(1-r))), \\ \Psi_2^S(F_2^{-1}(r)) & \text{if } r \leq F_2(\Psi_2^{S^{-1}}(F_1^{-1}(1-r))), \end{cases}$$

and for agent 2, any type in $\left[\Psi_2^{S^{-1}}(F_1^{-1}(1-r)), \Psi_2^{B^{-1}}(F_1^{-1}(1-r))\right]$ is worst-off, including $\omega_2 = F_1^{-1}(1-r)$. As set out in Appendix B, $u_i(\theta) = \theta(q_i(\theta) - r_i) - m_i(\theta)$ and, in the

IIB mechanism, $m_i(\theta) = \theta(q_i(\theta) - r_i) - \int_{\omega_i}^{\theta_i}(q_i(y) - r_i)dy - \eta_i\pi(\mathbf{Q}, \boldsymbol{\omega})$, where $\pi(\mathbf{Q}, \boldsymbol{\omega}) = \sum_{i\in\mathcal{N}} \mathbb{E}[\Psi_i(\theta_i, \omega_i)q_i(\theta_i)] - \sum_{i\in\mathcal{N}} \omega_i r_i$. For the case of $n = 2$ with $w = 1$, we have $\eta_1 = 1$ and so $u_1(\theta_1) = \int_{\omega_1}^{\theta_1}(q_1(y) - r)dy + \sum_{i=1}^{2} \mathbb{E}_{\theta_i}[\Psi_i(\theta_i, \omega_i)q_i(\theta_i)] - \omega_1 r - \omega_2(1 - r)$. Thus, agent 1's expected net payoff when $w = 1$ is $U_1(r, 1) = \int_0^1 \int_{\omega_1}^{\theta_1}(q_1(y) - r)dydF_1(\theta_1) + \sum_{i=1}^{2} \int_0^1 \Psi_i(\theta_i, \omega_i)q_i(\theta_i)dF_i(\theta_i) - \omega_1 r - \omega_2(1 - r)$. We can write the double integral in the expression for $U_1(r, 1)$ as $\int_{\omega_1}^1 \int_{\omega_1}^{\theta_1}(q_1(y)-r)dydF_1(\theta_1) - \int_0^{\omega_1} \int_{\theta_1}^{\omega_1}(q_1(y)-r)dydF_1(\theta_1) = \int_{\omega_1}^1(1-F_1(y))(q_1(y) - r)dy - \int_0^{\omega_1} F_1(y)(q_1(y) - r)dy$. Taking the case of r sufficiently close to 1 such that $\omega_1 = \Psi_2^B(F_2^{-1}(r)) > F_1^{-1}(1 - r)$, we can rewrite this as $\int_{\omega_1}^1(1 - F_1(y))(F_2(\Psi_2^{B^{-1}}(y)) - r)dy - \int_0^{F_1^{-1}(1-r)} F_1(y)(F_2(\Psi_2^{S^{-1}}(y)) - r)dy - \int_{F_1^{-1}(1-r)}^{\omega_1} F_1(y)(F_2(\Psi_2^{B^{-1}}(y)) - r)dy$, which has derivative with respect to r, evaluated at $r = 1$, of $1 - \mathbb{E}[\theta_1]$.[41] Turning to the summation term in the expression for $U_1(r, 1)$, for agent 1, and for r sufficiently close to 1 such that $\omega_1 = \Psi_2^B(F_2^{-1}(r)) > F_1^{-1}(1 - r)$, this is

$$\begin{aligned}
& \int_0^{\omega_i} \Psi_i^S(\theta_i)q_i(\theta_i)dF_i(\theta_i) + \int_{\omega_i}^1 \Psi_i^B(\theta_i)q_i(\theta_i)dF_i(\theta_i) \\
= \; & \int_0^{F_1^{-1}(1-r)} \Psi_1^S(\theta_1)F_2(\Psi_2^{S^{-1}}(\theta_1)dF_1(\theta_1) + \int_{F_1^{-1}(1-r)}^{\omega_1} \Psi_1^S(\theta_1)F_2(\Psi_2^{B^{-1}}(\theta_1))dF_1(\theta_1) \\
& + \int_{\omega_1}^1 \Psi_1^B(\theta_1)F_2(\Psi_2^{B^{-1}}(\theta_1))dF_1(\theta_1),
\end{aligned}$$

which has derivative with respect to r, evaluated at $r = 1$, of $2/f_2(1)$.[42] For agent 2, noting that we are working with r such that $\Psi_2^{B^{-1}}(F_1^{-1}(1-r)) \leq F_2^{-1}(r)$, the summation term for agent 2 is $\int_0^{\Psi_2^{S^{-1}}(F_1^{-1}(1-r))} \Psi_2^S(\theta_2)(F_1(\Psi_2^S(\theta_2))-(1-r))dF_2(\theta_2)+\int_{\Psi_2^{B^{-1}}(F_1^{-1}(1-r))}^1 \Psi_2^B(\theta_2)(F_1(\Psi_2^B(\theta_2))-(1 - r))dF_2(\theta_2) + (1 - r)\omega_2$. Differentiating with respect to r and evaluating at $r = 1$, we get $\int_{\Psi_2^{B^{-1}}(0)}^1 \Psi_2^B(\theta_2)dF_2(\theta_2)$.[43] Finally, note that for r sufficiently close to 1, the derivative of $-\omega_1 r - \omega_2(1 - r)$, evaluated at $r = 1$, is equal to $-1 - \frac{2}{f_2(1)}$. Thus, gathering the terms calculated above, for r sufficiently close to 1, the derivative of $U_1(r, 1)$ taken with respect to r and evaluated at $r = 1$, is

$$\begin{aligned}
U_{1r}(1, 1) &= 1 - \mathbb{E}[\theta_1] + 2/f_2(1) + \int_{\Psi_2^{B^{-1}}(0)}^1 \Psi_2^B(\theta_2)dF_2(\theta_2) - 1 - 2/f_2(1) \\
&= -\int_0^1 \theta_1 dF_1(\theta_1) + \int_{\Psi_2^{B^{-1}}(0)}^1 \theta_2 dF_2(\theta_2) - \int_{\Psi_2^{B^{-1}}(0)}^1 (1 - F_2(\theta_2))\, d\theta_2.
\end{aligned}$$

[41] Generally, for r sufficiently close to 1, we have $\omega_1 - \mathbb{E}[\theta_1] - \omega_1'(1 - F_1(\omega_1))(F_2(\Psi_2^{B^{-1}}(\omega_1)) - r) + F_1^{-1\prime}(1-r)(1 - r)\big[F_2(\Psi_2^{S^{-1}}(F_1^{-1}(1 - r))) - F_2(\Psi_2^{B^{-1}}(F_1^{-1}(1 - r)))\big]$.

[42] Generally, for r sufficiently close to 1, we have $\Psi_1^S(F_1^{-1}(1-r))\big[F_2(\Psi_2^{B^{-1}}(F_1^{-1}(1-r)))-F_2(\Psi_2^{S^{-1}}(F_1^{-1}(1-r))\big] + \Psi_2^{B\prime}(F_2^{-1}(r))F_2^{-1\prime}(r)r$.

[43] For general r sufficiently close to 1, we get $\int_0^{\Psi_2^{S^{-1}}(F_1^{-1}(1-r))} \Psi_2^S(\theta_2)dF_2(\theta_2) + \int_{\Psi_2^{B^{-1}}(F_1^{-1}(1-r))}^1 \Psi_2^B(\theta_2)dF_2(\theta_2) - \omega_2 + (1 - r)\omega_2'$.

If $F_1 = F_2 = F$, then we have

$$U_{1r}(1,1) \;=\; -\int_0^{\Psi^{B^{-1}}(0)} \theta_1 dF(\theta_1) - \int_{\Psi^{B^{-1}}(0)}^1 (1 - F(\theta_2))\, d\theta_2 < 0.$$

By analogous calculations, $U_{1r}(0,1) > 0$ if $F_1 = F_2$. By continuity, $U_{1r}(r,1) < 0$ for r sufficiently close to 1 and $U_{1r}(r,1) > 0$ for r sufficiently close to 0, assuming that $F_1 = F_2$.

For $w = 0$, we can reverse the roles of agents 1 and 2 in the analysis above and replace r with $1-r$, giving us the result that $U_{2r}(r,0) < 0$ for all r sufficiently large and $U_{2r}(r,0) > 0$ for all r sufficiently small, assuming that $F_1 = F_2$.

Now turn to effects for the agent without the bargaining power, starting with $w = 1$. For agent 2, we have $\eta_2 = 0$ and so $u_2(\theta_2) = \int_{\omega_2}^{\theta_2}(q_2(y) - (1-r))dy$ and $U_2(r,1) = \int_0^1 \int_{\omega_2}^{\theta_2}(q_2(y) - (1-r))dydF_2(\theta_2)$. This is analogous to the double integral term analyzed above. Replacing 1 with 2 and r with $1-r$ in the expression above, we have $U_2(r,1) = \int_{\omega_2}^1 (1 - F_2(y))(q_2(y) - (1-r))dy - \int_0^{\omega_2} F_2(y)(q_2(y) - (1-r))dy$. Thus, using the definition of q_2, we have$U_2(r,1) = \int_{\Psi_2^{B^{-1}}(F_1^{-1}(1-r))}^1 (1-F_2(y))(F_1(\Psi_2^B(y)) - (1-r))dy - \int_0^{\Psi_2^{S^{-1}}(F_1^{-1}(1-r))} F_2(y)(F_1(\Psi_2^S(y)) - (1-r))dy$. Differentiating with respect to r, we get

$$\begin{aligned} U_{2r}(r,1) &= \int_{\Psi_2^{B^{-1}}(F_1^{-1}(1-r))}^1 (1 - F_2(y))dy - \int_0^{\Psi_2^{S^{-1}}(F_1^{-1}(1-r))} F_2(y)dy \\ &= \begin{cases} -\int_0^{\Psi_2^{S^{-1}}(1)} F_2(y)dy < 0 & \text{if } r = 0, \\ \int_{\Psi_2^{B^{-1}}(0)}^1 (1 - F_2(y))dy > 0 & \text{if } r = 1. \end{cases} \end{aligned}$$

By continuity, $U_{2r}(r,1) > 0$ for all r sufficiently close to 1 and $U_{2r}(r,1) < 0$ for all r sufficiently close to 0. Analogously, $U_{1r}(r,0) > 0$ for r sufficiently close to 1 and $U_{1r}(r,0) < 0$ for r sufficiently close to 0. ■

Proof of Proposition 4. Concavity follows from the slope of the frontier, which we now prove. Let $\langle \mathbf{Q}^w, \mathbf{M}^w \rangle$ be the incomplete information mechanism for a given w. Letting $u_i^w(\theta_i)$ denote agent i's interim expected net payoff given w, away from ex post efficiency, the expected net payoff frontier is given by $(\mathbb{E}_{\theta_1}[u_1^w(\theta_1)], \mathbb{E}_{\theta_2}[u_2^w(\theta_2)])_{w\in[0,1]}$, where $\frac{d\mathbb{E}_{\theta_1}[u_1^w(\theta_1)]}{dw} > 0$, and so the frontier has slope $\frac{d\mathbb{E}_{\theta_2}[u_2^w(\theta_2)]}{dw} / \frac{d\mathbb{E}_{\theta_1}[u_1^w(\theta_1)]}{dw}$. By the envelope theorem, the derivative with respect to w of the optimized objective for the incomplete information bargaining problem satisfies

$$\begin{aligned} &\frac{d}{dw}\Big(\mathbb{E}_{\boldsymbol{\theta}}[w(Q_1^w(\boldsymbol{\theta})\theta_1 - M_1^w(\boldsymbol{\theta})) + (1-w)(Q_2^w(\boldsymbol{\theta})\theta_2 - M_2^w(\boldsymbol{\theta}))]\Big) \\ &= \mathbb{E}_{\boldsymbol{\theta}}[Q_1^w(\boldsymbol{\theta})\theta_1 - M_1^w(\boldsymbol{\theta}) - (Q_2^w(\boldsymbol{\theta})\theta_2 - M_2^w(\boldsymbol{\theta}))]. \end{aligned} \tag{A.1}$$

Using $u_i^w(\theta_i) = (q_i^w(\theta_i) - r_i)\theta_i - m_i^w(\theta_i)$, we can write $w\mathbb{E}_{\theta_1}[u_1^w(\theta_1)] + (1-w)\mathbb{E}_{\theta_2}[u_2^w(\theta_2)] = \mathbb{E}_{\boldsymbol{\theta}}[w(Q_1^w(\boldsymbol{\theta})\theta_1 - M_1(\boldsymbol{\theta})) + (1-w)(Q_2^w(\boldsymbol{\theta})\theta_2 - M_2(\boldsymbol{\theta}))] - \mathbb{E}_{\boldsymbol{\theta}}[wr\theta_1 + (1-w)(1-r)\theta_2]$. Differentiating the above equation with respect to w and using (A.1), we get

$$\begin{aligned}
& \mathbb{E}_{\theta_1}[u_1^w(\theta_1)] - \mathbb{E}_{\theta_2}[u_2^w(\theta_2)] + w\frac{d\mathbb{E}_{\theta_1}[u_1^w(\theta_1)]}{dw} + (1-w)\frac{d\mathbb{E}_{\theta_2}[u_2^w(\theta_2)]}{dw} \\
= \; & \mathbb{E}_{\boldsymbol{\theta}}[Q_1^w(\boldsymbol{\theta})\theta_1 - M_1(\boldsymbol{\theta}) - (Q_2^w(\boldsymbol{\theta})\theta_2 - M_2(\boldsymbol{\theta}))] - \mathbb{E}_{\boldsymbol{\theta}}[r\theta_1 - (1-r)\theta_2] \\
= \; & \mathbb{E}_{\theta_1}[u_1^w(\theta_1)] - \mathbb{E}_{\theta_2}[u_2^w(\theta_2)],
\end{aligned}$$

which gives $\frac{d\mathbb{E}_{\theta_2}[u_2^w(\theta_2)]}{dw} / \frac{d\mathbb{E}_{\theta_1}[u_1^w(\theta_1)]}{dw} = -\frac{w}{1-w}$. In contrast, when ex post efficiency is achieved, $\mathbb{E}_{\theta_1}[u_1^w(\theta_1)] + \mathbb{E}_{\theta_2}[u_2^w(\theta_2)]$ is constant, which implies that the slope of the frontier at ex post efficiency is -1. ■

Proof of Proposition 5. Consider the setup with $n = 2$ and decreasing marginal values. The second-best allocation rule is the pointwise maximizer of $\sum_{i\in\mathcal{N}} \rho\mathbb{E}_{\boldsymbol{\theta}}\big[Q_i\big(\overline{\Psi}_{i,\frac{1}{\rho}}(\theta_i,\omega_i) - \frac{1}{2}Q_i\big)\big]$, where, for convenience, we normalize bargaining weights to 1, with the implication that feasible ρ satisfy $\rho \geq 1$. Assuming that $r = 1$, worst-off types are $\omega_1 = \overline{\theta}_1$ and $\omega_2 = \underline{\theta}_2$, and so the objective reduces to $\mathbb{E}_{\boldsymbol{\theta}}\big[Q_1(\boldsymbol{\theta})\big(\Psi^S_{1,\frac{1}{\rho}}(\theta_1) - \frac{1}{2}Q_1(\boldsymbol{\theta})\big)\big] + \mathbb{E}_{\boldsymbol{\theta}}\big[Q_2(\boldsymbol{\theta})\big(\Psi^B_{2,\frac{1}{\rho}}(\theta_2) - \frac{1}{2}Q_2(\boldsymbol{\theta})\big)\big]$. If $\Psi^S_{1,\frac{1}{\rho}}(\theta_1) + \Psi^B_{2,\frac{1}{\rho}}(\theta_2) \leq 1$, the second-best allocation is $Q_1(\boldsymbol{\theta}) = \min\{\theta_1, \Psi^S_{1,\frac{1}{\rho}}(\theta_1)\} = \theta_1$ and $Q_2(\boldsymbol{\theta}) = \min\{\theta_2, \max\{0, \Psi^B_{2,\frac{1}{\rho}}(\theta_2)\}\} = \max\{0, \Psi^B_{2,\frac{1}{\rho}}(\theta_2)\}$. If $\Psi^S_{1,\frac{1}{\rho}}(\theta_1) + \Psi^B_{2,\frac{1}{\rho}}(\theta_2) > 1$, then, ignoring the free-disposal constraint for the moment, the second-best allocation is $(Q^*_\rho(\boldsymbol{\theta}), 1 - Q^*_\rho(\boldsymbol{\theta}))$, where $Q^*_\rho(\boldsymbol{\theta})$ is the value of Q_ρ that maximizes $Q_\rho\big(\Psi^S_{1,\frac{1}{\rho}}(\theta_1) - \frac{1}{2}Q_\rho\big) + (1-Q_\rho)\big(\Psi^B_{2,\frac{1}{\rho}}(\theta_2) - \frac{1}{2}(1-Q_\rho)\big)$, which has first-order condition

$$Q^*_\rho(\boldsymbol{\theta}) \equiv \frac{1}{2}\left(1 + \Psi^S_{1,\frac{1}{\rho}}(\theta_1) - \Psi^B_{2,\frac{1}{\rho}}(\theta_2)\right) = \frac{1+\theta_1-\theta_2}{2} + \frac{\rho-1}{2\rho}\left(\frac{F_1(\theta_1)}{f_1(\theta_1)} + \frac{1-F_2(\theta_2)}{f_2(\theta_2)}\right) \geq \frac{1+\theta_1-\theta_2}{2},$$

with a strict inequality if $\rho > 1$ and either $\theta_1 > \underline{\theta}_1$ or $\theta_2 < \overline{\theta}_2$. Accounting for free-disposal, we have $Q_1(\boldsymbol{\theta}) = \min\{\theta_1, Q^*_\rho(\boldsymbol{\theta})\}$ and $Q_2(\boldsymbol{\theta}) = \min\{\theta_2, 1 - Q_1(\boldsymbol{\theta})\}$. Note that $\Psi^S_{1,\frac{1}{\rho}}(\theta_1) + \Psi^B_{2,\frac{1}{\rho}}(\theta_2) = \theta_1 + \theta_2 + \frac{\rho-1}{\rho}\left(\frac{F_1(\theta_1)}{f_1(\theta_1)} - \frac{1-F_2(\theta_2)}{f_2(\theta_2)}\right)$, so, depending on the distributions, we can have four possible cases:

	Case	$\mathbf{Q}^e(\boldsymbol{\theta})$	**Second-best** $\mathbf{Q}(\boldsymbol{\theta})$
1.	$\theta_1 + \theta_2 \leq 1$ and $\Psi^S_{1,\frac{1}{\rho}}(\theta_1) + \Psi^B_{2,\frac{1}{\rho}}(\theta_2) \leq 1$	(θ_1, θ_2)	$(\theta_1, \max\{0, \Psi^B_{2,\frac{1}{\rho}}(\theta_2)\})$
2.	$\theta_1 + \theta_2 > 1$ and $\Psi^S_{1,\frac{1}{\rho}}(\theta_1) + \Psi^B_{2,\frac{1}{\rho}}(\theta_2) \leq 1$	$(\frac{1+\theta_1-\theta_2}{2}, \frac{1-\theta_1+\theta_2}{2})$	$(\theta_1, \max\{0, \Psi^B_{2,\frac{1}{\rho}}(\theta_2)\})$
3.	$\theta_1 + \theta_2 \leq 1$ and $\Psi^S_{1,\frac{1}{\rho}}(\theta_1) + \Psi^B_{2,\frac{1}{\rho}}(\theta_2) > 1$	(θ_1, θ_2)	$(\min\{\theta_1, Q^*_\rho(\boldsymbol{\theta})\}, \min\{\theta_2, 1 - Q_1(\boldsymbol{\theta})\})$
4.	$\theta_1 + \theta_2 > 1$ and $\Psi^S_{1,\frac{1}{\rho}}(\theta_1) + \Psi^B_{2,\frac{1}{\rho}}(\theta_2) > 1$	$(\frac{1+\theta_1-\theta_2}{2}, \frac{1-\theta_1+\theta_2}{2})$	$(\min\{\theta_1, Q^*_\rho(\boldsymbol{\theta})\}, \min\{\theta_2, 1 - Q_1(\boldsymbol{\theta})\})$

If ex post efficiency is not possible, then $\rho > 1$, and so in case 1, $\theta_2 \geq \max\{0, \Psi^B_{2,\frac{1}{\rho}}(\theta_2)\}$, with a strict inequality for all $\theta_2 \in (\underline{\theta}_2, \overline{\theta}_2)$. In case 2, $\frac{1+\theta_1-\theta_2}{2} \neq \theta_1$ unless $\theta_1 = 1 - \theta_2$, which is a zero-measure set of types. In case 4, as we showed, $Q^*_\rho(\boldsymbol{\theta}) \geq \frac{1+\theta_1-\theta_2}{2}$, with a strict inequality for all but a zero measure set of types, and given $\theta_1+\theta_2 > 1$, we have $\frac{1+\theta_1-\theta_2}{2} < \theta_1$, so $\frac{1+\theta_1-\theta_2}{2} = \min\{\theta_1, Q^*_\rho(\boldsymbol{\theta})\}$ for at most a zero-measure set of types.

This leaves us with case 3. In order to have the conditions for case 3, we require $\theta_1+\theta_2 \leq 1$ and $\Psi^S_{1,\frac{1}{\rho}}(\theta_1) + \Psi^B_{2,\frac{1}{\rho}}(\theta_2) = \theta_1 + \theta_2 + \frac{\rho-1}{\rho}\left(\frac{F_1(\theta_1)}{f_1(\theta_1)} - \frac{1-F_2(\theta_2)}{f_2(\theta_2)}\right) > 1$. Thus, we require that $\frac{1-F_2(\theta_2)}{f_2(\theta_2)} < \frac{F_1(\theta_1)}{f_1(\theta_1)}$, and this must hold for an open set of types such that $\theta_1 + \theta_2 \leq 1$. Under the condition that $\frac{F_1(x)}{f_1(x)} \leq \frac{1-F_2(x)}{f_2(x)}$ for all $x \in [0, 1/2]$, this cannot hold. Under case 3, one can have $\mathbf{Q}^e(\boldsymbol{\theta}) = \mathbf{Q}(\boldsymbol{\theta})$ for an open set of types for example if $F_1(\theta) = \theta$ and $F_2(\theta) = 2\theta - \theta^2$, both with support $[0, 1]$. This completes the proof. ■