Efficient trade and ownership on networks*

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Abstract

Consider the problem of placing a valuable resource on a network before demands are realized, anticipating costs of subsequent transportation. We say that a node *reach dominates* another if it has cumulatively more neighbors for any distance such that transportation could be ex post efficient. We show that under IID demands, the placement that maximizes expected social surplus is confined to reach dominant nodes. Given IID demands, the same holds for an authority that maximizes expected profits and, if the resource is indivisible, for an authority that uses a second-best reallocation mechanism. Stochastic reach dominance generalizes reach dominance to account for different distributions and distances between nodes. A universal impossibility result obtains: for sufficiently high transportation costs, there is no initial placement that permits ex post efficient reallocation, assuming that placement confers ownership and that the reallocation mechanism must respect incentive compatibility, individual rationality, and no-deficit constraints.

Keywords: mechanism design, reach dominance, partnership models, optimal auctions on networks, transaction costs, supply chains

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1 Introduction

Suppose some resources have to be placed at nodes in a network before demands are realized, anticipating that reallocation will involve costly transportation. Examples range from the placement of medical or military personnel and equipment before the onslaught of a pandemic or an attack by enemy forces, to the placement of production and storage facilities, to the allocation of resources within an organization or across different firms, to the more mundane choice of hotel location by a tourist who does not know which activities will be most attractive. Where should the resources be placed? This is the question we address in this paper.

We assume that nodes in the network represent agents whose private values are independent draws from commonly known distributions. Transportation occurs along the edges of the network and involves a constant marginal cost, which, like the network structure, is commonly known, and for most of the analysis, we assume that the resources are divisible. To fix ideas, consider first the problem with identical distributions, assuming that ex post efficient reallocation is always possible. This implies that for any given initial resource placement, the resources are shipped to the agent with the highest value net of transportation costs, which involves the possibility that they are not shipped at all.

To see what governs the placement that maximizes expected social surplus, consider first a network in the shape of the letter H, which has six nodes. The nodes at the opposite ends of the horizontal line can be thought of as “uptown” and “downtown” locations. If the transportation cost is so large that the resources are only ever shipped the length of one edge, then the number of immediate neighbors of a node entirely determines the value of placing the resources at that node. Because the uptown and downtown nodes each has three immediate neighbors, while all the other nodes only have one, it follows that the optimal placement can be confined to these nodes when the marginal cost of transportation is sufficiently large. As this cost decreases, eventually shipping the length of two nodes will sometimes be ex post efficient. But, because these two nodes have more and closer neighbors than any of the other nodes, optimal placement remains confined to them for any smaller marginal cost of transportation.

More formally, we call the vector that contains the fraction of other nodes at various distances a node’s *reach vector*. In the example of the H-network, the reach vectors is $(3/5, 2/5)$ for each of the nodes at the opposite ends of the horizontal line, and it is $(1/5, 2/5, 2/5)$ for all the other nodes, using the convention of not displaying elements that are zero. The nodes at the opposite ends of the horizontal line *reach dominate* the other ones for any marginal cost of transportation because cumulatively they have a higher fraction of nodes at distances of no more than $k \in \{1, 2, 3\}$ links away. As we show, if there are nodes that are reach dominant, then optimal placements are confined to these, and optimal placements never involve nodes that are reach dominated. A completely connected node has a reach vector that is simply (1), which implies that when there are completely connected nodes, optimal placements are confined to them. This means that, for example, in a star or wheel network, it is optimal to place all resources at the hub.

In general, whether a node is reach dominant depends on the marginal cost of transportation. To see this, consider a wide H-network in which there is an additional node—a "midtown" node— in the middle of the horizontal line, so that there are seven nodes. The reach vectors of the downtown and uptown nodes are $(3/6, 1/6, 2/6)$, the reach vector of the midtown node is $(2/6, 4/6)$, and the reach vector for all other nodes is $(1/6, 2/6, 1/6, 2/6)$. When the marginal cost of transportation is so large that only shipments of length one are ever ex post efficient, the downtown and uptown nodes are reach dominant. As shipments of length two or more become ex post efficient, these nodes and the midtown node can no longer be ranked by reach dominance. However, they always reach dominate the peripheral nodes, implying that resources are never optimally placed at a peripheral node. One can show that for uniformly distributed values on the unit interval, the resources are optimally placed at downtown and uptown nodes when the marginal cost of transportation is sufficiently large and otherwise at the midtown node. This has the interesting and perhaps counterintuitive feature that a reduction in transportation (or transaction) costs leads to optimal placements in midtown, which has fewer close neighbors, but more neighbors at intermediate distances than uptown and downtown.

If each agent is privately informed about its values and if placement does not confer control, then the placement problem faced by an authority that maximizes profits, subject to the agent's incentive compatibility and individual rationality constraints, is isomorphic to that faced by a social surplus maximizer. In particular, with identical distributions, reach dominance governs the optimal placement. This follows because profit maximization is equivalent to social surplus maximization, with true values replaced by virtual values.

If placing resources at a node confers control or ownership rights over the resources to that node, then ex post efficient reallocation may not be possible, subject to incentive compatibility, individual rationality, and no-deficit constraints. We show that ex post efficiency is never possible with extremal ownership and, more surprisingly, for any ownership structure if the marginal cost of transportation is sufficiently large. With this in mind, we first derive the constrained-efficient reallocation mechanism and then use this mechanism to determine the optimal ownership structure, which, loosely speaking, finds a balance between incentive and transportation costs. While, in general, this optimal ownership structure is not determined via reach dominance, we show that under IID, the reach dominance arguments extend to optimal ownership if the marginal cost of transportation is sufficiently large or if the resource is indivisible, as in the case of, say, a production plant.

Assuming that ex post efficient reallocation is always possible and that the authority maximizes social surplus, the arguments underlying reach dominance extend to heterogeneous distributions and links between nodes that are of different lengths. The key is to replace the fraction of agents at a given distance from a node by the distribution of the highest draw from the agents not further away from a node than some given distance (including the agent at that node). The ranking is then based on stochastic dominance, which is why we refer to this generalization as *stochastic reach dominance*. We also show that the analysis extends to the case in which the cost of transportation is a fixed cost, which seems an appropriate description, for example, when the cost relates to difficulties of communication. While we use the interpretation of transport costs, these costs can equivalently and more broadly be interpreted as transaction costs above and beyond those associated with private information, with nodes at further distances from each other representing, for example, firms that have fewer ongoing transactions with one another.[1]

This paper relates to the literature on the (im)possibility of ex post efficient trade initiated by Vickrey (1961) and Myerson and Satterthwaite (1983) and debates surrounding the Coase Theorem (Coase, 1960). That extremal ownership prevents ex post efficient trade follows from an extension of the impossibility theorem of Myerson and Satterthwaite to costly transportation.[2] That ex post efficient trade is not possi-

[1] Interpreted this way, our framework provides a network-based formalization of the transaction costs and reallocation problems discussed in Cramton et al. (1998).

[2] It is also an extension of the bilateral trade setup to settings with multiple buyers and one

ble for any ownership vector if the marginal cost of transportation is sufficiently large is, to our knowledge, a new impossibility theorem. The fact that, if the marginal cost of transportation is small, then the designer trades off incentive costs against transportation costs, builds on the insight from the partnership literature that, without costly transportation, ex post efficient trade is possible with appropriately structured ownership; see, for example, Cramton et al. (1987), Che (2006), Segal and Whinston (2011), or Figueroa and Skreta (2012). We show that this insight extends to costly transportation, provided that the marginal cost of transportation is sufficiently small. A precursor to our paper is Salant and Siegel (2015), which considers partnership dissolution problems with reallocation costs. They analyze a more general family of transportation costs, but do not model a network structure. Moreover, like the aforementioned papers on partnership problems, they do not analyze second-best or constrained-efficient mechanisms.

To solve the problem involving constrained-efficient reallocation mechanisms, our paper builds on the work related to optimal trading mechanisms for asset markets—problems in which each agent's trading positions (buy, sell, remain inactive) are determined endogenously—and partnership models by Lu and Robert (2001), Loertscher and Wasser (2019), and Liu et al. (2023).[3] The related literature on networks includes Akbarpour and Jackson (2018), which examines how diffusion patterns depend on the network placement of heterogeneous agents, and Houde et al. (2023), which shows that incentives for tax avoidance led Amazon to distort its distribution network in a way that increased transportation costs.[4]

The remainder of this paper is structured as follows. Section 2 contains the setup together with the definitions of the various problems of interest and basic results. Sections 3 analyzes the problem when resource placement does not confer control and Section 4 the problem when it does. Extensions are presented in Section 5, and Section 6 concludes the paper.

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seller, but that extension is already in the literature (see, e.g., Gresik and Satterthwaite, 1989).

[3]The term "asset market" has been used by Loertscher and Marx (2020, 2023), Delacrétaz et al. (2022), and Liu et al. (2023). Analyses of asset market problems (that do not use that label) are also provided by Lu and Robert (2001) and Li and Dworczak (2021).

[4]Condorelli et al. (2017) examine trade on a network, as do we, but their focus is on dynamic bilateral bargaining with binary types and no transportation costs. For an analysis of the positioning of disaster relief supplies at hospitals, see Wang et al. (2022), and on the positioning of granaries, see Shiue (2004).

2 Setup

There are n agents indexed by $i \in \mathcal{N} \equiv \{1, \ldots, n\}$ and a resource whose total supply is 1. Each agent $i \in \mathcal{N}$ is located at a node in an undirected graph that connects all agents, where $d_{ij} \in \{0, 1, \ldots, n-1\}$ is the length of the minimum path through the network between agents i and j (for all $i \in \mathcal{N}$, $d_{ii} = 0$).[5] Before trade occurs, each agent $i \in \mathcal{N}$ holds a resource amount $r_i \in [0, 1]$ with $\sum_{i \in \mathcal{N}} r_i = 1$. Each agent i has constant marginal value v_i for the resource, which is independently drawn from the distribution F_i. The support of F_i, denoted $\mathcal{V}_i$, is bounded by 0 and 1 and contains both 0 and 1. Some of our analysis assumes that all agents draw their values from the same distribution F, which is the IID case referred to above. When analyzing incentive problems, like profit maximization or optimal ownership problems, we assume that each agent i is privately informed about the realization of its value v_i and that for each $i \in \mathcal{N}$, F_i is a continuous distribution with support $\mathcal{V}_i = [0, 1]$ and density $f_i > 0$. Let $\mathcal{V} \equiv \prod_{i \in \mathcal{N}} \mathcal{V}_i$ denote the type space.[6]

We assume that the cost of transporting $x \in [0, r_i]$ units of the resource from agent i to agent j is xcd_{ij}, where $c \geq 0$ is the commonly known marginal transportation cost per edge traveled. The $n \times n$ symmetric matrix $C = (C_{ij})_{i,j \in \mathcal{N}}$ is the transportation cost matrix, with component C_{ij} representing the transportation cost between agents i and j, where for all $i, j \in \mathcal{N}$, $C_{ij} = C_{ji} = cd_{ij}$.

Two agents i and j are directly connected if $d_{ij} = 1$. A network is *complete* if every agent is directly connected to every other agent. Agent i is said to be *completely connected* if agent i is directly connected to every other agent, and we say that agent i is *maximally connected* if no other agent is directly connected to a larger number of other agents than is agent i. For example, the *star* network with $n \geq 3$ agents is defined as having agent 1 as the hub and a transportation cost matrix such that for $i > 1$, $C_{1i} = c$, and for $1 < i < j$, $C_{ij} = 2c$. A *wheel* network with $n \geq 5$ agents is a star augmented by a "ring road," that is, agent 1 is the hub, and each agent at the periphery is also directly linked to exactly two other peripheral nodes.

[5] Our results extend to directed graphs (see Section 5.3).

[6] These assumptions are imposed because the independent private values model has the property of giving rise to a tradeoff between profits and social surplus based on the primitives of the model. As Loertscher and Marx (2022a) note, in some sense, it is the only model that has this property: without independence, Crémer and McLean (1985, 1988) show that full surplus extraction is possible (see also Myerson, 1981) and, as shown by (Mezzetti, 2004, 2007), the same is true if one allows for interdependent values even when maintaining the assumption of independent types.

A trading mechanism $\langle \mathbf{Q}, \mathbf{M} \rangle$ consists of an allocation rule $\mathbf{Q} = (Q_i)_{i \in \mathcal{N}}$, where $Q_i : \mathcal{V} \to [0,1]$ with $\sum_{i \in \mathcal{N}} Q_i(\mathbf{v}) \leq 1$, and a payment rule $\mathbf{M}$, where $M_i : \mathcal{V} \to \mathbb{R}$. Given types $\mathbf{v}$, $Q_i(\mathbf{v})$ specifies agent i's consumption after trade, and $M_i(\mathbf{v})$ specifies the payment made by agent i to the operator of the mechanism.

2.1 Ex post efficient trade

Given realized types $\mathbf{v}$, define the $n \times n$ binary matrix $V^e(\mathbf{v})$, each of whose rows sums to 1, by

$$V_{ij}^e(\mathbf{v}) \equiv \begin{cases} 1 & \text{if } v_j - C_{ij} \geq \max_\ell v_\ell - C_{i\ell} \text{ and } v_j - C_{ij} > \max_{\ell < j} v_\ell - C_{i\ell}, \\ 0 & \text{otherwise,} \end{cases}$$

which uses the tie-breaking rule of prioritizing agents with a lower index. This choice is arbitrary but without loss of generality. By definition, $V_{ij}^e(\mathbf{v}) = 1$ only if moving agent i's resources to agent j maximizes value net of transportation costs.

The ex post efficient allocation rule assigns to agent i the resources of agents j with $V_{ji}^e(\mathbf{v}) = 1$, that is $Q_{i,\mathbf{r}}^e(\mathbf{v}) \equiv \sum_{j=1}^n V_{ji}^e(\mathbf{v}) r_j$. Maximized social surplus is therefore

$$SS_{\mathbf{r}}^e(\mathbf{v}) = \sum_{i=1}^n \sum_{j=1}^n (v_i - C_{ji}) \, V_{ji}^e(\mathbf{v}) r_j, \tag{1}$$

with expected value

$$ss_{\mathbf{r}}^e \equiv \mathbb{E}_{\mathbf{v}}[SS_{\mathbf{r}}^e(\mathbf{v})] = \sum_{j=1}^n r_j \mathbb{E}_{\mathbf{v}} \Big[\sum_{i=1}^n (v_i - C_{ji}) \, V_{ji}^e(\mathbf{v}) \Big]. \tag{2}$$

Expected transportation costs under the ex post efficient allocation rule are

$$t_{\mathbf{r}}^e \equiv \sum_{j=1}^n r_j \mathbb{E}_{\mathbf{v}} \Big[\sum_{i=1}^n C_{ji} V_{ji}^e(\mathbf{v}) \Big]. \tag{3}$$

2.2 Placement problems

In a *placement problem*, a central authority or social planner places resources with agents before types are realized. In this setting, we refer to $\mathbf{r}$ as the placement vector. The planner retains control over the resources, meaning that after agents' types are realized, the planner can direct the reallocation of resources subject to incentive compatibility and individual rationality constraints, where each agent's outside option is zero. Reallocation in the placement problem is thus a one-sided allocation problem in which the agents are buyers. Because the planner can, for example, run a second-price auction, where each agent's bid is adjusted for the required transportation cost, the reallocation phase always permits an incentive compatible and individually rational solution that does not run a deficit.[7]

We consider objectives for the planner of either social surplus maximization or profit maximization. In either case, the planner first places resources on the network and then implements an incentive compatible, individually rational mechanism that reallocates the resources and collects payment from the agents, with the planner paying for the associated transportation costs. A social-surplus-maximizing planner uses the incentive compatible, individually rational mechanism that maximizes expected social surplus net of transportation costs. A profit-maximizing planner uses the incentive compatible, individually rational mechanism that maximizes its expected revenue net of transportation costs. As mentioned, when analyzing profit maximization, we assume that for each i, F_i has support $[0, 1]$ and a positive density f_i.

2.3 Ownership problems

In an *ownership problem*, a market designer determines resource ownership by the agents, where ownership gives an agent property rights or control over the resources. In this setting, we refer to $\mathbf{r}$ as the ownership vector. Then, following type realizations, the designer implements an incentive compatible, individually rational reallocation mechanism that does not run a deficit, i.e., the expected revenue to the designer is sufficient to cover expected transportation costs. The individual rationality constraints affecting the mechanism vary with the resource ownership because each agent's outside option is to consume its owned resources. Thus, an ownership problem is more constrained than a placement problem, where the agents' outside options are zero.

In an ownership problem, if there is extremal resource ownership, i.e., $r_i = 1$ for some agent i, then the reallocation phase is a two-sided allocation problem with one seller (agent i) and $n - 1$ buyers. In contrast, if resource ownership is dispersed

[7]By standard arguments, it can always be made to balance the budget; see, e.g., Börgers and Norman (2009). A placement problem also arises if agents' realized values are commonly known.

among multiple agents, then the reallocation phase becomes what is sometimes called an "asset market" because the trading positions of the agents—buy, sell, or do not trade—depend, in general, on their own realized types and the realized types of all other agents.

For some, but not all, ownership vectors, there exists an incentive compatible, individually rational, deficit-free trade mechanism that implements ex post efficient trade. For ownership vectors for which ex post efficient trade is not possible, a social-surplus maximizing designer specifies the constrained-efficient mechanism, i.e., the mechanism that maximizes expected social surplus subject to incentive compatibility, individual rationality, and revenues that at least cover the transportation costs. A profit-maximizing designer specifies the incentive compatible, individually rational mechanism that maximizes the designer's expected revenue from the agents net of the transportation costs.

Our framework encompasses various, extensively studied and used models as special cases. For $c = 0$, it collapses to a homogeneous good model in which the network structure plays no role, implying that any placement permits the first-best and that an optimal ownership structure exists that permits the first-best (see, e.g., Cramton et al., 1987; Che, 2006; Figueroa and Skreta, 2012). With a complete network and identical distributions, for $c \geq 0$, any placement permits the first-best. A star network preserves the property of the homogeneous good model that there is always only one buyer (and hence only one cluster of trade) under ex post efficiency.

2.4 Virtual type functions

For the analysis involving constrained-efficient and profit-maximizing reallocation mechanisms (defined below), we assume, beyond continuous distributions with densities $f_i > 0$ on $[0, 1]$, that each agent i's virtual type functions,

$$\Psi_i^B(v) \equiv v - \frac{1 - F_i(v)}{f_i(v)} \quad \text{and} \quad \Psi_i^S(v) \equiv v + \frac{F_i(v)}{f_i(v)},$$

are increasing. Despite this monotonicity of the virtual types, which corresponds to what Myerson (1981) calls the "regular" case, the mechanism design problem in the reallocation phase will not be regular away from extremal ownership.

3 Optimal placement

In this section, we first characterize the optimal placement for a social-surplus-maximizing planner, and then we consider the case of a profit-maximizing planner.

3.1 Optimal placement under social-surplus maximization

We first show that under social surplus maximization, extremal placement is always optimal.

Extremal placement is always optimal

The linearity in $\mathbf{r}$ of social surplus under ex post efficient trade, $SS^e_{\mathbf{r}}(\mathbf{v})$, which is defined in (1), implies that its expectation, $ss^e_{\mathbf{r}}$ defined in (2), is also linear in $\mathbf{r}$.[8] This in turn implies that an extremal placement, i.e., $r_i = 1$ for some $i \in \mathcal{N}$, is always optimal. We state this in the following proposition:

Proposition 1. *The social-surplus maximizing placement problem has a solution involving extremal placement followed by ex post efficient trade.*

Proof. See Appendix A.

We refer to the social-surplus maximizing placement followed by ex post efficient trade as the *first-best*. While extremal placement is not necessarily uniquely optimal, and while in general not any extremal placement will be optimal, one extremal placement always will be. Specifically, note that $r_n = 1-\sum_{i=1}^{n-1} r_i$ and write expected social surplus as a function of $\mathbf{r}_{-n} \equiv (r_1, \dots, r_{n-1}, 1-\sum_{i=1}^{n-1} r_i)$. If for some $j \in \{1, \dots, n-1\}$ we have $\frac{\partial ss^e_{\mathbf{r}_{-n}}}{\partial r_j} = \max_{h\in\{1,\dots,n-1\}} \frac{\partial ss^e_{\mathbf{r}_{-n}}}{\partial r_h} \geq 0$, then the extremal placement $r_j = 1$ is optimal. Otherwise, that is, if $\frac{\partial ss^e_{\mathbf{r}_{-n}}}{\partial r_j} < 0$ for all $j \in \{1, \dots, n-1\}$, then the extremal placement $r_n = 1$ is uniquely optimal. If $\frac{\partial ss^e_{\mathbf{r}_{-n}}}{\partial r_j} = 0$ for all $j \in \{1, \dots, n-1\}$, as is the case, for example, if $c = 0$ or in a complete network with identical distributions, then any $\mathbf{r}$ is optimal.

[8] Linearity of $SS^e_{\mathbf{r}}(\mathbf{v})$ and $ss^e_{\mathbf{r}}$ in $\mathbf{r}$ means that for any $\mathbf{r}, \mathbf{r}'$ and any $\alpha \in [0, 1]$, we have $SS^e_{\alpha\mathbf{r}+(1-\alpha)\mathbf{r}'}(\mathbf{v}) = \alpha SS^e_{\mathbf{r}}(\mathbf{v}) + (1-\alpha)SS^e_{\mathbf{r}'}(\mathbf{v})$ and $ss^e_{\alpha\mathbf{r}+(1-\alpha)\mathbf{r}'} = \alpha ss^e_{\mathbf{r}} + (1-\alpha)ss^e_{\mathbf{r}'}$, respectively.

Reach dominance and the first-best

For the remainder of this subsection, we assume that $F_i = F$ for all $i \in \mathcal{N}$. This is mainly done for illustrative purposes since, as we show in Section 5.2, the key insights and mechanics hold more generally.

To develop an understanding of what determines social surplus maximizing placement, assume first that $c \in (1/2, 1)$. In this case, only the immediate neighbors of an agent are candidate trading partners for that agent. We denote by $n_i(1)$ the *degree centrality* of agent i. This is the number of immediate neighbors of agent i divided by $n-1$, where $n-1$ is the maximum possible number of immediate neighbors (Jackson, 2008, p. 38). It then follows that $r_i = 1$ is optimal if and only if i has the maximum number of immediate neighbors, that is,

$$n_i(1) = \max_{j \in \mathcal{N}} n_j(1).$$

If there are multiple agents with the maximum number of immediate neighbors, then optimal placement can be dispersed across these agents. For c less than $1/2$, we also have to take into account agents other than the immediate neighbors of agent i when determining the value of having $r_i = 1$. For any given $c \in (0, 1)$, the maximum reach that needs to be considered is $\min\{\lceil 1/c \rceil, n-1\}$, where $\lceil x \rceil$ denotes the largest integer no larger than x.

For any agent i, we let $\mathbf{n}_i = (n_i(1), \ldots, n_i(n-1))$ be the $n-1$-dimensional vector where for $\ell \in \{1, \ldots, n-1\}$,

$$n_i(\ell) \equiv \frac{1}{n-1} \left| \{ j \in \mathcal{N} \mid d_{ij} = \ell \} \right|.$$

In words, $n_i(\ell)$ is the number of agents at distance ℓ from agent i, normalized by $n-1$. We refer to $\mathbf{n}_i$ as the *reach vector* of agent i. Then for any $i \in \mathcal{N}$, we have

$$\sum_{j=1}^{n-1} n_i(j) = 1.$$

Drawing on the concept of reach centrality from the graph theory literature, given

$k \in \{1, \ldots, n-1\}$, agent i's *k-step reach centrality* is:[9]

$$\sigma_i(k) \equiv \sum_{j=1}^{k} n_i(j).$$

We can then employ k-step reach centrality to define reach dominance: agent i *reach dominates* (RDs) agent h given c if for all $k \in \{1, \ldots, \min\{\lceil 1/c \rceil, n-1\}\}$,

$$\sigma_i(k) \geq \sigma_h(k), \tag{4}$$

with a strict inequality for at least one k. Reach dominance induces an incomplete order and is equivalent to first-order stochastic dominance, with better outcomes having cumulatively higher probability.

Given c, let $\mathcal{D}(c)$ denote the set of of agents who are reach dominated by some other agent. That is,

$$\mathcal{D}(c) = \{h \in \mathcal{N} \mid \exists\, i \in \mathcal{N} \text{ s.t. } i \text{ RDs } h \text{ given } c\}.$$

If follows that if $\mathbf{r}$ is part of the solution of the planner's problem, then $r_j = 0$ for all $j \in \mathcal{D}(c)$.

We define the (possibly empty) set of *reach dominant* agents given c, denoted $\mathcal{T}(c)$, to be the set of agents such that any agent not in $\mathcal{T}(c)$ is reach dominated given c by every agent in $\mathcal{T}(c)$ and for all $i, j \in \mathcal{T}(c)$ and all $\ell \in \{1, \ldots, \min\{\lceil 1/c \rceil, n-1\}\}$, (4) holds with equality. For example, if $c \in (1/2, 1)$, $\mathcal{T}(c)$ is the set of agents with the maximum number of immediate neighbors.

Proposition 2. *Assume identical distributions. Under the first-best, resources are never placed with reach dominated agents. If the set of reach dominant agents is non-empty, then under the first-best, resources are only placed with reach dominant agents.*

Proof. See Appendix A.

A first corollary to Proposition 2, is, as foreshadowed above:

[9] See, e.g., Borgatti et al. (2018, Chapter 10.3.5), Sosnowska and Skibski (2018).

Corollary 1. *Assume identical distributions. If* $c \in (1/2, 1)$*, then the set of reach dominant agents is non-empty and consists of the agents with the maximum number of immediate neighbors. Further, if the set of completely connected agents is non-empty, then for any* $c \in (0, 1)$*, an agent is reach dominant if and only if it is completely connected.*

The reason is simple: completely connected agents are reach dominant for any $c \in (0, 1)$. Consequently, we have for the star and wheel networks:

Corollary 2. *Assume identical distributions. For star and wheel networks with* $c \in (0, 1)$*, it is uniquely optimal to place all resources at the hub.*

As an illustration and application, begin by considering a line network with five nodes. We label the three inner nodes from left to right as 1, 2 and 3 and the peripheral nodes as 0 and 4 as shown in Figure 1(a). For this network, the reach vectors are:

$$\mathbf{n}_1 = \mathbf{n}_3 = (2/4, 1/4, 1/4)\,, \quad \mathbf{n}_2 = (2/4, 2/4) \quad \text{and} \quad \mathbf{n}_0 = \mathbf{n}_4 = (1/4, 1/4, 1/4, 1/4)\,.$$

It follows that agents 0 and 4 are reach dominated for any c by agents 1, 2, and 3. For $c \in (1/2, 1)$, $\mathcal{T}(c) = \{1, 2, 3\}$, and for $c < 1/2$, $\mathcal{T}(c) = \{2\}$. Thus, in this network, the set of first-best placements is determined entirely by reach dominance. While $\mathcal{T}(c)$ varies with c, it does so monotonically in the sense of set inclusion.

To see what happens beyond networks with this property, we amend the line network by adding a link to node 1 and a link to node 3, which we denote by $0'$ and $4'$, respectively, as shown in Figure 1(b). For this "wide H network," the reach vectors are:

$$\mathbf{n}_1 = \mathbf{n}_3 = (3/6, 1/6, 2/6)\,, \quad \mathbf{n}_2 = (2/6, 4/6)\,, \text{ and } \mathbf{n}_i = (1/6, 2/6, 1/6, 2/6)\,,$$

for $i \in \{0, 0', 4, 4'\}$. The set of reach dominated agents now consists of the nodes $\{0, 0', 4, 4'\}$. For $c \in (1/2, 1)$, $\mathcal{T}(c) = \{1, 3\}$, and otherwise $\mathcal{T}(c) = \varnothing$. Thus, for $c < 1/2$, determining the optimal placement depends on c and requires computation. It will, of course, be confined to agents that are not reach dominated. For example, for F uniform on $[0, 1]$, $r_1 = 1$ and $r_3 = 1$ (and $r_1 + r_3 = 1$) are optimal for $c > 0.09$, and $r_2 = 1$ is optimal otherwise. Thus, the set of nodes at which resources are optimally placed varies with c in a nonmonotonic way. For c large, resources are placed with

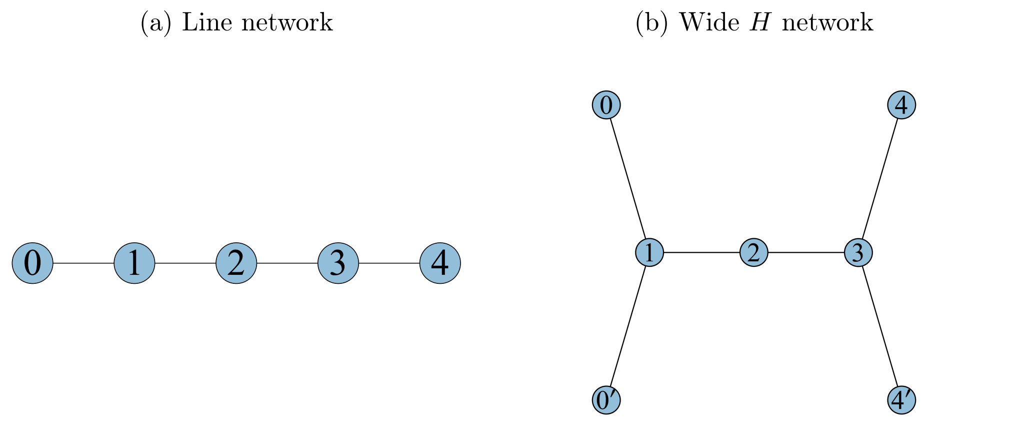

Figure 1: Panel (a): A network in which first-best placement varies monotonically with c. Panel (b): A network in which first-best placement varies nonmonotonically with c.

agents 1 or 3, which have the most immediate neighbors. In contrast, when c is small, agent 2, which has few immediate neighbors but has all agents within a distance of two, is optimally the sole initial holder of the resources.

3.2 Optimal placement under profit maximization

While it is sensible to think of a planner as maximizing social surplus, it is also conceivable that a planner, i.e., an entity that retains control of the resources after placement, maximizes its expected profit, subject to the agents' incentive compatibility and individual rationality constraints. With that in mind, we now examine the profit-maximizing mechanism and placement for the planner. Its profit is defined as payments from the agents minus transportation costs. We begin by taking the placement as given and deriving the profit-maximizing mechanism, and then we optimize over the placement. Throughout this subsection, we assume that for all $i \in \mathcal{N}$, F_i is a continuous distribution with support $[0, 1]$ and density $f_i > 0$ that exhibits an increasing virtual value function $\Psi_i^B(v)$.

Because the planner retains control over the resources, it acts as a seller with all agents trading as buyers, including the agent with whom the resources are initially placed. Thus, the planner's optimal mechanism reallocates units to agents in order according to their virtual values net of transportation costs if and only if the net virtual value is positive. Specifically, the planner's profit-maximizing allocation rule is given by

$$Q_{i,\mathbf{r}}^{\text{profit}}(\mathbf{v}) \equiv \sum_{j=1}^{n} V_{ji}^{\text{profit}}(\mathbf{v}) r_j,$$

where

$$V_{ij}^{\text{profit}}(\mathbf{v}) \equiv \begin{cases} 1 & \text{if } \Psi_j^B(v_j) - C_{ij} = \max_\ell \Psi_\ell^B(v_\ell) - C_{i\ell} \geq 0 \\ & \quad \text{and } \Psi_j^B(v_j) - C_{ij} > \max_{\ell<j} \Psi_\ell^B(v_\ell) - C_{i\ell}, \\ 0 & \text{otherwise,} \end{cases}$$

which uses, again, the tie-breaking rule that prioritizes agents with a lower index. To define the planner's expected profit, it will be useful to have the following lemma, which follows from standard mechanism design arguments:

Lemma 1. *For the planner's problem, given an incentive compatible mechanism* $\langle \mathbf{Q}, \mathbf{M} \rangle$ *with individual rationality binding for the worst-off types, agent i's expected payment to the mechanism is* $\mathbb{E}_{\mathbf{v}}[M_i(\mathbf{v})] = \mathbb{E}_{\mathbf{v}}\left[\Psi_i^B(v_i) Q_i(\mathbf{v})\right]$.

Using Lemma 1, the expected profit to the planner not including transportation costs is

$$\Pi_{\mathbf{r}}^{\text{profit}} \equiv \mathbb{E}_{\mathbf{v}}\Big[\sum_{i=1}^{n} \Psi_i^B(v_i) Q_{i,\mathbf{r}}^{\text{profit}}(\mathbf{v})\Big] = \sum_{j=1}^{n} r_j \mathbb{E}_{\mathbf{v}}\Big[\sum_{i=1}^{n} \Psi_i^B(v_i) V_{ji}^{\text{profit}}(\mathbf{v})\Big],$$

and expected transportation costs are

$$t_{\mathbf{r}}^{\text{profit}} \equiv \sum_{j=1}^{n} r_j \mathbb{E}_{\mathbf{v}}\Big[\sum_{i=1}^{n} C_{ji} V_{ji}^{\text{profit}}(\mathbf{v})\Big],$$

giving the planner maximized expected profit conditional on $\mathbf{r}$ of $\Pi_{\mathbf{r}}^{\text{profit}} - t_{\mathbf{r}}^{\text{profit}}$, which, notably, is linear in $\mathbf{r}$.

The key observation regarding optimal placement under profit maximization is a simple isomorphism between maximizing expected social surplus and profit. To see this, let $\tilde{F}_i(\psi) = F_i(\Psi_i^{B^{-1}}(\psi))$ for $\psi \in (0,1]$, and $\tilde{F}_i(\psi) = F_i(\Psi_i^{B^{-1}}(0))$ otherwise, be the distribution of i's virtual value, conditional on its being positive. Note that, just like the values, the nonnegative virtual values are independent random variables with support $[0,1]$. The only difference is that the distributions of the nonnegative virtual values are $\tilde{F}_i$ rather than F_i. Letting $\boldsymbol{\psi} = (\psi_1, \ldots, \psi_n)$, we then have $V_{ji}^{\text{profit}}(\mathbf{v}) =$

$V^e_{ji}(\boldsymbol{\psi})$ and thus

$$\sum_{i=1}^{n}\Big(\mathbb{E}_{\mathbf{v}}[(\Psi^B_i(v_i)-C_{ji})V^{\text{profit}}_{ji}(\mathbf{v})]\Big)=\sum_{i=1}^{n}\Big(\mathbb{E}_{\boldsymbol{\psi}}[(\psi_i-C_{ji})V^e_{ji}(\boldsymbol{\psi})]\Big).$$

This basic observation yields the corollaries that follow.[10] The first one is an implication of Proposition 1:

Corollary 3. *Extremal ownership is optimal for a profit-maximizing planner.*

In the case of identical distributions, that is, $F_i = F$ for all $i \in \mathcal{N}$, we have $\tilde{F}_i(\psi) = \tilde{F}(\psi)$ for all $i \in \mathcal{N}$ and all $\psi \in [0,1]$, where $\tilde{F}(\psi) = F(\Psi^{B^{-1}}_i(\psi))$ for $\psi \in (0,1]$, and $\tilde{F}(\psi) = F(\Psi^{B-1}_i(0))$ otherwise. Proposition 2 then yields the next corollary:

Corollary 4. *Assuming identical distributions and a profit-maximizing planner, resources are never placed with reach dominated agents, and if the set of reach dominant agents is non-empty, then resources are only placed with reach dominant agents.*

Our earlier Corollaries 1 and 2 also extend to the setting with profit maximization. For example, in the wide H network with $n = 7$ shown in Figure 1(b) and uniformly distributed types, for a social-surplus-maximizing planner, $r_1 = 1$ and $r_3 = 1$ are both optimal for $c \in (0.09, 1)$, but only $r_2 = 1$ is optimal for $c \in (0, 0.09)$. In the case of a profit-maximizing planner, the range where $r_2 = 1$ is uniquely optimal extends to all $c \in (0, 0.175)$. Thus, the profit-maximizing planner places resources with the agent with fewer immediate neighbors (but with all agents within a distance of 2) for a larger range of costs. As intuition, notice that the planner's expected profit in the profit-maximizing mechanism is the same as in the ex post efficient mechanism, but with types drawn from different distributions, i.e., with agent i's type drawn from the distribution of $\Psi^B_i(v_i)$. Because $\Psi^B_i(v_i) < v_i$ for $v_i \in [0,1)$, it is as if the profit-maximizing planner faces agents with a worse distribution. When facing a worse distribution, the planner values having the extra "draws" within close range (specifically within a distance of 2), that come with having the resources placed with

[10] The observation that profit and social surplus maximization are isomorphic in this sense was made and exploited by Loertscher et al. (2022). Whether that was the first explicit formalization of that fact we do not know. Clearly, it is implicit in the analysis of regular mechanism design problems, such as optimal auctions or bilateral trade problems à la Myerson (1981) and Myerson and Satterthwaite (1983).

agent 2, rather than with agents 1 or 3. As the distribution becomes worse, having access to additional type realizations becomes more valuable.[11]

4 Optimal ownership

We now consider ownership problems, which as mentioned above, arise when agents are endowed with ownership of resources at their nodes. We first derive conditions for ex post efficiency to be (im)possible. Then we derive the constrained-efficient mechanism when ex post efficient trade is not possible and characterize conditions under which only constrained-efficient trade is possible. We then derive the optimal ownership for a social-surplus-maximizing and a profit-maximizing designer. Throughout this section, assume that for all $i \in \mathcal{N}$, F_i is a continuous distribution with support $[0, 1]$ and density $f_i > 0$.

4.1 Impossibility of ex post efficient trade

We begin by establishing two sets of impossibility results. The first, in the tradition of Vickrey (1961) and Myerson and Satterthwaite (1983), is based on the observation that ex post efficient trade is impossible under extremal ownership. This implies that the first-best is not possible when the first-best dictates extremal ownership, which is, for example, the case for the star and wheel networks with identical distributions. Second, we show that ex post efficiency is impossible for any ownership vector when $c \geq 1/2$.

Impossibility with extremal ownership

Consider an extremal ownership vector in which $r_1 = 1$, so that agent 1 is the seller whenever there is trade. Trade between agent 1 and agent $i \in \{2, \dots, n\}$ is ex post efficient if and only if $v_i - C_{1i} = \max_{j \in \{2,\dots,n\}} v_j - C_{1j}$ and $v_i - C_{1i} > v_1$. We denote

[11] For example, consider ex post efficient trade and types drawn from $F_i = F$, where F is the uniform distribution on $[\ell, 1]$, where $\ell < 1$. Then for $\ell = 0$, the problem is one of a social-surplus-maximizing planner facing types drawn from the uniform distribution on $[0, 1]$. For $\ell = -1$, the problem is one of a profit-maximizing planner facing types drawn from the uniform distribution on $[0, 1]$ because in that case the agents' virtual values are uniformly distributed on $[-1, 1]$. For the wide H network, $r_2 = 1$ is uniquely optimal for $c \in (0, \overline{c}(\ell))$, and $r_1 = 1$ and $r_3 = 1$ are optimal for $c \in (\overline{c}(\ell), 1)$, where $\overline{c}(\ell)$ is decreasing in ℓ, implying that the range of costs for which $r_2 = 1$ is optimal decreases with ℓ.

by i_{eff} the index of such an agent i. Consider then the market-clearing (Walrasian) prices that establish ex post efficient trade given types $\mathbf{v}$. Without loss of generality, we let the seller bear the transportation cost.

If $(p_1^W, \ldots, p_n^W)$ is a Walrasian price vector, then it has to satisfy $v_1 + C_{1i_{\text{eff}}} \leq p_{i_{\text{eff}}}^W \leq v_{i_{\text{eff}}}$ so that agent 1 is willing to pay the transportation cost $C_{1i_{\text{eff}}}$ to sell to agent i_{eff} at price $p_{i_{\text{eff}}}^W$ and so that agent i_{eff} is willing to buy at this price. In addition, for $j \in \{2, \ldots, n\} \setminus \{i_{\text{eff}}\}$, Walrasian prices require that $v_j \leq p_j^W$ so that agent j does not want to buy at price p_j^W and $p_j^W - C_{1j} \leq p_{i_{\text{eff}}}^W - C_{1i_{\text{eff}}}$ so that agent 1 does not want to sell to agent j instead of agent i_{eff}. Putting all of this together, the smallest and largest Walrasian price, denoted $\underline{p}^W$ and $\overline{p}^W$, are given as

$$\underline{p}^W \equiv \max_{j \in \{2,\ldots,n\} \setminus \{i_{\text{eff}}\}} \{v_1, v_j - C_{1j}\} + C_{1i_{\text{eff}}} \quad \text{and} \quad \overline{p}^W \equiv v_{i_{\text{eff}}},$$

respectively. As is reasonably well known and easily established, a trading buyer's payment in the VCG mechanism is $\underline{p}^W$ and a trading seller's payment is $\overline{p}^W$ (see e.g. Delacrétaz et al., 2022).[12] Consequently, if trade occurs under ex post efficiency, then the revenue of the VCG mechanism is

$$\underline{p}^W - \overline{p}^W \leq 0,$$

where the inequality is strict unless $\max_{j \in \{2,\ldots,n\} \setminus i_{\text{eff}}} v_j - C_{1j} = v_{i_{\text{eff}}} - C_{1i_{\text{eff}}}$. Because ties have probability 0 with continuous distributions, it follows that the VCG mechanism almost always runs a deficit when trade is ex post efficient (and never a budget surplus). Consequently, in expectation, the VCG mechanism runs a deficit. Because the ex post and hence interim expected payoffs are zero for buyers of type 0 and for the seller of type 1, it follows that the VCG mechanism satisfies the interim individual rationality constraints with equality. By the payoff equivalence theorem, this implies that no other ex post efficient, (Bayesian or dominant strategy) incentive compatible, and interim individually rational mechanism runs a smaller deficit. Because the VCG

[12] The gains from trade with agent i_{eff} present, but excluding its value for the allocation, are $-v_1 - C_{1i_{\text{eff}}}$, whereas gains from trade with agent i_{eff} reporting a value of 0 are $\max\{0, \max_{j \in \{2,\ldots,n\} \setminus \{i_{\text{eff}}\}} v_j - C_{1j} - v_1\}$. Hence, the VCG transfer of agent i_{eff} is $\max\{0, \max_{j \in \{2,\ldots,n\} \setminus \{i_{\text{eff}}\}} v_j - C_{1j} - v_1\} - (-v_1 - C_{1i_{\text{eff}}}) = \max\{v_1, \max_{j \in \{2,\ldots,n\} \setminus \{i_{\text{eff}}\}} v_j - C_{1j}\} + C_{1i_{\text{eff}}} = \underline{p}^W$. Similarly, gains from trade with agent 1 present, but without its value for the allocation, are $v_{i_{\text{eff}}}$, whereas they are 0 with the seller reporting a value of 1. Hence, the VCG payment that agent 1 receives is $v_{i_{\text{eff}}} = \overline{p}^W$.

mechanism runs a deficit, it follows that ex post efficiency is impossible for any network when ownership is extremal. We summarize this in the following result:

Proposition 3. *For any ownership problem with $r_i = 1$ and $c \in [0, 1)$, ex post efficient trade is impossible.*

Propositions 2 and 3 imply immediately:

Corollary 5. *Assuming identical distributions and $c \in (0, 1)$, the first-best cannot be achieved in an ownership problem for any network with a single completely connected agent.*

Universal impossibility

Proposition 3 and Corollary 5 are, as foreshadowed, impossibility results in the tradition of Vickrey (1961) and Myerson and Satterthwaite (1983) because they depend on extremal ownership. They imply that one will need to consider constrained-efficient trade if $r_i = 1$ for some $i \in \mathcal{N}$. But they leave open the question of whether there exist nonextremal ownership vectors that permit ex post efficient trade. Intuition based on Cramton et al. (1987) may suggest that the answer is affirmative.

With that in mind, our next result is probably unexpected because it states that for $c \geq 1/2$, ex post efficient trade is impossible for *any* ownership vector.[13] As intuition for the result, note that under ex post efficiency, any agent of type $v \leq 1 - c$ ever only trades as a seller, and any agent of type $v \geq c$ ever only trades as a buyer. Consequently, for $c \geq 1/2$, agents with types $v \in [1 - c, c]$ never trade and have payoffs of 0. As a result, for $c \geq 1/2$, the trading problem is not only ex post two-sided but already *ad interim*—knowing only its type, every agent knows whether it will trade as a buyer (if $v > c$) or as a seller (if $v < 1 - c$) if it trades and agents with types between $1 - c$ and c know that they will never trade. As in the proof of Proposition 3, it therefore suffices to verify that transportation costs are not covered under VCG transfers, and that the VCG mechanism satisfies the agents' ex post individual rationality constraints with equality, which means that it also satisfies interim individual rationality with equality.

Proposition 4. *In an ownership problem, for $c \geq 1/2$, ex post efficient reallocation is impossible for any network and any ownership vector.*

[13] This uses our assumption that the support of the agents' type distribution is $[0, 1]$. For a more general support of $[\underline{v}, \overline{v}]$, the required condition on transportation costs is that $c \geq (\overline{v} + \underline{v})/2$.

Proof. See Appendix A.

Proposition 4 provides a simple sufficient condition for ex post efficient reallocation to be impossible for any ownership structure. The result is thus a form of "Coase-on-networks Theorem": private information and sufficiently high transportation costs combine to create insurmountable transactions costs because there is no ownership structure that would permit efficient reallocation.

4.2 Possibility for small costs and dispersed ownership

We now provide a necessary and sufficient condition for ex post efficient trade to be possible. To do so, it is useful to begin with two lemmas that characterize, for any (Bayesian) incentive compatible mechanism, agents' worst-off types and expected payments to the mechanism.

Consider an incentive compatible mechanism $\langle \mathbf{Q}, \mathbf{M} \rangle$. Let $q_i(v_i) \equiv \mathbb{E}_{\mathbf{v}_{-i}}[Q_i(v_i, \mathbf{v}_{-i})]$ and $m_i(v_i) \equiv \mathbb{E}_{\mathbf{v}_{-i}}[M_i(v_i, \mathbf{v}_{-i})]$ denote the interim expected allocation and payment of agent i when its type is v_i. Accordingly, $u_i(v) \equiv q_i(v)v - m_i(v) - r_i v$ denotes agent i's interim expected gains from participation in the mechanism, net of its outside option. Incentive compatibility implies that q_i is nondecreasing, from which it follows that the first-order condition $u_i'(v) = q_i(v) - r_i = 0$ characterizes a global minimum for agent i's interim expected payoff, provided that it is satisfied for some v. The following lemma, a version of which was first established by Cramton et al. (1987), characterizes the set of worst-off types for any allocation rule such that q_i is nondecreasing:

Lemma 2. *Given an incentive compatible, individually rational mechanism* $\langle \mathbf{Q}, \mathbf{M} \rangle$*, if there is a* v_i *such that* $q_i(v_i) = r_i$*, then the set of worst-off types for agent* i *is* $\{v_i \mid q_i(v_i) = r_i\}$*. If* $q_i(v_i) \neq r_i$ *for all* $v_i \in [0, 1]$*, then the set of worst-off types for agent* i *is the singleton set* $\{v_i \mid q_i(v) < r_i \; \forall v < v_i \text{ and } q_i(v) > r_i \; \forall v > v_i\}$.

As observed by Cramton et al. (1987), intuitively, the worst-off type of an agent expects on average to be neither a net buyer nor a net seller, and therefore an agent with the worst-off type has no incentive to overstate or understate its valuation and so does not need to be compensated to induce truthful reporting, which is why it is the worst-off type.

Given an incentive compatible mechanism, we can use standard mechanism design techniques to write an agent's expected payment to the mechanism in terms of its worst-off type and its virtual type functions. Defining

$$\Psi_i(v;\omega) \equiv \begin{cases} \Psi_i^S(v) & \text{if } v \leq \omega, \\ \Psi_i^B(v) & \text{if } v > \omega, \end{cases} \tag{5}$$

we have:

Lemma 3. *Given ownership* $\mathbf{r}$ *and an incentive compatible trade mechanism* $\langle \mathbf{Q}, \mathbf{M} \rangle$, *for any agent* i *and* $\omega_i \in [0,1]$, *agent* i*'s expected payment to the mechanism can be written as*

$$\mathbb{E}_{\mathbf{v}}[M_i(\mathbf{v})] = \mathbb{E}_{\mathbf{v}}\left[\Psi_i(v_i;\omega_i)Q_i(\mathbf{v})\right] - r_i\omega_i - u_i(\omega_i).$$

Proof. See Appendix A.

Letting $\omega_{i,\mathbf{r}}^e$ denote agent i's worst-off type (or one of its worst-off types) under the ex post efficient allocation rule $Q_{i,\mathbf{r}}^e$, and using Lemma 3, we obtain an expression for the expected budget surplus (not including transportation costs) of an ex post efficient reallocation mechanism that satisfies the agents' individual rationality constraints with equality[14]:

$$\Pi_{\mathbf{r}}^e \equiv \mathbb{E}_{\mathbf{v}}\Big[\sum_{i=1}^{n} \Psi_i(v_i;\omega_{i,\mathbf{r}}^e)Q_{i,\mathbf{r}}^e(\mathbf{v})\Big] - \sum_{i=1}^{n} \omega_{i,\mathbf{r}}^e r_i.$$

Thus, ex post efficient reallocation is possible without running a deficit if and only if $\Pi_{\mathbf{r}}^e \geq t_{\mathbf{r}}^e$. It follows that the necessary and sufficient condition for the possibility of ex post efficient reallocation is

$$\mathbb{E}_{\mathbf{v}}\Big[\sum_{i=1}^{n}\sum_{j=1}^{n}(\Psi_i(v_i;\omega_{i,\mathbf{r}}^e) - C_{ji})V_{ji}^e(\mathbf{v})r_j\Big] \geq \sum_{i=1}^{n} \omega_{i,\mathbf{r}}^e r_i. \tag{6}$$

Condition (6) implicitly defines the set of combinations of ownership vectors $\mathbf{r}$ and transportation cost matrices C such that ex post efficient trade is possible. We can use (6) to calculate, for a given $\mathbf{r}$, the maximum c such that ex post efficient trade is possible, denoted by $c_n^{max}(\mathbf{r})$ (and defined to be $-\infty$ if no such c exists). Further, for

[14] Saying that the individual rationality constraints are satisfied with equality is shorthand for saying that these constraints are satisfied with equality for the worst-off type of each agent.

each $c \leq \max_{\mathbf{r}} c_n^{max}(\mathbf{r})$, the boundary of the set of ownership vectors such that ex post efficient trade is possible is defined by vectors $\mathbf{r}$ that satisfy (6) with equality. For example, if we consider a star or wheel network with $\mathbf{r} = (r, (1-r)/(n-1), \ldots, (1-r)/(n-1))$, then for each r and each $c \leq \max_{\mathbf{r}} c_n^{max}(\mathbf{r})$, we can calculate the maximum r, such that ex post efficient trade is possible, denoted by $\overline{r}_n(c)$. We illustrate this in Figure 2 for a star network and uniformly distributed types.

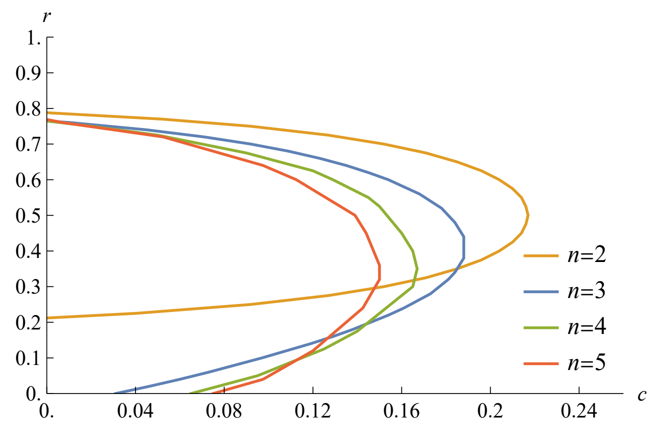

Figure 2: First-best permitting region: values for (r, c) that permit ex post efficient reallocation for star networks. As illustrated, $\overline{r}_2(0) = 0.7887$, $\overline{r}_3(0) = 0.7654$, $\overline{r}_4(0) = 0.7647$, and $\overline{r}_5(0) = 0.7689$. Further, $c_2^{max} = 0.217$, $c_3^{max} = 0.187$, $c_4^{max} = 0.1675$, and $c_5^{max} = 0.150$. Assumes $\mathbf{r} = (r, \frac{1-r}{n-1}, \ldots, \frac{1-r}{n-1})$ and uniformly distributed types.

As illustrated in Figure 2, for a star network with $n = 2$ and $c = 0$, the first-best is possible for all $r \in [0.21, 0.79]$, which corresponds to the values obtained by Cramton et al. (1987). While Figure 2 illustrates that $\overline{r}_n(0)$ need not be monotone in n, if one properly accounts for the expansion in the number of agents by calculating the resources accounted for by the first $x \in [0, 1]$ share of agents, giving us a distribution of resources G_n defined by

$$G_n(x) \equiv \begin{cases} nx\overline{r}_n(0) & \text{if } x \leq 1/n, \\ 1 - (1-x)n\frac{1-\overline{r}_n(0)}{n-1} & \text{if } x > 1/n, \end{cases}$$

then one finds that, at least for uniformly distributed types, G_n first-order stochastically dominates $G_{n'}$ if $n < n'$ (see the Online Appendix). Thus, with more agents, the boundary of $\mathcal{R}(0)$ shifts towards greater concentration at the hub.

4.3 Constrained-efficient reallocation mechanism

An implication of Proposition 3 is that in an ownership problem, either ownership is nonextremal or trade is not ex post efficient, or both. As shown above, for some transportation costs, ex post efficient trade is not an option for any ownership vector. This motivates us to characterize constrained-efficient reallocation mechanisms, which we do now. As mentioned, this analysis assumes that for each $i \in \mathcal{N}$, the virtual types functions $\Psi_i^B(v)$ and $\Psi_i^S(v)$ are increasing.

The constrained-efficient reallocation mechanism maximizes the sum of the agents' expected surpluses subject to incentive compatibility, individual rationality, and no deficit, which requires that the expected budget surplus of the mechanism must be sufficient to cover the expected transportation costs. To define the mechanism, it is useful to introduce the notion of weighted virtual types and their ironed counterparts. For $a \in [0,1]$ and threshold type $\omega \in [0,1]$, we let

$$\Psi_{i,a}(v;\omega) \equiv av + (1-a)\Psi_i(v;\omega)$$

denote the weighted virtual type of agent i, with $\Psi_i(v;\omega)$ being given by (5). The ironed weighted virtual type of an agent with type v and threshold type ω, denoted $\overline{\Psi}_{i,a}(v;\omega)$, is defined as

$$\overline{\Psi}_{i,a}(v;\omega) \equiv \begin{cases} \Psi_{i,a}^S(v) & \text{if } \Psi_{i,a}^S(v) < z, \\ z & \text{if } \Psi_{i,a}^B(v) \le z \le \Psi_{i,a}^S(v), \\ \Psi_{i,a}^B(v) & \text{if } z < \Psi_{i,a}^B(v), \end{cases}$$

where $\Psi_{i,a}^S(v) \equiv av + (1-a)\Psi_i^S(v)$ and $\Psi_{i,a}^B(v) \equiv av + (1-a)\Psi_i^B(v)$ are agent i's weighted virtual cost and virtual value functions and where the ironing parameter z satisfies

$$\int_0^\omega \max\{0, \Psi_{i,a}^S(v) - z\} dF_i(v) = \int_\omega^1 \max\{0, z - \Psi_{i,a}^B(v)\} dF_i(v). \tag{7}$$

The constrained-efficient mechanism, as shown by Loertscher and Wasser (2019), is the solution to a saddle-point problem that simultaneously chooses the allocation rule to maximize expected social surplus given agents' worst-off types, subject to constraints, and chooses the agents' worst-off types to minimize their expected payoffs given the allocation rule.

Focusing on the maximization problem for the moment, let ω_i denote agent i's worst-off type. Then letting ρ be the Lagrange multiplier on the no-deficit constraint and μ_i be the Lagrange multiplier on agent i's individual rationality constraint, and using Lemma 3, we have the Lagrangian

$$\begin{aligned}\mathcal{L} \ \equiv\ & \mathbb{E}_{\mathbf{v}}\Big[\sum_{i=1}^{n}\left(Q_i(\mathbf{v})v_i - Q_i(\mathbf{v})\Psi_i(v_i;\omega_i) + r_i\omega_i + u_i(\omega_i)\right)\\ & +\rho\Big(\sum_{i=1}^{n}\left(Q_i(\mathbf{v})\Psi_i(v_i;\omega_i) - r_i\omega_i - u_i(\omega_i)\right) - T_{\mathbf{r}}(\mathbf{Q}(\mathbf{v}))\Big)\Big] + \sum_{i=1}^{n}\mu_i u_i(\omega_i),\end{aligned}$$

where $T_{\mathbf{r}}(\mathbf{Q}(\mathbf{v}))$ is the total transportation cost under allocation rule $\mathbf{Q}$ and type vector $\mathbf{v}$ when the ownership vector is $\mathbf{r}$. Rearranging this, we have

$$\mathcal{L} = \rho\mathbb{E}_{\mathbf{v}}\Big[\sum_{i=1}^{n} Q_i(\mathbf{v})\Psi_{i,\frac{1}{\rho}}(v_i;\omega_i) - T_{\mathbf{r}}(\mathbf{Q}(\mathbf{v}))\Big] + (1-\rho)\sum_{i=1}^{n} r_i\omega_i + \sum_{i=1}^{n}(1-\rho+\mu_i)u_i(\omega_i).$$

Given $\boldsymbol{\omega}$ and ρ, we can then solve for $\mathbf{Q}$ pointwise, subject to the constraint that $\mathbf{Q}$ is nondecreasing (thus, requiring ironing). Specifically, given Lagrange multiplier ρ and worst-off types $\boldsymbol{\omega}$, the constrained-efficient reallocation rule for agent i is given by

$$Q^{ce}_{i,\mathbf{r}}(\mathbf{v};\rho,\boldsymbol{\omega}) \equiv \sum_{j=1}^{n} V^{ce}_{ji}(\mathbf{v};\rho,\boldsymbol{\omega})r_j,$$

where V^{ce} is defined analogously to V^{e}, but with actual types replaced by ironed virtual types:

$$V^{ce}_{ij}(\mathbf{v};\rho,\boldsymbol{\omega}) \equiv \begin{cases} 1 & \text{if } \overline{\Psi}_{j,1/\rho}(v_j;\omega_j) - C_{ij} \geq \max_{\ell}\overline{\Psi}_{\ell,1/\rho}(v_\ell;\omega_\ell) - C_{i\ell} \\ & \quad \text{and } \overline{\Psi}_{j,1/\rho}(v_j;\omega_j) - C_{ij} > \max_{\ell<j}\overline{\Psi}_{\ell,1/\rho}(v_\ell;\omega_\ell) - C_{i\ell}, \\ 0 & \text{otherwise,}\end{cases}$$

where, as before, ties are broken in favor of agents with a lower index. Using Lemma 3, the expected budget surplus under binding interim individual rationality is

$$\Pi^{ce}_{\mathbf{r}}(\rho,\boldsymbol{\omega}) \ \equiv\ \mathbb{E}_{\mathbf{v}}\Big[\sum_{i=1}^{n}\Psi_i(v_i;\omega_i)Q^{ce}_{i,\mathbf{r}}(\mathbf{v};\rho,\boldsymbol{\omega})\Big] - \sum_{i=1}^{n}\omega_i r_i.$$

Expected transportation costs under the constrained-efficient allocation rule are:

$$t_{\mathbf{r}}^{ce}(\rho, \boldsymbol{\omega}) \equiv \mathbb{E}_{\mathbf{v}}\Big[\sum_{i=1}^{n}\sum_{j=1}^{n} C_{ji} V_{ji}^{ce}(\mathbf{v}; \rho, \boldsymbol{\omega}) r_j\Big].$$

Given this, we can state the following result:

Proposition 5. *The constrained-efficient reallocation rule is the same as the ex post efficient reallocation rule if* $\Pi_{\mathbf{r}}^{e} \geq t_{\mathbf{r}}^{e}$*, and otherwise it is defined by* $\mathbf{Q}_{\mathbf{r}}^{ce}(\mathbf{v}; \rho^*, \boldsymbol{\omega}^*)$*, where* $\boldsymbol{\omega}^*$ *and* ρ^* *are such that for all* $i \in \mathcal{N}$*,* $\mathbb{E}_{\mathbf{v}_{-i}}[Q_{i,\mathbf{r}}^{ce}(\omega_i^*, \mathbf{v}_{-i}; \rho^*, \boldsymbol{\omega}^*)] = r_i$ *and* $\rho^* = \arg\min_{\rho}\{\rho \geq 1 \mid \Pi_{\mathbf{r}}^{ce}(\rho, \boldsymbol{\omega}^*) \geq t_{\mathbf{r}}^{ce}(\rho, \boldsymbol{\omega}^*)\}$.

The constrained-efficient reallocation mechanism has the allocation rule specified by Proposition 5 along with the payment rule given by Lemma 3, with $\boldsymbol{\omega}$ equal to $\boldsymbol{\omega}^*$. Figure 3 illustrates the contrast between the ex post efficient and constrained-efficient reallocation rules for the case of two agents. In setups with identical distributions and no transportation costs, the constrained-efficient reallocation rule coincides with the ex post efficient reallocation when both agents have small values and when both agents have large values if ironing occurs in the interior (see Loertscher and Wasser, 2019; Loertscher and Marx, 2022b). To see this, assume $v_1 > v_2$ and observe that under the optimal mechanism with $c = 0$ trade occurs if and only if $\Psi_{1,a}^{S}(v_1) > \Psi_{2,a}^{S}(v_2)$ when both types are small, respectively $\Psi_{1,a}^{B}(v_1) > \Psi_{2,a}^{B}(v_2)$ when both types are large. With identical distributions this is equivalent to $v_1 > v_2$.

Interestingly, this feature does not extend to a settings with positive transportation costs, in which case it is easy to obtain, locally, more trade than under ex post efficiency. To see this, assume ironing occurs in the interior and consider v_1 and v_2 with $v_1 > v_2$, both of which are sufficiently small so that trade of r_2 occurs if and only if $\Psi_{1,a}^{S}(v_1) > \Psi_{2,a}^{S}(v_2) + c$, which is equivalent to

$$v_1 > v_2 + (1-a)\left[\frac{F(v_2)}{f(v_2)} - \frac{F(v_1)}{f(v_1)}\right] + c. \tag{8}$$

Under the constrained-efficient mechanism, $a = 1/\rho^* < 1$, and so if F/f is increasing, then the right side of (8) is smaller than $v_2 + c$, which is the condition for trade under ex post efficiency. (And when both types are large, trade of r_2 occurs if and only if $v_1 > v_2 + (1-a)\left[\frac{1-F(v_1)}{f(v_1)} - \frac{1-F(v_2)}{f(v_2)}\right] + c$, whose right side is less than $v_2 + c$ if $(1-F)/f$ is decreasing.) These hazard rate properties are satisfied, for example, by the uniform distribution. This possibility of locally excessive trade is illustrated in Figure 3(c) for uniform distributions. At the boundaries, the contours of the constrained-efficient reallocation lie “inside” the contours for ex post efficiency.[15]

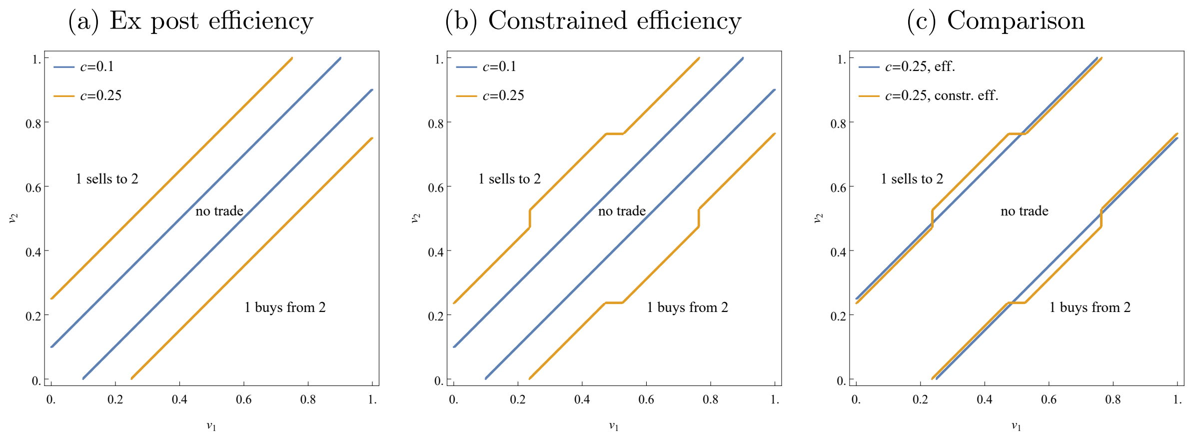

Figure 3: Efficient and constrained-efficient reallocation rules. Assumes $n = 2$, $\mathbf{r} = (0.5, 0.5)$, and uniformly distributed types. The constrained-efficient results for $c = 0.25$ use $\rho^* = 1.06$ and ironing parameters $z_1^* = z_2^* = 0.5$. For $c = 0.1$, ex post efficiency is possible.

Figure 3 illustrates two of the three effects that reductions in the marginal cost of transportation have. First, as shown in Panel (a), decreasing c increases trade under ex post efficiency and thereby social surplus. Second, in an ownership problem reductions in c make the market work better. This is illustrated in Panel (b), where ex post efficiency is possible for $c = 0.1$ but not for $c = 0.25$. The third effect that reductions in the marginal transportation cost have is via the optimal ownership structure, which is what we analyze next.

4.4 Optimal ownership under social-surplus maximization

It is useful to define three sets of ownership vectors, parameterized by the transportation cost by c. First, let $\mathcal{R}(c)$ be the possibly empty set of ownership vectors satisfying (6), where ex post efficient trade is possible. Second, let $\mathcal{R}^P(c)$ denote

[15]This possibility of locally excessive trade depends simultaneously on transportation costs and on the ironing ranges being interior. For example, if $c > 0$ is the fixed cost of producing a public good, then in the optimal mechanism, production occurs if and only if $\sum_{i=1}^n \Psi_{i,a}^B(v_i) > c$, which for any $a < 1$ is more restrictive than the condition for production under ex post efficiency. Likewise, if c is a transportation cost but ironing ranges are at the bounds, for example because $r_2 = 1$, then trade occurs if and only if $\Psi_{1,a}^B(v_1) > \Psi_{2,a}^S(v_2) + c$, where for any any $a < 1$, the left side is less than v_1 and the right side is larger than $v_2 + c$.

the set of optimal placement vectors, and, third, let $\mathcal{R}^O(c)$ denote the set of optimal ownership vectors.

We consider different levels of transportation cost in turn, beginning with a result for the case of zero transportation costs. In that case, any ownership vector that allows ex post efficient trade is optimal in the ownership problem:

Proposition 6. *For $c = 0$, the ownership problem with a social-surplus-maximizing designer is solved by any $\mathbf{r} \in \mathcal{R}(0)$, i.e., $\mathcal{R}^O(0) = \mathcal{R}(0)$.*

Further, for a network with one completely connected agent, for $c \in (0, 1)$, $\mathcal{R}^P(c)$ contains only extremal ownership, and using Proposition 3, $\mathcal{R}(0)$ does not contain any extremal ownership. Thus, by continuity, in such a network, for $c > 0$ sufficiently close to zero, the set of solutions to the ownership problem has an empty intersection with the set of solutions to the placement problem:

Proposition 7. *For a network with one completely connected agent, there exists $\hat{c} \in (0, 1)$ such that for all $c \in (0, \hat{c})$, $\mathcal{R}^O(c) \cap \mathcal{R}^P(c) = \varnothing$.*

Turning to the case of sufficiently high transportation costs, we begin by noting that this case is simplified by each agent having the same worst-off type.

Lemma 4. *If $c \geq 1/2$, then $1/2$ is a worst-off type for every agent.*

Proof. See Appendix A.

Using Proposition 5, the maximized objective under the constrained-efficient reallocation rule can be written as:

$$\begin{aligned} \mathcal{L}^*(\mathbf{r}) \quad \equiv \quad & \rho^* \mathbb{E}_{\mathbf{v}} \left[\sum_{i=1}^{n} \sum_{j=1}^{n} \left(\Psi_{i, \frac{1}{\rho^*}}(v_i; \omega_i^*) - C_{ji} \right) V_{ji}^{ce}(\mathbf{v}; \rho^*, \boldsymbol{\omega}^*) r_j \right] \\ & + (1 - \rho^*) \sum_{j=1}^{n} \omega_j^* r_j + \sum_{j=1}^{n} (1 - \rho^* + \mu_j^*) u_j(\omega_j^*). \end{aligned} \tag{9}$$

For $c \geq 1/2$, we have $\omega_1^* = \cdots = \omega_n^* = 1/2$, so in that case $\sum_{j=1}^{n} \omega_j^* r_j = 1/2$, and the only direct effects of $\mathbf{r}$ occur in the expression in (9) in square brackets. Further, as the following lemma shows, we can rewrite the expectation of the term in square brackets in (9) in terms of the ironed rather than unironed weighted virtual types:

Lemma 5. *For $c \geq 1/2$ and $F_i = F$ for all $i \in \mathcal{N}$, the maximized objective $\mathcal{L}^*(\mathbf{r})$ can be written as:*

$$\mathcal{L}^*(\mathbf{r}) = \rho^* \mathbb{E}_{\mathbf{v}} \Big[\sum_{j=1}^{n} \sum_{i=1}^{n} \Big(\overline{\Psi}_{i,\frac{1}{\rho^*}}(v_i; 1/2) - C_{ji} \Big) V_{ji}^{ce}(\mathbf{v}; \rho^*, \mathbf{1/2}) r_j \Big] + \frac{1-\rho^*}{2} + \sum_{j=1}^{n} (1 - \rho^* + \mu_j^*) u_j(1/2).$$

Proof. See Appendix A.

By the envelope theorem, only the direct effects of $\mathbf{r}$ matter. Using Lemma 5, these are captured in the terms in square brackets. By an isomorphism that mirrors the one observed above for optimal placement, we see that the expression in square brackets in (9) (and in Lemma 5) is the same as the objective for the unconstrained problem of maximizing

$$\mathbb{E}_{\mathbf{x}} \Big[\sum_{j=1}^{n} \sum_{i=1}^{n} (x_i - C_{ji}) \, V_{ji}^{e}(\mathbf{x}) r_j \Big],$$

where x_i is drawn from the distribution of $\overline{\Psi}_{i,\frac{1}{\rho^*}}(v_i; 1/2)$. Thus, we have the following result:

Proposition 8. *Assume identical distributions. Given $c \geq 1/2$, optimal ownership under social-surplus maximization is confined to reach dominant agents.*

Because for $c \geq 1/2$, the set reach of dominant agents is non-empty, Proposition 8 shows that optimal ownership under social-surplus maximization is pinned down by reach dominance for sufficiently large marginal costs of transportation and identical distributions. For example, for the star and the wheel network, it implies that ownership is concentrated at the hub for $c \geq 1/2$ and identical distributions.

In cases in which ex post efficient trade is possible, we have $\rho^* = 1$, and so

$$\mathcal{L}^*(r) = \mathbb{E}_{\mathbf{v}} \Big[\sum_{j=1}^{n} \sum_{i=1}^{n} (v_i - C_{ji}) \, V_{ji}^{e}(\mathbf{v}) r_j \Big] + \sum_{j=1}^{n} \mu_j^* u_j(\omega_j^*),$$

where again the term in square brackets is the objective for the unconstrained problem and so solved with extremal ownership. Thus, for a star or wheel network with solution to the ownership problem of $\mathbf{r}^O = (r, \frac{1-r}{n-1}, \ldots, \frac{1-r}{n-1})$ and identical distributions, if the solution to the ownership problem involves ex post efficient trade, then the optimal ownership vector is on the boundary of the region permitting ex post efficient trade, i.e., if $\mathbf{r}^O \in \mathcal{R}(c)$, then $r_1^O = \max\{r_1 \mid \mathbf{r}^O \in \mathcal{R}(c)\}$.

Combining these results, we see that for star and wheel networks, for a range of intermediate values for c, the solution to the ownership problem has $\mathbf{r}^O$ that does not permit ex post efficient trade, and is not extremal, and so the solution is intermediate between the solution to the ownership problem with $c = 0$ and with $c \geq 1/2$. We illustrate this in Figure 4.

(a) $\mathcal{R}(c)$ and $r^O(c)$ for a star network with $n = 3$

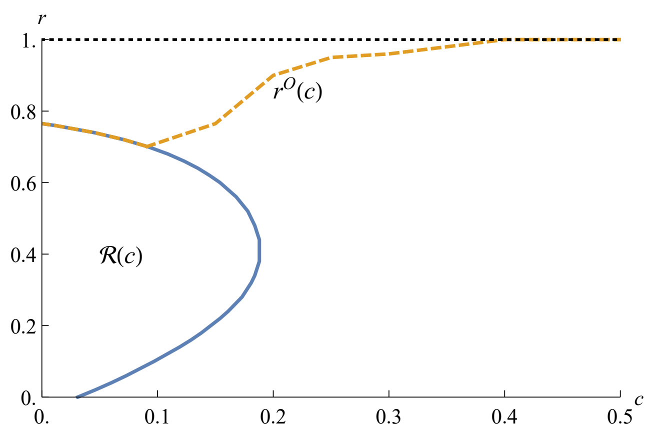

Figure 4: Values for (r, c) that permit ex post efficient trade and $r^O(c)$. Assumes a star network with $n = 3$, $\mathbf{r} = (r, \frac{1-r}{2}, \frac{1-r}{2})$, and uniformly distributed types.

4.5 Optimal ownership under profit maximization

We conclude this section with an analysis of a designer that seeks to maximize its expected profit. This analysis continues to assume that for each $i \in \mathcal{N}$, the virtual types functions $\Psi_i^B(v)$ and $\Psi_i^S(v)$ are increasing.

Given worst-off types $\boldsymbol{\omega}$, the designer's profit-maximizing allocation rule $\mathbf{Q}^D$ is defined analogously to $\mathbf{Q}^{ce}$, but with the ironed weighted virtual types replaced by the ironed unweighted (i.e., weight equal to zero) virtual types:

$$Q_{i,\mathbf{r}}^D(\mathbf{v}; \boldsymbol{\omega}) \equiv \sum_{j=1}^{n} V_{ji}^D(\mathbf{v}; \boldsymbol{\omega}) r_j,$$

where V^D is defined by

$$V^D_{ij}(\mathbf{v};\boldsymbol{\omega}) \equiv \begin{cases} 1 & \text{if } \overline{\Psi}_{j,0}(v_j;\omega_j) - C_{ij} \geq \max_\ell \overline{\Psi}_{\ell,0}(v_\ell;\omega_\ell) - C_{i\ell} \\ & \quad \text{and } \overline{\Psi}_{j,0}(v_j;\omega_j) - C_{ij} > \max_{\ell<j} \overline{\Psi}_{\ell,0}(v_\ell;\omega_\ell) - C_{i\ell}, \\ 0 & \text{otherwise,} \end{cases}$$

as usual with tie-breaking in favor of agents with a lower index. Then we have the following result:

Proposition 9. *The designer's profit-maximizing allocation rule is* $\mathbf{Q}^D_{\mathbf{r}}(\mathbf{v};\boldsymbol{\omega}^*)$*, where* $\boldsymbol{\omega}^*$ *is such that for all* $i \in \mathcal{N}$*,* $\mathbb{E}_{\mathbf{v}_{-i}}[Q^D_{i,\mathbf{r}}(\omega^*_i, \mathbf{v}_{-i};\boldsymbol{\omega}^*)] = r_i$*.*

Using Lemma 3, given $\boldsymbol{\omega}$, the expected profit to the designer not including transportation costs is

$$\Pi^D_{\mathbf{r}}(\boldsymbol{\omega}) \equiv \mathbb{E}_{\mathbf{v}}\Big[\sum_{i=1}^n \Psi_i(v_i;\omega_i) Q^D_{i,\mathbf{r}}(\mathbf{v};\boldsymbol{\omega})\Big] - \sum_{i=1}^n \omega_i r_i,$$

and expected transportation costs are:

$$t^D_{\mathbf{r}}(\boldsymbol{\omega}) \equiv \mathbb{E}_{\mathbf{v}}\Big[\sum_{i=1}^n \sum_{j=1}^n C_{ji} V^D_{ji}(\mathbf{v};\boldsymbol{\omega}) r_j\Big].$$

Thus, the designer's maximized expected profit is

$$\Pi^D_{\mathbf{r}}(\boldsymbol{\omega}^*) - t^D_{\mathbf{r}}(\boldsymbol{\omega}^*),$$

where $\boldsymbol{\omega}^*$ is as defined in Proposition 9.

Turning to the optimal ownership vector for a profit-maximizing designer, note that ownership affects not only transportation costs, but also the individual rationality constraint, which affects the profit-maximizing allocation rule. The tradeoffs differ somewhat from the case of a social-surplus-maximizing designer because it is as if the profit-maximizing designer faces agents with worse distributions, i.e., sellers with higher types and buyers with lower types.

5 Extensions

In this section, we provide extensions that consider the designer's problem with an indivisible resource, define stochastic reach dominance for problems with heterogeneous distributions and distances, allow for directed networks, and allow for fixed costs of transportation per edge.

5.1 Reach dominance and optimal ownership

Characterizing the optimal ownership is, in general, plagued by the problem that optimal ownership may be shared, which implies that the constrained-efficient mechanism varies nontrivially with $\mathbf{r}$. The "asset market" nature of this mechanism renders characterizing optimal ownership difficult in general. However, the problem simplifies if the resource is indivisible, as is the case, for example, for a network with a single production plant. In this case, $r_i = 1$ for some $i \in \mathcal{N}$. In this case, the constrained-efficient reallocation mechanism is simply the constrained-efficient mechanism for a two-sided allocation problem in which agent i is the seller and all other agents are buyers. This constrained-efficient mechanism is an extension of the second-best mechanism derived by Myerson and Satterthwaite (1983) to a setting with multiple buyers and costly transportation. We now show that, with identical distributions and an indivisible resource, the optimal ownership is governed by reach dominance in the same way as is the optimal placement.

Proposition 10. *Assuming an indivisible resource and identical distributions, the optimal ownership is confined to the set of reach dominant agents provided that this set is non-empty, and reach dominated agents are never given positive ownership.*

Proof. See Appendix A.

The proof shows that, with identical distributions, if agent i reach dominates agent j, then the expected social surplus under the constrained-efficient reallocation mechanism is larger when resources are owned by agent i. The argument relies on a revealed preference argument that shows that the mechanism with ownership by i could treat the agents the same way it does with ownership by j, thereby either directly generating more social surplus or generating positive revenue, which can then be used to increase social surplus by reoptimizing.[16]

[16]The argument is similar to the proof in Loertscher and Marx (2019) that shows that a merger

5.2 Stochastic reach dominance

Our analysis of optimal placement based on reach dominance rested on the assumption that the agents' distributions were identical and, like the entire analysis up to this point, that links were of equal length, implying that the cost of transportation between any two neighbors is the same. We now show that both of these assumptions can be dropped simultaneously without qualitatively altering the conclusions or mechanics at work.

To this end, let us first allow for heterogeneous distributions while keeping the length of each edge the same. For every agent i, let $A_i(k)$ be the set of agents that are $k \in \{0, 1, \ldots, \min\{\lceil 1/c \rceil, n-1\}\}$ links away from i, and let $\mathcal{A}_i(k) \equiv \cup_{h=1}^{k} A_i(h)$ be the set of agents that are within k links of agent i. Denote by

$$L_k^i(v) \equiv \prod_{j \in \mathcal{A}_i(k)} F_j(v)$$

the distribution of the highest draw among all neighbors of agent i that are not farther away than k links. Accordingly agent (or node) i is said to *stochastically reach dominate (SRD)* agent j given c if for all $k \in \{0, 1, \ldots, \min\{\lceil 1/c \rceil, n-1\}\}$ and all $v \in [0, 1]$

$$L_k^i(v) \leq L_k^j(v) \tag{10}$$

holds, with a strict inequality for some v and k. Stochastic reach dominance extends the insight that more and closer neighbors are better, which holds under identical distributions, to something like "stronger agents with stronger, more, and closer neighbors are better." With identical distributions, reach dominance and stochastic reach dominance are equivalent because having more draws and stochastic dominance are equivalent with identical distributions.

To see that the concept of stochastic reach dominance extends straightforwardly to settings in which links between agents are not necessarily of equal length, for any distance $x \in (0, 1/c)$ from i, one can define $\mathcal{A}_i(x)$ to be the set of agents not farther away from i than x and, correspondingly, define $L_x^i(v) = \prod_{j \in \mathcal{A}_i(x)} F_j(v)$ to be the distribution of the highest draw among i's neighbors that are not farther away than x. Then we say that i SRDs j if $L_x^i(v) \leq L_x^j(v)$ holds for all $x \in (0, 1/c)$ and all $v \in [0, 1]$, with strict inequality for some x and v. Replacing "reach dominance" with

between two suppliers harms a powerful buyer.

"stochastic reach dominance," Proposition 2 then extends to settings with heterogeneous distributions and distances.

5.3 Directed networks

The networks we analyzed thus far were undirected graphs. This means that resources can be shipped in either direction along any given edge. For some applications, it may be more appropriate to think of links as being directed, meaning that resources can only be shipped in one direction, in which case each node is characterized by a number of inbound and a number of outbound links. (Undirected graphs are thus the special case of directed graphs with the property that every outbound edge is paired with an inbound edge.) For example, it may be that resources can only be shipped downhill, which would typically be the case for water. So if the resource placement corresponds to setting up a dam to store water and c captures the amount of water that is lost due to, say, evaporation, then a directed graph would be an appropriate model. A moment's reflection then reveals that our results pertaining to reach dominance and stochastic reach dominance extend to directed graphs by confining attention to outbound links and nodes and distributions along outbound links.

5.4 Fixed cost of communication

In some applications, it is more appropriate to think of transportation as involving a fixed cost per edge that is independent of the amount being shipped. For example, the agent shipping and the agent receiving the good may need to communicate about the specifics of the shipment and what it requires. This communication may be costly due to lack of a common language or cultural differences, but once the cost is borne and a common understanding is established, the shipment is free and hence the cost does not vary with the quantity shipped. This kind of problem is pervasive for resource (re-)allocation within organizations, where different units and departments have their own culture and language.

If agents i and j are directly linked, then it is ex post efficient for agent i to ship r_i to agent j if and only if $c < r_i(v_j - v_i)$ or equivalently

$$v_i + \frac{c}{r_i} < v_j.$$

The larger is r_i, the more likely is agent i thus to ship r_i units to agent j under ex post efficiency. Moreover, the gains from trade $r_i(v_j - v_i) - c$ increase in r_i.

If $c \in [1/2, 1)$, then with nonextremal placement, at most one agent will be able to ship because $c/r_i \geq 1$ for any $r_i \leq 1/2$. If $c \in [0, 1/2)$, then the analysis above related to reach dominant agents applies. This gives us the following result:

Proposition 11. *Assume identical distributions. In the model with a fixed cost per edge, if* $c \in [1/2, 1)$*, then optimal placement gives all resources to one agent; and if* $c \in [0, 1/2)$*, then optimal placement gives all resources to one reach dominant agent whenever such an agent exists.*

Proof. See Appendix A.

Proposition 11 provides conditions under which the placement problem with fixed costs is solved by extremal placement when agents have identical distributions. More generally, if one of the solutions to the placement problem under constant marginal cost of transportation involves extremal placement, then with fixed costs, extremal placement is uniquely optimal. In all of these cases in which the first-best requires extremal placement, we of course obtain the result that in the ownership problem, the first-best is impossible.

6 Conclusions

We analyze the optimal placement of a resource on a network that occurs before demands are realized, anticipating costly reallocation. We show that in the case of identical distributions, both the social surplus and profit maximizing placements are governed by reach dominance. With identical distributions, reach dominance also governs the optimal ownership chosen by a designer whose reallocation mechanism is constrained efficient, provided that the resource is indivisible or the marginal cost of transportation is sufficiently large. Stochastic reach dominance is the generalization beyond the case of identical distributions and equal distances.

There are numerous avenues for potentially fruitful future research. Here we discuss three. First, because we restrict attention to the case in which each agent has constant marginal value for the entire resource, a natural generalization would be to allow binding maximum demands, implying that marginal values decrease to zero beyond some number of units. The analysis of such "multi-unit" cases is simple in the case with identical distributions when there are at least as many reach dominant nodes as there are units and these nodes are isolated (insofar as they do not share any common neighbors that would ever be served under ex post efficiency if the resources were placed at these reach dominant nodes). In situations like these, all resources are placed with some of these nodes. The nontrivial generalization thus concerns problems in which there are not sufficiently many of these nodes. Second, rather than using the constrained-efficient reallocation that we study, one could develop detail-free trade-sacrifice mechanisms for the second stage that never run a deficit, endow the agents with dominant strategies, respect their individual rationality constraints ex post, and reallocate, loosely speaking, close to ex post efficiently. Given their conceptual simplicity, such mechanisms may be of considerable practical relevance. Finally, for the planner's problem, the assumption of independently distributed values could be dropped, with the form of correlation across the network being given by the application of interest.

A Proofs

Proof of Proposition 1. Letting $v^e_{ji} = \mathbb{E}_{\mathbf{v}}[V^e_{ji}(\mathbf{v})]$, we have $t^e_{\mathbf{r}} = \sum_{i=1}^n \sum_{j=1}^n C_{ji} v^e_{ji} r_j$ and $ss^e_{\mathbf{r}} = \mathbb{E}_{\mathbf{v}}\left[\sum_{i=1}^n \sum_{j=1}^n v_i V^e_{ji}(\mathbf{v}) r_j\right] - t^e_{\mathbf{r}}$. Using $r_n = 1 - \sum_{\ell=1}^{n-1} r_\ell$, we have

$$\begin{aligned} ss^e_{\mathbf{r}} &= \mathbb{E}_{\mathbf{v}}\Big[\sum_{i=1}^n \sum_{j=1}^{n-1} v_i V^e_{ji}(\mathbf{v}) r_j + \sum_{i=1}^n v_i V^e_{ni}(\mathbf{v})\Big(1 - \sum_{\ell=1}^{n-1} r_\ell\Big)\Big] \\ &\quad - \sum_{i=1}^n \sum_{j=1}^{n-1} C_{ji} v^e_{ji} r_j - \sum_{i=1}^n C_{ni} v^e_{ni}\Big(1 - \sum_{\ell=1}^{n-1} r_\ell\Big), \end{aligned}$$

so for $j \in \{1, \ldots, n-1\}$, we have

$$\frac{\partial ss^e_{\mathbf{r}}}{\partial r_j} = \mathbb{E}_{\mathbf{v}}\Big[\sum_{i=1}^n v_i \left(V^e_{ji}(\mathbf{v}) - V^e_{ni}(\mathbf{v})\right)\Big] - \sum_{i=1}^n \left(C_{ji} v^e_{ji} - C_{ni} v^e_{ni}\right),$$

which is independent of r_j (and any other r_i). This implies that an extremal ownership vector is *always* optimal, independently of network structure and distributions. ■

Proof of Proposition 2. For any $h \in \mathcal{N}$, we use $\overline{\mathbf{n}}_h$ to denote the unnormalized reach vector for agent $h \in \mathcal{N}$. That is, for all $\ell \in \{1, \ldots, n-1\}$, $\overline{n}_h(\ell) \equiv n_h(\ell)(n-1)$. We let $\mathbf{e}_i$ be the vector of length n with coordinate i equal to 1 and all other coordinates equal to 0. Let $\ell^{\max} \equiv \min\{\lceil 1/c \rceil, n-1\}$. The expected social surplus under the ex post efficient allocation rule associated with placing all resources with agent i is

$$\begin{aligned} ss^e_{\mathbf{e}_i} &= \mathbb{E}_{\mathbf{v}}\Big[\sum_{\ell=1}^n (v_\ell - C_{i\ell})\, V^e_{i\ell}(\mathbf{v})\Big] \\ &= \mathbb{E}_{\mathbf{v}}\Big[\sum_{\ell=1}^n (v_\ell - cd_{i\ell}) \cdot \mathbf{1}_{v_\ell - cd_{i\ell} \geq \max_h v_h - cd_{ih}}\Big] \\ &= \mathbb{E}_{\mathbf{v}}\left[\max\{v_\ell - cd_{i\ell}\}_{\ell \in \{1,\ldots,n\}}\right] \\ &= \mathbb{E}_{\mathbf{v}}\left[\max\{v_\ell - cd_{i\ell}\}_{\ell \in \{1,\ldots,n \mid d_{i\ell} \leq \ell^{\max}\}}\right] \\ &= \mathbb{E}_{\tilde{\mathbf{v}}_i}\left[\max \tilde{\mathbf{v}}_i\right], \end{aligned}$$

where the fourth inequality uses $v_i - cd_{ii} = v_i \geq 0$ and $v_\ell \leq 1$, which implies that

$v_\ell - cd_{i\ell}$ cannot be maximal if $d_{i\ell} > \ell^{\max}$, and where

$$\begin{aligned} \tilde{\mathbf{v}}_i &\equiv \left\{ \{\tilde{v}^m_\ell - c\ell\}_{m=1}^{\overline{\mathbf{n}}_i(\ell)} \right\}_{\ell=0}^{\ell=\ell^{\max}} \\ &= (\tilde{v}^1_0,\ \underbrace{\tilde{v}^1_1 - c, \ldots, \tilde{v}^{\overline{\mathbf{n}}_i(1)}_1 - c}_{\overline{\mathbf{n}}_i(1) \text{ agents at distance 1 from } i}, \ldots, \underbrace{\tilde{v}^1_{\ell^{\max}} - c\ell^{\max}, \ldots, \tilde{v}^{\overline{\mathbf{n}}_i(\ell^{\max})}_{\ell^{\max}} - c\ell^{\max}}_{\overline{\mathbf{n}}_i(\ell^{\max}) \text{ agents at distance } \ell^{\max} \text{ from } i}), \end{aligned}$$

with each $\tilde{v}^m_\ell$ independently drawn from F.

Thus, the expected social surplus associated with placing all resources with agent i depends only on the first $\ell^{\max}$ coordinates of the vector $\overline{\mathbf{n}}_i$, and it decreases if any of the first $\ell^{\max}$ coordinates of the vector $\overline{\mathbf{n}}_i$ is decremented, and it also decreases if the ℓ-th coordinate of $\overline{\mathbf{n}}_i$ is decreased by $x \in \{1, \ldots, \overline{n}_i(\ell)\}$ while the ℓ'-th coordinate is increased by x, with $\ell < \ell' \leq \ell^{\max}$.

With these preliminary results in hand, suppose that agent i reach dominates agent j given c. We need to show that it is never optimal to place the resources with agent j. We know from Proposition 1 that extremal placement is optimal, so it is sufficient to show that placing all resources with agent i results in greater expected social surplus than placing all resources with agent j. Given our preliminary result, it is sufficient to show that the first $\ell^{\max}$ coordinates of $\overline{\mathbf{n}}_j$ can be obtained from the first $\ell^{\max}$ coordinates of $\overline{\mathbf{n}}_i$ through a finite number of steps, where at each step we either decrement one of the coordinates or decrease coordinate ℓ while increasing coordinate ℓ' by the same amount, with $\ell < \ell' \leq \ell^{\max}$. We now provide such an algorithm.

Recall that because agent i reach dominates agent j, either $\sum_{\ell=1}^{\ell^{\max}} \overline{n}_i(\ell) = \sum_{\ell=1}^{\ell^{\max}} \overline{n}_j(\ell)$ or $\sum_{\ell=1}^{\ell^{\max}} \overline{n}_i(\ell) > \sum_{\ell=1}^{\ell^{\max}} \overline{n}_j(\ell)$.

0. Let $\tilde{\mathbf{n}}_0 \equiv \overline{\mathbf{n}}_i$ and let $k = 1$ be the iteration index.

1. If $\sum_{\ell=1}^{\ell^{\max}} \tilde{n}_{k-1}(\ell) = \sum_{\ell=1}^{\ell^{\max}} \overline{n}_j(\ell)$ skip to step 2. Otherwise, let h be the largest index in $\{1, \ldots, \ell^{\max}\}$ such that $\tilde{n}_{k-1}(h) > 0$ and decrement $\tilde{n}_{k-1}(h)$ by 1, i.e., let $\tilde{n}_k(h) \equiv \tilde{n}_{k-1}(h) - 1$, and for $\ell \neq h$, let $\tilde{n}_k(\ell) \equiv \tilde{n}_{k-1}(\ell)$. Increment k by 1 and return to step 1.

2. If $\tilde{n}_{k-1}(\ell) = \overline{n}_j(\ell)$ for all $\ell \in \{1, \ldots, \ell^{\max}\}$, then we are done, otherwise continue to step 3.

3. In this case, we have $\sum_{\ell=1}^{\ell^{\max}} \tilde{n}_{k-1}(\ell) = \sum_{\ell=1}^{\ell^{\max}} \overline{n}_j(\ell)$.

(a) let h be the largest index in $\{1, \ldots, \ell^{\max}\}$ such that $\tilde{n}_{k-1}(h) > \overline{n}_j(h)$ and let h' be the smallest index in $\{h+1, \ldots, \ell^{\max}\}$ such that $\tilde{n}_{k-1}(h') < \overline{n}_j(h')$;

(b) define

$$\tilde{n}_k(h) \equiv \begin{cases} \overline{n}_j(h) & \text{if } \tilde{n}_{k-1}(h) - \overline{n}_j(h) < \overline{n}_j(h') - \tilde{n}_{k-1}(h'), \\ \tilde{n}_{k-1}(h) + \tilde{n}_{k-1}(h') - \overline{n}_j(h') & \text{otherwise,} \end{cases}$$

which implies that $\tilde{n}_k(h) < \tilde{n}_{k-1}(h)$, and define

$$\tilde{n}_k(h') \equiv \begin{cases} \tilde{n}_{k-1}(h') + \tilde{n}_{k-1}(h) - \overline{n}_j(h) & \text{if } \tilde{n}_{k-1}(h) - \overline{n}_j(h) < \overline{n}_j(h') - \tilde{n}_{k-1}(h'), \\ \overline{n}_j(h') & \text{otherwise,} \end{cases}$$

which implies that $\tilde{n}_k(h') > \tilde{n}_{k-1}(h')$;

(c) for $\ell \notin \{h, h'\}$, define $\tilde{n}_k(\ell) \equiv \tilde{n}_{k-1}(\ell)$;

(d) increment k by 1 and return to step 2.

Because this algorithm ends in a finite number $\overline{k}$ of steps with $\tilde{n}_{\overline{k}}(\ell) = \overline{n}_j(\ell)$ for all $\ell \in \{1, \ldots, \ell^{\max}\}$, and because at each step expected social surplus decreases, we have completed the proof that resources are never placed with agents that are reach dominated, which is the first part of the proposition.

If the set $\mathcal{T}(c)$ of reach dominant agents is non-empty, then any agent not in $\mathcal{T}(c)$ is reach dominated, implying that resources are only placed with agents in $\mathcal{T}(c)$, which completes the proof of the second part of the proposition. ■

Proof of Proposition 4. Suppose $c \geq 1/2$, in which case agents only ever trade with their immediate neighbors. Let $\mathcal{N}_i \subset \mathcal{N}$ be the set of agent i's immediate neighbors. Consider the mechanism in which agent i can buy from agent $j \in \mathcal{N}_i$ at (per-unit) price $\max\{v_j + c, \max_{h \in \mathcal{N}_j} v_h\}$ and agent i can sell to agent $j \in \mathcal{N}_i$ at (per-unit) price $v_j - c$. This mechanism induces agent i to demand r_j units from agent j if $v_i > \max\{v_j + c, \max_{h \in \mathcal{N}_j} v_h\}$, and zero units otherwise, and it induces agent i to offer r_i units to agent j if $v_i < v_j - c$ and $v_j = \max_{h \in \mathcal{N}_i} v_h$, and zero units otherwise. Thus, this mechanism induces the ex post efficient trade with trading buyers paying the lowest Walrasian price and trading sellers receiving the highest Walrasian price. Agents with type 1/2 do not trade and have zero payments. These types are worst-off, implying that worst-off types satisfy ex post and interim individual rationality constraints with equality. Turning to the budget surplus of this mechanism, if $v_i > \max\{v_j + c, \max_{h\in\mathcal{N}_j} v_h\}$, then agent i purchases r_j units from agent j and makes a payment of $r_j \max\{v_j + c, \max_{h\in\mathcal{N}_j} v_h\}$, while agent j is paid $r_j(v_i - c)$. Thus, the budget surplus associated with trades involving agent i is

$$\begin{aligned} &\sum_{\ell\in\mathcal{N}_i \text{ s.t. } v_i > \max\{v_\ell + c, \max_{h\in\mathcal{N}_\ell} v_h\}} \left(r_\ell \max\{v_\ell + c, \max_{h\in\mathcal{N}_\ell} v_h\} - r_\ell\,(v_i - c) \right) \\ = &\sum_{\ell\in\mathcal{N}_i \text{ s.t. } v_i > \max\{v_\ell + c, \max_{h\in\mathcal{N}_\ell} v_h\}} r_\ell \left(\max\{v_\ell - v_i + 2c, \max_{h\in\mathcal{N}_\ell} v_h - v_i + c\} \right), \end{aligned}$$

where $v_\ell - v_i + 2c = (v_\ell + c) - v_i + c < v_i - v_i + c = c$ and $\max_{h\in\mathcal{N}_\ell} v_h - v_i + c < v_i - v_i + c = c$, which says that the transportation costs are not covered (on a trade-by-trade basis). This then proves the impossibility of ex post efficient trade. ■

Proof of Lemma 3. The individual rationality condition can be stated as for all $v \in [\underline{v}, \overline{v}]$, $u_i(v) \geq 0$. By incentive compatibility, $u_i(v) = \max_{\hat{v}} q_i(\hat{v})v - m_i(\hat{v}) - r_i v$, which implies that u_i is differentiable almost everywhere and by the envelope theorem, whenever it is differentiable, we have $u_i'(v) = q_i(v) - r_i$. Thus, for all $\omega \in [\underline{v}, \overline{v}]$,

$$u_i(v) = \int_\omega^v (q_i(x) - r)dx + u_i(\omega).$$

From this, it follows that $m_i(v) = q_i(v)v - r_i v - \int_\omega^v (q_i(x) - r)dx - u_i(\omega)$, implying

$$\begin{aligned} \mathbb{E}_{v_i}[m_i(v_i)] &= \int_{\underline{v}}^{\overline{v}} (q_i(x) - r_i)\, x dF_i(x) - \int_{\underline{v}}^{\overline{v}} \int_\omega^y (q_i(x) - r) f_i(y) dx dy - u_i(\omega) \\ &= \int_{\underline{v}}^{\overline{v}} (q_i(x) - r_i)\, \Psi_i(x;\omega) dF_i(x) - u_i(\omega) = \int_{\underline{v}}^{\overline{v}} q_i(x)\Psi_i(x;\omega)dF_i(x) - r_i\omega - u_i(\omega) \\ &= \mathbb{E}_{v_i}[q_i(v_i)\Psi_i(v_i;\omega)] - r_i\omega - u_i(\omega) = \mathbb{E}_{\mathbf{v}}[Q_i(\mathbf{v})\Psi_i(v_i;\omega)] - r_i\omega - u_i(\omega), \end{aligned}$$

where the last equality uses the definition of $q_i(v_i)$. By definition $m_i(v_i)$, we have $\mathbb{E}_{v_i}[m_i(v_i)] = \mathbb{E}_{\mathbf{v}}[M_i(\mathbf{v})]$. This completes the proof. ■

Proof of Lemma 4. For $v \geq 1 - c$, we have $q_i(v) \geq r_i$ because i can never trade as a seller. For $v \leq c$, we have $q_i(v) \leq r_i$ because i can never trade as a buyer. This implies that for $c > 1/2$, in which case we have $1 - c \leq c$, the agent's interim expected allocation satisfies $q_i(v) = r_i$ for all $v \in [1-c, c]$. Hence, for $c \geq 1/2$, all types $v \in [1-c, c]$ will be worst-off. ■

Proof of Lemma 5. Assume that $F_i = F$ for all $i \in \mathcal{N}$, and so drop the agent subscripts on the virtual type functions. Define the function $\hat{z}(\omega, a)$ to be the implicit solution for the ironing parameter z that solves (7) (this is the same for all i given the assumption that $F_i = F$ for all $i \in \mathcal{N}$). When $c \geq 1/2$, ex post efficient trade is not possible by Proposition 4, so we have $\rho^* > 1$. Thus, $\hat{z}(1/2, 1/\rho^*) \in (0, 1)$, and we can let $\hat{c} \in [1/2, 1)$ be such that for all $c \geq \hat{c}$, we have $1 - c < \hat{z}(1/2, 1/\rho^*) < c$. Focusing on the expression in (9) in square brackets, if $1 - c < z(1/\rho^*, \omega^*) \equiv z^* < c$, then in order to have $V^e_{ij} = 1$ for $i \neq j$, we require that $\overline{\Psi}_{\frac{1}{\rho^*}}(v_i; \omega^*_i) < z^*$, which implies that $\overline{\Psi}_{\frac{1}{\rho^*}}(v_i; \omega^*_i) = \Psi_{\frac{1}{\rho^*}}(v_i; \omega^*_i)$, and $\overline{\Psi}_{\frac{1}{\rho^*}}(v_j; \omega^*_j) > z^*$, which implies that $\overline{\Psi}_{\frac{1}{\rho^*}}(v_j; \omega^*_j) = \Psi_{\frac{1}{\rho^*}}(v_j; \omega^*_j)$. So the term in square brackets can be written as

$$\sum_{i=1}^{n} \sum_{j \in \mathcal{N} \setminus \{i\}} \left(\overline{\Psi}_{\frac{1}{\rho^*}}(v_i; \omega^*_i) - C_{ji} \right) V^{ce}_{ji}(\mathbf{v}; \rho^*, \boldsymbol{\omega}^*) r_j + \sum_{i=1}^{n} \Psi_{\frac{1}{\rho^*}}(v_i; \omega^*_i) V^{ce}_{ii}(\mathbf{v}; \rho^*, \boldsymbol{\omega}^*) r_i.$$

But notice that, dropping the arguments on the V^{ce}_{ij} terms,

$$\begin{aligned}
& \mathbb{E}_{\mathbf{v}} \left[\left(\overline{\Psi}_{\frac{1}{\rho^*}}(v_i; \omega^*_i) - \Psi_{\frac{1}{\rho^*}}(v_i; \omega^*_i) \right) V^{ce}_{ii} \right] \\
= \; & \mathbb{E}_{\mathbf{v}} \left[\left(z^* - \Psi_{\frac{1}{\rho^*}}(v_i; \omega^*_i) \right) V^{ce}_{ii} \mid \overline{\Psi}_{\frac{1}{\rho^*}}(v_i; \omega^*_i) = z^* \right] \Pr \left(\overline{\Psi}_{\frac{1}{\rho^*}}(v_i; \omega^*_i) = z^* \right) \\
= \; & \mathbb{E}_{\mathbf{v}} \left[z^* - \Psi_{\frac{1}{\rho^*}}(v_i; \omega^*_i) \mid \overline{\Psi}_{\frac{1}{\rho^*}}(v_i; \omega^*_i) = z^*, \, V^{ce}_{ii} = 1 \right] \Pr \left(\overline{\Psi}_{\frac{1}{\rho^*}}(v_i; \omega^*_i) = z^*, \, V^{ce}_{ii} = 1 \right) \\
= \; & \mathbb{E}_{\mathbf{v}} \left[z^* - \Psi_{\frac{1}{\rho^*}}(v_i; \omega^*_i) \mid \overline{\Psi}_{\frac{1}{\rho^*}}(v_i; \omega^*_i) = z^* \right] \Pr \left(\overline{\Psi}_{\frac{1}{\rho^*}}(v_i; \omega^*_i) = z^* \right) \\
= \; & 0,
\end{aligned}$$

where the first equality uses the fact that the ironed and unironed virtual types are identical outside of the ironing range, the second equality uses the binary nature of V^{ce}_{ij}, the third equality uses the result that if v_i is in the ironing range and $c > \hat{c}$, then it is not possible for agent i to trade and so $V^{ce}_{ii} = 1$, and the final equality uses the definition of the ironing parameter given in (7). Thus, the expectation of the expression in (9) in square brackets is

$$\mathbb{E}_{\mathbf{v}}\Big[\sum_{j=1}^{n}\sum_{i=1}^{n}\Big(\overline{\Psi}_{\frac{1}{\rho^*}}(v_i;\omega_i^*)-C_{ji}\Big)\,V_{ji}^{ce}(\mathbf{v};\rho^*,\boldsymbol{\omega}^*)r_j\Big],$$

which completes the proof. ∎

Proof of Proposition 10. Because $F_i = F$ for all $i \in \mathcal{N}$, we can drop the agent subscripts on the virtual type functions. Moreover, because the resource is indivisible, we have $r_j = 1$ for some $j \in \mathcal{N}$ and $r_i = 0$ for all $i \neq j$. As noted in the text, this implies that the constrained-efficient reallocation mechanism is a generalization of the second-best mechanism of Myerson and Satterthwaite (1983) to a setting with costly transportation with $n-1$ buyers whose worst-off types are 0 and agent j as the seller whose worst-off type is 1.

Suppose that agent i reach dominates agent j. We show that the expected constrained-efficient social surplus from giving ownership to agent i is greater than from giving it to agent j. Note that if agent j has ownership of all resources, i.e., $\mathbf{r} = \mathbf{e}_j$, where $\mathbf{e}_j$ is the unit vector of length n with j-th coordinate equal to 1, and $\mathbf{Q}$ is an incentive compatible reallocation rule, then the expected budget surplus net of transportation costs under binding individual rationality is

$$\mathbb{E}_{\mathbf{v}}\left[\Psi^S(v_j)Q_j(\mathbf{v})+\sum_{\ell\neq i}\left(\Psi^B(v_\ell)-cd_{j\ell}\right)Q_\ell(\mathbf{v})\right]-1$$

because, in this case, agent j can trade only as a seller and all other agents can trade only as buyers, implying that the worst-off type of agent j is 1 and the worst-off types of each other agent is 0 (the final "-1" reflects the outside option of agent j's worst-off type).

Let $k_j \equiv 1+\sum_{\ell=1}^{\min\{\lceil 1/c\rceil,n-1\}} n_j(\ell)(n-1)$ be the number of agents within distance $\min\{\lceil 1/c\rceil, n-1\}$ of agent j (including agent j itself), and order those agents from 1 to k_j starting with agent j and then next the agents at distance 1 from agent j (ordered arbitrarily), then the agents at distance 2 from agent j (ordered arbitrarily), and so on, up to the agents at distance $\min\{\lceil 1/c\rceil, n-1\}$ from agent j (ordered arbitrarily). For $h \in \{1,\ldots,k_j\}$, let $I_j(h)$ be the identity of the h-th agent in this ordering. Analogously, let $k_i \equiv 1+\sum_{\ell=1}^{\min\{\lceil 1/c\rceil,n-1\}} n_i(\ell)(n-1)$ be the number of agents within distance $\min\{\lceil 1/c\rceil, n-1\}$ of agent i, and order those agents from 1 to k_i starting with agent i and then the agents at distance 1 from agent i (ordered arbitrarily), and so on, up to the agents at distance $\min\{\lceil 1/c\rceil, n-1\}$ from agent i (ordered arbitrarily). For $h \in \{1, \ldots, k_i\}$, let $I_i(h)$ be the identity of the h-th agent in this ordering. Note that $I_j(1) = j$ and $I_i(1) = i$. Because agent i reach dominates agent j, $k_i \geq k_j$ and for $h \in \{1, \ldots, k_j\}$, $d_{iI_i(h)} \leq d_{jI_j(h)}$, with a strict inequality for at least one h.

We let $\mathbf{x} \in [0,1]^{k_j}$ denote the vector of types of the agents within distance $\min\{\lceil 1/c\rceil, n-1\}$ of agent j, in the same order as the ordering just defined, all of which are independent draws from F. Analogously, we let $\mathbf{y} \in [0,1]^{k_i}$ denote the vector of types of the agents within distance $\min\{\lceil 1/c\rceil, n-1\}$ of agent i, in the same order as the ordering just defined, again all independent draws from F. Further, we let $\overline{\mathbf{y}}$ denote the first k_j coordinates of $\mathbf{y}$. We refer to $\mathbf{x}$ ($\mathbf{y}$) as the j-centered (i-centered) types.

Given ownership $\mathbf{e}_j$, we can write the constrained-efficient reallocation rule as a function of the types of agents within distance $\min\{\lceil 1/c\rceil, n-1\}$ from agent j because the constrained-efficient rule never reallocates to agents at a distance greater than $\min\{\lceil 1/c\rceil, n-1\}$ from agent j. We denote by $\hat{Q}_{I_j(h),\mathbf{e}_j}(\mathbf{x})$ the constrained-efficient allocation for agent $I_j(h)$ when the j-centered types are $\mathbf{x}$. Ignoring ties and letting $\rho_{\mathbf{e}_j}$ be the Lagrange multiplier on the no-deficit constraint when the ownership is $\mathbf{e}_j$, we have for $\mathbf{x} \in [0,1]^{k_j}$,

$$\hat{Q}_{I_j(1),\mathbf{e}_j}(\mathbf{x}) \equiv \begin{cases} 1 & \text{if } \Psi^S_{1/\rho_{\mathbf{e}_j}}(x_1) \geq \max\limits_{h \in \{2,\ldots,k_j\}} \Psi^B_{1/\rho_{\mathbf{e}_j}}(x_h) - cd_{jI_j(h)}, \\ 0 & \text{otherwise,} \end{cases}$$

and for $h \in \{2, \ldots, k_j\}$,

$$\hat{Q}_{I_j(h),\mathbf{e}_j}(\mathbf{x}) \equiv \begin{cases} 1 & \text{if } \Psi^B_{1/\rho_{\mathbf{e}_j}}(x_h) - cd_{jI_j(h)} \geq \max\limits_{h' \in \{2,\ldots,k_j\}} \{\Psi^S_{1/\rho_{\mathbf{e}_j}}(x_1), \Psi^B_{1/\rho_{\mathbf{e}_j}}(x_{h'}) - cd_{jI_j(h')}\}, \\ 0 & \text{otherwise.} \end{cases}$$

The expected budget surplus net of transportation costs under binding individual rationality is

$$\pi_{\mathbf{e}_j} \equiv \mathbb{E}_{\mathbf{x}}\Big[\Psi^S(x_1)\hat{Q}_{I_j(1),\mathbf{e}_j}(\mathbf{x}) + \sum_{h=2}^{k_j} \left(\Psi^B(x_h) - cd_{jI_j(h)}\right) \hat{Q}_{I_j(h),\mathbf{e}_j}(\mathbf{x})\Big] - 1.$$

By the assumption of identical distributions, we can "apply" $\hat{\mathbf{Q}}_{\mathbf{e}_j}$ to the i-centered types $\overline{\mathbf{y}}$ by defining for each $h \in \{1, \ldots, k_j\}$, $\tilde{Q}_{I_i(h),\mathbf{e}_j}(\overline{\mathbf{y}}) \equiv \hat{Q}_{I_j(h),\mathbf{e}_j}(\overline{\mathbf{y}})$. This gives us

$$\begin{aligned} \pi_{\mathbf{e}_j} &= \mathbb{E}_{\overline{\mathbf{y}}}\Big[\Psi^S(x_1)\tilde{Q}_{I_j(1),\mathbf{e}_j}(\overline{\mathbf{y}}) + \sum_{h=2}^{k_j} \left(\Psi^B(x_h) - cd_{jI_j(h)}\right) \tilde{Q}_{I_j(h),\mathbf{e}_j}(\overline{\mathbf{y}})\Big] - 1 \\ &< \mathbb{E}_{\overline{\mathbf{y}}}\Big[\Psi^S(y_1)\tilde{Q}_{I_i(1),\mathbf{e}_j}(\overline{\mathbf{y}}) + \sum_{h=2}^{k_j} \left(\Psi^B(y_h) - cd_{iI_i(h)}\right) \tilde{Q}_{I_i(h),\mathbf{e}_j}(\overline{\mathbf{y}})\Big] - 1, \end{aligned}$$

where the inequality uses the implication of reach dominance that $d_{iI_i(h)} \leq d_{jI_j(h)}$ with a strict inequality for at least one h. This implies that when reallocation rule $\tilde{\mathbf{Q}}_{\mathbf{e}_j}$ is applied under ownership $\mathbf{e}_i$, the expected budget surplus net of transportation costs under binding individual rationality is greater than $\pi_{\mathbf{e}_j}$.

By an analogous argument, with Ψ^S and Ψ^B replaced by the identity function, expected social surplus under $\hat{\mathbf{Q}}_{\mathbf{e}_j}$ is

$$\mathbb{E}_{\mathbf{x}}\Big[x_1\hat{Q}_{I_j(1),\mathbf{e}_j}(\mathbf{x}) + \sum_{h=2}^{k_j} \left(x_h - cd_{jI_j(h)}\right) \hat{Q}_{I_j(h),\mathbf{e}_j}(\mathbf{x})\Big] < \mathbb{E}_{\overline{\mathbf{y}}}\Big[y_1\tilde{Q}_{I_i(1),\mathbf{e}_j}(\overline{\mathbf{y}}) + \sum_{h=2}^{k_j} \left(y_h - cd_{iI_i(h)}\right) \tilde{Q}_{I_i(h),\mathbf{e}_j}(\overline{\mathbf{y}})\Big],$$

which says that when the reallocation rule $\tilde{\mathbf{Q}}_{\mathbf{e}_j}$ is applied under ownership $\mathbf{e}_i$, the expected social surplus is greater than in the constrained-efficient outcome under ownership $\mathbf{e}_j$. Thus, we have shown that under ownership $\mathbf{e}_i$, one can achieve greater expected social surplus than in the constrained-efficient outcome under ownership $\mathbf{e}_j$, while continuing to satisfying all of the constraints. Further, optimizing the reallocation rule for ownership $\mathbf{e}_i$ and applying it to $\mathbf{y}$ (rather than only $\overline{\mathbf{y}}$), expected social surplus further weakly increases, completing the proof that with indivisible resources, ownership by reach dominated agents is not optimal under constrained efficiency. ■

Proof of Proposition 11. First consider the case with $c \in [1/2, 1)$. If $r_j \leq c$ for all j, then there is no trade and social surplus is simply $\mathbb{E}[v]$. If $r_i > c$ for some agent i, then social surplus is $\mathbb{E}[v] + r_i GFT_i(r_i)$, where $GFT_i(r_i)$ is the expected gain in social surplus associated with trades involving agent i, necessarily as a seller, given r_i. Because $GFT_i(r_i)$ is positive and increasing in r_i (because c/r_i is less than one and decreases with r_i), social surplus is maximized for $r_i \in (c, 1]$ at $r_i = 1$. It then remains to choose the agent i to maximize $GFT_i(1)$.

Now consider the case with $c \in [0, 1/2)$. Suppose that agent i reach dominates agent j given cost c, and consider the gains from trade associated with trades involving agents i and j. To allow for the possibility that there are gains from trade, we assume that the sum of ownership shares of agents i and j satisfies $r_i + r_j > c$, which makes sure that for sufficiently extremal ownership (e.g. by giving all of $r_i + r_j$ to agent i), there are positive gains from trade. Then we have

$$\begin{aligned} r_i GFT_i(r_i) + r_j GFT_j(r_j) &\leq r_i GFT_i(r_i) + r_j GFT_i(r_j) \\ &< r_i GFT_i(r_i + r_j) + r_j GFT_i(r_i + r_j) \\ &= (r_i + r_j) GFT_i(r_i + r_j), \end{aligned}$$

where the first inequality uses that agent i reach dominates agent j, and the second inequality is strict because $GFT_i(\cdot)$ is strictly increasing in its argument when it is positive. Thus, social surplus is increased by shifting ownership towards the reach dominating agent. It then follows that if the set of reach dominant agents is non-empty, then the optimal ownership places resources only with reach dominant agents. Further, if there are two reach dominant agents, then it is optimal to place all resources with a single one of them because of the increasing returns to scale in the gains from trade. ■

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ONLINE APPENDIX

Efficient trade and ownership on networks*

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1 Details for ordering of $\overline{r}_n(0)$

This appendix provides details for the ordering of $\overline{r}_n(0)$, as illustrated in Figure 2.

Assume that $F_i = F$ for all $i \in \mathcal{N}$ and drop agent-specific subscripts on the virtual type functions, writing Ψ^S, Ψ^B, and Ψ instead. For arbitrary n and $c = 0$, we have $q_i^e(v) = H_n(v)$, where $H_n(v) \equiv F^{n-1}(v)$, so $\omega_i^e = H_n^{-1}(r_i)$. Using (6), $\overline{\mathbf{r}}(0)$ is defined by the $\mathbf{r}$ that solves

$$\mathbb{E}_{\mathbf{v}}\left[\sum_{i=1}^{n} \Psi(v_i;\omega_i^e) q_i^e(v_i)\right] = \sum_{i=1}^{n} \omega_i^e r_i,$$

which we can write as

$$\begin{aligned}
0 &= \mathbb{E}_{\mathbf{v}}\left[\sum_{i=1}^{n} \Psi(v_i; H_n^{-1}(r_i)) H_n(v_i)\right] - \sum_{i=1}^{n} H_n^{-1}(r_i) r_i \\
&= \sum_{i=1}^{n}\left(\int_0^{H_n^{-1}(r_i)} \Psi^S(x) H_n(x) f(x) dx + \int_{H_n^{-1}(r_i)}^{1} \Psi^B(x) H_n(x) f(x) dx - H_n^{-1}(r_i) r_i\right) \\
&= \sum_{i=1}^{n}\left(\int_0^{H_n^{-1}(r_i)} \left(x + \frac{F(x)}{f(x)}\right) H_n(x) f(x) dx + \int_{H_n^{-1}(r_i)}^{1} \left(x - \frac{1-F(x)}{f(x)}\right) H_n(x) f(x) dx - H_n^{-1}(r_i) r_i\right) \\
&= \sum_{i=1}^{n}\left(\int_0^{1} x H_n(x) f(x) dx + \int_0^{1} H_n(x) F(x) dx - \int_{H_n^{-1}(r_i)}^{1} H_n(x) dx - H_n^{-1}(r_i) r_i\right) \\
&= 1 + (n-1)\int_0^1 F^n(x) dx - \sum_{i=1}^{n}\left(\int_{H_n^{-1}(r_i)}^{1} H_n(x) dx + H_n^{-1}(r_i) r_i\right) \\
&= 1 + (n-1)\int_0^1 F^n(x) dx - \sum_{i=1}^{n}\left(1 - \int_{H_n^{-1}(r_i)}^{1} x dH_n(x)\right) \\
&= (n-1)\left(\int_0^1 F^n(x) dx - 1 + \frac{1}{n-1}\sum_{i=1}^{n}\int_{H_n^{-1}(r_i)}^{1} x dH_n(x)\right).
\end{aligned}$$

Thus, for a star network with ownership vector $(r, \frac{1-r}{n-1}, \ldots, \frac{1-r}{n-1})$ and $c = 0$, we have

$$\int_0^1 F^n(x) dx = 1 - \frac{1}{n-1}\int_{(F^{n-1})^{-1}(r)}^{1} x dF^{n-1}(x) - \int_{(F^{n-1})^{-1}(\frac{1-r}{n-1})}^{1} x dF^{n-1}(x),$$

which for the uniform distribution can be written as

$$\frac{n}{n+1} = r^{\frac{n}{n-1}} + (n-1)^{\frac{-1}{n-1}} (1-r)^{\frac{n}{n-1}}.$$

Note that the left side is constant in r and the right side is convex in r and increasing in r at $r = 1$. Further, the right side is equal to 1 at $r = 1$, and so greater than the left side, and the right side is less than the left side at $r = 0$ for $n > 2$. So for all $n > 2$, there is a unique solution. Solving this (and taking the maximum solution for $n = 2$), we have:

n	$\overline{r}_n(0)$	$n\frac{1-r_1}{n-1}$
2	0.788675	0.433650
3	0.765431	0.351853
4	0.764689	0.313749
5	0.768943	0.288821
6	0.774292	0.270849
7	0.779633	0.257095
8	0.784635	0.246131
9	0.789221	0.237126
10	0.793397	0.229559

While $\overline{r}_n(0)$ is not monotone in n, that comparison does not properly account for the expansion in the number of agents. Here we show that the cumulative distributions $G_n(i) \equiv \sum_{\ell=1}^{i} r_\ell$ defined for $i \in \{0, 1, \ldots, n\}$ satisfy FOSD in n with G_n first-order stochastically dominating $G_{n'}$ if $n < n'$. More precisely, define the continuous functions

$$G_n(x) \equiv \begin{cases} nr_1 x & \text{if } x \leq 1/n, \\ 1 - \frac{1-r_1}{1-1/n} + \frac{1-r_1}{1-1/n}x & \text{if } x > 1/n. \end{cases}$$

The slope at $x = 1$, $\frac{1-r_1}{1-1/n}$, is decreasing in n, which is sufficient to prove FOSD.