Online Appendix

to accompany

"Bargaining among partners"

by

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Sections in this Online Appendix are labeled starting with "B" to distinguish these appendices from Appendix A in the paper.

B Mechanism design details

In the interest of providing a self-contained description, in this appendix we repeat some of the definitions provided in the body of the paper.

B.1 Preliminaries

The incomplete information bargaining mechanism is a direct mechanism $\langle \mathbf{Q}, \mathbf{M} \rangle$ that consists of an allocation rule $\mathbf{Q} : \Theta \to \mathbb{R}^n_+$ satisfying $Q_i(\boldsymbol{\theta}) \in [0,1]$ and $\sum_{i \in \mathcal{N}} Q_i(\boldsymbol{\theta}) \leq 1$, and a payment rule $\mathbf{M} : \Theta \to \mathbb{R}^n$, where $\Theta \equiv \times_{i \in \mathcal{N}} [\underline{\theta}_i, \overline{\theta}_i]$. For reports $\boldsymbol{\theta} \in \Theta$, $Q_i(\boldsymbol{\theta})$ specifies the quantity allocated to agent i, and $M_i(\boldsymbol{\theta})$ specifies the payment from agent i to the mechanism.[1] We focus on Bayesian incentive compatible, interim individually rational mechanisms that have no deficit in expectation.[2] Agent i's outside option from not participating in the mechanism is $\theta_i r_i$.

For a fixed mechanism $\langle \mathbf{Q}, \mathbf{M} \rangle$, we denote agent i's interim expected allocation and payments, respectively, by

$$q_i(\theta_i) \equiv \mathbb{E}_{\boldsymbol{\theta}_{-i}}[Q_i(\theta_i, \boldsymbol{\theta}_{-i})] \quad \text{and} \quad m_i(\theta_i) \equiv \mathbb{E}_{\boldsymbol{\theta}_{-i}}[M_i(\theta_i, \boldsymbol{\theta}_{-i})].$$

The interim expected net payoff of agent i from participating in the mechanism when its type is θ and when it reports its type truthfully, with "net" meaning net of the outside option $r_i\theta$, is denoted by

$$u_i(\theta) \equiv \theta(q_i(\theta) - r_i) - m_i(\theta).$$

[1]By the Revelation Principle, a focus on direct mechanisms is without loss of generality. Constraint $Q_i \in [0,1]$ is for convenience and can be dropped by replacing $Q_i(\boldsymbol{\theta})$ with $\min\{1, Q_i(\boldsymbol{\theta})\}$ in i's payoff.

[2]As shown in footnote 3, the focus on no deficit in expectation is also without loss of generality within the class of mechanisms that satisfy interim individual rationality.

The direct mechanism $\langle \mathbf{Q}, \mathbf{M} \rangle$ is *Bayesian incentive compatible* (IC) if for all $i \in \mathcal{N}$ and all $\theta, x \in [\underline{\theta}_i, \overline{\theta}_i]$,

$$u_i(\theta) \geq \theta(q_i(x) - r_i) - m_i(x), \tag{B.1}$$

and $\langle \mathbf{Q}, \mathbf{M} \rangle$ is *interim individually rational* (IR) if for all $i \in \mathcal{N}$ and all $\theta \in [\underline{\theta}_i, \overline{\theta}_i]$,

$$u_i(\theta) \geq 0. \tag{B.2}$$

Type $\hat{\theta}_i \in [\underline{\theta}_i, \overline{\theta}_i]$ is called a *worst-off type* of agent i if $u_i(\theta) \geq u_i(\hat{\theta}_i)$ for all $\theta \in [\underline{\theta}_i, \overline{\theta}_i]$. The no-deficit constraint is satisfied if:[3]

$$\sum_{i \in \mathcal{N}} \mathbb{E}_{\theta_i}\left[m_i(\theta_i)\right] \geq 0. \tag{B.3}$$

The incomplete information bargaining mechanism maximizes the weighted sum of the agents' ex ante expected payoffs subject to IC, IR, and no deficit:

$$\max_{\mathbf{Q}, \mathbf{M}} \mathbb{E}_{\boldsymbol{\theta}}\Big[\sum_{i \in \mathcal{N}} w_i\big(\theta_i Q_i(\boldsymbol{\theta}) - M_i(\boldsymbol{\theta})\big)\Big] \text{ subject to (B.1)–(B.3).} \tag{B.4}$$

By the standard characterization of Bayesian incentive compatibility (see, e.g., Myerson, 1981), we have:

Lemma B.1. *Incentive compatibility holds if and only if:*

$$q_i \textit{ is nondecreasing} \tag{B.5}$$

and for all $\theta, \theta' \in [\underline{\theta}_i, \overline{\theta}_i]$,

$$u_i(\theta) = u_i(\theta') + \int_{\theta'}^{\theta} (q_i(y) - r_i) dy. \tag{B.6}$$

Proof. To see that incentive compatibility implies that q_i is nondecreasing, consider two types $\theta, \theta' \in [\underline{\theta}_i, \overline{\theta}_i]$. Incentive compatibility for type θ and θ' requires, respectively, $q_i(\theta)\theta - m_i(\theta) \geq q_i(\theta')\theta - m_i(\theta')$ and $q_i(\theta)\theta' - m_i(\theta) \leq q_i(\theta')\theta' - m_i(\theta')$. Subtracting the latter from the former

[3] Condition (B.3) only requires no deficit in expectation, but this is without loss of generality because for any Bayesian incentive compatible mechanism $\langle \mathbf{Q}, \mathbf{M} \rangle$ such that $\sum_{i \in \mathcal{N}} \mathbb{E}_{\theta_i}[m_i(\theta_i)] = \kappa$ holds for some $\kappa \in \mathbb{R}$, there is a Bayesian incentive compatible mechanism $\langle \mathbf{Q}, \tilde{\mathbf{M}} \rangle$ with the same allocation rule and the same interim expected payments m_i whose revenue is κ ex post, i.e., that satisfies $\sum_{i \in \mathcal{N}} \tilde{M}_i(\boldsymbol{\theta}) = \kappa$ for all $\boldsymbol{\theta}$: let $\tilde{M}_i(\boldsymbol{\theta}) = m_i(\theta_i) - \sum_{j \neq i} m_j(\theta_j)/(n-1) + c_i$ with $c_i = (\mathbb{E}_{\theta_i}[m_i(\theta_i)] - \kappa)/(n-1)$. It further follows that if the mechanism $\langle \mathbf{Q}, \mathbf{M} \rangle$ satisfies interim individual rationality, then so does $\langle \mathbf{Q}, \tilde{\mathbf{M}} \rangle$. For more on equivalences of this form, see Börgers and Norman (2009).

implies that $q_i(\theta)(\theta - \theta') \geq q_i(\theta')(\theta - \theta')$, which is equivalent to q_i being nondecreasing. For more background on mechanism design, see Börgers (2015) and Krishna (2010). By the envelope theorem (Milgrom and Segal, 2002, Corollary 1), $u_i'(\theta) = q_i(\theta) - r_i$ wherever u_i is differentiable (incentive compatibility implies that u_i is a maximum of a family of affine functions, which implies that u_i is convex and so absolutely continuous and differentiable almost everywhere in the interior of its domain), implying that (B.6) holds. ■

Using the definition of $u_i(\theta)$, we can rewrite equation (B.6) as saying that for all $\theta, \theta' \in [\underline{\theta}_i, \overline{\theta}_i]$,

$$m_i(\theta) = \theta(q_i(\theta) - r_i) - \int_{\theta'}^{\theta} (q_i(x) - r_i)dx - u_i(\theta'). \tag{B.7}$$

Reiterating the virtual type definitions from the body of the paper, for all agents $i \in \mathcal{N}$, we define

$$\Psi_i^S(\theta) \equiv \theta + \frac{F_i(\theta)}{f_i(\theta)} \quad \text{and} \quad \Psi_i^B(\theta) \equiv \theta - \frac{1 - F_i(\theta)}{f_i(\theta)}$$

where Ψ_i^S and Ψ_i^B are referred to as agent i's virtual cost and virtual value functions, respectively.For results involving the second-best, we impose regularity (Myerson, 1981), assuming that F_i is such that Ψ_i^S and Ψ_i^B are increasing. The overall virtual type function with critical type x is

$$\Psi_i(\theta, x) \equiv \begin{cases} \Psi_i^S(\theta) & \text{if } \theta \in [\underline{\theta}_i, x), \\ \Psi_i^B(\theta) & \text{if } \theta \in [x, \overline{\theta}_i]. \end{cases} \tag{B.8}$$

Because q_i is nondecreasing, it follows that the first-order condition $u_i'(\theta) = q_i(\theta) - r_i = 0$ characterizes a global minimum, provided that it is satisfied for some θ. The following lemma, a version of which was first established by Cramton et al. (1987), characterizes the set of worst-off types for any allocation rule such that q_i is nondecreasing:

Lemma B.2. *Given an IC, IR mechanism $\langle \mathbf{Q}, \mathbf{M} \rangle$, agent i's set of* worst-off types $\Omega_i(\mathbf{Q}) \equiv \arg\min_{\theta \in [\underline{\theta}_i, \overline{\theta}_i]} u_i(\theta)$ *satisfies*

$$\Omega_i(\mathbf{Q}) = \begin{cases} \{\theta_i \mid q_i(\theta_i) = r_i\} & \textit{if there exists } \theta_i \textit{ such that } q_i(\theta_i) = r_i, \\ \{\theta_i \mid q_i(\theta) < r_i \; \forall \theta < \theta_i \textit{ and } q_i(\theta) > r_i \; \forall \theta > \theta_i\} & \textit{otherwise,} \end{cases}$$

where $\Omega_i(\mathbf{Q})$ is a possibly degenerate interval in the first case and a singleton in the second case; further, given $\hat{\theta}_i \in \Omega_i(\mathbf{Q})$,

$$\mathbb{E}_{\theta_i}[m_i(\theta_i)] = \mathbb{E}_{\theta_i}\left[\Psi_i(\theta_i, \hat{\theta}_i) q_i(\theta_i)\right] - \hat{\theta}_i r_i - u_i(\hat{\theta}_i), \tag{B.9}$$

where $u_i(\hat{\theta}_i) \geq 0$ (and $u_i(\hat{\theta}_i) = 0$ if IR binds for agent i's worst-off types).

Proof. The characterization of worst-off types follows from Cramton et al. (1987). To obtain the expression for $\mathbb{E}_{\theta_i}[m_i(\theta_i)]$, note that using the expression for $m_i(\theta)$ in (B.7) with θ' replaced by $\hat{\theta}_i$, we have $\int_{\underline{\theta}_i}^{\overline{\theta}_i} m_i(\theta)dF_i(\theta) = \int_{\underline{\theta}_i}^{\overline{\theta}_i}(q_i(\theta)-r_i)\theta dF_i(\theta) - \int_{\hat{\theta}_i}^{\overline{\theta}_i}\int_{\hat{\theta}_i}^{\theta}(q_i(x)-r_i)dxdF_i(\theta) + \int_{\underline{\theta}_i}^{\hat{\theta}_i}\int_{\theta}^{\hat{\theta}_i}(q_i(x)-r_i)dxdF_i(\theta) - u_i(\hat{\theta}_i)$. Changing the order of integration in the double integrals and substituting the virtual type functions and noting that $\mathbb{E}_{\theta_i}[\Psi_i(\theta_i, \hat{\theta}_i)] = \hat{\theta}_i$ gives the result. ■

As observed by Cramton et al. (1987), intuitively, the worst-off type expects on average to be neither a buyer nor a seller, and therefore an agent with the worst-off type has no incentive to overstate or understate its valuation and so does not need to be compensated to induce truthful reporting, which is why it is the worst-off type.[4]

B.2 Optimal mechanisms

As shown in Loertscher and Marx (2024), the incomplete information bargaining mechanism is found by solving a saddle-point problem, simultaneously choosing the allocation rule to maximize a Lagrangian taking as given the agents' worst-off types (inner optimization problem) and finding the worst-off types that minimize agents' interim expected payoffs (outer optimization problem).[5] This saddle-point property means that even though the worst-off types are endogenous to the allocation rule, the allocation and payment rule are still separable in the sense that one can first derive the optimal allocation rule and then adjust payments to satisfy IR.[6]

[4] In simpler mechanism design settings, such as a sales auction, procurement auction, or two-sided setting as in Myerson and Satterthwaite (1983) or Gresik and Satterthwaite (1989), Lemma B.2 together with the payoff equivalence theorem, (B.6), and the monotonicity of the allocation due to IC imply that, for any $\mathbf{Q}$ satisfying (B.5), $\Omega_i(\mathbf{Q})$ contains 0 if $r_i = 0$ and 1 if $r_i = 1$. Thus, for settings like these, it is known a priori for which type of an agent the IR constraint will bind, irrespective of the specifics of the allocation rule. This means that when looking for optimal mechanisms, the market maker can focus on the allocation rule without worrying about repercussions on the IR constraint. In contrast, if $0 < r_i < 1$, then agent i's worst-off types will typically be interior and depend on the allocation rule. For example, under the first-best allocation rule $\mathbf{Q}^e$, which allocates 1 to the agents with the highest types, $\Omega_i(\mathbf{Q}^e)$ is a singleton $\hat{\theta}_i \in (0,1)$ if $0 < r_i < 1$. Of course, because the allocation rule is fixed under the first-best and so is not a choice variable, the worst-off types only depend on $\mathbf{r}$ and the distributions. Away from the first-best, this will, however, not be the case.

[5] The solution builds on and generalizes the earlier results by Lu and Robert (2001), who assume identical distributions and maximum demands (not necessarily equal to 1), and Loertscher and Wasser (2019), who study a partnership problem, i.e., they assume that each agent's maximum demand is equal to 1, allowing for heterogeneous distributions but with a common support. The setup of Loertscher and Wasser (2019) is more general in that they allow interdependent values.

[6] It is difficult to see how one could solve these problems in any degree of generality without the saddle-point property. As a case in point, not being aware of it, Kittsteiner (2003) found what turns out to be the optimal dissolution mechanism for a partnership problem with two agents without being able to prove optimality (see his footnote 19).

Building on the insights of Loertscher and Wasser (2019), we show that optimal mechanisms have a saddle point property. To state this property, recall that we define agent i's *weighted virtual type* with weight $\alpha \in [0,1]$ by $\Psi_{i,\alpha}(\theta, x) \equiv \alpha\theta_i + (1-\alpha)\Psi_i(\theta, x)$. This allows us to define the agents' *virtual net surplus* by

$$\tilde{W}_\rho(\mathbf{Q}, \hat{\boldsymbol{\theta}}) \equiv \rho\mathbb{E}_{\boldsymbol{\theta}}\left[\sum_{i\in\mathcal{N}}(Q_i(\boldsymbol{\theta}) - r_i)\Psi_{i,\frac{w_i}{\rho}}(\theta_i, \hat{\theta}_i)\right].$$

As we now show, for a given value of $\rho \geq \max \mathbf{w}$ and a given vector of worst-off types $\boldsymbol{\omega}^*$, the optimal allocation rule, denoted $\mathbf{Q}^*$, maximizes $\tilde{W}_\rho(\mathbf{Q}, \boldsymbol{\omega}^*)$ over monotone allocation rules $\mathbf{Q}$, while $\boldsymbol{\omega}^*$ is a type vector $\hat{\boldsymbol{\theta}}$ that minimizes $\tilde{W}_\rho(\mathbf{Q}^*, \hat{\boldsymbol{\theta}})$.

Defining

$$W_\rho(\mathbf{Q}, \mathbf{M}) \equiv \sum_{i\in\mathcal{N}} w_i\mathbb{E}_{\boldsymbol{\theta}}[\theta_i Q_i(\boldsymbol{\theta}) - M_i(\boldsymbol{\theta})] + \rho\sum_{i\in\mathcal{N}}\mathbb{E}_{\boldsymbol{\theta}}\left[M_i(\boldsymbol{\theta})\right], \tag{B.10}$$

we can use standard techniques to obtain the following lemma:

Lemma B.3. *If the mechanism $\langle \mathbf{Q}, \mathbf{M}\rangle$ satisfies IC, i.e., $(B.5)$ and $(B.6)$ hold, then for all $\hat{\boldsymbol{\theta}} \in \Theta$,*

$$W_\rho(\mathbf{Q}, \mathbf{M}) = \tilde{W}_\rho(\mathbf{Q}, \hat{\boldsymbol{\theta}}) - \sum_{i\in\mathcal{N}}(\rho - w_i)u_i(\hat{\theta}_i) + \sum_{i\in\mathcal{N}} r_i w_i \mathbb{E}_{\theta_i}[\theta_i], \tag{B.11}$$

and

$$\Omega(\mathbf{Q}) = \arg\min_{\boldsymbol{\theta}\in\Theta} \tilde{W}_\rho(\mathbf{Q}, \boldsymbol{\theta}). \tag{B.12}$$

Proof. Using the expression for $\mathbb{E}_{\theta_i}[m_i(\theta_i)]$ from equation (B.9) in Lemma B.2 and

$$\mathbb{E}_{\theta_i}\left[\Psi_{i,\frac{w_i}{\rho}}(\theta_i; \hat{\theta}_i)\right] = \frac{w_i}{\rho}\mathbb{E}_{\theta_i}[\theta_i] + (1 - \frac{w_i}{\rho})\hat{\theta}_i, \tag{B.13}$$

we can write:

$$\begin{aligned}
W_\rho(\mathbf{Q}, \mathbf{M}) &= \sum_{i\in\mathcal{N}} w_i\mathbb{E}_{\theta_i}[\theta_i q_i(\theta_i) - m_i(\theta_i)] + \rho\sum_{i\in\mathcal{N}}\mathbb{E}_{\theta_i}\left[m_i(\theta_i)\right] \\
&= \sum_{i\in\mathcal{N}} \rho\mathbb{E}_{\theta_i}\left[\Psi_{i,\frac{w_i}{\rho}}(\theta_i; \hat{\theta}_i) q_i(\theta_i)\right] + \sum_{i\in\mathcal{N}}(w_i - \rho)\left(u_i(\hat{\theta}_i) + \hat{\theta}_i r_i\right) \\
&= \sum_{i\in\mathcal{N}} \rho\mathbb{E}_{\theta_i}\left[\Psi_{i,\frac{w_i}{\rho}}(\theta_i; \hat{\theta}_i)\left(q_i(\theta_i) - r_i\right)\right] + \sum_{i\in\mathcal{N}}(w_i - \rho)\left(u_i(\hat{\theta}_i) + \hat{\theta}_i r_i\right) + \sum_{i\in\mathcal{N}}\rho\mathbb{E}_{\theta_i}\left[\Psi_{i,\frac{w_i}{\rho}}(\theta_i; \hat{\theta}_i) r_i\right] \\
&= \tilde{W}_\rho(\mathbf{Q}, \hat{\boldsymbol{\theta}}) + \sum_{i\in\mathcal{N}}(w_i - \rho)\, u_i(\hat{\theta}_i) + \sum_{i\in\mathcal{N}} r_i w_i \mathbb{E}_{\theta_i}[\theta_i],
\end{aligned}$$

where the first equality uses the definition of W_ρ, the second uses (B.9), the third uses the definition of $\Psi_{i,\frac{w_i}{\rho}}$, which implies that

$$\mathbb{E}_{\theta_i}\left[\left(w_i\theta_i + (\rho - w_i)\Psi_{i,0}(\theta_i;\hat{\theta}_i)\right) q_i(\theta_i)\right] = \rho\mathbb{E}_{\theta_i}\left[\Psi_{i,\frac{w_i}{\rho}}(\theta_i;\hat{\theta}_i)q_i(\theta_i)\right],$$

and the last uses (B.13). This establishes (B.11).

According to (B.11), for any exogenously fixed critical types $\hat{\boldsymbol{\theta}}$, we can write $W_\rho(\mathbf{Q},\mathbf{M})$ as being equal to $\tilde{W}_\rho(\mathbf{Q},\hat{\boldsymbol{\theta}})$ minus $\sum_{i\in\mathcal{N}}(\rho - w_i)u_i(\hat{\theta}_i)$ and plus $\sum_{i\in\mathcal{N}} r_i w_i \mathbb{E}_{\theta_i}[\theta_i]$, which does not depend on $\hat{\boldsymbol{\theta}}$. Because for a given allocation rule $\mathbf{Q}$, (B.11) is constant over all $\hat{\boldsymbol{\theta}}$, the sets of critical types that minimize $u_i(\cdot)$ for each i must also be the sets of critical types that minimize $\tilde{W}_\rho(\mathbf{Q},\hat{\boldsymbol{\theta}})$, implying (B.12). ■

Writing the payment rule as a function of the allocation rule

Defining

$$\mathcal{Q} \equiv \{\mathbf{Q} \mid q_i \text{ is nondecreasing for each } i \in \mathcal{N}\}$$

and using Lemma B.1, we can replace the IC constraint (B.1) with $\mathbf{Q} \in \mathcal{Q}$ and (B.6), or, equivalently, with $\mathbf{Q} \in \mathcal{Q}$ and (B.7). Further, we can replace the IR constraint (B.2) with the requirement that for all $i \in \mathcal{N}$ and $\omega_i \in \Omega(\mathbf{Q})$, we have $u_i(\omega_i) \geq 0$.

Thus, if we consider an allocation rule $\mathbf{Q} \in \mathcal{Q}$ and worst-off types $\boldsymbol{\omega} = (\omega_1, \ldots, \omega_n) \in \Omega(\mathbf{Q})$, then our constrained optimization problem (B.4) amounts to maximizing

$$\sum_{i\in\mathcal{N}} w_i \mathbb{E}_{\boldsymbol{\theta}}[\theta_i Q_i(\boldsymbol{\theta}) - M_i(\boldsymbol{\theta})]$$

subject to (B.7) (which together with $\mathbf{Q} \in \mathcal{Q}$ gives us IC), the IR constraint that $u_i(\omega_i) \geq 0$ for all $i \in \mathcal{N}$, and the no-deficit constraint that $\sum_{i\in\mathcal{N}} \mathbb{E}_{\boldsymbol{\theta}}[M_i(\boldsymbol{\theta})] \geq 0$. The associated Lagrangian is then

$$\begin{aligned}
\mathcal{L} &= \sum_{i\in\mathcal{N}} w_i \mathbb{E}_{\boldsymbol{\theta}}[\theta_i Q_i(\boldsymbol{\theta}) - M_i(\boldsymbol{\theta})] + \rho \sum_{i\in\mathcal{N}} \mathbb{E}_{\boldsymbol{\theta}}[M_i(\boldsymbol{\theta})] + \sum_{i\in\mathcal{N}} \mu_i u_i(\omega_i) \text{ s.t. (B.7)} \\
&= W_\rho(\mathbf{Q},\mathbf{M}) + \sum_{i\in\mathcal{N}} \mu_i u_i(\omega_i) \text{ s.t. (B.7)} \\
&= \tilde{W}_\rho(\mathbf{Q},\boldsymbol{\omega}) + \sum_{i\in\mathcal{N}} (w_i - \rho + \mu_i)\, u_i(\omega_i) + \sum_{i\in\mathcal{N}} r_i w_i \mathbb{E}_{\theta_i}[\theta_i],
\end{aligned} \tag{B.14}$$

where $u_i(\omega_i)$ can be treated parametrically.[7] The constants $u_i(\omega_i)$ play the role of distributing all the expected budget surplus that remains after IR constraints are satisfied with equality

[7]Because $u_i(\omega_i) = \omega_i(q_i(\omega_i) - r_i) - m_i(\omega_i)$, no matter what the pointwise maximizer implies for $q_i(\omega_i)$, one can achieve any value for $u_i(\omega_i)$ by appropriately varying the fixed payment in $m_i(\omega_i)$.

to the agents with the maximum bargaining weight, in proportion to shares $\eta_1, \ldots, \eta_n$ if multiple agents have the maximum bargaining weight.

As is apparent from the final line of (B.14), feasibility requires that $\rho \geq \max \mathbf{w}$,[8] because otherwise the solution would specify that an infinite amount be transferred from low-bargaining-weight agents to high-bargaining-weight agents, in violation of IR.[9]

Using the result of Lemma B.2 that $\mathbb{E}_{\theta_i}[m_i(\theta_i)] = \mathbb{E}_{\theta_i}[\Psi_{i,0}(\theta_i;\omega_i)q_i(\theta_i)] - \omega_i r_i - u_i(\omega_i)$, the expected revenue when IR is satisfied with equality for agents' worst-off types is

$$\pi(\mathbf{Q},\boldsymbol{\omega}) \equiv \sum_{i\in\mathcal{N}} \mathbb{E}_{\theta_i}[\Psi_i(\theta_i,\omega_i)q_i(\theta_i)] - \sum_{i\in\mathcal{N}} \omega_i r_i.$$

If $\pi(\mathbf{Q},\boldsymbol{\omega}) > 0$, then the objective in (B.4) is maximized by allocating $\pi(\mathbf{Q},\boldsymbol{\omega})$ among the agents with bargaining weights equal to $\max \mathbf{w}$, which is accomplished by having interim expected payoffs to the agents' worst-off types of

$$u_i(\omega_i) = \eta_i \pi(\mathbf{Q},\boldsymbol{\omega}),$$

where $\eta_i \in [0,1]$, $\sum_{i\in\mathcal{N}} \eta_i = 1$, and $\eta_i = 0$ for any agent that does not have the maximum bargaining weight, i.e., $\eta_i = 0$ if $w_i < \max \mathbf{w}$.[10] This implies $\mathbf{M}$ such that interim expected payments satisfy, for all $i \in \mathcal{N}$,

$$m_i(\theta_i) = \theta_i\left(q_i(\theta_i) - r_i\right) - \int_{\omega_i}^{\theta_i} (q_i(y) - r_i)dy - \eta_i \pi(\mathbf{Q},\boldsymbol{\omega}).$$

Given this, we can turn our attention to the allocation rule.

[8]There is an equivalence between working with a bargaining weight of $w \in [0,1]$ for agent 1 and $1-w$ for agent 2 and, alternatively, assuming that $w_1 - w_2 = \Delta \in [-1,1]$, where $\max\{w_1, w_2\} = 1$. We have:

$$w(\Delta) = \begin{cases} \frac{1}{2-\Delta} & \text{if } \Delta \in [0,1], \\ \frac{1+\Delta}{2+\Delta} & \text{if } \Delta \in [-1,0], \end{cases} \quad \text{and} \quad \Delta(w) = \begin{cases} \frac{2w-1}{w} & \text{if } w \geq 1/2, \\ \frac{2w-1}{1-w} & \text{if } w < 1/2. \end{cases}$$

The expected net payoff frontier for a given r has slope $\frac{-w}{1-w}$ for $w \in [0,1]$ or equivalently $\frac{-1}{1-\Delta}$ if $\Delta \in [0,1]$ and $-(1+\Delta)$ if $\Delta \in [-1,0]$.

[9]The result follows from the same arguments that were first developed in the working paper version of Gresik and Satterthwaite (1989) and that were first used in published form in Myerson and Satterthwaite (1983).

[10]For completeness, note that when $\pi(\mathbf{Q}^*,\boldsymbol{\omega}^*) > 0$, we have $\rho^* = 1$ and, by stationarity, $\mu_i^* = \rho^* - w_i$, which implies that $\mu_i^* = 0$ if $w_i = \max \mathbf{w} = 1$ and $u_i(\omega^*) = 0$ if $w_i < 1$, ensuring that the associated complementary slackness condition is satisfied. If $\pi(\mathbf{Q}^*,\boldsymbol{\omega}^*) = 0$, then $\rho^* \geq 1$, $\mu_i^* = \rho^* - w_i \geq 0$, and $u_i(\omega^*) = 0$, so again complementary slackness is satisfied.

Determining the allocation rule

It remains to determine the allocation rule. Because the second term on the right side of (B.14) can be treated parametrically and the third term is independent of the allocation rule, we can restrict attention to maximizing $\tilde{W}_\rho(\mathbf{Q}, \boldsymbol{\omega}) = \min_{\hat{\boldsymbol{\theta}}} \tilde{W}_\rho(\mathbf{Q}, \hat{\boldsymbol{\theta}})$, where the equality follows from (B.12) in Lemma B.3. Consequently, an optimal allocation rule $\mathbf{Q}_\rho$ has to satisfy

$$\mathbf{Q}_\rho \in \arg\max_{\mathbf{Q}\in\mathcal{Q}} \min_{\hat{\boldsymbol{\theta}}\in\Theta} \tilde{W}_\rho(\mathbf{Q}, \hat{\boldsymbol{\theta}}). \tag{B.15}$$

Instead of directly solving the max-min problem in (B.15), we look for a *saddle point* $(\mathbf{Q}^*, \boldsymbol{\omega}^*)$ of $\tilde{W}_\rho$ that satisfies

$$\mathbf{Q}^* \in \arg\max_{\mathbf{Q}\in\mathcal{Q}} \tilde{W}_\rho(\mathbf{Q}, \boldsymbol{\omega}^*) \tag{B.16}$$

$$\boldsymbol{\omega}^* \in \arg\min_{\hat{\boldsymbol{\theta}}\in\Theta} \tilde{W}_\rho(\mathbf{Q}^*, \hat{\boldsymbol{\theta}}). \tag{B.17}$$

For a saddle point, (B.16) requires that the allocation rule $\mathbf{Q}^*$ maximizes the virtual objective $\tilde{W}_\rho$ under critical types $\boldsymbol{\omega}^*$, whereas (B.17) requires that the critical types $\boldsymbol{\omega}^*$ are worst-off types under allocation rule $\mathbf{Q}^*$, i.e., $\boldsymbol{\omega}^* \in \Omega(\mathbf{Q}^*)$.

If a saddle point $(\mathbf{Q}^*, \boldsymbol{\omega}^*)$ exists, then $\mathbf{Q}_\rho$ solves the problem in (B.15) if and only if $(\mathbf{Q}_\rho, \boldsymbol{\omega}^*)$ is a saddle point.[11] Next, in Section B.3, we show that a saddle point $(\mathbf{Q}^*, \boldsymbol{\omega}^*)$ exists, and then in Section B.4, we characterize the second-best mechanism.

B.3 Existence of a saddle point

Here we establish the existence of the second-best mechanism through an application of Sion's minimax theorem (Sion, 1958).

To begin, note that $\Theta \equiv \times_{i\in\mathcal{N}}[\underline{\theta}_i, \overline{\theta}_i]$ is a compact convex subset of a linear topological space with the usual Euclidian distance, vector addition, and scalar multiplication. Recall that $\mathcal{Q} \equiv \{\mathbf{Q} \mid q_i \text{ is nondecreasing for each } i \in \mathcal{N}\}$, and define addition and scalar multiplication on $\mathcal{Q}$ pointwise.[12] Further define $\mathcal{Q}^\Delta \equiv \{\mathbf{Q} \in \mathcal{Q} \mid \mathbf{Q}(\boldsymbol{\theta}) \in \Delta \text{ for all } \boldsymbol{\theta} \in \Theta\}$. It is straightforward to show that $\mathcal{Q}^\Delta$ is convex.[13] We can define a distance measure on $\mathcal{Q}$ based

[11] As noted by Loertscher and Wasser (2019, footnote 11): Suppose that $(\mathbf{Q}^*, \boldsymbol{\omega}^*)$ satisfies (B.16) and (B.17). Then $\min_{\hat{\boldsymbol{\theta}}} \tilde{W}_\rho(\mathbf{Q}^*, \hat{\boldsymbol{\theta}}) = \tilde{W}_\rho(\mathbf{Q}^*, \boldsymbol{\omega}^*) \geq \tilde{W}_\rho(\mathbf{Q}, \boldsymbol{\omega}^*) \geq \min_{\hat{\boldsymbol{\theta}}} \tilde{W}_\rho(\mathbf{Q}, \hat{\boldsymbol{\theta}})$ for all $\mathbf{Q} \in \mathcal{Q}$ and hence $\mathbf{Q}^*$ solves the problem in (B.15). Conversely, for all $\mathbf{Q}_\rho$ that satisfy (B.15), the above has to hold with equality, implying that $(\mathbf{Q}_\rho, \boldsymbol{\omega}^*)$ is a saddle point.

[12] That is, given $\mathbf{Q}^1, \mathbf{Q}^2 \in \mathcal{Q}$, $\mathbf{Q}^1 + \mathbf{Q}^2$ is defined by $(\mathbf{Q}^1 + \mathbf{Q}^2)_i(\boldsymbol{\theta}) = Q_i^1(\boldsymbol{\theta}) + Q_i^2(\boldsymbol{\theta})$ for all $i \in \mathcal{N}$, and given $\lambda \in \mathbb{R}$, $\lambda\mathbf{Q}^1$ is defined by $(\lambda\mathbf{Q}^1)_i(\boldsymbol{\theta}) = \lambda Q_i^1(\boldsymbol{\theta})$ for all $i \in \mathcal{N}$.

[13] Because if $\mathbf{Q}^1, \mathbf{Q}^2 \in \mathcal{Q}$ and $\lambda \in (0,1)$, then $\mathbb{E}_{\boldsymbol{\theta}_{-i}}[\lambda Q_i^1(\boldsymbol{\theta}) + (1-\lambda)Q_i^2(\boldsymbol{\theta})] = \lambda q_i^1(\theta_i) + (1-\lambda)q_i^2(\theta_i)$, which is nondecreasing because q_i^1 and q_i^2 are nondecreasing and $(\lambda\mathbf{Q}^1 + (1-\lambda)\mathbf{Q}^2)(\boldsymbol{\theta})$ is contained in the

on the sup norm by

$$d(\mathbf{Q}^0,\mathbf{Q}^1)\equiv\sup_{i\in\mathcal{N},\boldsymbol{\theta}\in\Theta}\left|Q_i^0(\boldsymbol{\theta})-Q_i^1(\boldsymbol{\theta})\right|.$$

Because d is a metric,[14] it follows that $\mathcal{Q}^\Delta$ together with the sup norm metric is a convex subset of a linear topological space. One can further show that $\mathcal{Q}^\Delta$ is compact, giving us the following result:

Lemma B.4. *$\mathcal{Q}^\Delta$ with the sup norm metric is a compact convex subset of a linear topological space.*

Proof. Given the arguments above, it remains to show only that $\mathcal{Q}^\Delta$ is compact. To show this, first note that $\mathcal{Q}^\Delta$ is bounded by the simplex. Second, note that given any convergent sequence $(\mathbf{Q}^j)_{j=1}^\infty$ in $\mathcal{Q}^\Delta$ with limit $\mathbf{Q}^*$, the sequence is eventually contained in an open ball of radius $\delta>0$ around $\mathbf{Q}^*$. It is straightforward to show that q_i^* is nondecreasing for all $i\in\mathcal{N}$, which implies that $\mathbf{Q}^*\in\mathcal{Q}$, and that $\mathbf{Q}^*(\boldsymbol{\theta})$ is in the simplex for all $\boldsymbol{\theta}\in\Theta$, giving us the result that $\mathbf{Q}^*\in\mathcal{Q}^\Delta$, completing the proof that $\mathcal{Q}^\Delta$ is compact. ■

Recalling that given $\mathbf{w}\in[0,1]^n$ and $\rho\geq\max\mathbf{w}$, we define $\tilde{W}_\rho:\mathcal{Q}^\Delta\times\Theta\to\mathbb{R}$ by

$$\tilde{W}_\rho(\mathbf{Q},\hat{\boldsymbol{\theta}})\equiv\rho\mathbb{E}_{\boldsymbol{\theta}}\left[\sum_{i\in\mathcal{N}}(Q_i(\boldsymbol{\theta})-r_i)\Psi_{i,\frac{w_i}{\rho}}(\theta_i,\hat{\theta}_i)\right],$$

we have the following result:

Lemma B.5. *Given $\hat{\boldsymbol{\theta}}\in\Theta$, $\tilde{W}_\rho(\cdot,\hat{\boldsymbol{\theta}})$ is upper semicontinuous and quasiconcave on $\mathcal{Q}^\Delta$, and given $\mathbf{Q}\in\mathcal{Q}^\Delta$, $\tilde{W}_\rho(\mathbf{Q},\cdot)$ is lower semicontinuous and quasiconvex on Θ.*

Proof. Take $\mathbf{Q}\in\mathcal{Q}^\Delta$ as given. Using the definitions of $q_i(\theta_i)$, which is nondecreasing and so differentiable almost everywhere, and of$\Psi_{i,\frac{w_i}{\rho}}(\theta_i;\hat{\theta}_i)$, we can write

$$\tilde{W}_\rho(\mathbf{Q},\hat{\boldsymbol{\theta}})=\rho\sum_{i\in\mathcal{N}}\left(\int_{\underline{\theta}_i}^{\hat{\theta}_i}(q_i(\theta_i)-r_i)\Psi^S_{i,\frac{w_i}{\rho}}(\theta_i)dF_i(\theta_i)+\int_{\hat{\theta}_i}^{\bar{\theta}_i}(q_i(\theta_i)-r_i)\Psi^B_{i,\frac{w_i}{\rho}}(\theta_i)dF_i(\theta_i)\right).$$

It then follows that $\tilde{W}_\rho(\mathbf{Q},\hat{\boldsymbol{\theta}})$ is continuous (indeed differentiable) in $\hat{\boldsymbol{\theta}}$ and so it is lower

simplex for all $\boldsymbol{\theta}\in\Theta$, implying that $\lambda\mathbf{Q}^1+(1-\lambda)\mathbf{Q}^2\in\mathcal{Q}^\Delta$.

[14]To see this, note that $d(\mathbf{Q}^0,\mathbf{Q}^1)\geq 0$, $d(\mathbf{Q}^0,\mathbf{Q}^1)=d(\mathbf{Q}^1,\mathbf{Q}^0)$, $\mathbf{Q}^0\neq\mathbf{Q}^1$ implies $d(\mathbf{Q}^0,\mathbf{Q}^1)>0$, and $d(\mathbf{Q}^0,\mathbf{Q}^2)\leq d(\mathbf{Q}^0,\mathbf{Q}^1)+d(\mathbf{Q}^1,\mathbf{Q}^2)$. To see that this last triangle inequality holds, note that $d(\mathbf{Q}^0,\mathbf{Q}^2)=\sup_{i\in\mathcal{N},\boldsymbol{\theta}\in\Theta}\left|Q_i^0(\boldsymbol{\theta})-Q_i^2(\boldsymbol{\theta})\right|\leq\sup_{i\in\mathcal{N},\boldsymbol{\theta}\in\Theta}\left(\left|Q_i^0(\boldsymbol{\theta})-Q_i^1(\boldsymbol{\theta})\right|+\left|Q_i^1(\boldsymbol{\theta})-Q_i^2(\boldsymbol{\theta})\right|\right)\leq\sup_{i\in\mathcal{N},\boldsymbol{\theta}\in\Theta}\left|Q_i^0(\boldsymbol{\theta})-Q_i^1(\boldsymbol{\theta})\right|+\sup_{i\in\mathcal{N},\boldsymbol{\theta}\in\Theta}\left|Q_i^1(\boldsymbol{\theta})-Q_i^2(\boldsymbol{\theta})\right|=d(\mathbf{Q}^0,\mathbf{Q}^1)+d(\mathbf{Q}^1,\mathbf{Q}^2)$.

semicontinuous in $\hat{\boldsymbol{\theta}}$. We have

$$\frac{\partial \tilde{W}_\rho(\mathbf{Q},\hat{\boldsymbol{\theta}})}{\partial \hat{\theta}_j} = \rho(q_j(\hat{\theta}_j)-r_j)\left(\Psi^S_{j,\frac{w_j}{\rho}}(\hat{\theta}_j)-\Psi^B_{j,\frac{w_j}{\rho}}(\hat{\theta}_j)\right)f_j(\hat{\theta}_j) = \rho(q_j(\hat{\theta}_j)-r_j)(1-w_j/\rho).$$

Thus, for $j \neq \ell$, $\frac{\partial^2 \tilde{W}_\rho(\mathbf{Q},\hat{\boldsymbol{\theta}})}{\partial \hat{\theta}_j \partial \hat{\theta}_\ell} = 0$. Given j and $\hat{\boldsymbol{\theta}}$ such that q_j is differentiable at $\hat{\theta}_j$, $\frac{\partial^2 \tilde{W}_\rho(\mathbf{Q},\hat{\boldsymbol{\theta}})}{\partial \hat{\theta}_j^2} = \rho q_j'(\hat{\theta}_j)(1-\frac{w_j}{\rho}) \geq 0$. If q_j is not differentiable at $\hat{\theta}_j$, then q_j jumps up at $\hat{\theta}_j$, implying a jump up in $\frac{\partial \tilde{W}_\rho(\mathbf{Q},\hat{\boldsymbol{\theta}})}{\partial \hat{\theta}_j}$. Thus $\tilde{W}_\rho(\mathbf{Q},\hat{\boldsymbol{\theta}})$ is convex and so quasiconvex with respect to $\hat{\boldsymbol{\theta}}$.

Now take $\hat{\boldsymbol{\theta}} \in \Theta$ as given and consider how $\tilde{W}_\rho(\mathbf{Q},\hat{\boldsymbol{\theta}})$ varies with $\mathbf{Q} \in \mathcal{Q}^\Delta$. We can write

$$\tilde{W}_\rho(\mathbf{Q},\hat{\boldsymbol{\theta}}) = \rho \int_{\underline{\theta}_1}^{\overline{\theta}_1} \cdots \int_{\underline{\theta}_n}^{\overline{\theta}_n} \left(\sum_{i\in\mathcal{N}} (Q_i(\boldsymbol{\theta})-r_i)\Psi_{i,\frac{w_i}{\rho}}(\theta_i,\hat{\theta}_i)\right) d\mathbf{F}(\boldsymbol{\theta}).$$

To see that $\tilde{W}_\rho(\mathbf{Q},\hat{\boldsymbol{\theta}})$ is quasiconcave in $\mathbf{Q}$, note that for all $\lambda \in (0,1)$, we have

$$\begin{aligned}
\tilde{W}_\rho(\lambda\mathbf{Q}^0+(1-\lambda)\mathbf{Q}^1,\hat{\boldsymbol{\theta}}) &= \rho \int_{\underline{\theta}_1}^{\overline{\theta}_1} \cdots \int_{\underline{\theta}_n}^{\overline{\theta}_n} \left(\sum_{i\in\mathcal{N}} (\lambda Q_i^0(\boldsymbol{\theta})+(1-\lambda)Q_i^1(\boldsymbol{\theta})-r_i)\Psi_{i,\frac{w_i}{\rho}}(\theta_i,\hat{\theta}_i)\right) d\mathbf{F}(\boldsymbol{\theta}) \\
&= \lambda\rho \int_{\underline{\theta}_1}^{\overline{\theta}_1} \cdots \int_{\underline{\theta}_n}^{\overline{\theta}_n} \left(\sum_{i\in\mathcal{N}} (Q_i^0(\boldsymbol{\theta})-r_i)\Psi_{i,\frac{w_i}{\rho}}(\theta_i,\hat{\theta}_i)\right) d\mathbf{F}(\boldsymbol{\theta}) \\
&\quad +(1-\lambda)\rho \int_{\underline{\theta}_1}^{\overline{\theta}_1} \cdots \int_{\underline{\theta}_n}^{\overline{\theta}_n} \left(\sum_{i\in\mathcal{N}} (Q_i^1(\boldsymbol{\theta})-r_i)\Psi_{i,\frac{w_i}{\rho}}(\theta_i,\hat{\theta}_i)\right) d\mathbf{F}(\boldsymbol{\theta}) \\
&= \lambda\tilde{W}_\rho(\mathbf{Q}^0,\hat{\boldsymbol{\theta}}) + (1-\lambda)\tilde{W}_\rho(\mathbf{Q}^1,\hat{\boldsymbol{\theta}}) \\
&\geq \min\left\{\tilde{W}_\rho(\mathbf{Q}^0,\hat{\boldsymbol{\theta}}), \tilde{W}_\rho(\mathbf{Q}^1,\hat{\boldsymbol{\theta}})\right\}.
\end{aligned}$$

To see that $\tilde{W}_\rho(\mathbf{Q},\hat{\boldsymbol{\theta}})$ is upper semicontinuous in $\mathbf{Q}$, let $\mathbf{Q}^0 \in \mathcal{Q}^\Delta$ be given and let real number y be such that $y > \tilde{W}_\rho(\mathbf{Q}^0,\hat{\boldsymbol{\theta}})$. We show that there exists a neighborhood $\mathcal{U}$ of $\mathbf{Q}^0$ such that $\tilde{W}_\rho(\mathbf{Q},\hat{\boldsymbol{\theta}}) < y$ for all $\mathbf{Q} \in \mathcal{U}$. To see this, let $\mathcal{U}$ be an open ball of radius $\delta > 0$ around $\mathbf{Q}^0$, i.e., $\mathcal{U} \equiv \{\mathbf{Q} \in \mathcal{Q}^\Delta \mid d(\mathbf{Q},\mathbf{Q}^0) < \delta\}$, where

$$\delta = \frac{y-\tilde{W}_\rho(\mathbf{Q}^0,\hat{\boldsymbol{\theta}})}{\rho\sum_{i\in\mathcal{N}}\int_{\underline{\theta}_i}^{\overline{\theta}_i}\left|\Psi_{i,\frac{w_i}{\rho}}(\theta_i,\hat{\theta}_i)\right| dF_i(\theta_i)} > 0.$$

Then for all $\mathbf{Q} \in \mathcal{U}$, we have

$$\begin{aligned}
\tilde{W}_\rho(\mathbf{Q}, \hat{\boldsymbol{\theta}}) - \tilde{W}_\rho(\mathbf{Q}^0, \hat{\boldsymbol{\theta}}) &= \rho \int_{\underline{\theta}_1}^{\bar{\theta}_1} \cdots \int_{\underline{\theta}_n}^{\bar{\theta}_n} \left(\sum_{i \in \mathcal{N}} (Q_i(\boldsymbol{\theta}) - Q_i^0(\boldsymbol{\theta})) \Psi_{i, \frac{w_i}{\rho}}(\theta_i, \hat{\theta}_i) \right) d\mathbf{F}(\boldsymbol{\theta}) \\
&\leq \rho \int_{\underline{\theta}_1}^{\bar{\theta}_1} \cdots \int_{\underline{\theta}_n}^{\bar{\theta}_n} \left(\sum_{i \in \mathcal{N}} \left| Q_i(\boldsymbol{\theta}) - Q_i^0(\boldsymbol{\theta}) \right| \left| \Psi_{i, \frac{w_i}{\rho}}(\theta_i, \hat{\theta}_i) \right| \right) d\mathbf{F}(\boldsymbol{\theta}) \\
&< \delta\rho \sum_{i \in \mathcal{N}} \int_{\underline{\theta}_i}^{\bar{\theta}_i} \left| \Psi_{i, \frac{w_i}{\rho}}(\theta_i, \hat{\theta}_i) \right| dF_i(\theta_i) \\
&= y - \tilde{W}_\rho(\mathbf{Q}^0, \hat{\boldsymbol{\theta}}),
\end{aligned}$$

which completes the proof. ■

Using Lemmas B.4 and B.5 together with Sion's Minimax Theorem,[15] we have:

Theorem B.1. *There exists a* saddle point $(\mathbf{Q}^*, \boldsymbol{\omega}^*)$ *of* $\tilde{W}_\rho$ *that satisfies* $\mathbf{Q}^* \in \arg\max_{\mathbf{Q} \in \mathcal{Q}^\Delta} \tilde{W}_\rho(\mathbf{Q}, \boldsymbol{\omega}^*)$ *and* $\boldsymbol{\omega}^* \in \arg\min_{\hat{\boldsymbol{\theta}} \in \Theta} \tilde{W}_\rho(\mathbf{Q}^*, \hat{\boldsymbol{\theta}})$.

Theorem B.1 establishes the existence of the second-best mechanism. In what follows we provide a characterization.

B.4 Characterizing the second-best mechanism

Consider the optimization problem in (B.16). Pointwise maximization of

$$\tilde{W}_\rho(\mathbf{Q}, \boldsymbol{\omega}^*) \equiv \rho \mathbb{E}_{\boldsymbol{\theta}} \Big[\sum_{i \in \mathcal{N}} (Q_i(\boldsymbol{\theta}) - r_i) \Psi_{i, \frac{w_i}{\rho}}(\theta; \omega_i^*) \Big]$$

would require allocating the supply to one of the agents with the highest weighted virtual types, $\{\Psi_{i, \frac{w_i}{\rho}}(\theta; \omega_i^*)\}_{i \in \mathcal{N}}$. However, if $\omega_i^* \in (\underline{\theta}_i, \bar{\theta}_i)$ and $w_i < \rho$, then $\Psi_{i, \frac{w_i}{\rho}}(\theta; \omega_i^*)$ is nonmonotone (with a jump downward at ω_i^*), resulting in a violation of the monotonicity constraint $\mathbf{Q} \in \mathcal{Q}$. The solution to (B.16) therefore involves ironing (Myerson 1981), with the resources being allocated to one the agents with the highest *ironed* weighted virtual types, which we define below in (B.18).

[15] Using our notation, Sion's Minimax Theorem (as formulated by Komiya (1988)) states: If Θ is a compact convex subset of a linear topological space, $\mathcal{Q}^\Delta$ is a convex subset of a linear topological space, and $\tilde{W}_\rho$ is a real-valued function on $\mathcal{Q}^\Delta \times \Theta$ with $\tilde{W}_\rho(\cdot, \boldsymbol{\theta})$ upper semicontinuous and quasi-concave on $\mathcal{Q}^\Delta$ for all $\boldsymbol{\theta} \in \Theta$ and $\tilde{W}_\rho(\mathbf{Q}, \cdot)$ lower semicontinuous and quasi-convex on Θ for all $\mathbf{Q} \in \mathcal{Q}^\Delta$, then $\min_{\boldsymbol{\theta} \in \Theta} \sup_{\mathbf{Q} \in \mathcal{Q}^\Delta} \tilde{W}_\rho(\mathbf{Q}, \boldsymbol{\theta}) = \sup_{\mathbf{Q} \in \mathcal{Q}^\Delta} \min_{\boldsymbol{\theta} \in \Theta} \tilde{W}_\rho(\mathbf{Q}, \boldsymbol{\theta})$ (Sion, 1958, Corollary 3.3; Komiya, 1988). We show that, in addition to the requirements for Sion's Minimax Theorem, $\mathcal{Q}^\Delta$ is compact, giving us the result that $\min_{\boldsymbol{\theta} \in \Theta} \max_{\mathbf{Q} \in \mathcal{Q}^\Delta} \tilde{W}_\rho(\mathbf{Q}, \boldsymbol{\theta}) = \max_{\mathbf{Q} \in \mathcal{Q}^\Delta} \min_{\boldsymbol{\theta} \in \Theta} \tilde{W}_\rho(\mathbf{Q}, \boldsymbol{\theta})$.

Ironing

For the discussion of ironing, it will be useful to define the weighted virtual cost and weighted virtual value functions: for $\alpha \in [0,1]$,

$$\Psi^S_{i,\alpha}(\theta) \equiv \alpha\theta + (1-\alpha)\Psi^S_i(\theta) \quad \text{and} \quad \Psi^B_{i,\alpha}(\theta) \equiv \alpha\theta + (1-\alpha)\Psi^B_i(\theta).$$

The assumed monotonicity of $\Psi^S_i(\theta)$ and $\Psi^B_i(\theta)$ implies monotonicity of $\Psi^S_{i,\alpha}(\theta)$ and $\Psi^B_{i,\alpha}(\theta)$. However, as noted above, the overall weighted virtual type, $\Psi_{i,\alpha}(\theta,\omega)$, is not monotone if $\omega \in (\underline{\theta}_i, \overline{\theta}_i)$ and $\alpha \in [0,1)$.

Thus, to satisfy monotonicity, the allocation rule of the second-best mechanism will be based on the *ironed* weighted virtual type functions,

$$\overline{\psi}_{i,\alpha}(\theta, z_i) \equiv \begin{cases} \Psi^S_{i,\alpha}(\theta) & \text{if } \Psi^S_{i,\alpha}(\theta) < z_i, \\ z_i & \text{if } \Psi^B_{i,\alpha}(\theta) \leq z_i \leq \Psi^S_{i,\alpha}(\theta), \\ \Psi^B_{i,\alpha}(\theta) & \text{if } z_i < \Psi^B_{i,\alpha}(\theta), \end{cases} \tag{B.18}$$

where for $\omega_i \in [\underline{\theta}_i, \overline{\theta}_i]$, the ironing parameter $z_i \in [\Psi^B_{i,\alpha}(\omega_i), \Psi^S_{i,\alpha}(\omega_i)]$ is the unique solution to

$$\mathbb{E}_{\theta_i}\left[\Psi_{i,\alpha}(\theta_i, \omega_i)\right] = \mathbb{E}_{\theta_i}\left[\overline{\psi}_{i,\alpha}(\theta_i, z_i)\right],$$

which defines $z^*_{i,\alpha}(\omega_i)$. In the body of the paper, we write the ironed virtual type as $\overline{\Psi}_{i,\alpha}(\theta, \omega_i)$, where

$$\overline{\Psi}_{i,\alpha}(\theta, \omega_i) \equiv \overline{\psi}_{i,\alpha}(\theta, z^*_{i,\alpha}(\omega_i)). \tag{B.19}$$

According to (B.19), there is a one-to-one relation between the critical type ω^*_i and the corresponding ironing parameter z_i, which we write as

$$\omega^*_i = \omega_{i,\rho}(z_i). \tag{B.20}$$

Note that $\omega_{i,\rho}(\cdot)$ is a continuous and strictly increasing function.

Observe that for any $\alpha_i \in [0,1]$ and any $z_i \in [\Psi^B_{i,\alpha}(\underline{\theta}_i), \Psi^S_{i,\alpha}(\overline{\theta}_i)]$, the function $\overline{\psi}_{i,\alpha_i}(\theta, z_i)$ is monotone in θ because $\Psi^S_{i,\alpha_i}(\theta)$ and $\Psi^B_{i,\alpha_i}(\theta)$ are monotone. Consequently, any allocation rule $\mathbf{Q}_{\boldsymbol{\alpha}}$ that for each $\boldsymbol{\theta}$ allocates the resources to an agent with the highest ironed weighted virtual type, with ties among agents with the same ironed weighted virtual type broken randomly, satisfies the monotonicity requirement (B.5). As we show, the incomplete information bargaining mechanism always uses such an allocation rule.[16]

[16] Under the first-best, we have $\alpha_i = 1$ for all i and the ironing parameters $\mathbf{z}$ can be chosen arbitrarily

Tie-breaking rules

We must address the possibility that $z_i^* = z_j^*$ for some i and j, in which case ties between the ironed weighted virtual types arise with positive probability.[17] As in Loertscher and Wasser (2019), let H denote the set of all $n!$ permutations $(h_1, h_2, \ldots, h_{n!})$ of $(1, 2, \ldots, n)$. We call each $h \in H$ a hierarchy among the agents in $\mathcal{N}$. A *hierarchical tie-breaking rule* breaks ties in favor of the agent that is the highest in the hierarchy. Under a *split hierarchical tie-breaking rule* $\mathbf{a} \in \Delta^{n!-1}$, one hierarchy h is randomly selected from H according to the probability distribution $\mathbf{a}$ over H and then ties are broken according to h. The outcome in terms of the interim expected allocation of any tie-breaking rule can equivalently be obtained by a split hierarchical tie-breaking rule $\mathbf{a}$.

Ironed virtual type allocation rule

The *ironed virtual type allocation rule* $Q_{i,\rho}^{\mathbf{z},\mathbf{a}}(\boldsymbol{\theta})$ with ironing parameters $\mathbf{z}$ and split hierarchical tie-breaking rule $\mathbf{a}$ is then given by:

$$Q_{i,\rho}^{\mathbf{z},\mathbf{a}}(\boldsymbol{\theta}) \equiv \begin{cases} 0 & \text{if } \overline{\psi}_{i,\frac{w_i}{\rho}}(\theta_i, z_i) \text{ is not maximal in } \mathcal{P} \equiv \{\overline{\psi}_{j,\frac{w_j}{\rho}}(\theta_j, z_j)\}_{j\in\mathcal{N}} \\ 1 & \text{if } \overline{\psi}_{i,\frac{w_i}{\rho}}(\theta_i, z_i) \text{ is uniquely maximal in } \mathcal{P} \\ \sum_{h\in H} a_h \cdot \mathbf{1}_{h(i)=\max_{j\in\mathcal{T}} h(j)} & \text{otherwise, where } \mathcal{T} \subseteq \mathcal{N} \text{ is the set of agents tied in } \mathcal{P}, \end{cases} \tag{B.21}$$

where, when tie-breaking is required, hierarchy h is chosen with probability a_h and the resources are allocated to the tied agent that is highest in the randomly selected hierarchy.

For a given $\boldsymbol{\omega}^*$, the allocation rule $\mathbf{Q}^* = \mathbf{Q}_\rho^{\mathbf{z},\mathbf{a}}$ solves the problem in (B.16) for $\mathbf{z} = (\omega_{1,\rho}^{-1}(\omega_1^*), \ldots, \omega_{n,\rho}^{-1}(\omega_n^*))$ and any tie-breaking rule $\mathbf{a} \in \Delta^{n!-1}$. Having established that all allocation rules consistent with (B.16) are equivalent to ironed weighted virtual type allocation rules $\mathbf{Q}_\rho^{\mathbf{z},\mathbf{a}}$, we now turn to the second requirement for a saddle point. Condition (B.17) requires that the critical types $\boldsymbol{\omega}^*$ are worst-off types under allocation rule $\mathbf{Q}^*$. A simultaneous solution to (B.16) and (B.17) hence corresponds to a vector of ironing parameters $\mathbf{z}$ and a tie-breaking rule $\mathbf{a}$ such that $\omega_{i,\rho}(z_i) \in \Omega_i(\mathbf{Q}_\rho^{\mathbf{z},\mathbf{a}})$ for each agent $i \in \mathcal{N}$.

To define the payment rule in the second-best mechanism, we use $\mathbf{Q}_\rho^{\mathbf{z},\mathbf{a}}$, defined in (B.21), and we say that a payment rule $\mathbf{M}_\rho^{\mathbf{z},\mathbf{a}}$ corresponds to $\mathbf{Q}_\rho^{\mathbf{z},\mathbf{a}}$ if interim expected payments

because $\overline{\psi}_{i,1}(\theta, z) = \overline{\psi}_{i,1}(\theta, z') = \theta$. Away from the first-best, the ironing parameters are uniquely pinned down.

[17] While the ironing procedure is as in Myerson (1981), in contrast to Myerson, the ties here cannot be broken arbitrarily because of the need to respect the IR constraints.

satisfy for all $i \in \mathcal{N}$,

$$m_{i,\rho}^{\mathbf{z},\mathbf{a}}(\theta_i) = \theta_i(q_{i,\rho}^{\mathbf{z},\mathbf{a}}(\theta_i) - r_i) - \int_{\omega_{i,\rho}(z_i)}^{\theta_i} (q_{i,\rho}^{\mathbf{z},\mathbf{a}}(y) - r_i)dy - \eta_i \pi(\mathbf{Q}_\rho^{\mathbf{z},\mathbf{a}}, \omega_{1,\rho}(z_1), \ldots, \omega_{n,\rho}(z_n)).$$

We then have the following result:

Theorem B.2. *For each $\rho \geq \max \mathbf{w} = 1$, given a vector of ironing parameters $\mathbf{z}$ and a tie-breaking rule $\mathbf{a}$ such that $\omega_{i,\rho}(z_i) \in \Omega_i(\mathbf{Q}_\rho^{\mathbf{z},\mathbf{a}})$ for each $i \in \mathcal{N}$, $\mathbf{Q}_\rho^{\mathbf{z},\mathbf{a}}$ with a corresponding payment rule $\mathbf{M}_\rho^{\mathbf{z},\mathbf{a}}$ solves $\max_{\mathbf{Q},\mathbf{M}} W_\rho(\mathbf{Q}, \mathbf{M})$ subject to IC and IR.*

To complete the characterization of the bargaining mechanism, which must also satisfy the no-deficit constraint, note that in the limit at ρ goes to infinity, the allocation rule approaches that for the mechanism that maximizes the market maker's revenue. But, of course, that mechanism has positive expected revenue because the market maker can, for example, propose the mechanism that matches pairs of agents where one agent is willing to buy a unit at the fixed price of 2/3 and the other agent is willing to sell a unit at the fixed price of 1/3, with randomization used to select which agents trade if the number of willing buyers and willing sellers at those prices differ. The mechanism is individually rational because an agent could report a type of 1/2 and guarantee that it does not trade. Thus, there exists finite ρ^* such that the expected revenue is positive. The continuity and monotonicity of our problem then guarantee that $\rho^* = \min\{\rho \geq \max \mathbf{w} \mid \sum_{i \in \mathcal{N}} \mathbb{E}[M_{i,\rho}^{\mathbf{z},\mathbf{a}}(\boldsymbol{\theta})] \geq 0\}$ is well defined and unique.

We summarize in the following proposition:

Proposition B.1. *The incomplete information bargaining allocation rule assigns the supply to an agent with the maximal ironed weighted virtual type, $\{\overline{\Psi}_{i,\frac{w_i}{\rho^*}}(\theta_i, \omega_i^*)\}_{i \in \mathcal{N}}$, with a tie-breaking rule such that ω_i^* is a worst-off type for each agent i and ρ^* is the smallest feasible value such that the no-deficit constraint is satisfied.*

C Omitted proofs

C.1 Proof of Lemma A.2

Proof of Lemma A.2. Let $\mathcal{N}_U$ denote the set of $n_U \geq 1$ agents with distribution support $[0, 1]$ and $\mathcal{N}_D$ denote the set of $n_D \geq 1$ agents with distribution support $[\underline{\theta}, 1 + \underline{\theta}]$. Define $\mathcal{N} \equiv \mathcal{N}_U \cup \mathcal{N}_D$.

As noted in the proof of Lemma B.1, an implication of IC is that $u_i'(\theta) = q_i(\theta) - r_i$ wherever u_i is differentiable, which by IC is almost everywhere. Given this, the monotonicity of u_i implies the following characterization of the set of worst-off types for agent i, denoted by $\Omega_i \equiv \arg\min_{\theta_i \in [\underline{\theta}_i, \overline{\theta}_i]} u_i(\theta_i)$ (see also Cramton et al. (1987, Lemma 2) and Loertscher and Wasser (2019)):

$$\Omega_i = \begin{cases} \{\theta_i \in [\underline{\theta}_i, \overline{\theta}_i] \mid q_i(\theta_i) = r_i\} & \text{if } \exists \theta_i \in [\underline{\theta}_i, \overline{\theta}_i] \text{ s.t. } q_i(\theta_i) = r_i, \\ \{\theta_i \in [\underline{\theta}_i, \overline{\theta}_i] \mid q_i(z) < r_i \ \forall z < \theta_i \text{ and } q_i(z) > r_i \ \forall z > \theta_i\} & \text{otherwise.} \end{cases}$$

In the first case, in which there exists $\theta_i \in [\underline{\theta}_i, \overline{\theta}_i]$ such that $q_i(\theta_i) = r_i$, the set Ω_i is a (possibly degenerate) interval, and in the second case, Ω_i is a singleton.

Using this characterization of worst-off types, in our setting with $\underline{\theta} < 1$, for any agent $j \in \mathcal{N}_U$, $q_j^e(\underline{\theta}) = 0$ and $q_j^e(1) \leq 1$ (strict if $\underline{\theta} > 0$), which implies that either (i) $\hat{\theta}_j^e(r_j) \in [0,1]$ and $q_j^e(\hat{\theta}_j^e(r_j)) = r_j$ or (ii) $\hat{\theta}_j^e(r_j) = 1$ and $q_j^e(1) < r_j$. If $n_D \geq 2$, then for all $i \in \mathcal{N}_D$, $q_i^e(\underline{\theta}) = 0$ and $q_i^e(1 + \underline{\theta}) = 1$. This implies that for all $i \in \mathcal{N}_D$, $q_i^e(\hat{\theta}_i^e(r_i)) = r_i$. If $n_D = 1$ and $\underline{\theta} > 0$, then for agent $i \in \mathcal{N}_D$, we have $q_i^e(\underline{\theta}) > 0$. This implies that for the sole agent in $\mathcal{N}_D$, either (i) $q_i^e(\hat{\theta}_i^e(r_i)) = r_i$ or (ii) $\hat{\theta}_i^e(r_i) = \underline{\theta}$ and $r_i < q_i^e(\underline{\theta})$.

Given this, the proposition follows from the combination of Lemmas C.1–C.4, which show that $\Pi^e(\mathbf{r})$ is concave in $\mathbf{r}$ (Lemma C.1), there exists ownership $\mathbf{r}^*$ that equalizes the agents worst-off types (Lemma C.2), $\Pi^e(\mathbf{r})$ is maximized at $\mathbf{r}^*$ (Lemma C.3), and $\Pi^e(\mathbf{r}^*) > 0$ (Lemma C.4). We now state and prove these lemmas.

Lemma C.1. *Given* $\underline{\theta} < 1$, $\Pi^e(\mathbf{r})$ *is concave in* $\mathbf{r}$.

Proof. Given $\mathbf{r}$, let $\hat{\theta}_i^e(r_i)$ denote a worst-off type of agent i under the ex post efficient allocation rule and ownership $\mathbf{r}$. To show that Π^e is concave, we show that for all $\lambda \in (0,1)$ and $\mathbf{r}, \hat{\mathbf{r}} \in \Delta$,

$$\Pi^e(\lambda \mathbf{r} + (1-\lambda)\hat{\mathbf{r}}) \geq \lambda \Pi^e(\mathbf{r}) + (1-\lambda)\Pi^e(\hat{\mathbf{r}}). \tag{C.22}$$

Note that Π^e can be written as $\Pi^e(\mathbf{r}) = \sum_{i \in \mathcal{N}} \Pi_i^e(r_i)$, where

$$\Pi_i^e(r_i) \equiv \int_{\underline{\theta}_i}^{\hat{\theta}_i^e(r_i)} \Psi_i^S(\theta) q_i^e(\theta) dF_i(\theta) + \int_{\hat{\theta}_i^e(r_i)}^{\overline{\theta}_i} \Psi_i^B(\theta) q_i^e(\theta) dF_i(\theta) - r_i \hat{\theta}_i^e(r_i).$$

Using the definition of $\Pi_i^e(r_i)$ and $(\Psi_i^S(\theta) - \Psi_i^B(\theta)) f_i(\theta) = 1$, we have

$$\begin{aligned} \Pi_i^e(\lambda r_i + (1-\lambda)\hat{r}_i) \;=\; & \lambda \Pi_i^e(r_i) + (1-\lambda)\Pi_i^e(\hat{r}_i) \\ & + \lambda \int_{\hat{\theta}_i^e(r_i)}^{\hat{\theta}_i^e(\lambda r_i + (1-\lambda)\hat{r}_i)} q_i^e(\theta) d\theta - (1-\lambda) \int_{\hat{\theta}_i^e(\lambda r_i + (1-\lambda)\hat{r}_i)}^{\hat{\theta}_i^e(\hat{r}_i)} q_i^e(\theta) d\theta \\ & + \lambda r_i \hat{\theta}_i^e(r_i) + (1-\lambda)\hat{r}_i \hat{\theta}_i^e(\hat{r}_i) - (\lambda r_i + (1-\lambda)\hat{r}_i)\, \hat{\theta}_i^e(\lambda r_i + (1-\lambda)\hat{r}_i). \end{aligned} \tag{C.23}$$

Consider an agent i and let $\lambda \in (0,1)$ and $r_i, \hat{r}_i \in [0,1]$ with $0 \leq r_i < \hat{r}_i \leq 1$ be given. There are five possibilities for agent i's worst-off types: (i) $\hat{\theta}_i^e(r_i) < \hat{\theta}_i^e(\hat{r}_i)$ and $r_i = q_i^e(\hat{\theta}_i^e(r_i)) < q_i^e(\hat{\theta}_i^e(\hat{r}_i)) = \hat{r}_i$; (ii) agent $i \in \mathcal{N}_U$, $\hat{\theta}_i^e(r_i) < \hat{\theta}_i^e(\hat{r}_i) = 1$, and $r_i = q_i^e(\hat{\theta}_i^e(r_i)) < q_i^e(\hat{\theta}_i^e(\hat{r}_i)) < \hat{r}_i$; (iii) agent i is the only agent in $\mathcal{N}_D$, $\underline{\theta} = \hat{\theta}_i^e(r_i) < \hat{\theta}_i^e(\hat{r}_i)$, and $r_i < q_i^e(\underline{\theta}) < q_i^e(\hat{\theta}_i^e(\hat{r}_i)) = \hat{r}_i$; (iv) agent $i \in \mathcal{N}_U$, $\hat{\theta}_i^e(r_i) = \hat{\theta}_i^e(\hat{r}_i) = 1$, and $q_i^e(1) < r_i < \hat{r}_i$; or (v) agent i is the only agent in $\mathcal{N}_D$, $\underline{\theta} = \hat{\theta}_i^e(r_i) = \hat{\theta}_i^e(\hat{r}_i)$, and $r_i < \hat{r}_i < q_i^e(\underline{\theta})$.

Under possibilities (i), (ii), and (iii),

$$\int_{\hat{\theta}_i^e(r_i)}^{\hat{\theta}_i^e(\lambda r_i + (1-\lambda)\hat{r}_i)} q_i^e(\theta) d\theta > r_i \left(\hat{\theta}_i^e(\lambda r_i + (1-\lambda)\hat{r}_i) - \hat{\theta}_i^e(r_i) \right) \tag{C.24}$$

and

$$\int_{\hat{\theta}_i^e(\lambda r_i + (1-\lambda)\hat{r}_i)}^{\hat{\theta}_i^e(\hat{r}_i)} q_i^e(\theta) d\theta \leq \hat{r}_i \left(\hat{\theta}_i^e(\hat{r}_i) - \hat{\theta}_i^e(\lambda r_i + (1-\lambda)\hat{r}_i) \right). \tag{C.25}$$

Thus, under possibilities (i), (ii), and (iii),

$$\begin{aligned}
&\Pi_i^e(\lambda r_i + (1-\lambda)\hat{r}_i) > \lambda \Pi_i^e(r_i) + (1-\lambda)\Pi_i^e(\hat{r}_i) \\
&+\lambda r_i \left(\hat{\theta}_i^e(\lambda r_i + (1-\lambda)\hat{r}_i) - \hat{\theta}_i^e(r_i) \right) - (1-\lambda)\hat{r}_i \left(\hat{\theta}_i^e(\hat{r}_i) - \hat{\theta}_i^e(\lambda r_i + (1-\lambda)\hat{r}_i) \right) \\
&+\lambda r_i \hat{\theta}_i^e(r_i) + (1-\lambda)\hat{r}_i \hat{\theta}_i^e(\hat{r}_i) - (\lambda r_i + (1-\lambda)\hat{r}_i)\, \hat{\theta}_i^e(\lambda r_i + (1-\lambda)\hat{r}_i),
\end{aligned}$$

which simplifies to

$$\Pi_i^e(\lambda r_i + (1-\lambda)\hat{r}_i) > \lambda \Pi_i^e(r_i) + (1-\lambda)\Pi_i^e(\hat{r}_i).$$

In contrast, under possibilities (iv) and (v),

$$\Pi_i^e(\lambda r_i + (1-\lambda)\hat{r}_i) = \lambda \Pi_i^e(r_i) + (1-\lambda)\Pi_i^e(\hat{r}_i).$$

Thus, $\Pi_i^e(\lambda r_i + (1-\lambda)\hat{r}_i) \geq \lambda \Pi_i^e(r_i) + (1-\lambda)\Pi_i^e(\hat{r}_i)$ for each agent, implying that (C.22) holds and proving that Π^e is concave. □

Lemma C.2. *Assume $\underline{\theta} < 1$, and let $\mathbf{r}^*$ be defined by for all $i \in \mathcal{N}$,*

$$r_i^* \equiv \begin{cases} q_i^e(\hat{\theta}) & \text{if } \sum_{i \in \mathcal{N}} q_i^e(1) \geq 1, \\ q_i^e(1) & \text{if } \sum_{i \in \mathcal{N}} q_i^e(1) < 1 \text{ and } i \in \mathcal{N}_D, \\ \left(1 - \sum_{i \in \mathcal{N}_D} q_i^e(1)\right)/N_U & \text{if } \sum_{i \in \mathcal{N}} q_i^e(1) < 1 \text{ and } i \in \mathcal{N}_U, \end{cases}$$

where $\hat{\theta} \in (\underline{\theta}, 1]$ is uniquely defined by $(q_1^e(\hat{\theta}), \ldots, q_n^e(\hat{\theta})) \in \Delta$. Then $\mathbf{r}^* \in \Delta$ and $\mathbf{r}^*$ equal-

izes the agents' worst-off types under ex post efficiency at $\hat{\theta}$ *if* $\sum_{i\in\mathcal{N}} q_i^e(1) > 1$ *and at* 1 *if* $\sum_{i\in\mathcal{N}} q_i^e(1) \leq 1$.

Proof of Lemma C.2. By the definition of ex post efficiency, for all $i \in \mathcal{N}$ and $\theta \in [\underline{\theta}, 1]$, $q_i^e(\theta_i) = \prod_{j\in\mathcal{N}\setminus\{i\}} F_j(\theta_i)$, which is continuous and increasing in θ_i on $[\underline{\theta}, 1]$. We first show that $\sum_{i\in\mathcal{N}} q_i^e(\underline{\theta}) < 1$. Given our assumption of $n_D \geq 1$, each agent $i \in \mathcal{N}_U$ has $q_i^e(\underline{\theta}) = 0$, which implies that $\sum_{i\in\mathcal{N}} q_i^e(\underline{\theta}) = \sum_{i\in\mathcal{N}_D} q_i^e(\underline{\theta})$. So, it is sufficient to show that

$$\sum_{i\in\mathcal{N}_D} q_i^e(\underline{\theta}) < 1. \tag{C.26}$$

If $n_D = 1$, then $q_i^e(\underline{\theta}) < 1$, and so (C.26) holds. If $n_D > 1$, then for $i \in \mathcal{N}_D$, $q_i^e(\underline{\theta}) = 0$, so again (C.26) holds. Thus, we have established (C.26).

Case 1: $\sum_{i\in\mathcal{N}} q_i^e(1) \geq 1$. Using (C.26), we have

$$\sum_{i\in\mathcal{N}} q_i^e(\underline{\theta}) < 1 \leq \sum_{i\in\mathcal{N}} q_i^e(1).$$

By the continuity and monotonicity of $q_i^e(\cdot)$ on $[\underline{\theta}, 1]$, there exists a unique $\hat{\theta} \in (\underline{\theta}, 1]$ such that $\sum_{i\in\mathcal{N}} q_i^e(\hat{\theta}) = 1$. Further, $q_i^e(\hat{\theta}) \in [0, 1]$ for all θ. So, $(q_1^e(\hat{\theta}), \ldots, q_n^e(\hat{\theta})) \in \Delta$. It then follows that $\mathbf{r}^* \in \Delta$ and, since for all $i \in \mathcal{N}$, $q_i^e(\hat{\theta}) = r_i$, all agents have a common worst-off type $\hat{\theta}$ under ex post efficiency and ownership $\mathbf{r}^*$.

Case 2: $\sum_{i\in\mathcal{N}} q_i^e(1) < 1$. In this case, we have $\sum_{j\in\mathcal{N}_D} q_j^e(1) < 1$. Also, $N_D \geq 2$ because with only one agent j in $\mathcal{N}_D$, $q_j^e(1) = 1$, which contradicts $\sum_{j\in\mathcal{N}_D} q_j^e(1) < 1$. It then follows that $\mathbf{r}^* \in \Delta$. Since for $j \in \mathcal{N}_D$, $q_j^e(1) = r_j^*$, agents in $\mathcal{N}_D$ have a worst-off type of 1. For all $i \in \mathcal{N}_U$, $q_i^e(1) = \prod_{j\in\mathcal{N}_D} F_j(1)$, which is the same for all agents in $\mathcal{N}_U$, implying that for any $i \in \mathcal{N}_U$, $N_U q_i^e(1) = \sum_{j\in\mathcal{N}_U} q_j^e(1)$. Thus, for any $i \in \mathcal{N}_U$,

$$\begin{aligned}
r_i^* - q_i^e(1) &= \left(1 - \sum_{i\in\mathcal{N}_D} q_i^e(1)\right)/N_U - q_i^e(1) = \frac{1}{N_U}\left(1 - \sum_{i\in\mathcal{N}_D} q_i^e(1) - N_U q_i^e(1)\right) \\
&= \frac{1}{N_U}\left(1 - \sum_{i\in\mathcal{N}_D} q_i^e(1) - \sum_{j\in\mathcal{N}_U} q_j^e(1)\right) = \frac{1}{N_U}\left(1 - \sum_{i\in\mathcal{N}} q_i^e(1)\right) \\
&> 0.
\end{aligned}$$

Because for all $\theta \in [0, 1]$, $q_i^e(1) < r_i^*$, it follows that 1 is also the worst-off type for each agent in $\mathcal{N}_U$. □

Lemma C.3. *Given* $\underline{\theta} \in [0, 1)$, *ownership* $\mathbf{r}^*$ *maximizes* $\Pi^e(\mathbf{r})$ *subject to* $\mathbf{r} \in \Delta$.

Proof. By Lemma C.1, $\Pi^e(\mathbf{r})$ is concave, so we need only confirm that necessary conditions for a maximum are satisfied at $\mathbf{r}^*$. For each agent i, let $\hat{\theta}_i^e : [0,1] \to [\underline{\theta}_i, \overline{\theta}_i]$ map r_i onto a worst-off type for agent i. For an agent $i \in \mathcal{N}_U$, $\hat{\theta}_i^e(r_i)$ is uniquely defined unless $r_i = 0$, in which case $\underline{\theta}$ is a worst-off type. So we can define a worst-off type function for $i \in \mathcal{N}_U$ by

$$\hat{\theta}_i^e(r_i) \equiv \begin{cases} \underline{\theta} & \text{if } r_i = 0, \\ q_i^{e^{-1}}(r_i) & \text{if } r_i \in (0, q_i^e(1)], \\ 1 & \text{if } r_i > q_i^e(1), \end{cases}$$

which is continuous and nondecreasing. In addition, it is differentiable everywhere if $\underline{\theta} = 0$ and otherwise differentiable except at $r_i = q_i^e(1) < 1$. For agent $j \in \mathcal{N}_D$, $\hat{\theta}_j^e(r_j)$ is uniquely defined unless $n_D = 1$ and $r_j = 1$, in which case 1 is a worst-off type. So we can define a worst-off type function for $j \in \mathcal{N}_D$ by

$$\hat{\theta}_j^e(r_j) \equiv \begin{cases} \underline{\theta} & \text{if } r_j < q_j^e(\underline{\theta}), \\ q_j^{e^{-1}}(r_j) & \text{if } r_j \in [q_j^e(\underline{\theta}), 1), \\ 1 & \text{if } r_j = 1, \end{cases}$$

which is continuous, nondecreasing, and, except at $r_j = q_j^e(\underline{\theta}) < 1$, differentiable.

Note that Π^e can be written as

$$\Pi^e(\mathbf{r}) = \sum_{i \in \mathcal{N}} \left(\int_{\underline{\theta}_i}^{\hat{\theta}_i^e(r_i)} \Psi_i^S(\theta) q_i^e(\theta) dF_i(\theta) + \int_{\hat{\theta}_i^e(r_i)}^{\overline{\theta}_i} \Psi_i^B(\theta) q_i^e(\theta) dF_i(\theta) - r_i \hat{\theta}_i^e(r_i) \right).$$

Consider the maximization problem $\max_{\mathbf{r}} \Pi^e(\mathbf{r})$ subject to $\mathbf{r} \in \Delta$, i.e., subject to $r_i \geq 0$ for all $i \in \mathcal{N}$ and $\sum_{i \in \mathcal{N}} r_i = 1$. This has Lagrangian

$$\mathcal{L}(\mathbf{r}) = \Pi^e(\mathbf{r}) + \lambda\Big(\sum_{i \in \mathcal{N}} r_i - 1\Big) + \sum_{i \in \mathcal{N}} \mu_i r_i = \Pi^e(\mathbf{r}) - \lambda + \sum_{i \in \mathcal{N}} (\lambda + \mu_i) r_i,$$

where for all $i \in \mathcal{N}$, $\mu_i \geq 0$. Wherever $\hat{\theta}_i^e(r_i)$ is differentiable, we have

$$\frac{\partial \mathcal{L}(\mathbf{r})}{\partial r_i} = \hat{\theta}_i^{e\prime}(r_i) \left(q_i^e(\hat{\theta}_i^e(r_i)) - r_i \right) - \hat{\theta}_i^e(r_i) + \lambda + \mu_i.$$

For a maximum, we require the KKT conditions that for all $i \in \mathcal{N}$, $\frac{\partial \mathcal{L}}{\partial r_i} = 0$, $\mu_i \geq 0$, $r_i \geq 0$, and $\mu_i r_i = 0$, and that $\sum_{i \in \mathcal{N}} r_i = 1$.

Using Lemma C.2, if $\Sigma_{i \in \mathcal{N}} q_i^e(1) \geq 1$, then for all $i \in \mathcal{N}$, $\hat{\theta}_i^e(r_i^*) = \hat{\theta} \in (\underline{\theta}, 1]$ and $q_i^e(\hat{\theta}) = r_i^*$, which implies that $\hat{\theta}_i^{e\prime}(r_i^*)$ exists and is nonnegative and that all KKT conditions are satisfied at $\mathbf{r} = \mathbf{r}^*$, $\mu_i = 0$, and $\lambda = \hat{\theta}$. Also using Lemma C.2, if $\Sigma_{i\in\mathcal{N}} q_i^e(1) < 1$, then $n_D \geq 2$ and for all $i \in \mathcal{N}$, $\hat{\theta}_i^e(r_i^*) = 1$, which implies that $\hat{\theta}_i^{e\prime}(r_i^*)$ exists and is nonnegative. Further, for all $i \in \mathcal{N}_U$, $\hat{\theta}_i^{e\prime}(r_i^*) = 0$, and for all $j \in \mathcal{N}_D$, $q_j^e(1) = r_j^*$. Thus, all KKT conditions are satisfied at $\mathbf{r} = \mathbf{r}^*$, $\mu_i = 0$, and $\lambda = 1$. This completes the proof that $\mathbf{r}^*$ maximizes $\Pi^e(\mathbf{r})$. □

Lemma C.4. *For* $\underline{\theta} < 1$, $\Pi^e(\mathbf{r}^*) > 0$.

Proof. An efficient, IC mechanism for which the interim individual rationality constraint of each agent is binding when evaluated at the agent's worst-off type is a Vickrey-Clarke-Groves (VCG) mechanism (see Vickrey, 1961; Clarke, 1971; Groves, 1973). Standard mechanism design results imply that the VCG mechanism maximizes ex ante expected revenue subject to ex post efficiency, (dominant-strategy) incentive compatibility, and interim individual rationality.[18] By the payoff equivalence theorem (see, e.g., Börgers, 2015), any ex post efficient mechanism that is incentive compatible and that satisfies the interim individual rationality constraints with equality for the worst-off types induces the same interim payoffs, and thus ex ante expected payoffs, as the VCG mechanism. Consequently, focusing on the VCG mechanism is without loss of generality within the context of ex post efficient mechanisms.

We denote the VCG mechanism by $\langle \mathbf{Q}^e, \mathbf{T_r} \rangle$. For a given vector of reported types $\boldsymbol{\theta}$, maximal social surplus, denoted $W(\boldsymbol{\theta})$, is

$$W(\boldsymbol{\theta}) = \max_{\hat{\boldsymbol{Q}}\in\Delta} \sum_{i\in\mathcal{N}} \theta_i \hat{Q}_i(\boldsymbol{\theta}) = \sum_{i\in\mathcal{N}} \theta_i Q_i^e(\boldsymbol{\theta}),$$

where the second equality holds because $\mathbf{Q}^e$ is the ex post efficient allocation rule. Given worst-off type for agent i, $\hat{\theta}_i \in [\underline{\theta}_i, \overline{\theta}_i]$, the VCG transfer from agent i is

$$T_{\mathbf{r},i}(\boldsymbol{\theta}) = W(\hat{\theta}_i, \boldsymbol{\theta}_{-i}) - (W(\boldsymbol{\theta}) - \theta_i Q_i^e(\boldsymbol{\theta})) - \hat{\theta}_i r_i.$$

Accordingly, the ex post budget surplus, denoted $\Pi^e(\boldsymbol{\theta}, \mathbf{r})$, is

$$\Pi^e(\boldsymbol{\theta}, \mathbf{r}) = \sum_{i\in\mathcal{N}} T_{\mathbf{r},i}(\boldsymbol{\theta}) = \sum_{i\in\mathcal{N}} W(\hat{\theta}_i, \boldsymbol{\theta}_{-i}) - (n-1)W(\boldsymbol{\theta}) - \sum_{i\in\mathcal{N}} \hat{\theta}_i r_i, \tag{C.27}$$

where the second equality follows from substituting $T_{\mathbf{r},i}(\boldsymbol{\theta})$. The transfer $T_{\mathbf{r},i}(\boldsymbol{\theta})$ depends

[18] The notion of incentive compatibility is not material here. The VCG mechanism being dominant-strategy incentive compatible implies that it is also Bayesian incentive compatible. With independent private values, for any ex post efficient mechanism that is Bayesian incentive compatible, there exists an equivalent mechanism that is dominant strategy incentive compatible.

both directly on r_i, as is evident from the definition, and indirectly because $\hat{\theta}_i$ is also a function of r_i.

Take $\boldsymbol{\theta}$ and $\mathbf{r}$ as given. Let $\hat{\theta}_i$ be agent i's worst-off type under the VCG mechanism. Fixing an agent $i \in \mathcal{N}$, when the type profile is $(\hat{\theta}_i, \boldsymbol{\theta}_{-i})$, social welfare under the efficient allocation for that type profile, $\mathbf{Q}^e(\hat{\theta}_i, \boldsymbol{\theta}_{-i})$, is at least as large as the social welfare under the allocation $\mathbf{Q}^e(\boldsymbol{\theta})$, because, by construction, $\mathbf{Q}^e(\hat{\theta}_i, \boldsymbol{\theta}_{-i})$ maximizes social welfare given the type profile $(\hat{\theta}_i, \boldsymbol{\theta}_{-i})$. That is, $W(\hat{\theta}_i, \boldsymbol{\theta}_{-i}) \geq W(\boldsymbol{\theta}) + (\hat{\theta}_i - \theta_i)Q_i^e(\boldsymbol{\theta})$, which is equivalent to $W(\boldsymbol{\theta}) - W(\hat{\theta}_i, \boldsymbol{\theta}_{-i}) \leq (\theta_i - \hat{\theta}_i)Q_i^e(\boldsymbol{\theta})$. Summing up yields

$$\sum_{i\in\mathcal{N}} \Big(W(\boldsymbol{\theta}) - W(\hat{\theta}_i, \boldsymbol{\theta}_{-i})\Big) \leq W(\boldsymbol{\theta}) - \sum_{i\in\mathcal{N}} \hat{\theta}_i Q_i^e(\boldsymbol{\theta}). \tag{C.28}$$

Using (C.27) and (C.28), we have

$$\begin{aligned}
\Pi^e(\boldsymbol{\theta}, \mathbf{r}) &= \sum_{i\in\mathcal{N}} W(\hat{\theta}_i, \boldsymbol{\theta}_{-i}) - (n-1)W(\boldsymbol{\theta}) - \sum_{i\in\mathcal{N}} \hat{\theta}_i r_i \\
&= W(\boldsymbol{\theta}) - \sum_{i\in\mathcal{N}} \hat{\theta}_i Q_i^e(\boldsymbol{\theta}) - \sum_{i\in\mathcal{N}} \Big(W(\boldsymbol{\theta}) - W(\hat{\theta}_i, \boldsymbol{\theta}_{-i})\Big) \\
&\quad + \sum_{i\in\mathcal{N}} \hat{\theta}_i Q_i^e(\boldsymbol{\theta}) - \sum_{i\in\mathcal{N}} \hat{\theta}_i r_i \\
&\geq \sum_{i\in\mathcal{N}} \hat{\theta}_i Q_i^e(\boldsymbol{\theta}) - \sum_{i\in\mathcal{N}} \hat{\theta}_i r_i.
\end{aligned} \tag{C.29}$$

Let $\hat{\theta}_i^*$ be agent i's worst-off type under the VCG mechanism $\langle \mathbf{Q}^e, \mathbf{T}_{\mathbf{r}^*} \rangle$, where $\mathbf{r}^*$ is as defined in Lemma C.2. We now show that $\mathbb{E}_{\boldsymbol{\theta}}\left[\sum_{i\in\mathcal{N}} \hat{\theta}_i^* Q_i^e(\boldsymbol{\theta}) - \sum_{i\in\mathcal{N}} \hat{\theta}_i^* r_i^*\right] > 0$, which, using (C.29), implies that $\Pi^e(\mathbf{r}^*) = \mathbb{E}_{\boldsymbol{\theta}}\left[\Pi^e(\boldsymbol{\theta}, \mathbf{r}^*)\right] > 0$.

Using Lemma C.2, agents have a common worst-off type under $\mathbf{r}^*$ and $\mathbf{Q}^e$, i.e., $\hat{\theta}_i^* = \hat{\theta}$ for all $i \in \mathcal{N}$. Thus,

$$\Pi^e(\boldsymbol{\theta}, \mathbf{r}^*) \geq \sum_{i\in\mathcal{N}} \hat{\theta}_i^* Q_i^e(\boldsymbol{\theta}) - \sum_{i\in\mathcal{N}} \hat{\theta}_i^* r_i^* = \hat{\theta} \sum_{i\in\mathcal{N}} (Q_i^e(\boldsymbol{\theta}) - r_i^*) = 0 \tag{C.30}$$

because $\sum_{i\in\mathcal{N}} Q_i(\boldsymbol{\theta}) = 1$. The inequality in (C.30) is strict for a positive measure of types (to see this, note that (C.30) holds with equality only if $Q_i^e(\boldsymbol{\theta}) = Q_i^e(\hat{\theta}, \boldsymbol{\theta}_{-i})$, which only holds for a restricted set of $\boldsymbol{\theta}$ given that $\underline{\theta} < 1$ and $f_i > 0$ for all i on $(\underline{\theta}, 1)$), and so $\Pi^e(\mathbf{r}^*) > 0$ follows. □

Combining all the elements above completes the proof of Lemma A.2. ■

C.2 Proof of Proposition 7

Proof of Proposition 7. For every pair i and j of agents in $\mathcal{N}_D$, it is possible for either to have the highest type, so the ranking of those agents matters for efficiency. In addition, for $\underline{\theta} < 1$, for every pair of agents, one in $\mathcal{N}_U$ and one in $\mathcal{N}_D$, it is possible for either to have the highest type, so the ranking of those agents matters for efficiency.

We complete the proof with two lemmas. In Lemma C.5, we show that for two agents with overlapping supports, the ranking of their types cannot be the same as the ranking of their ironed weighted virtual types for all types in an open interval of the support overlap if the bargaining weights differ. This implies that for ex post efficiency, all agents in $\mathcal{N}_D$ must have the same bargaining weight and that when $\underline{\theta} < 1$, all agents must have the same bargaining weight. Then we address the case of $\underline{\theta} \geq 1$ in Lemma C.6. It is convenient as part of Lemma C.6 to also state an additional result for the case of one agent in $\mathcal{N}_D$, and so the proof of this lemma also serves as the proof of Proposition 2, which assumes $n_U = 1$ and $n_D = 1$.

Lemma C.5. *The ironed weighted virtual type functions for two agents cannot be the same for all types in an open interval in the support of both agents' type distributions if the agents' bargaining weights differ.*

Proof. Suppose that $w_i \neq w_j$ and let A be an open interval that is a subset of the interval of overlap in the supports of the distributions of agents i and j. Suppose that for all $\theta \in A$, the ranking of agents i and j according to their types is the same as the ranking of those agents according to their weighted ironed virtual types under incomplete information bargaining:

$$\overline{\Psi}_{j,\frac{w_j}{\rho^*}}(\theta_j;\omega_j^*) > \overline{\Psi}_{i,\frac{w_i}{\rho^*}}(\theta_i;\omega_i^*) \quad \Leftrightarrow \quad \theta_j > \theta_i.$$

Then we require that for all $\theta \in A$, $\overline{\Psi}_{j,w_j/\rho^*}(\theta;\omega_j^*) = \overline{\Psi}_{i,w_i/\rho^*}(\theta;\omega_i^*)$ and that the ironed weighted virtual type functions are increasing in this region (if they are equal, but constant, as in the ironed portions, then the allocation is random and so ex post efficiency is not achieved). Given that the ironed weighted virtual type functions are increasing, they are either equal to the weighted virtual value or the weighted virtual cost, and, as we now show, these cannot be the same on an open interval if the weights differ.

To see this, suppose that for $\theta \in A$, $\overline{\Psi}_{j,w_j/\rho^*}(\theta;\omega_j^*) = \Psi^B_{j,w_j/\rho^*}(\theta) = \overline{\Psi}_{i,w_i/\rho^*}(\theta;\omega_i^*) = \Psi^S_{i,w_i/\rho^*}(\theta)$. Then we have

$$\theta - (1 - \frac{w_j}{\rho^*})\frac{1 - F_j(\theta)}{f_j(\theta)} = \theta + (1 - \frac{w_i}{\rho^*})\frac{F_i(\theta)}{f_i(\theta)},$$

which we can rewrite as

$$\frac{F_i(\theta)f_j(\theta)}{(1-F_j(\theta))\,f_i(\theta)+F_i(\theta)f_j(\theta)}=\frac{\rho^*-w_j}{w_i-w_j}\equiv C$$

and, alternatively, as

$$\frac{(1-F_j(\theta))\,f_i(\theta)}{F_i(\theta)f_j(\theta)+(1-F_j(\theta))\,f_i(\theta)}=\frac{\rho^*-w_i}{w_j-w_i}\equiv C'$$

Using $\rho^* \geq \max\{w_j, w_i\}$, if $w_j < w_i$, then $C \geq 1$, and we require that for all $\theta \in A$, $\frac{1-F_j(\theta)}{f_j(\theta)}/\frac{F_i(\theta)}{f_i(\theta)} = \frac{1-C}{C}$, which is a contradiction because the left side is positive and $(1-C)/C \leq 0$; and if $w_j > w_i$, then $C' \geq 1$, and we require that $\frac{F_i(\theta)}{f_i(\theta)}/\frac{(1-F_j(\theta))}{f_j(\theta)} = \frac{1-C'}{C'}$, which is similarly a contradiction.

Now suppose that $\overline{\Psi}_{j,w_j/\rho^*}(\theta;\omega_j^*) = \Psi^B_{j,w_j/\rho^*}(\theta) = \overline{\Psi}_{i,w_2/\rho^*}(\theta;\omega_i^*) = \Psi^B_{i,w_i/\rho^*}(\theta)$. Then we require that $\theta-(1-\frac{w_j}{\rho^*})\frac{1-F_j(\theta)}{f_j(\theta)} = \theta-(1-\frac{w_j}{\rho^*})\frac{1-F_i(\theta)}{f_i(\theta)}$. We can write this as $\frac{1-F_i(\theta)}{f_i(\theta)}/\left(\frac{1-F_j(\theta)}{f_j(\theta)}-\frac{1-F_i(\theta)}{f_i(\theta)}\right) = \frac{\rho^*-w_j}{w_j-w_i} \equiv C''$, which implies that $\frac{1-F_j(\theta)}{f_j(\theta)}/\frac{1-F_i(\theta)}{f_i(\theta)} = \frac{1+C''}{C''}$, giving us a contradiction for $w_j < w_i$ because the left side is positive and the right side is negative. An analogous contradiction obtains if $w_j > w_i$. The remaining case, $\overline{\Psi}_{j,w_j/\rho^*}(\theta;\omega_j^*) = \Psi^S_{j,w_j/\rho^*}(\theta) = \overline{\Psi}_{i,w_2/\rho^*}(\theta;\omega_i^*) = \Psi^S_{i,w_i/\rho^*}(\theta)$, provides an analogous contradiction, which completes the proof. □

Lemma C.6. *Letting $\overline{w}_U \equiv \max_{j\in\mathcal{N}_U \text{ s.t. } r_j>0} w_j$, given agent $i \in \mathcal{N}_U$ with $r_i > 0$, if $\underline{\theta} \geq 1$, then ex post efficiency requires $\frac{\max\{\overline{w}_U,w_D\}-w_i}{\max\{\overline{w}_U,w_D\}} \leq (\underline{\theta}-1)\,f_i(1)$. Further, if there is only one agent j in $\mathcal{N}_D$, then ex post efficiency is possible if and only if $1+(1-\frac{w_i}{\max\{w_i,w_j\}})/f_i(1) \leq \underline{\theta}-(1-\frac{w_j}{\max\{w_i,w_j\}})/f_j(\underline{\theta})$.*

Proof. Given $\underline{\theta} \geq 1$, ex post efficiency requires that resources move from agent $i \in \mathcal{N}_U$ with $r_i > 0$ to an agent in $\mathcal{N}_D$ for all type realizations. For incomplete information bargaining to result in resources moving from agent i to an agent in $\mathcal{N}_D$, we require that for some agent $j \in \mathcal{N}_D$,

$$\overline{\Psi}_{i,\frac{w_i}{\rho^*}}(\theta_i;\omega_i^*) \leq \overline{\Psi}_{j,\frac{w_j}{\rho^*}}(\theta_j;\omega_j^*). \tag{C.31}$$

Under ex post efficiency, agent i is a seller with worst-off type 1, all agents in $\mathcal{N}_D$ have common bargaining weight w_D, and $\rho^* = \overline{w} \equiv \max\{\overline{w}_U, w_D\}$. So, (C.31) becomes $\Psi^S_{i,\frac{w_i}{\overline{w}}}(\theta_i) \leq \overline{\Psi}_{j,\frac{w_D}{\overline{w}}}(\theta_j;\omega_j^*)$, and this must hold even for $\theta_i = 1$ and even if all agents in $\mathcal{N}_D$ have type $\underline{\theta}$, so we have

$$1+\frac{1-\frac{w_i}{\overline{w}}}{f_i(1)} \leq \overline{\Psi}_{j,\frac{w_D}{\overline{w}}}(\underline{\theta};\omega_j^*) \in (\Psi^B_{j,\frac{w_D}{\overline{w}}}(\underline{\theta}), \Psi^S_{j,\frac{w_D}{\overline{w}}}(\underline{\theta})),$$

which, using $\Psi^S_{j,\frac{w_D}{\overline{w}}}(\underline{\theta}) = \underline{\theta}$, implies that $1+\left(1-\frac{w_i}{\overline{w}}\right)/f_i(1) \leq \underline{\theta}$. Thus, for all agents $i \in \mathcal{N}_U$ with $r_i > 0$, $\frac{\overline{w}-w_i}{\overline{w}} \leq (\underline{\theta}-1)\,f_i(1)$, which completes the proof of the first part of the lemma.

If there is only one agent j in $\mathcal{N}_D$, then under ex post efficiency, agent j is a buyer with worst-off type $\underline{\theta}$, and so $\overline{\Psi}_{j,w_D/\overline{w}}(\underline{\theta};\omega_j^*) = \Psi^B_{j,w_D/\overline{w}}(\underline{\theta}) = \underline{\theta} - \frac{1-\frac{w_D}{\overline{w}}}{f_j(\underline{\theta})}$, which gives us

$$1 + \frac{1-\frac{w_i}{\overline{w}}}{f_i(1)} \leq \underline{\theta} - \frac{1-\frac{w_D}{\overline{w}}}{f_j(\underline{\theta})}. \tag{C.32}$$

Under condition (C.32), trade occurs for all type realizations, and so the expected budget surplus under ex post efficiency and binding IR for the agents' worst-off types is

$$\begin{aligned} \Pi^e &\equiv \mathbb{E}\big[\Psi_j^B(\theta_j)q_j^e(\theta_j) + \textstyle\sum_{i\in\mathcal{N}\setminus\{j\}} \Psi_i^S(\theta_i)q_i^e(\theta_i)\big] - r_j\underline{\theta} - \textstyle\sum_{i\in\mathcal{N}\setminus\{j\}} r_i \\ &= \underline{\theta} - r_j\underline{\theta} - \textstyle\sum_{i\in\mathcal{N}\setminus\{j\}} r_i \\ &\geq 0, \end{aligned}$$

where the second equality uses $q_j^e(\theta_j) = 1$, $q_i^e(\theta_i) = 0$ for $i \neq j$, and $\mathbb{E}\left[\Psi_j^B(\theta_j)\right] = \underline{\theta}$,and the inequality uses $\underline{\theta} \geq 1$ and $\sum_{i\in\mathcal{N}} r_i = 1$. Because Π^e is nonnegative, it follows that (C.32) is necessary and sufficient for ex post efficiency. Efficient trade can be achieved in an IC, IR, no-deficit way and that allocates Π^e to the agent with the higher bargaining weight using a posted price of 1 if $w_i > w_j$ and $\underline{\theta}$ if $w_j > w_i$; and if $w_i = w_j$, then Π^e can be allocated in any proportions to the agents using posted prices between 1 and $\underline{\theta}$. This completes the proof of the second part of the lemma. □

Combining these results completes the proof of Proposition 7. ■

D Double-auction implementation

In this appendix, we provide a double-auction implementation of the incomplete information bargaining mechanism for two agents with $w = 1$ and $r \in (0,1)$, assuming that types are uniformly distributed on $[0,1]$. Thus, we extend the k-double auction of Chatterjee and Samuelson (1983), which assumes $r \in \{0,1\}$, to a partnership setting. For this appendix, we assume that $\underline{\theta} = 0$.

As noted in the main text, to implement optimal mechanisms away from ex post efficiency with shared ownership $r \in (0,1)$, one can use appropriately adjusted variants of the k-DA. These variants continue to allocate the resources to the high bidder, and at a per-unit price defined by k times agent 1's bid plus $1-k$ times agent 2's bid; however, additional payments not related to the amount traded are required to maintain IR.

Specifically, if the bids are b_1 and b_2, then agent 1 pays $-\tau_1(b_1)+\tau_2(b_2)$ (possibly negative) to agent 2, independently of which agent wins the double auction, where we refer to τ_1

and τ_2 as *tax payment* functions. For example, with equal bargaining weights, $r = 0.9$, and uniformly distributed types on $[0, 1]$, the allocation rule shown in Figure 2(c) can be implemented by a 1/2-double auction with appropriately chosen tax payment functions. Specifically, there exist τ_1 and τ_2 (displayed in Figure D.1(c)) that induce the equilibrium bid functions depicted in panels (a) and (b) of Figure D.1, which then induce the allocation rule shown in Figure 2(c).

(a) $\beta_1(\theta)$ (b) $\beta_2(\theta)$ (c) Tax payment functions

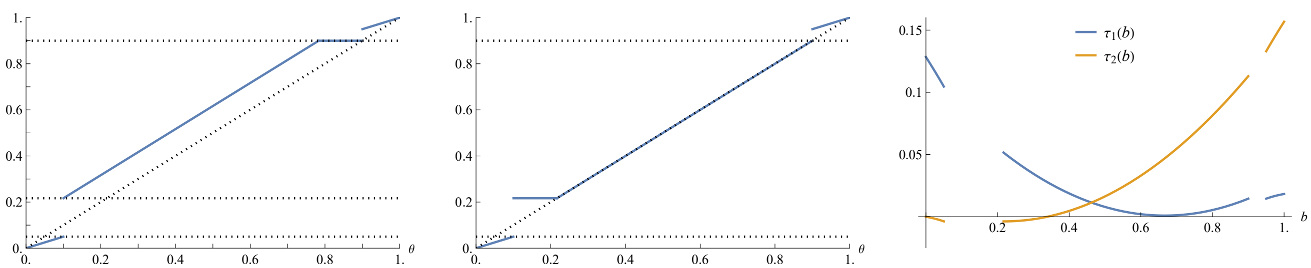

Figure D.1: Equilibrium bid functions for the 1/2-double auction and associated tax payment functions for $r = 0.9$. In panel (c), gaps in the tax functions indicate bids that have a sufficiently high penalty that they are never optimal. Assumes uniformly distributed types on $[0, 1]$.

D.1 Benchmark incomplete information bargaining allocation rule

We next characterize the incomplete information bargaining allocation rule for this setting:

Proposition D.1. *With two agents, $w = 1$, $r \in [0, 1]$, and uniformly distributed types on $[0, 1]$, the incomplete information bargaining allocation rule is*

$$Q_1(\theta_1, \theta_2) = \begin{cases} 1 & \text{if } 2\theta_2 < \theta_1 \text{ and } \theta_1 \leq 1 - r, \\ 1 & \text{if } 2\theta_2 - 1 < \theta_1 \text{ and } \theta_1 > 1 - r, \\ 0 & \text{otherwise,} \end{cases}$$

and $Q_2(\theta_1, \theta_2) = 1 - Q_1(\theta_1, \theta_2)$.

Proof. See Appendix D.3 below.

D.2 Specification of the double-auction implementation

In a double-auction implementation, agents submit bids b_1 and b_2 and the good is allocated to agent 1 if $b_1 \geq b_2$ and to agent 2 if $b_1 < b_2$ (as we will see, ties will occur with zero probability). We are focused on the case with $w = 1$, which corresponds to the k-double auction with $k = 1$. In this auction, the per-unit payment is equal to agent 1's bid. In our adaptation to $r \in (0,1)$, we also require agent 1 to pay a "tax" $\tau(b_1)$ to agent 2 regardless of which agent has the higher bid.

Consider the bid functions:

$$\beta_1(\theta_1) = \begin{cases} \theta_1/2 & \text{if } \theta_1 \leq 1-r, \\ (\theta_1+1)/2 & \text{if } \theta_1 > 1-r, \end{cases} \tag{D.33}$$

and

$$\beta_2(\theta_2) = \theta_2 \tag{D.34}$$

and the tax payment function for agent 1 of

$$\tau(b_1) = \begin{cases} rb_1 - \frac{(3-2r)}{4}r & \text{if } b_1 \leq (1-r)/2, \\ -(1-r)b_1 + \frac{(3-2r)}{4}(1-r) & \text{if } b_1 > (1-r)/2. \end{cases}$$

Assuming for the moment that these bid functions form an equilibrium, the double auction has the feature that when $r = 1$, $\beta_1(\theta_1) = (\theta_1+1)/2$ and $\tau(\beta_1) = 0$, so the mechanism is equivalent to the optimal take-it-or-leave it sell offer from agent 1 (with no tax), and when $r = 0$, $\beta_1(\theta_1) = \theta_1/2$ and $\tau(\beta_1) = 0$, so the mechanism is equivalent to the optimal take-it-or-leave it buy offer from agent 1 (with no tax). Further, the bid functions implement the incomplete information bargaining allocation rule.

Notice that even though the the bid functions and tax payment functions are themselves discontinuous, the agents' interim expected net payoffs are continuous. For agent 1, we have

$$\begin{aligned} u_1(\theta_1) &= \int_0^{\beta_1(\theta_1)} (\theta_1 - \beta_1(\theta_1)(1-r))\, d\theta_2 + \int_{\beta_1(\theta_1)}^1 \beta_1(\theta_1) r d\theta_2 - \tau(\beta_1(\theta_1)) - r\theta_1 \\ &= \begin{cases} \frac{\theta_1^2}{4} - r\theta_1 + \frac{3-2r}{4}r & \text{if } \theta_1 \leq 1-r, \\ \frac{(\theta_1+1)^2}{4} - r\theta_1 - \frac{3-2r}{4}(1-r) & \text{if } \theta_1 > 1-r. \end{cases} \end{aligned} \tag{D.35}$$

Noting that $\lim_{\theta_1 \uparrow 1-r} u_1(\theta_1) = \lim_{\theta_1 \downarrow 1-r} u_1(\theta_1)$, we see that for a given r, $u_1(\theta_1)$ is continuous and convex. (This is illustrated in Figure D.2 below.) Similarly, $u_2(\theta_2)$, which is given by

$$u_2(\theta_2) = \int_0^1 \Big((\theta_2 - r\beta_1(\theta_1)) \cdot \mathbf{1}_{\beta_1(\theta_1)<\theta_2} + (1-r)\beta_1(\theta_1) \cdot \mathbf{1}_{\beta_1(\theta_1)>\theta_2} + \tau(\beta_1(\theta_1)) \Big) d\theta_1 - (1-r)\theta_2, \tag{D.36}$$

is continuous.

We now show that the bid functions above form an equilibrium and that IR is satisfied. The no-deficit constraint is satisfied for a double auction by construction because all payments are from one agent to the other.

Proposition D.2. *With two agents,* $w = 1$*,* $r \in [0, 1]$*, and uniformly distributed types on* $[0, 1]$*, the incomplete information bargaining allocation rule is implemented using a k-double auction with* $k = 1$ *(i.e., the per-unit payment is equal to agent 1's bid) and a tax payment function for agent* 1 *of* τ *given in* (D.2)*, which induces equilibrium bidding strategies* β_1 *and* β_2 *given in* (D.33) *and* (D.34)*.*

Proof. Given bids b_1 and b_2, because agent 2 pays agent 1's $b_1 r$ for agent 1's r units whenever it wins and receives $b_1(1-r)$ in exchange for its $1-r$ units whenever it loses (and receives the tax payment $\tau(b_1)$ regardless), it is optimal for agent 2 to bid its type, i.e., $\beta_2(\theta_2) = \theta_2$. We next check that it is optimal for agent 1 to bid according to $\beta_1(\theta_1)$ given in (D.33). Given agent 2's bid function, agent 1's interim expected payoff is$\max_{b_1} \int_0^{b_1} (\theta_1 - b_1(1-r))\, d\theta_2 + \int_{b_1}^1 b_1 r d\theta_2 - \tau(b_1)$. The first-order condition is $0 = \theta_1 - b_1(1-r) - b_1 r - b_1(1-r) + (1-b_1)r - \tau'(b_1)$, which we can rewrite as

$$b_1 = \frac{\theta_1 + r - \tau'(b_1)}{2} = \begin{cases} \frac{\theta_1}{2} & \text{if } b_1 \leq (1-r)/2, \\ \frac{\theta_1+1}{2} & \text{if } b_1 > (1-r)/2, \end{cases}$$

or as

$$\beta_1(\theta_1) = \begin{cases} \frac{\theta_1}{2} & \text{if } \theta_1 \leq 1-r, \\ \frac{\theta_1+1}{2} & \text{if } \theta_1 > 1-r. \end{cases}$$

Noting that the second-order condition is satisfied because $-2 - \tau''(\beta_1) = -2 < 0$ confirms that (D.33) is indeed agent 1's optimal bid function.

It remains to confirm that IR is satisfied. To do this, we need to identify the agents' worst-off types. To do this, we first calculate the interim expected allocations, $q_i(\theta_i) \equiv \mathbb{E}_{\theta_{-i}}[Q_i(\boldsymbol{\theta})]$. We obtain

$$q_1(\theta_1) = \begin{cases} \theta_1/2 & \text{if } \theta_1 \leq 1-r, \\ (\theta_1+1)/2 & \text{if } \theta_1 > 1-r, \end{cases}$$

and

$$q_2(\theta_2) = \begin{cases} 2\theta_2 & \text{if } \theta_2 < \frac{1-r}{2}, \\ 1-r & \text{if } \frac{1-r}{2} \leq \theta_2 \leq \frac{2-r}{2}, \\ 2\theta_2 - 1 & \text{if } \frac{2-r}{2} < \theta_2. \end{cases}$$

Using the interim expected allocation functions, we can derive the agents' worst-off types. For agent 2 it is easy because for all $r \in [0, 1]$, $q_2(1/2) = 1-r$, which implies that $1/2$ is a worst-off type for agent 2 for all r. Thus, we have $\omega_2 = 1/2$. For agent 1, solving $q_1(\omega_1) = r$ gives us

$$\omega_1 = \begin{cases} 2r & \text{if } r < 1/3, \\ 1-r & \text{if } 1/3 \le r \le 2/3, \\ 2r-1 & \text{if } 2/3 < r. \end{cases}$$

Evaluating agent 1's expected net payoff given in (D.35) at its worst-off type ω_1, we have

$$u_1(\omega_1) = \begin{cases} \frac{3}{4}(1-2r)r & \text{if } r < \frac{1}{3}, \\ \frac{1}{4}(1-3r+3r^2) & \text{if } \frac{1}{3} \le r \le \frac{2}{3}, \\ \frac{3}{4}(-1+3r-2r^2) & \text{if } r > \frac{2}{3}, \end{cases}$$

which is nonnegative for all $r \in [0,1]$ and zero for $r \in \{0,1\}$, implying that IR is satisfied for agent 1, with equality for its worst-off type when $r \in \{0,1\}$. For agent 2, referring to (D.36), we have $u_2(\omega_2) = u_2(1/2) = 0$, implying that IR is satisfied for agent 2, and with equality for its worst-off type, which completes the proof. ■

We illustrate the interim expected net payoffs associated with the double auction in Figure D.2.

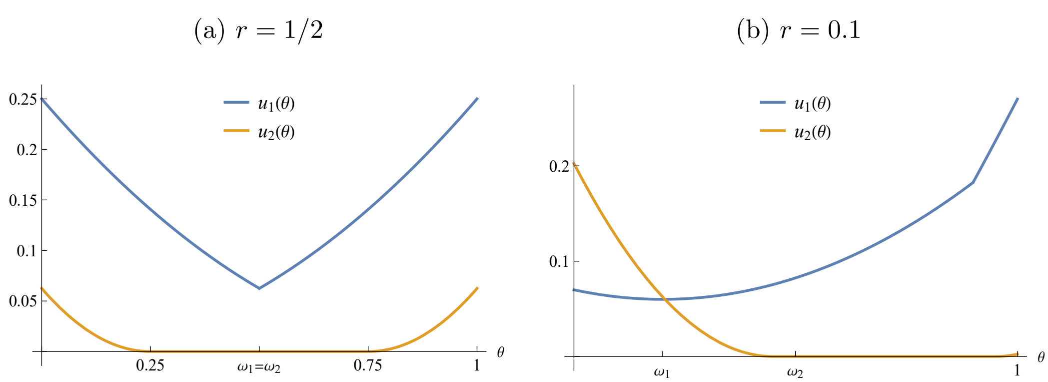

Figure D.2: Interim expected net payoffs under the double auction. Assumes $w = 1$ and uniformly distributed types.

D.3 Proof of Proposition D.1

Proof of Proposition D.1. When $w = 1$, referring to Proposition 1, the incomplete information bargaining allocation rule is such that for all $\boldsymbol{\theta}$, $Q_1(\boldsymbol{\theta}) = 1 - Q_2(\boldsymbol{\theta})$ and $Q_2(\boldsymbol{\theta}) = 1$ if and only if (ignoring ties for the moment) $\overline{\Psi}_{1,w/\rho}(\theta_1, \hat{\theta}_1) < \overline{\Psi}_{2,(1-w)/\rho}(\theta_2, \hat{\theta}_2)$, where $\hat{\theta}_i$ is a worst-off type for agent i and ρ is the smallest value greater than or equal to 1 such that the no-deficit constraint is satisfied, and $Q_2(\boldsymbol{\theta}) = 0$ otherwise. When $w = 1$, the mechanism is optimal for agent 1, so the no-deficit constraint does not bind, implying that $\rho = 1$. Thus, the allocation rule of the incomplete information bargaining mechanism is such that $Q_2(\boldsymbol{\theta}) = 1$ if and only if

$$\theta_1 < \overline{\Psi}_{2,0}(\theta_2, \hat{\theta}_2),$$

where $\hat{\theta}_2 \in [0, 1]$ is a worst-off type for agent 2 and where ties are safely ignored because they occur with probability zero, and $Q_2(\boldsymbol{\theta}) = 0$ otherwise. Given $\hat{\theta}_2$, the ironed weighted virtual type for agent 2 satisfies

$$\overline{\Psi}_{2,0}(\theta, \hat{\theta}_2) \equiv \begin{cases} \Psi^S_{2,0}(\theta) & \text{if } \Psi^S_{2,0}(\theta) < z_2, \\ z_2 & \text{if } \Psi^B_{2,0}(\theta) \leq z_2 \leq \Psi^S_{2,0}(\theta), \\ \Psi^B_{2,0}(\theta) & \text{if } z_2 < \Psi^B_{2,0}(\theta), \end{cases}$$

where the ironing parameter $z_2 \in [\Psi^B_{2,0}(\hat{\theta}_2), \Psi^S_{2,0}(\hat{\theta}_2)]$ is the unique solution to

$$\mathbb{E}_{\theta_2}\left[\Psi_{2,0}(\theta_2, \hat{\theta}_2)\right] = \mathbb{E}_{\theta_2}\left[\overline{\Psi}_{2,0}(\theta_2, \hat{\theta}_2)\right],$$

which, for uniformly distributed types on $[0, 1]$, we can write as $\int_{z_2/2}^{\hat{\theta}_2}(2\theta_2 - z_2)d\theta_2 = \int_{\hat{\theta}_2}^{(z_2+1)/2}(z_2 - 2\theta_2 + 1)d\theta_2$, or $4\hat{\theta}_2 = 1 + 2z_2$, which is satisfied, for example, for $\hat{\theta}_2 = 1/2 = z_2$.

For uniformly distributed types on $[0, 1]$, we have $Q_1(\boldsymbol{\theta}) = 1$ if and only if

$$\theta_1 > \begin{cases} 2\theta_2 & \text{if } 2\theta_2 < z_2, \\ z_2 & \text{if } 2\theta_2 - 1 \leq z_2 \leq 2\theta_2, \\ 2\theta_2 - 1 & \text{if } z_2 < 2\theta_2 - 1, \end{cases}$$

which holds if and only if

$$\theta_1 > \begin{cases} 2\theta_2 & \text{if } \theta_1 < z_2, \\ 2\theta_2 - 1 & \text{if } z_2 < \theta_1, \end{cases}$$

which implies that

$$q_2(\theta_2) = \begin{cases} 2\theta_2 & \text{if } \theta_2 < z_2/2, \\ z_2 & \text{if } z_2/2 < \theta_2 < (1 + z_2)/2, \\ 2\theta_2 - 1 & \text{if } (1 + z_2)/2 < \theta_2. \end{cases}$$

We need to solve for z_2 such that $q_2(\hat{\theta}_2) = q_2(\frac{1+2z_2}{4}) = 1 - r$, which is only satisfied if

$z_2 = 1 - r$. Thus, agent 2's worst-off type is $\hat{\theta}_2 = \frac{1+2z_2}{4} = \frac{1}{4} + \frac{1}{2}(1-r)$. This means that $Q_1(\boldsymbol{\theta}) = 1$ if and only if

$$\theta_1 > \begin{cases} 2\theta_2 & \text{if } \theta_1 < 1 - r, \\ 2\theta_2 - 1 & \text{if } 1 - r < \theta_1, \end{cases}$$

and $Q_1(\boldsymbol{\theta}) = 0$ otherwise. This completes the proof. ■

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