International Journal of Industrial Organization 50 (2017) 430–459

Contents lists available at ScienceDirect

International Journal of Industrial Organization

www.elsevier.com/locate/ijio

Club good intermediaries[☆]

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A R T I C L E   I N F O

Article history:
Available online 1 June 2016

JEL classification:
C72
D82
L13

Keywords:
Revenue maximization
Excludable public goods
Two-sided platforms
Optimal pricing
Digital goods

A B S T R A C T

The emergence and ubiquitous presence in everyday life of digital goods such as songs, movies, and e-books give renewed salience to the problem of providing public goods with exclusion. Because digital goods are typically traded via intermediaries like iTunes, Amazon, and Netflix, the question arises as to the optimal pricing mechanism for such club good intermediaries. We derive the direct Bayesian optimal mechanism for allocating club goods when the mechanism designer is an intermediary that neither produces nor consumes the goods, and we develop an indirect mechanism that implements this mechanism. We also derive sufficient conditions for the intermediary-optimal mechanism to be implementable with revenue sharing contracts, which are widely used in e-business.

1. Introduction

The issue of optimal public goods provision has achieved new salience with the emergence of digital goods like e-books, downloadable songs, and movies along with new technologies that make exclusion and distribution possible at negligible marginal costs.

[☆] We thank seminar participants at the 34th Australasian Economic Theory Workshop, Cédric Wasser, and an anonymous referee for helpful comments.
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http://dx.doi.org/10.1016/j.ijindorg.2016.05.007
0167-7187/

An aspect not present in the analysis of excludable public goods or "club goods" in the age of the internet is that consumers gain access to these digital goods via an intermediary such as Amazon, iTunes, Spotify, or Netflix rather than contracting directly with the producer of the club good as they might with the owner of a country club. In this paper, we provide an analysis of optimal mechanisms for *club good intermediaries*, which are brokers who intermediate between the producer of a public good with exclusion and consumers. Besides the aforementioned intermediaries for digital goods, crowdfunding platforms like Kickstarter or Indiegogo are other examples of club good intermediaries.[1]

We analyze club good intermediaries within the independent private values paradigm in Bayesian mechanism design, assuming that all agents draw their values and costs independently from some commonly known distributions and that they are privately informed about the realizations of their types. Among other things, the Bayesian mechanism design approach has the benefit of imposing no constraints on the optimal mechanism other than the usual incentive compatibility and individual rationality constraints.

We extend the methods and insights developed by Myerson (1981) and Myerson and Satterthwaite (1983) for private goods to the case of club goods.[2] Just as with a broker for private goods, the optimal mechanism for a club good intermediary can be stated in terms of virtual valuations and virtual costs, with trade occurring when the former exceed the latter. However, the conditions for trade for a club good intermediary are more intricate because the optimal allocation rule is such that trade occurs if and only if the sum of the virtual values of those buyers with nonnegative virtual values exceeds the seller's virtual (fixed) cost for producing the good.[3] Also, in the private good setting, the mechanism need only determine a single price to be paid by the winning buyer. But in the club good setting, the mechanism must determine a price for each buyer who receives the good, and these prices may differ across buyers as a function of the buyers' reported types and identities.

Besides characterizing the Bayesian optimal club good mechanism for an intermediary, we define a club good clock auction in which it is a dominant strategy for buyers to exit at their values. We provide generalizations that allow for congestion effects, nonmonotonic virtual types that require ironing, and an objective for the designer that includes weight on both profit and social surplus. We also derive sufficient conditions for the intermediary-optimal mechanism to be implementable with revenue sharing contracts. These are widely used in e-business and featured prominently in Apple's e-book case.[4]

[1] For more on crowdfunding platforms, see, for example, Marwell (2015).

[2] It is well known that under seemingly weak conditions (a lower bound support of the buyers' type distributions of zero) the Bayesian optimal mechanism allocates the public good with probability zero in the large economy (Rob, 1989). With n buyers, the sum of the virtual types of the buyers converges to n times the expected virtual type, which is equal to the lower bound of support of the buyers' type distribution, and hence to n times zero under the condition just stated. This provides further motivation to study Bayesian optimal provision of club goods.

[3] Bearing in mind the applications of digital goods, we assume that the marginal cost of delivering every additional copy is zero once the fixed cost of producing the good has been borne. In general, the constraint is that the sum of the virtual values of those buyers with virtual values no less than the marginal costs exceeds the seller's virtual fixed cost.

[4] "Ruling That Apple Led E-Book Pricing Conspiracy Is Upheld," *New York Times*, June 30, 2015.

In some settings, whether a good is offered as a private good or as a public good is a choice variable of the designer. For example, whether radio spectrum is used as a public good with exclusion or under rivalry of consumption as a private good depends on the technological standards, which may be a choice variable of the designer. Similarly, artists can choose between selling private goods in the form of paintings and club goods in the form of photographs. Under the assumptions that the demand and fixed cost of production for the two kinds of goods are the same, our analysis shows that a profit maximizing designer will choose to offer the good as a club good. In this way, our analysis provides a way of endogenizing the choice of technological format for applications like these.

First and foremost, our paper contributes to the large and growing literature on market making and intermediation by introducing club goods. Thus far, this literature has either focused on private goods – see, for example, Rubinstein and Wolinsky (1987), Stahl (1988), Gehrig (1993), Spulber (1996), Rust and Hall (2003), Loertscher (2007, 2008), or Loertscher and Niedermayer (2015) – or on the need of market making intermediaries to bring both sides on board, accounting for direct network effects, like Caillaud and Jullien (2001, 2003), Rochet and Tirole (2002, 2006), or Gomes (2014). With club goods, there is an indirect network effect even after both sides are on board because each buyer *benefits* from additional buyers because they are a source of revenue and therefore make the provision of the good more likely even though no buyer directly cares about any other buyer.[5]

Our paper is also related to the literature on the provision of excludable public goods. Perhaps the paper closest to ours in this strand of literature is Schmitz (1997). Schmitz studies a setup with one-sided private information and derives the optimal allocation for a club good (allowing for congestion effects) for a profit maximizing seller under the assumption of regularity. He shows that in the limit as the number of buyers goes to infinity (and the cost increases with the number of buyers as in Rob (1989)), the profit maximizing mechanism is a posted-price mechanism. Norman (2004) considers a similar environment to Schmitz's with an excludable, nonrivalrous good and with buyers whose values are privately known. However, Norman focuses on efficient provision subject to a constraint that the mechanism not run a deficit, whereas we consider a more general objective for the designer.[6] Norman shows that in his model the optimal mechanism can be approximated with a mechanism that provides a fixed quantity of the good and charges a fixed admission fee. Fang and Norman (2010) extend the analysis to

[5] As discussed in Marwell (2015), this effect also arises in all-or-nothing fundraising models, where fundraisers only collect donations if the total amount pledged exceeds a target and donors care about the amount raised. In concurrent work, Strausz (2015) analyzes the principal-agent problems that arise in crowdfunding. His work is complementary to ours, as we analyze the platform's optimal pricing problem absent moral hazard.

[6] The results of Norman (2004) continue to hold if the designer maximizes the weighted average of profit and consumer surplus as long as the weight on profit is no more than that on consumer surplus because, when consumer surplus has a greater or equal weight, the objective can be increased by returning any profits to consumers, in which case the constraint on nonnegative profit binds and any positive weight on profit has no effect. (See Norman, 2004, Section 2.3.)

include multiple excludable public goods. Hellwig (2010) considers a similar environment to Norman (2004) and shows that with sufficient inequality aversion, the optimal mechanism may involve randomized admissions. Ledyard and Palfrey (2007) characterize interim efficient allocation rules that satisfy interim incentive compatibility and interim individual rationality constraints.

The remainder of this paper is organized as follows. In Section 2, we describe our setup. Section 3 characterizes the Bayesian optimal direct mechanism. Section 4 describes dominant strategy implementations of the mechanism. Section 5 presents asymptotic results. Section 6 generalizes the model to allow for congestion effects and nonmonotonic virtual types that require ironing. Section 7 concludes.

2. Model

We consider a setup in which a seller has one unit of a good that can be allocated to one buyer at marginal cost $k \geq 0$ and to additional buyers at marginal cost of zero. We assume that there are no congestion effects. We relax this in the extensions section. We denote by $\mathbb{N}$ the set of buyers and $n \geq 2$ the number of buyers. We assume buyers have unit demands.

We consider both a one-sided and a two-sided setup. In the one-sided setup, the designer knows the seller's cost k. In the two-sided setup, the designer acts as an intermediary between the seller and the buyers, paying the seller for its input and collecting payments from buyers.

Many club goods, including digital goods like songs, movies, and e-books, are not directly sold from the producer to final consumers but rather traded via an intermediary. It is therefore of interest and relevance to understand the Bayesian optimal mechanism for a broker who is uncertain both about buyers' valuations and the seller's cost. To this end, we also consider the two-sided setup. In that two-sided setup, the seller's cost k is the seller's private information and is viewed by the designer as drawn from the distribution G with support $[\underline{k}, \overline{k}]$ and positive density g.

The designer's objective is to maximize the weighted sum of revenue and social surplus, with weight $\alpha \in [0, 1]$ on revenue. Thus, in the two-sided setup, if $\alpha = 1$, then the designer is a profit maximizing intermediary, paying the seller for its input and collecting payments from buyers. A buyer's payoff is zero if he does not trade and is equal to his value minus the price he pays to the designer if he does trade. The seller's payoff is the sum of the payments received from the designer minus the cost of providing the good when there is trade. If there is no trade, the seller's payoff is zero.

Each buyer i draws his value v_i independently from the distribution F_i, with support $[\underline{v}_i, \overline{v}_i]$ and positive density f_i. Denote the weighted virtual value function for bidder i by

$$\Phi_i(v) \equiv v - \alpha \frac{1 - F_i(v)}{f_i(v)}, \tag{1}$$

and denote the weighted virtual cost function for the seller by

$$\Gamma(k) \equiv k + \alpha \frac{G(k)}{g(k)}.$$

Note that $\Gamma(\underline{k}) = \underline{k}$ and $\Phi_i(\overline{v}_i) = \overline{v}_i$. As observed by Bulow and Roberts (1989), the virtual cost can be interpreted as a marginal cost of production accounting for the informational rents while virtual values can be interpreted as marginal revenue. We assume that

$$\sum_{i=1}^{n} \Phi_i(\underline{v}_i) < \Gamma(\overline{k}) \quad \text{and} \quad \sum_{i=1}^{n} \overline{v}_i > \underline{k}, \tag{2}$$

which ensures that trade is sometimes optimal for the designer in the two-sided setup.

3. Optimal direct mechanisms

3.1. One-sided setup

We begin with the one-sided setup and focus on the regular case, where Φ_i is a monotonically increasing function for all i. In Section 6, we consider the extension to the nonregular case, which requires ironing.

Consider a direct mechanism and denote by $q_i(\boldsymbol{v})$ the probability that agent i receives access to the club good when the vector of reports is $\boldsymbol{v}$. Standard arguments (see Appendix A for details) imply that in any incentive compatible interim individually rational mechanism the designer's expected profit is

$$\Pi = E_{\mathbf{v}}\left[\sum_{i=1}^{n} \Phi_i(v_i) q_i(\boldsymbol{v}, k) - k \min\left\{1, \sum_{i=1}^{n} q_i(\boldsymbol{v}, k)\right\}\right], \tag{3}$$

minus a constant, which, however, can be set equal to 0 by making the individual rationality constraints bind for the worst-off types.

We look for incentive compatible, interim individually rational mechanisms that maximize the designer's profit Π. It is always possible to have $q_i(\boldsymbol{v}, k) = 0$ for all i and all $\mathbf{v}$, in which case $\Pi = 0$. Thus, avoiding deficits (ex ante, at least) is never a binding constraint for a profit maximizing mechanism.

In the tradition of Myerson, let us look at the term in square brackets in (3),

$$\sum_{i=1}^{n} \Phi_i(v_i) q_i(\boldsymbol{v}, k) - k \min\left\{1, \sum_{i=1}^{n} q_i(\boldsymbol{v}, k)\right\}, \tag{4}$$

to find out whether pointwise maximization will give us an incentive compatible mechanism. A first result for pointwise maximization is immediate: Set

$$q_j(\boldsymbol{v}, k) = 0 \quad \text{for all} \quad j \quad \text{with} \quad v_j \leq \Phi_j^{-1}(0). \tag{5}$$

Regardless of other agents' reports, these types only decrease the expression in (4) when $q_j > 0$. Let $\mathbb{I}(\boldsymbol{v})$ be the set of all agents i with $\Phi_i(v_i) > 0$:

$$\mathbb{I}(\boldsymbol{v}) \equiv \left\{i : v_i > \Phi_i^{-1}(0)\right\}.$$

For all agents $i \in \mathbb{I}(\boldsymbol{v})$, set $q_i(\boldsymbol{v}, k) = 1$ if

$$\sum_{h \in \mathbb{I}(\boldsymbol{v})} \Phi_h(v_h) - k > 0, \tag{6}$$

and set $q_i(\boldsymbol{v}, k) = 0$ otherwise. We can write this as:

$$q_i(\boldsymbol{v},k) \equiv \begin{cases} 1, & \text{if } v_i \in \mathbb{I}(\boldsymbol{v}) \text{ and } \sum_{h \in \mathbb{I}(\boldsymbol{v})} \Phi_h(v_h) > k \\ 0, & \text{otherwise.} \end{cases} \tag{7}$$

This is the allocation rule that maximizes Π pointwise. To see that it is monotone (which as is well known is necessary and sufficient for being implementable in a Bayesian incentive compatible way), notice first that an agent l with $v_l \leq \Phi_l^{-1}(0)$ never gets access and can hence only increase the probability of getting access by reporting something higher. When of type $v_l > \Phi_l^{-1}(0)$, agent l receives access with probability 1 if

$$\sum_{h \in \mathbb{I}_{-l}(\boldsymbol{v})} \Phi_h(v_h) - k > 0, \tag{8}$$

where

$$\mathbb{I}_{-l}(\boldsymbol{v}) \equiv \{j \in \mathbb{I}(\boldsymbol{v}) \backslash l\},$$

in which case reporting something higher will still give him access with probability 1, or he increases the probability of getting access by becoming **pivotal** in the sense that

$$\sum_{h \in \mathbb{I}_{-l}(\boldsymbol{v})} \Phi_h(v_h) - k \leq 0 < \sum_{h \in \mathbb{I}(\boldsymbol{v})} \Phi_h(v_h) - k. \tag{9}$$

Consider the example of the case with two buyers whose values are drawn from the uniform distribution on [0, 1], and focus on the case of a revenue maximizing designer ($\alpha = 1$). Assume $k = 0.25$. Fig. 1(a) shows the regions of trade when the good is private and so can only be allocated to one buyer. Fig. 1(b) shows the regions of trade when the good is a club good. In the areas indicated in the upper right quadrant of Fig. 1(b), both buyers consume the good.

As illustrated in Fig. 1, whenever a buyer trades in the private good case, he or she also trades in the club good case. In addition, in the club good case, sometimes both buyers trade and there are cases in which both buyers trade when there would be no trade at all in the private good case. This occurs when the buyers have positive virtual values that are less than k, but the sum of the buyers' virtual values is greater than k.

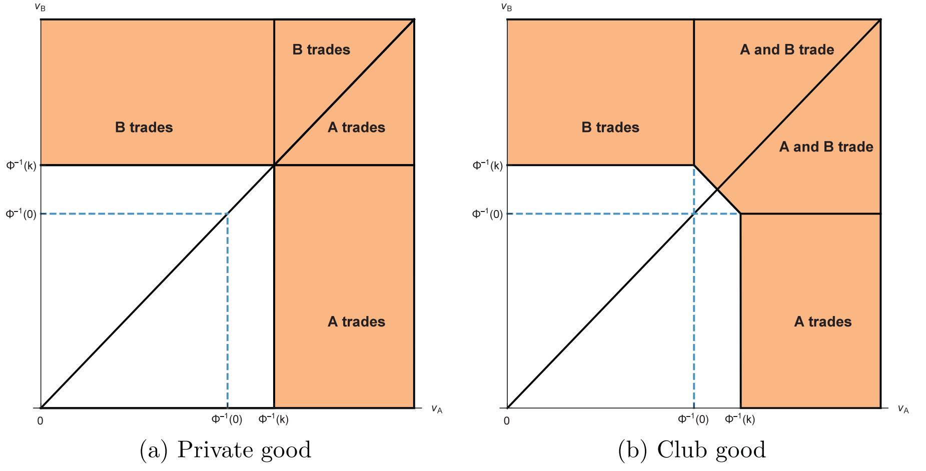


Fig. 1. Illustration of revenue maximizing allocation ($\alpha = 1$) for the one-sided setup with two buyers, labeled A and B, drawing values from the uniform distribution and $k = 0.25$.

As Fig. 1 suggests, whether there is more revenue in the optimal private good mechanism relative to the optimal club good mechanism may depend on the underlying distributions. Thus, when a revenue-maximizing designer has a choice as to whether a good should be allocated as a private good or as a club good, it may seem a priori unclear which option will maximize the designer's objective. However, as we now show, the club good dominates.

To define payments that can be used to implement the optimal allocation rule described above, we focus on the dominant strategy incentive compatible, ex post individually rational implementation. Our focus on dominant strategy mechanisms is motivated by a desire for mechanisms that are more robust in ways relevant for practical implementation. For example, concerns regarding the demanding nature of the assumption of common priors for practical purposes are raised by, among others, Ledyard (1986), Hagerty and Rogerson (1987) and Wilson (1987), the literature on robust mechanism design, e.g., Bergemann and Morris (2005), and the literature on prior-free mechanism design, e.g., Loertscher and Marx (2016). In addition, equilibrium strategies that can be explained to bidders in terms of dominance may be more transparent to bidders. As described in Kagel (1995), even dominant strategies can be difficult for novice bidders to comprehend, with even greater difficulty for Bayesian equilibrium strategies.

As shown by Mookherjee and Reichelstein (1992), the existence of a dominant strategy implementation of the allocation rule in (7) is guaranteed by the monotonicity of the allocation rule. Moreover, in our environment, given the assumption of regularity, an allocation is implementable in a Bayesian equilibrium if and only if it is impementable in a dominant strategy equilibrium (see Mookherjee and Reichelstein, 1992). Let $m_i(\mathbf{v},k)$ be the price that i has to pay. By ex post individual rationality, of course, $m_i(\boldsymbol{v}, k) = 0$

for all $i \notin \mathbb{I}(\boldsymbol{v})$. For any $l \in \mathbb{I}(\boldsymbol{v})$, we have

$$m_l(\boldsymbol{v}, k) = \begin{cases} \Phi_l^{-1}(0), & \text{if } \sum_{h \in \mathbb{I}_{-l}(\boldsymbol{v})} \Phi_h(v_h) - k > 0 \\ \Phi_l^{-1}\Big(k - \sum_{h \in \mathbb{I}_{-l}(\boldsymbol{v})} \Phi_h(v_h)\Big), & \text{if } l \text{ is pivotal ((9)holds)} \\ 0, & \text{otherwise.} \end{cases} \tag{10}$$

To see that ex post individual rationality is satisfied, note that buyer i only makes a positive payments when $v_i > \Phi_i^{-1}(0)$ and he trades, in which case he pays either $\Phi_i^{-1}(0)$, which provides positive surplus, or $\Phi_i^{-1}\Big(k - \sum_{h \in \mathbb{I}_{-i}(\boldsymbol{v})} \Phi_h(v_h)\Big)$. The latter payment applies only when i is pivotal, which implies that $\Phi_i(v_i) + \sum_{h \in \mathbb{I}_{-i}(\boldsymbol{v})} \Phi_h(v_h) > k$, and so $\Phi_i^{-1}\Big(k - \sum_{h \in \mathbb{I}_{-i}(\boldsymbol{v})} \Phi_h(v_h)\Big) > v_i$ and buyer i has positive surplus.

To see that dominant strategy incentive compatibility is satisfied, consider buyer i with value v_i. If $v_i \leq \Phi_i^{-1}(0)$, clearly there is no profitable deviation because the minimum payment conditional on trade is $\Phi_i^{-1}(0)$, which is greater than the buyer's value. Suppose $v_i > \Phi_i^{-1}(0)$. A report less than or equal to $\Phi_i^{-1}(0)$ causes buyer i not to trade. If buyer i reports a value greater than $\Phi_i^{-1}(0)$ but less than v_i, to the extent it changes the outcome for buyer i, it is because buyer i was pivotal and the change caused there to be no trade, leaving buyer i worse off. If buyer i reports a value r_i greater than v_i, to the extent it changes the outcome for buyer i, it is because buyer i becomes pivotal and causes there to be trade when there was no trade under truthful reporting, i.e.,

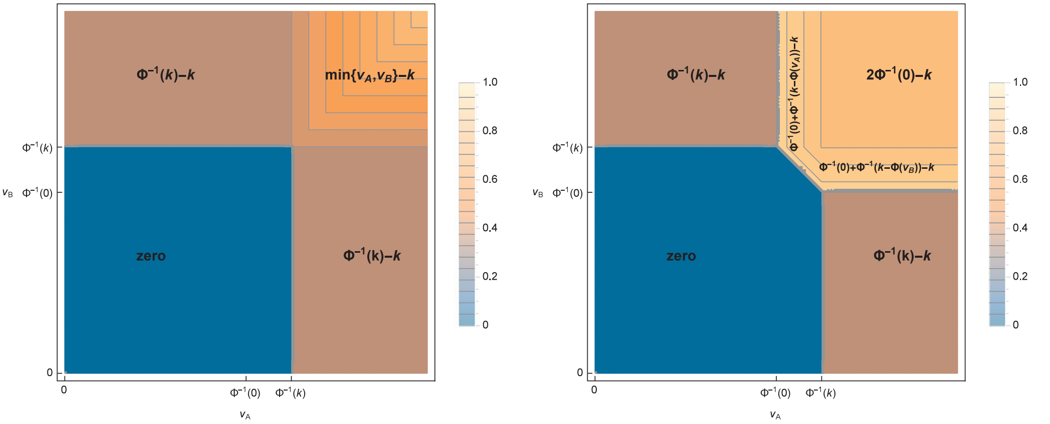
$$\Phi_i(v_i) + \sum_{h \in \mathbb{I}_{-i}(\boldsymbol{v})} \Phi_h(v_h) - k \leq 0 < \Phi_i(r_i) + \sum_{h \in \mathbb{I}_{-i}(\boldsymbol{v})} \Phi_h(v_h) - k,$$

which implies that $v_i \leq \Phi_i^{-1}\Big(k - \sum_{h \in \mathbb{I}_{-i}(\boldsymbol{v})} \Phi_h(v_h)\Big)$. Because the amount buyer i pays under the deviation is equal to the right side in this inequality, the deviation is not profitable.

We illustrate the payoffs to a revenue maximizing ($\alpha = 1$) designer associated with this payment rule in Fig. 2. The two panels contrast the corresponding mechanisms for the case of private and club goods. The figure shows that the revenue to the designer is weakly greater in the case of a club good. Revenue is the same in the region where only zero or one buyer trades in the club good case, but it otherwise is higher for the club good case.

Because providing the good as a private good – or, for that matter, as a public good – is an option that is available in the club good setting but revealed to be dominated by providing it as a club good, providing the good as a club good is better for the designer than providing it as either a private good or a public good. In this sense, the club good *dominates* both private and public goods.

As mentioned in the introduction, this is relevant when the designer can choose the technology with which the buyers use the asset he offers for sale. It also matters for artists who can choose between making a painting, which is a private good, or a photograph, which is a club good. It is well known that paintings command higher prices than



(a) Private good　　(b) Club good

Fig. 2. Illustration of payoffs for the revenue-maximizing designer ($\alpha = 1$) in the optimal dominant strategy incentive compatible, ex post individually rational mechanism for the one-sided setup with two buyers, labeled A and B, drawing values from the uniform distribution and $k = 0.25$.

photographs because photographs are replicable. Nevertheless, if the costs of production and the per buyer demand for the two works of art are the same, this analysis shows that the profit-maximizing artist will produce and sell photographs.

In contrast, the revenue comparison between private goods and public goods without exclusion is, as far as we are aware, an open question. The answer to this question is not trivial because the set of admissible allocations under a private good neither nests nor is nested by those for a public good without exclusion.

We summarize with the following proposition.

Proposition 1. *For the one-sided setup, in the Bayesian optimal club good mechanism, subject to incentive compatibility and interim individual rationality, the allocation rule is given by (7). In the dominant strategy incentive compatible, ex post individually rational implementation, payments are given by (10). Moreover, the club good dominates both private and public goods.*

As noted by Bulow and Roberts (1989), virtual values can be interpreted as marginal revenue, treating the (change in the) probability of trade as the (marginal change in) quantity. Following this insight, we can illustrate the basic tensions present in the optimal club good allocation problem by examining a standard monopoly pricing problem with decreasing marginal costs. Consider linear demand $P = a - Q$, with marginal revenue $P = a - 2Q$, and a cost function with marginal cost $k = 0.5$ up to a quantity of 0.25 and no incremental cost thereafter. Then the monopolist chooses the quantity that equates marginal revenue with marginal cost. As shown in first pair of graphs in Fig. 3, when a is sufficiently large, this occurs with $Q = 0.75$, where marginal revenue is equal to zero. As a decreases, we reach a range where there are two local maxima for profit, one

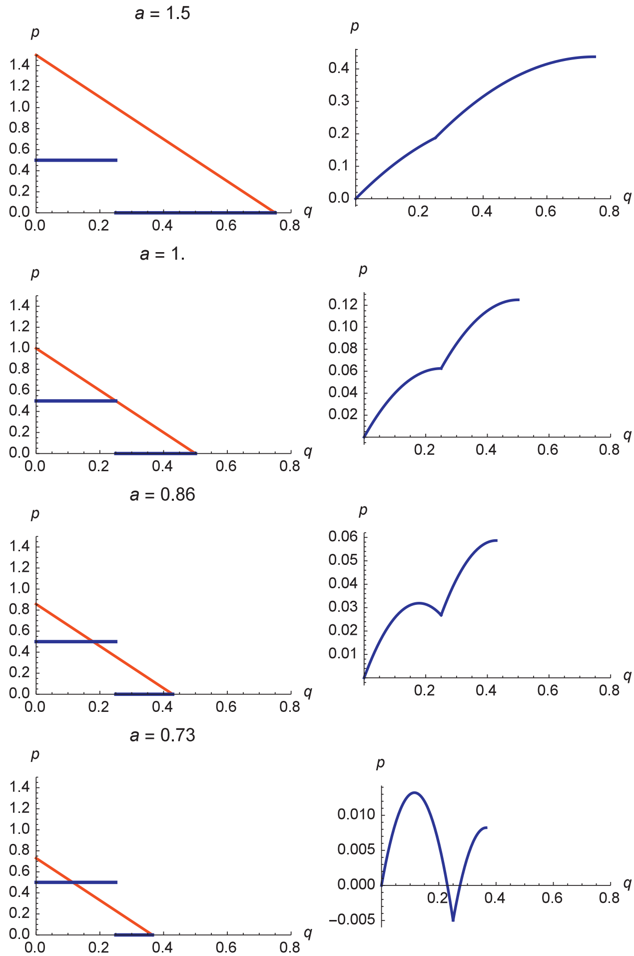


Fig. 3. Profit maximization with demand $P = a - Q$.

corresponding to marginal revenue equal to k and one corresponding to marginal revenue equal to zero. For a sufficiently small, such as in the final pair of graphs in Fig. 3, the global maximum for profit occurs where marginal revenue is equal to k.

In this monopoly pricing example, the monopolist either prices such that marginal revenue is equal to zero or such that marginal revenue is equal to k. As demand shifts to the left, the monopolist switches from pricing based on the marginal cost of zero to pricing based on a marginal cost of k. We observe a similar shift from allocating based on a marginal cost of zero to allocating based on a marginal cost of k in the Bayesian optimal mechanism for a club good. However, a difference is that in the Bayesian optimal mechanism for club goods, there is price discrimination, with different buyers potentially paying different amounts for the good.

Types	Virtual Values	Quantities	Payments	Profit	Types of Traders
0.764	0.528	1	$\frac{1}{2}$	1.4	3 nonpivotal
0.597	0.194	1	$\frac{1}{2}$		
0.524	0.048	1	$\frac{1}{2}$		
0.247	−0.506	0	0		
0.853	0.706	1	0.55	0.45	1 pivotal
0.375	−0.25	0	0		
0.274	−0.452	0	0		
0.225	−0.55	0	0		
0.856	0.712	1	0.534	0.934	1 pivotal, 1 nonpivotal
0.516	0.032	1	$\frac{1}{2}$		
0.363	−0.274	0	0		
0.289	−0.422	0	0		
0.53	0.06	1	0.523	0.943	2 pivotal
0.527	0.054	1	0.52		
0.426	−0.148	0	0		
0.406	−0.188	0	0		

Fig. 4. Illustration of revenue-maximizing outcomes ($\alpha = 1$) with $n = 4$, types drawn from the Uniform distribution on [0, 1], and $k = 0.1$.

In the Bayesian optimal mechanism for a club good, when many buyers have types that satisfy the criteria that $v_i \geq \Phi_i^{-1}(0)$, then each individual buyer is less likely to be pivotal in determining whether it is optimal for the designer to allocate the good. In the case in which no individual buyer is pivotal, each buyer i pays $\Phi_i^{-1}(0)$. When one or more buyers are pivotal (if buyers draw their values from the same distribution it will always be the buyers with the highest values that are pivotal), a pivotal buyer i pays more than $\Phi_i^{-1}(0)$ and a nonpivotal buyer i pays $\Phi_i^{-1}(0)$. If there is only one buyer i with $v_i \geq \Phi_i^{-1}(0)$ and $\Phi_i(v_i) \geq k$, then buyer i pays $\Phi_i^{-1}(k) \in (k, v_i)$.

When buyers are symmetric, the comparative statics for the monopoly pricing problem and the Bayesian optimal club good mechanism correspond. In the monopoly problem, for a sufficiently large, the optimal price corresponds to a marginal revenue of zero. In the optimal club good mechanism, for n sufficiently large, the optimal mechanism converges to a posted price mechanism with price equal to $\Phi^{-1}(0)$.

For the case of a revenue maximizing designer with $n = 4$, types drawn from the Uniform distribution on $[0, 1]$, and $k = 0.1$, the variety of possible outcomes is illustrated in Fig. 4.

Although Fig. 4 shows positive payoffs for the revenue-maximizing designer, the Bayesian optimal club good mechanism is not necessarily deficit free. To see this, consider a simple example with three buyers drawing from the Uniform distribution on [0, 1] and $k = 1.6$. If $v_1 = v_2 = v_3 = 1$, then no individual buyer is pivotal, so each pays only $\Phi^{-1}(0) = 1/2$, which implies that the designer does not cover her costs. As another example, suppose two buyers draw from distribution $F(x) = \sqrt{x}$ on [0, 1] and $k = 0.95$ (the virtual value does not require ironing for $x > \frac{4}{9}$). If $v_1 = v_2 = 0.98$, then each buyer is pivotal. Each pays $\Phi^{-1}(1 - \Phi(0.98)) = 0.47$, which implies that the designer does not cover her costs.

In a simulation of 1000 Bayesian optimal auctions for a club good with 5 buyers in each auction, values drawn from the Uniform distribution on [0, 1], and $k = 2$, we found that 7 out of the 1000 auctions resulted in negative profit to the designer, 872 auctions resulted in zero profit to the designer, and the remaining 121 resulted in positive profit to the designer.

One might think that for some environments the designer might prefer to commit to selling the good as a private good because doing so would eliminate the possibility of running a deficit. In fact, for some type realizations, revenue might be substantially larger if the designer committed to allocating the good to only one agent. For example, if $n = 2$ and types are, with high probability, either low or high, and if $\Phi^{-1}(k)$ is more than twice as large as $\Phi^{-1}(0)$, then revenue when both agents are of the high type, with neither being pivotal, is $\Phi^{-1}(k)$ with a private good and only $2\Phi^{-1}(0)$ with a club good. However, the expected payoff for the designer is lower with a private good because of the additional revenue in the cases when the good is allocated with only one agent being pivotal.

More formally, assume $n \geq 2$ with types drawn independently from F and $k = 0$. Expected revenue is $E_{\mathbf{v}}[\sum_{i=1}^{n} \Phi(\mathbf{v}_i) q_i(\mathbf{v},k)]$. This is maximized by setting $q_i = 1$ if $v_i \geq \Phi^{-1}(0)$ and $q_i = 0$ otherwise, which is the case with a club good. The restriction to a private good is a restriction that $\sum_{i=1}^{n} q_i(\mathbf{v},k) \leq 1$, which of course can only reduce the expected payoff.

3.2. Two-sided setup

The intermediary-optimal mechanism in the two-sided setup has the allocation rule (7) with k replaced by $\Gamma(k)$. Thus, under condition (2), it is sometimes but not always optimal for the good to be produced.

As in the one-sided setup, standard arguments imply that in any incentive compatible interim individually rational mechanism the designer's expected profit is

$$\Pi = E_{\mathbf{v},k}\left[\sum_{i=1}^{n} \Phi_i(v_i) q_i(\boldsymbol{v},k) - \Gamma(k) \min\left\{1, \sum_{i=1}^{n} q_i(\boldsymbol{v},k)\right\}\right],$$

minus a constant, which can be set equal to 0 by making the individual rationality constraints bind for the worst-off types. In an incentive compatible, interim individually rational mechanism that maximizes the designer's profit Π, it must be that $q_j(\boldsymbol{v},k) = 0$ for all j with $v_j \leq \Phi_j^{-1}(0)$. In addition, we require that there be no trade by any agent ($q_j(\boldsymbol{v},k) = 0$ for all j) if when one considers the set $\mathbb{I}(\boldsymbol{v})$, $\sum_{h \in \mathbb{I}(\boldsymbol{v})} \Phi_h(v_h) \geq \Gamma(k)$. Thus, in the two-sided setup,

$$q_i(\mathbf{v},k) \equiv \begin{cases} 1, & \text{if } v_i \in \mathbb{I}(\boldsymbol{v}) \text{ and } \sum_{h\in\mathbb{I}(\boldsymbol{v})} \Phi_h(v_h) > \Gamma(k) \\ 0, & \text{otherwise.} \end{cases} \tag{11}$$

As in the one-sided case, we have found the allocation rule that maximizes Π pointwise. Proceeding as in the one-sided case, say that agent h is pivotal if

$$\sum_{h\in\mathbb{I}_{-l}(\boldsymbol{v})} \Phi_h(v_h) - \Gamma(k) \le 0 < \sum_{h\in\mathbb{I}(\boldsymbol{v})} \Phi_h(v_h) - \Gamma(k). \tag{12}$$

one can show that in the dominant strategy, ex post individually rational implementation, for any $l \in \mathbb{I}(\boldsymbol{v})$, we have

$$m_l(\boldsymbol{v},k) = \begin{cases} \Phi_l^{-1}(0), & \text{if } \sum_{h\in\mathbb{I}_{-l}(\boldsymbol{v})} \Phi_h(v_h) - \Gamma(k) > 0 \\ \Phi_l^{-1}\Big(\Gamma(k) - \sum_{h\in\mathbb{I}_{-l}(\boldsymbol{v})} \Phi_h(v_h)\Big), & \text{if } l \text{ is pivotal ((12) holds)} \\ 0, & \text{otherwise} \end{cases} \tag{13}$$

and the payment to the seller when reports are $(\mathbf{v},k)$ is zero if there are no trades and otherwise is the worst type for the seller such that the seller continues to produce, i.e.,

$$m^S(\boldsymbol{v},k) = \begin{cases} \overline{k}, & \text{if } \Gamma(\overline{k}) \le \sum_{h\in\mathbb{I}(\boldsymbol{v})} \Phi_h(v_h) \\ \Gamma^{-1}\Big(\sum_{h\in\mathbb{I}(\boldsymbol{v})} \Phi_h(v_h)\Big), & \text{if } \Gamma(k) < \sum_{h\in\mathbb{I}(\boldsymbol{v})} \Phi_h(v_h) \le \Gamma(\overline{k}) \\ 0, & \text{otherwise.} \end{cases} \tag{14}$$

The same arguments as in the one-sided setup further imply that providing the good as a club good dominates its provision as either a private good or a public good. We summarize these results with the following proposition.

Proposition 2. *For the two-sided setup, in the Bayesian optimal club good mechanism, subject to incentive compatibility and interim individual rationality, the allocation rule is given by (11). In the dominant strategy, ex post individually rational implementation, payments are given by (13). Moreover, the club good dominates both private and public goods.*

As an illustration, assuming that the buyers draw their values from the distribution $F(v) = 1 - (1-v)^\beta$ with $v \in [0,1]$ and $\beta > 0$, we can obtain closed-form solutions because the virtual value function is linear. For example, for $\alpha = 1$, we have $\Phi(v) = ((\beta+1)v - 1)/\beta$, which implies that the inverse virtual value function is also is linear:

$\Phi^{-1}(y) = (\beta y + 1)/(\beta + 1)$. Consequently, the price buyer l faces at $(\boldsymbol{v}, k)$ is

$$m_l(\boldsymbol{v}, k) = \begin{cases} \dfrac{1}{\beta+1}, & \text{if } \sum_{h \in \mathbb{I}_{-l}(\boldsymbol{v})} \Phi_h(v_h) - \Gamma(k) > 0 \\ \dfrac{1 + \beta\Gamma(k) + I_{-l}}{\beta+1} - \sum_{h \in \mathbb{I}_{-l}(\boldsymbol{v})} v_h, & \text{if } l \text{ is pivotal ((12) holds)} \\ 0, & \text{otherwise,} \end{cases} \tag{15}$$

while the seller's price is as in (14), with $\sum_{h \in \mathbb{I}(\boldsymbol{v})} \Phi_h(v_h) = \frac{1}{\beta}\Big((\beta+1)\sum_{i \in \mathbb{I}(\boldsymbol{v})} v_i - I\Big)$.

4. Implementation

4.1. One sided: club good clock auction

In this section we define a club good clock auction that implements the Bayesian optimal mechanism in the one-sided setup. To do so, we adapt the definition of a clock auction in Loertscher and Marx (2016) and Milgrom and Segal (2015) to the club good setting. In a clock auction, active agents have the option to exit as the clock price increases, but agents who exit remain inactive forever after exiting. In contrast to a clock auction for a private good, in the club good setting, agents who exit may trade.

We focus on an increasing clock auction, where the price begins low with all buyers active and gradually increases. As we discuss below, this mechanism has an advantage over a decreasing clock in that it has the potential to preserve the privacy of some or all trading agents. In contrast, with a decreasing clock, the mechanism must elicit the types of all buyers with positive virtual types (to identify whether they are in the trading set) whenever there is trade, so a decreasing clock never preserves the privacy of any trading agents. See Milgrom and Segal (2015) and Loertscher and Marx (2016) on the value of privacy preservation for practical implementation.

The club good clock auction must elicit two types of information. It must elicit whether trade is optimal for the designer, and, if so, it must identify whether agents are pivotal or not. The auction proceeds by first increasing the clock until it is determined that trade is optimal for the designer. This is stage 1. Then the auction proceeds to stage 2, which continues until either there are no longer any active bidders or it is determined that the second-highest valuing buyer is not pivotal. If the second-highest valuing buyer is pivotal, then the value of the highest is required to determine prices, and so the auction must continue until all exit. It is always the case that buyers with higher values are more “at risk” for being pivotal, so the determination of pivotal buyers can proceed based on a series of thresholds.

In the interest of reducing notation, we assume that buyers are symmetric in that they all draw their types from the same distribution F. Thus, we drop the agent subscript on the distribution and virtual value function.

At period $t \in \{0, 1, ...\}$ of a *club good clock auction*, the state is $\boldsymbol{\omega}_t = (\text{stage}, \mathbb{N}^A, \mathbf{x}, \mathbf{z}, p)$. The first component of the state, stage $\in$ {stage 1, stage 2, trade, no trade}, specifies whether the auction has not yet identified the trading set (stage 1), has identified the trading set but not yet all the information required to determine prices (stage 2), has ended with trade (trade), or has ended without trade (no trade). The remaining components of the state are: the set of active buyers $\mathbb{N}^A \subseteq \mathbb{N}$ with cardinality n^A, the vector $\mathbf{x}$ of exit prices greater than $\Phi^{-1}(0)$ for the nonactive buyers who have exited at such prices, the vector $\mathbf{z}$ of indices of the nonactive buyers corresponding to the exit prices in $\mathbf{x}$, and the clock price $p \in \mathcal{R}$. Let Ω be the set of all possible states.

A club good clock auction starts in state $\boldsymbol{\omega}_0 \equiv (\text{ stage } 1, \mathbb{N}, \emptyset, \emptyset, \underline{p})$ with $\underline{p} \leq \underline{v}$.

The transition from state $\boldsymbol{\omega}_t$ to $\boldsymbol{\omega}_{t+1}$ relies on the trade determination function $\tau : \Omega \to \mathcal{R}$ and the pivotal agent determination function $\rho : \Omega \to \mathcal{R}$, where we can write the functions as depending only on the set of active agents and vector of exit prices greater than $\Phi^{-1}(0)$, rather than the entire state, as follows:

$$\tau(\mathbb{N}^A, \mathbf{x}) \equiv \Phi^{-1}\left(\frac{k}{n^A} - \frac{1}{n^A}\sum_{h=1}^{\dim \mathbf{x}} \Phi(x_h)\right)$$

and for $n^A \geq 2$,

$$\rho(\mathbb{N}^A, \mathbf{x}) \equiv \Phi^{-1}\left(\frac{k}{n^A - 1} - \frac{1}{n^A - 1}\sum_{h=1}^{\dim \mathbf{x}} \Phi(x_h)\right).$$

To understand the usefulness of these functions, consider a point in a clock auction with a clock price greater than $\Phi^{-1}(0)$, where n^A buyers remain active, $\dim \mathbf{x}$ buyers have exited at prices $\mathbf{x}$, all greater than $\Phi^{-1}(0)$, and the remaining buyers have exited at prices less than or equal to $\Phi^{-1}(0)$. If the n^A active buyers continue to be active at a clock price of $\tau(\mathbb{N}^A, \mathbf{x})$, then we can conclude that the condition for trade, $\sum_{h \in \mathbb{I}(\mathbf{v})} \Phi(v_h) \geq k$, is satisfied. To see this, note that in this case, the set $\mathbb{I}(\mathbf{v})$ consists of the $\dim \mathbf{x}$ buyers with types $\mathbf{x}$ as well as the n^A active buyers whose types are at least $\tau(\mathbb{N}^A, \mathbf{x})$, which implies that

$$\sum_{h \in \mathbb{I}(\mathbf{v})} \Phi(v_h) \geq \sum_{h=1}^{\dim \mathbf{x}} \Phi(x_h) + n^A \Phi(\tau(\mathbb{N}^A, \mathbf{x})) = \sum_{h=1}^{\dim \mathbf{x}} \Phi(x_h) + n^A\left(\frac{k}{n^A} - \frac{1}{n^A}\sum_{h=1}^{\dim \mathbf{x}} \Phi(x_h)\right) = k,$$

which is the condition for trade.

Furthermore, if the condition for trade is satisfied and $n^A \geq 2$ active buyers continue to be active at a clock price of $\rho(\mathbb{N}^A, \mathbf{x})$, then we can conclude that no agent is pivotal. To see this, note that for any active bidder l, if $n^A \geq 2$ so that there is at least one other active bidder, then each active bidder's value is at least $\Phi(\rho(\mathbb{N}^A, \mathbf{x}))$, and so

$$\sum_{h\in\mathbb{I}(\mathbf{v})\setminus l}\Phi(v_h)\geq\sum_{h=1}^{\dim\mathbf{x}}\Phi(x_h)+(n^A-1)\Phi(\rho(\mathbb{N}^A,\mathbf{x}))$$
$$=\sum_{h=1}^{\dim\mathbf{x}}\Phi(x_h)+(n^A-1)\left(\frac{k}{n^A-1}-\frac{1}{n^A-1}\sum_{h=1}^{\dim\mathbf{x}}\Phi(x_h)\right)=k,$$

so no such bidder is pivotal. Because the $\dim\mathbf{x}$ bidders that exited at prices greater than $\Phi^{-1}(0)$ have lower values than the active bidders, none of those bidders is pivotal either.

If the condition for trade is satisfied and $n^A=1$, then the final active bidder is pivotal if and only if $\sum_{i=1}^{\dim\mathbf{x}}\Phi(x_i)\leq k$. It is only necessary to have this final bidder reveal his type if there are other pivotal bidders. It is sufficient to check whether the exited bidder with the highest value would be pivotal if the active bidder's value were equal to the current clock price. The last bidder to exit may be pivotal if $\sum_{i=1}^{\dim\mathbf{x}-1}\Phi(x_i)+p\leq k$. If this is the case, we must continue to increase the clock price until either the final active bidder exits or the last bidder to exit is revealed not to be pivotal. This is necessary in the context of dominant strategy implementation because the designer needs to know the values for all other trading bidders in order to determine the price for a pivotal bidder.

A club good clock auction continues until a state is reached that has a first component equal to either "no trade" or "trade." In the case of "no trade," the auction ends with no trade. In the case of "trade," the auction ends with trade by the remaining active bidders as well as by those identified in the vector $\mathbf{z}$ as exiting at prices above $\Phi^{-1}(0)$. The prices are given by the pricing rule described below. The auction stays in stage 1 until either all buyers exit, in which case there is no trade, or the clock price reaches the level specified by the trade determination function $\tau(\mathbb{N}^A,\mathbf{x})$ and moves to stage 2, in which case there will be trade. The auction stays in stage 2 until either all buyers exit, only 2 active buyers remain and the clock reaches the level specified by the pivotal agent determination function $\rho(\mathbb{N}^A,\mathbf{x})$, or only 1 active buyer remains but it is possible that the second-highest valuing bidder might be pivotal.

- **Stage 1**: For $t\in\{0,1,...\}$, if $\boldsymbol{\omega}_t=(\text{stage }1,\mathbb{N}^A,\mathbf{x},\mathbf{z},p)$, then $\boldsymbol{\omega}_{t+1}$ is determined as follows:

  If $n^A=0$, $\boldsymbol{\omega}_{t+1}=(\text{no trade},\mathbb{N}^A,\mathbf{x},\mathbf{z},p)$.

  If $n^A>0$ and $p<\Phi^{-1}(0)$, then increase the clock from p until either a buyer i exits at clock price $\hat{p}\leq\Phi^{-1}(0)$ or the clock reaches $\Phi^{-1}(0)$ with no exit. If there is an exit, define $\boldsymbol{\omega}_{t+1}=(\text{stage }1,\mathbb{N}^A\setminus i,\mathbf{x},\mathbf{z},\hat{p})$, and otherwise $\boldsymbol{\omega}_{t+1}=(\text{stage }1,\mathbb{N}^A,\mathbf{x},\mathbf{z},\Phi^{-1}(0))$.

  If $n^A>0$ and $p\geq\Phi^{-1}(0)$, then increase the clock from p until either a buyer i exits at clock price $\hat{p}\leq\tau(\mathbb{N}^A,\mathbf{x})$ or the clock price reaches $\hat{p}\geq\tau(\mathbb{N}^A,\mathbf{x})$ with no exit.[7] If there is an exit, define $\boldsymbol{\omega}_{t+1}=(\text{stage }1,\mathbb{N}^A\setminus i,(\mathbf{x},\hat{p}),(\mathbf{z},i),\hat{p})$, and otherwise $\boldsymbol{\omega}_{t+1}=(\text{stage }2,\mathbb{N}^A,\mathbf{x},\mathbf{z},\hat{p})$.

[7] If the clock price is initially greater than or equal to the threshold, then the clock stops immediately.

- **Stage 2:** For $t \in \{1, 2, ...\}$, if $\boldsymbol{\omega}_t = (\text{stage } 2, \mathbb{N}^A, \mathbf{x}, \mathbf{z}, p)$, then $\boldsymbol{\omega}_{t+1}$ is determined as follows:

  If $n^A = 0$, $\boldsymbol{\omega}_{t+1} = (\text{trade}, \mathbb{N}^A, \mathbf{x}, \mathbf{z}, p)$.

  If $n^A \geq 2$, then increase the clock from p until either a buyer i exits at clock price $\hat{p} \leq \rho(\mathbb{N}^A, \mathbf{x})$ or the clock price reaches $\hat{p} \geq \rho(\mathbb{N}^A, \mathbf{x})$ with no exit. If there is an exit, define $\boldsymbol{\omega}_{t+1} = (\text{ stage } 2, \mathbb{N}^A \backslash i, (\mathbf{x}, \hat{p}), (\mathbf{z}, i), \hat{p})$, and otherwise $\boldsymbol{\omega}_{t+1} = (\text{trade}, \mathbb{N}^A, \mathbf{x}, \mathbf{z}, \hat{p})$.

  If $n^A = 1$:

  If $k < \sum_{i=1}^{\dim \mathbf{x}-1} \Phi(x_i) + \Phi(p)$, then the second-highest valuing bidder is revealed not to be pivotal, so the auction can end: $\boldsymbol{\omega}_{t+1} = (\text{trade}, \mathbb{N}^A, \mathbf{x}, \mathbf{z}, p)$.

  Otherwise, increase the clock from p until either the remaining buyer i exits at clock price $\hat{p} \leq \Phi^{-1}(k - \sum_{i=1}^{\dim \mathbf{x}-1} \Phi(x_i))$ or the clock price reaches $\hat{p} \geq \Phi^{-1}(k - \sum_{i=1}^{\dim \mathbf{x}-1} \Phi(x_i))$ with no exit. If there is an exit, define $\boldsymbol{\omega}_{t+1} = (\text{trade}, \mathbb{N}^A \backslash i, (\mathbf{x}, \hat{p}), (\mathbf{z}, i), \hat{p})$, and otherwise $\boldsymbol{\omega}_{t+1} = (\text{trade}, \mathbb{N}^A, \mathbf{x}, \mathbf{z}, \hat{p})$.

- **Pricing rule**: For $t \in \{1, 2, ...\}$, if $\boldsymbol{\omega}_t = (\text{trade}, \mathbb{N}^A, \mathbf{x}, \mathbf{z}, p)$, then buyers in $\mathbb{T} \equiv \mathbb{N}^A \cup \{z_1\} \cup ... \cup \{z_{\dim \mathbf{z}}\}$ trade, with buyer $l \in \mathbb{T}$ paying $m_l(\hat{\mathbf{v}}, k)$, where $\hat{\mathbf{v}}$ is defined by, for $l \in \mathbb{N}$,

$$\hat{v}_l \equiv \begin{cases} \underline{v}, & \text{if } l \notin \mathbb{T} \\ p, & \text{if } l \in \mathbb{N}^A \\ x_i, & \text{otherwise, where } l = z_i. \end{cases}$$

Thus, conditional on using an increasing clock auction to implement the Bayesian optimal club good mechanism in dominant strategies, this club good clock auction elicits the minimum information required. That information includes sufficient detail about types to discern whether $\sum_{h \in \mathbb{I}(\boldsymbol{v})} \Phi_h(v_h)$ is greater than k or not, to determine whether trade should occur, and sufficient detail about the types of trading agents to determine which, if any, agents are pivotal and the corresponding price for any pivotal agents. For example, if the types are such that no agent is pivotal, then the clock price stops at the point when this is apparent and does not require full revelation of agents' types, preserving the privacy of high-valuing nonpivotal agents.

4.1.1. Incentives in the club good clock auction

Bidders in the club good clock auction have an incentive to exit at their values. Because prices for trading agents are at least $\Phi^{-1}(0)$ and because bidders who exit a prices less than or equal to $\Phi^{-1}(0)$ do not trade, a bidder with a value less than or equal to $\Phi^{-1}(0)$ has an incentive to exit at his value, and bidders with a value greater than $\Phi^{-1}(0)$ have an incentive to stay active at least until the price exceeds $\Phi^{-1}(0)$.

Case 1: $v_i > \Phi^{-1}(0)$, $\sum_{h \in \mathbb{I}(\boldsymbol{v})} \Phi(v_h) > k$, i is not pivotal: With truthful bidding, there is trade and bidder i pays $\Phi^{-1}(0)$. If bidder i exits at a price greater than $\Phi^{-1}(0)$ but less than v_i or at a price greater than v_i, then there is no change. Because there is trade and i is not pivotal, a different exit point greater than $\Phi^{-1}(0)$ does not affect whether there is trade or whether i is pivotal.

Case 2: $v_i > \Phi^{-1}(0)$, $\sum_{h\in\mathbb{I}(\boldsymbol{v})}\Phi(v_h) > k$, i is pivotal: With truthful bidding, there is trade and bidder i pays $\Phi^{-1}\left(k - \sum_{h\in\mathbb{I}_{-i}(\boldsymbol{v})}\Phi(v_h)\right)$. If bidder i exits at a price greater than $\Phi^{-1}(0)$ but less than v_i, then either there is no change or there is no longer trade, in which case bidder i is worse off. If bidder i exits at a price greater than v_i, then there is still trade and bidder i is still pivotal, and so the amount bidder i pays is unchanged.

Case 3: $v_i > \Phi^{-1}(0)$, $\sum_{h\in\mathbb{I}(\boldsymbol{v})}\Phi(v_h) \le k$: With truthful bidding, there is no trade. If bidder i exits at a price less than v_i, then there continues to be no trade. If bidder i exits at a price greater than v_i, either there is no change or bidder i becomes pivotal and there is trade. In that case, bidder i pays $\Phi^{-1}\left(k - \sum_{h\in\mathbb{I}_{-i}(\boldsymbol{v})}\Phi(v_h)\right)$. Because for this case we assume, $\Phi(v_i) \le k - \sum_{h\in\mathbb{I}_{-i}(\boldsymbol{v})}\Phi(v_h)$, it follows that

$$\Phi^{-1}\left(k - \sum_{h\in\mathbb{I}_{-i}(\boldsymbol{v})}\Phi(v_h)\right) \ge \Phi^{-1}(\Phi(v_i)) = v_i,$$

and so deviator pays an amount at least v_i, implying that the deviation is not profitable.

Thus, it is a dominant strategy for each bidder to exit the auction when the clock price reaches his value. We summarize with the following proposition.

Proposition 3. *The club good clock auction implements the Bayesian optimal club good allocation in dominant strategies.*

As defined by Li (2015), a strategy s_i is obviously dominant if, for any deviating strategy s_0, starting from any earliest information set where s_0 and s_i disagree, the best possible outcome from s_0 is no better than the worst possible outcome from s_i. In the club good clock auction, the strategy for a buyer of exiting at its value is not an obviously dominant strategy.[8]

4.1.2. Privacy preservation in the club good clock auction

In the club good clock auction, we can show that the clock price never exceeds $\Phi^{-1}(k)$, so buyers with values greater than $\Phi^{-1}(k)$ are never asked to reveal any more than the fact that their value exceeds that threshold. The intuition for this result is that if the clock price reaches $\Phi^{-1}(k)$ with two or more active buyers, then it is revealed that the condition for trade is satisfied and that no buyer is pivotal, and so the auction ends. If

[8] Consider a buyer with value $v_i > \Phi^{-1}(0)$ who considers deviating by exiting at price $\hat{v} < v_i$. Suppose the clock price reaches $\hat{v}$. Then the buyers's payoff under the deviation will be $q_i(\hat{v}, \mathbf{v}_{-i})(v_i - p_i(\hat{v}, \mathbf{v}_{-i}))$. In the best possible outcome from the deviation, $q_i(\hat{v}, \mathbf{v}_{-i}) = 1$ and i is not pivotal and so pays $\Phi^{-1}(0)$, for surplus of $v_i - \Phi^{-1}(0)$. At the information set in which the clock price is $\hat{v}$, it may be unclear as to whether there will be trade. In particular, it may be that there is no trade even when buyer i exits at v_i. Thus, the best possible outcome from exit at $\hat{v}$ exceeds the worst possible outcome from exit at v_i from the perspective of that information set.

the clock price reaches $\Phi^{-1}(k)$ with only one active buyer, then it is again revealed that the condition for trade is satisfied and that buyers other than the one active buyer are not pivotal, so their purchase price is $\Phi^{-1}(0)$. The purchase price for the one active buyer is determined by the exit prices of the other buyers, which have already been revealed. Thus, if the clock price reaches $\Phi^{-1}(k)$, no further information is required to determine whether the condition for trade is satisfied or the purchase prices of any of the buyers.

Formally, if $n^A \geq 1$, then the trade determination price is

$$\tau(\mathbb{N}^A, \mathbf{x}) = \Phi^{-1}\left(\frac{k}{n^A} - \frac{1}{n^A}\sum_{h=1}^{\dim \mathbf{x}} \Phi(x_h)\right) \leq \Phi^{-1}\left(\frac{k}{n^A}\right) \leq \Phi^{-1}(k),$$

so the trade determination price never exceeds $\Phi^{-1}(k)$. Thus, the clock price never exceeds $\Phi^{-1}(k)$ during stage 1 of the auction. In stage 2, as long as two or more buyers are active, the clock price never exceeds $\rho(\mathbb{N}^A, \mathbf{x})$, which is no greater than $\Phi^{-1}(k)$:

$$\rho(\mathbb{N}^A, \mathbf{x}) = \Phi^{-1}\left(\frac{k}{n^A - 1} - \frac{1}{n^A - 1}\sum_{h=1}^{\dim \mathbf{x}} \Phi(x_h)\right) \leq \Phi^{-1}\left(\frac{k}{n^A - 1}\right) \leq \Phi^{-1}(k).$$

Finally, if $n^A = 1$ in stage 2 of the auction, then the clock price does not exceed $\Phi^{-1}(k - \sum_{i=1}^{\dim \mathbf{x}-1} \Phi(x_i)) \leq \Phi^{-1}(k)$. Thus, the clock price never exceeds $\Phi^{-1}(k)$.

We summarize in the following proposition.

Proposition 4. *The club good clock auction is privacy preserving for all buyers with values greater than* $\Phi^{-1}(k)$.

4.2. Two sided: revenue sharing contracts

In a variety of real-world situations, intermediaries who enable trade between producers and buyers of club goods may need to contract with the producer before buyers are present. Most naturally, this occurs when production precedes sales and marketing, for example because of the innovative nature of the good, as would typically be the case with works of art. We next show that this variation in the timing of events changes the intermediary's mechanism design problem from a club good problem to a standard procurement problem even though it does not affect the nonrivalrous nature of the good in question. We also show that it gives raise to a contractual arrangement that is widely used in practice, for example, by iTunes for songs and e-books, namely shared revenue between the intermediary and the seller.

To be specific, we suppose that the intermediary and the seller have to contract in stage 1 before any buyer arrives in stage 2. There is no loss in generality by assuming that there is no discounting, as will be shown shortly. Let n be the number of buyers who draw their types independently from the regular distribution F. To conserve on notation, for $x \geq \Gamma(\overline{k})$, define $\Gamma^{-1}(x)$ to be $\overline{k}$.

From Propositions 1 and 2, we know that it is optimal to sell the good as a club good. Denote the maximizer of $x(1 - F(x))$ over x by $\Phi^{-1}(0)$. The maximized expected per buyer revenue is thus $\Phi^{-1}(0)(1 - F(\Phi^{-1}(0)))$. Let R denote expected revenue from stage 2.[9] With this seemingly innocuous change in assumptions, the intermediary's problem becomes a standard procurement problem, with the willingness to pay of the risk-neutral, patient intermediary being given by R. Facing a seller whose cost k is drawn from the regular distribution G with virtual cost $\Gamma(k)$, the intermediary-optimal mechanism consists of the take-it-or-leave-it offer to the seller $p_S \equiv \Gamma^{-1}(R)$, as is well known from standard mechanism design theory.

Under the assumptions that the support of G is $[0, \overline{k}]$ and that $G(k)$ is a generalized Pareto distribution of the form $G(k) = (k/\overline{k})^\sigma$,[10] we have $\Gamma^{-1}(k) = \sigma k/(\sigma + 1)$ and

$$p_S = \frac{\sigma}{\sigma + 1} R.$$

That is, the optimal take-it-or-leave-it offer is a percentage of revenue. Interestingly, rather than setting a take-it-or-leave-it offer, the intermediary can equivalently implement this optimal contract by offering the percentage $\sigma/(\sigma + 1)$ of realized revenue. Moreover, as long as the intermediary and the seller are equally informed about demand and both believe that buyers draw their types independently from F, it is immaterial whether the intermediary or the seller sets the buyers' price $p_B = \Phi^{-1}(0)$. Thus, under the assumption of a generalized Pareto distribution, the predictions of this model are consistent with Apple's controversial e-books contracting model. This consisted of a revenue sharing agreement with publishers that left the authority to set the buyers' prices with the sellers. On the surface, this arrangement may appear to be different from Apple's business model for songs and movies on iTunes, which also involves revenue sharing but leaves the right to set buyers' prices with Apple. However, as observed by Loertscher and Niedermayer (2008) in the context of private goods, the broker-optimal fee is independent of the buyers' distribution F if the seller's distribution is a generalized Pareto distribution. Therefore, it does not even matter whether the intermediary and the seller are equally well informed about demand.

The result that the optimal procurement contract can be implemented with a fee levied on realized revenue generalizes to seller's type distributions of the form $G(k) = \left(\frac{k-\underline{k}}{\overline{k}-\underline{l}}\right)^\sigma$. As shown by Loertscher and Niedermayer (2008), these distributions are equivalent to linear virtual cost functions $\Gamma(k) = \frac{\sigma k+1-\underline{k}}{\sigma}$. This linearity obviously implies linearity of the inverse virtual cost function: $\Gamma^{-1}(y) = \frac{\sigma y+\underline{k}}{\sigma+1}$. Any risk-neutral seller will be indifferent between receiving p_S or $E[\Gamma^{-1}(Y)]$ if and only if Γ^{-1} is linear, where Y is realized revenue, whose distribution is such that $R = E[Y]$. As before, whether the seller or the

[9] All that matters in this setup is expected discounted future revenue, which is why the assumption that there is no discounting is without loss. If the discount factor were some $\delta > 0$ and expected stage 2 revenue were $\hat{R}$, we would simply have $R = \delta\hat{R}$.

[10] See Loertscher and Niedermayer (2015) for a micro-foundation for why generalized Pareto distributions may be a particularly good approximation of the supply side in thin markets.

intermediary sets the reserve price buyers face is immaterial. These results are related to but distinct from the observations of Loertscher and Niedermayer (2008; 2015), who analyze intermediary-optimal auctions for a seller of a private good. They notice that for this family of distributions, the optimal fee function is linear and independent of the buyers' distribution. With a private good, however, the seller has to set the reserve price that the buyers face because the optimal reserve price varies with the seller's private information about his cost.

5. Asymptotics in the two-sided setup

In some two-sided setups, it is reasonable to think of the number of buyers as being very large, for example for iTunes and other digital products. Thus, we consider the asymptotic properties of our mechanisms. For the purposes of this section, for simplicity assume that buyers are symmetric.

Schmitz (1997) provides asymptotic results for the one-sided setup. He allows the number of buyers to go to infinity and assumes, as in Rob (1989) that $k_n = \kappa n$, where $\kappa \in (0,\ 1)$. Schmitz (1997, Proposition 3) shows that in the optimal mechanism the club good is provided if and only if $E[\max\{0, \Phi(v_i)\}] > \kappa$, which implies that when this condition holds, the club good is provided with probability one in the limit. This is in contrast to the results of Rob (1989) and Mailath and Postlewaite (1990) for the case of nonexcludable public goods. They show that in their setup, the good is provided with probability zero in the limit.

We extend this result to the two-sided setup. We assume that the cost when there are n buyers is $k_n \equiv \kappa n k$, where $\kappa \in (0,\ 1)$ and k is drawn from distribution G. Thus, k_n has distribution $H(k_n) \equiv G(k_n/(n\kappa))$ on $[\kappa n \underline{k}, \kappa n \overline{k}]$ with density $h(k_n) = \frac{1}{n\kappa} g(k_n/(n\kappa))$. Then the weighted virtual cost associated with distribution H is $\hat{\Gamma}(k_n) \equiv k_n + \alpha \frac{H(k_n)}{h(k_n)} = k_n + \alpha n \kappa \frac{G(k_n/(n\kappa))}{h(k_n/(n\kappa))}$, and so $\hat{\Gamma}(k_n) = n\kappa\Gamma(k_n/(n\kappa)) = n\kappa\Gamma(k)$.

Under this assumption, in the two-sided case, when there are n buyers, it is optimal for the good to be provided if and only if $\sum_{h\in\mathbb{I}(\boldsymbol{v})} \Phi_h(v_h) > \hat{\Gamma}(k_n)$. Note that

$$\begin{aligned}
& \lim_{n\to\infty} \Pr\left(\sum_{h\in\mathbb{I}(\boldsymbol{v})} \Phi(v_h) > \hat{\Gamma}(k_n)\right) = 1 \\
\Leftrightarrow\quad & \lim_{n\to\infty} \Pr\left(\sum_{h\in\mathbb{I}(\boldsymbol{v})} \Phi(v_h) > n\kappa\Gamma(k)\right) = 1 \\
\Leftrightarrow\quad & \lim_{n\to\infty} \Pr\left(\frac{1}{\Gamma(k)}\frac{1}{n}\sum_{h\in\mathbb{I}(\boldsymbol{v})} \Phi(v_h) > \kappa\right) = 1
\end{aligned}$$

$$
\begin{aligned}
\Leftrightarrow \quad & \lim_{n\to\infty} E\left[\frac{1}{\Gamma(k)}\frac{1}{n}\sum_{h\in\mathbb{I}(\boldsymbol{v})}\Phi(v_h)\right] > \kappa \\
\Leftrightarrow \quad & \lim_{n\to\infty} E\left[\frac{1}{\Gamma(k)}\frac{1}{n}\sum_{i=1}^{n}\max\{0,\Phi(v_i)\}\right] > \kappa \\
\Leftrightarrow \quad & E\left[\frac{1}{\Gamma(k)}\right]\lim_{n\to\infty} E\left[\frac{1}{n}\sum_{i=1}^{n}\max\{0,\Phi(v_i)\}\right] > \kappa \\
\Leftrightarrow \quad & E[\max\{0,\Phi(v)\}] > \kappa E[\Gamma(k)].
\end{aligned}
$$

Thus, we have the following result.

Proposition 5. *In the optimal mechanism in the two-sided setup, the club good is provided with probability one in the limit as the number of buyers goes to infinity and* $k = \kappa n$*, where* $\kappa \in (0, 1)$*, if and only if* $E[\max\{0, \Phi(v)\}] > \kappa E[\Gamma(k)]$.

Thus, under the condition of Proposition 5, the club good intermediary facilitates trade with probability one in the limit. Using $E[\max\{0, \Phi(v)\}] = \Phi^{-1}(0)(1 - F(\Phi^{-1}(0)))$ and $E[\Gamma(k)] = \overline{k}$,[11] we can restate the necessary and sufficient condition of Proposition 5 as

$$\Phi^{-1}(0)(1 - F(\Phi^{-1}(0))) > \kappa\overline{k}.$$

For example, with values drawn from the uniform distribution on [0, 1] and $\overline{k} = 1$, the condition is $\kappa < \frac{1}{4}$.

6. Extensions

6.1. Congestion effects

We consider two types of congestion effects, those with costs borne by the seller and those that diminish the value to buyers.

We allow for the possibility that the seller incurs incremental costs to support additional buyers by assuming that if the designer allocates the good to ℓ buyers, then the seller's cost is $k + c(\ell)$, where k is drawn from distribution G and $c(\cdot)$ is a nonnegative nondecreasing function with $c(0) = 0$. Thus, the cost of the first unit is $k + c(1)$ and the incremental cost associated with selling ℓ units rather than one unit is $c(\ell) - c(1)$. The case of $c(\cdot) \equiv 0$ corresponds to no congestion costs for the seller. Given ℓ, the seller's cost $k + c(\ell)$ has distribution $\hat{G}(x; \ell) \equiv G(x - c(\ell))$, with associated virtual cost

[11] These follow using $E[\max\{0,\Phi(v)\}] = \int_{\underline{v}}^{\overline{v}} \max\{0,\Phi(v)\}f(v)dv = \int_{\Phi^{-1}(0)}^{\overline{v}}\left(v - \frac{1-F(v)}{f(v)}\right)f(v)dv$ and $E[\Gamma(k)] = \int_{\underline{k}}^{\overline{k}}\left(k + \frac{G(k)}{g(k)}\right)g(k)dk$ and using integration by parts.

$\hat{\Gamma}(x;\ell) \equiv x + \frac{\hat{G}(x;\ell)}{\hat{g}(x;\ell)} = x + \frac{G(x-c(\ell))}{g(x-c(\ell))}$, so $\hat{\Gamma}(k+c(\ell);\ell) = \Gamma(k) + c(\ell)$. Thus, we can view the seller as having virtual cost $\Gamma(k) + c(\ell)$.

We also allow for the possibility that buyers' values are diminished by increases in the number of buyers by assuming that buyer i with type v_i has value $v_i\theta(\ell)$ for the good if the good is supplied to ℓ buyers, where $\theta(\cdot)$ is a nonincreasing function with value in [0, 1] and $\theta(1) = 1$. The case of $\theta(\cdot) \equiv 1$ corresponds to no congestion costs for buyers.

In any incentive compatible interim individually rational mechanism the designer's expected profit is

$$\Pi = E_{\mathbf{v},k}\left[\sum_{i=1}^{n} \Phi_i(v_i)\theta\left(\sum_{j=1}^{n} q_j(\boldsymbol{v},k)\right) q_i(\boldsymbol{v},k) - \Gamma(k)\min\left\{1,\ \sum_{i=1}^{n} q_i(\boldsymbol{v},k)\right\} \right.$$
$$\left. - c\left(\sum_{i=1}^{n} q_i(\boldsymbol{v},k)\right)\right],$$

minus a constant, which, as before, can be set equal to 0 by making the individual rationality constraints bind for the worst-off types.

Recall that $\mathbb{I}(\boldsymbol{v}) \equiv \left\{i : v_i > \Phi_i^{-1}(0)\right\}$. Relabel the buyers in $\mathbb{I}(\boldsymbol{v})$ as $1, ..., |\mathbb{I}(\boldsymbol{v})|$ so that $\Phi_1(v_1) \geq ... \geq \Phi_{|\mathbb{I}(\boldsymbol{v})|}(v_{|\mathbb{I}(\boldsymbol{v})|})$, breaking ties at random. Relabel the buyers not in $\mathbb{I}(\boldsymbol{v})$ with indices greater than $|\mathbb{I}(\boldsymbol{v})|$. If $\mathbb{I}(\boldsymbol{v}) = \emptyset$, let $\overline{q} = 0$ and otherwise let $\overline{q}$ be such that

$$\overline{q} \in \arg \max_{h\in\{1,\ldots,|\mathbb{I}(\boldsymbol{v})|\}} \sum_{i=1}^{h} \Phi_i(v_i)\theta(h) - c(h).$$

Thus, allocation of the object to (relabeled) buyers $1, ..., \overline{q}$ maximizes the sum of the congestion adjusted virtual values of the trading buyers. The pointwise maximum requires trade with buyers $1, ..., \overline{q}$ if and only if trade with these buyers is optimal for the designer:

$$q_i(\mathbf{v},k) \equiv \begin{cases} 1, & \text{if } i \leq \overline{q} \text{ and } \sum_{h=1}^{\overline{q}} \Phi_h(v_h)\theta(\overline{q}) - \Gamma(k) - c(\overline{q}) > 0 \\ 0, & \text{otherwise.} \end{cases}$$

In the dominant strategy implementation, agents pay the lowest type they could report and yet still trade.

In some situations, congestion costs may be constant marginal costs that are borne by the intermediary. For example, in the case of digital goods like songs, books, and movies, costly server capacity is provided by the broker. If the intermediary and the seller contract before buyers arrive, as assumed in Section 4.2, then the intermediary-optimal contract will still be a take-it-or-leave-it offer $p_S = \Gamma^{-1}(R^{net})$, with the twist that R^{net} is now expected net revenue. If, in addition, the seller's virtual type function is linear, the intermediary-optimal contract can be implemented with a linear fee function of realized net revenue Y^{net} that satisfies $R^{net} = E[Y^{net}]$.

6.2. The case without regularity

If the virtual types are not monotone, then we must "iron." Given distribution F_i with positive density f_i (implying that F_i is invertible) and nonmonotone virtual value Φ_i, the ironed virtual value $\overline{\Phi}_i$ is defined as follows: Let $\kappa_i(x) \equiv \Phi_i(F_i^{-1}(x))$ and let $K_i(x) \equiv \int_{\underline{v}}^{x} \kappa_i(y)dy$ and let $K_i^{co}(x)$ define the convex hull of $K_i(x)$ on $[\underline{v}, \overline{v}]$. Define $\overline{\kappa}_i(x) \equiv K_i^{co\prime}(x)$. Then the ironed virtual value is

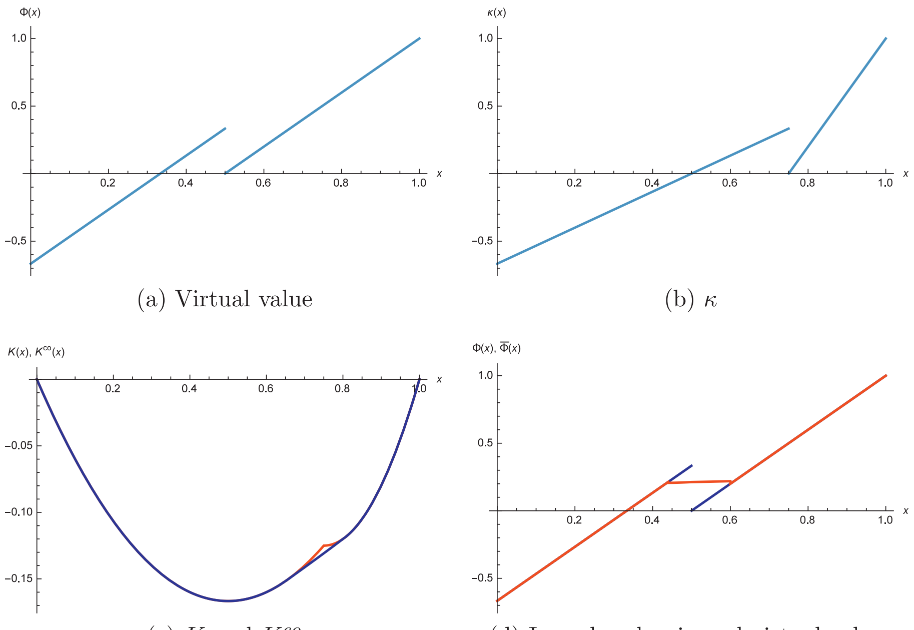
$$\overline{\Phi}_i \equiv \overline{\kappa}_i(F_i(x)).$$

For example, consider

$$F(x) \equiv \begin{cases} 3/2\ x, & \text{if } 0 \leq x \leq 1/2 \\ 1/2 + 1/2\ x, & \text{if } 1/2 < x \leq 1 \end{cases}$$

on [0, 1]. Then the virtual value is nonmonotone, as shown in Fig. 5(a). We show the construction of κ and K and its convex hull in Fig. 5(b) and (c). The original and ironed virtual values are shown in Fig. 5(d).

The Bayesian optimal club good mechanism and club good clock auction are then as defined above, but replacing the virtual types with ironed the virtual types and replacing



(a) Virtual value

(b) κ

(c) K and K^{co}

(d) Ironed and unironed virtual values

Fig. 5. Illustration of the ironing of the virtual value.

the inverse of the virtual types with the generalized inverse:

$$\overline{\Phi}_i^{-1}(x) \equiv \inf\left\{v_i \in [\underline{v}, \overline{v}] \mid \overline{\Phi}_i(v_i) \geq x\right\}.$$

In the private good environment, ironing introduces a need for tie breaking because buyers with different values can have the same ironed virtual type. In a rivalrous environment, tie breaking arises with positive probability when regularity is not satisfied. However, in a club good setting, if one buyer with a particular virtual type trades at a given price, then all buyers with that virtual type also trade at that same price, so tie breaking is not necessary.

7. Conclusions

The emergence of the internet and of digital goods have renewed interest in the optimal mechanisms for the provision of public goods with exclusion, also known as club goods. Because digital goods like e-books, songs, and movies are typically traded via intermediaries such as Amazon, iTunes, and Netflix, a new question of considerable practical relevance is what pricing mechanisms are optimal for a club good intermediary. In this paper, we have analyzed this question using a Bayesian mechanism design framework with independent private values, which beyond incentive compatibility and individual rationality imposes no constraints on the choice of mechanism.

Dynamics are an important aspect of real life that are not captured by our model. In many situations, not all buyers are present at the outset when the production decisions have to be made. Extending our one-shot model to a dynamic setup that incorporates features like these seems a valuable avenue for future research. Crowdfunding and crowdfunding platforms are another example in which uncertainty about aggregate demand, which is only lifted over time, is arguably important. Dynamics also play a pertinent role in the economics of innovation and R&D, with ideas being a quintessential example of a consumption good without rivalry and patents being one way of making this sort of public good excludable. In this vein, one might consider the optimal mechanism in a setup in which the seller first has to invest, for example developing a new drug or producing a new movie (with effort being noncontractible). One can then consider how large is the optimal weight on seller profit in this environment.

Appendix A. Standard arguments

As stated in the text, standard arguments imply that in any incentive compatible interim individually rational mechanism the designer's expected profit is

$$\Pi = E_{\mathbf{v},k}\left[\sum_{i=1}^{n} \Phi_i(v_i) q_i(\boldsymbol{v}, k) - \Gamma(k) \min\left\{1,\ \sum_{i=1}^{n} q_i(\boldsymbol{v}, k)\right\}\right]$$

minus a constant, which, however, can be set equal to 0 by making the individual rationality constraints bind for the worst-off types.

By the Revelation Principle, we can focus attention on direct mechanisms ($\mathbf{q}$, $\mathbf{m}$,m^S), where $\mathbf{q} : \times_{i=1}^{n}[\underline{v},\overline{v}] \times [\underline{k},\overline{k}] \to \times_{i=1}^{n}[0,1]$, $\mathbf{m} : \times_{i=1}^{n}[\underline{v},\overline{v}] \times [\underline{k},\overline{k}] \to \mathcal{R}^n$, and $m^S : \times_{i=1}^{n}[\underline{v},\overline{v}] \times [\underline{k},\overline{k}] \to \mathcal{R}$. This differs from the private good case in that $\mathbf{q}$ maps into $\times_{i=1}^{n}[0,1]$, whereas in the private good case it maps into the n-dimensional simplex. The expected quantity produced by the seller is $q^S(\mathbf{v},k) \equiv 1 - \Pi_{i=1}^{n}(1 - q_i(\boldsymbol{v}, k))$. The proof provided by Krishna (2002, Proposition 5.1) for the private good case is easily adapted to our club good environment. The following arguments, which are standard for the private good case (see, e.g., Krishna, 2002, Chapter 5.1), apply equally to the club good case.

Given a direct mechanism ($\mathbf{q}$, $\mathbf{m}$,m^S) and letting $f(\boldsymbol{v}) \equiv f_1(v_1) \dots f_n(v_n)$, define

$$\hat{q}_i(z_i) = \int_{[\underline{k},\overline{k}]} \int_{\times_{j\neq i}[\underline{v},\overline{v}]} q_i(z_i, \mathbf{v}_{-i}, k) f_{-i}(\mathbf{v}_{-i}) g(k) \mathbf{dv}_{-i} dk$$

to be the probability that i gets the object when he reports z_i and all other buyers and the seller report their types truthfully. Similarly, define

$$\hat{m}_i(z_i) = \int_{[\underline{k},\overline{k}]} \int_{\times_{j\neq i}[\underline{v},\overline{v}]} m_i(z_i, \mathbf{v}_{-i}, k) f_{-i}(\mathbf{v}_{-i}) g(k) \mathbf{dv}_{-i} dk$$

to be buyer i's expected payment when he reports z_i and others report truthfully. Because we assume independent draws, $\hat{q}_i(z_i)$ and $\hat{m}_i(z_i)$ depend only on the report z_i and not on the buyer i's true value v_i. The expected payoff of buyer i is then $\hat{q}_i(z_i)v_i - \hat{m}_i(z_i)$.

Similarly, define the expected quantity and payment to the seller to be

$$\hat{q}^S(k) = \int_{\times_{i=1}^{n}[\underline{v},\overline{v}]} q^S(\mathbf{v}, k) f(\mathbf{v}) \mathbf{dv}$$

and

$$\hat{m}^S(k) \equiv \int_{\times_{i=1}^{n}[\underline{v},\overline{v}]} m^S(\mathbf{v}, k) f(\mathbf{v}) \mathbf{dv}.$$

The direct mechanism is incentive compatible if for all i, x_i, and z_i,

$$U_i(v_i) \equiv \hat{q}_i(v_i)v_i - \hat{m}_i(v_i) \geq \hat{q}_i(z_i)v_i - \hat{m}_i(z_i)$$

and for all k and k',

$$U^S(k) \equiv \hat{m}^S(k) - k\hat{q}^S(k) \geq \hat{m}^S(k') - k\hat{q}^S(k').$$

Focusing on the buyers, this implies that $U_i(v_i) = \max_{z_i \in [\underline{v},\overline{v}]} \{\hat{q}_i(z_i)v_i - \hat{m}_i(z_i)\}$, i.e., U_i is a maximum of a family of affine functions, which implies that U_i is convex and so absolutely continuous and differentiable almost everywhere in the interior of its domain.[12] In addition, incentive compatibility implies that $U_i(z_i) \geq \hat{q}_i(v_i)z_i - \hat{m}_i(v_i) = U_i(v_i) + \hat{q}_i(v_i)(z_i - v_i)$, which for $\delta > 0$ implies

$$\frac{U_i(v_i+\delta) - U_i(v_i)}{\delta} \geq \hat{q}_i(v_i)$$

and for $\delta < 0$ implies

$$\frac{U_i(v_i+\delta) - U_i(v_i)}{\delta} \leq \hat{q}_i(v_i),$$

so taking the limit as δ goes to zero, at every point v_i where U_i is differentiable, $U_i'(v_i) = \hat{q}_i(v_i)$. Because U_i is convex, this implies that $\hat{q}_i(v_i)$ is nondecreasing.[13] Because every absolutely continuous function is the definite integral of its derivative,

$$U_i(v_i) = U_i(\underline{v}) + \int_{\underline{v}}^{v_i} \hat{q}_i(t_i)dt_i,$$

which implies that, up to an additive constant, a buyer's expected payoff in an incentive compatible direct mechanism depends only on the allocation rule.

Turning to the seller, $U^S(k) = \max_{k' \in [\underline{k}, \overline{k}]} \left\{\hat{m}^S(k') - k\hat{q}^S(k')\right\}$, implying, as with the buyers, that U^S is convex and so absolutely continuous and differentiable almost everywhere in the interior of its domain. In addition, at every point k where U^S is differentiable, $U^{S\prime}(k) = \hat{q}^S(k)$, and this is nondecreasing. Finally, again as above, we can write

$$U^S(k) = U^S(\underline{k}) + \int_{\underline{k}}^{k} \hat{q}^S(k')dk'.$$

Thus, the usual Revenue Equivalence result for private goods applies here: If the direct mechanism $(\mathbf{q},\mathbf{m},m^S)$ is incentive compatible, then for all i and v_i, the expected payment by buyer i is

$$\begin{aligned}
\hat{m}_i(v_i) &= \hat{q}_i(v_i)v_i - U_i(v_i) \\
&= \hat{q}_i(v_i)v_i - U_i(\underline{v}) - \int_{\underline{v}}^{v_i} \hat{q}_i(t_i)dt_i \\
&= \hat{m}_i(\underline{v}) - \hat{q}_i(\underline{v})\underline{v} + \hat{q}_i(v_i)v_i - \int_{\underline{v}}^{v_i} \hat{q}_i(t_i)dt_i
\end{aligned}$$

[12] A function $f : [\underline{v}, \overline{v}] \rightarrow \mathcal{R}$ is absolutely continuous if for all $\varepsilon > 0$ there exists $\delta > 0$ such that whenever a finite sequence of pairwise disjoint sub-intervals (v_k, v_k') of $[\underline{v}, \overline{v}]$ satisfies $\sum_k (v_k' - v_k) < \delta$, then $\sum_k \left|f(v_k') - f(v_k)\right| < \varepsilon$. One can show that absolute continuity on compact interval $[a,\ b]$ implies that f has a derivative f' almost everywhere, the derivative is Lebesgue integrable, and that $f(x) = f(a) + \int_a^x f'(t)dt$ for all $x \in [a, b]$. For example, the Cantor function is uniformly continuous but not absolutely continuous (the derivative of the Cantor function is zero almost everywhere and so $0 = \int_0^1 F'(t)dt < F(1) - F(0) = 1$).

[13] To see that a nondecreasing q_i implies incentive compatibility, note that a sufficient condition for incentive compatibility is that for all v_i and z_i, $U_i(z_i) \geq U_i(v_i) + q_i(v_i)(z_i - v_i)$, which because every absolutely continuous function is the definite integral of its derivative, we can write as $\int_{v_i}^{z_i} q_i(t_i)dt_i \geq q_i(v_i)(z_i - v_i)$, which holds if q_i is nondecreasing.

and the expected payment to the seller is

$$\begin{aligned}\hat{m}^S(k) &= U^S(k) - k\hat{q}^S(k)\\ &= U^S(\overline{k}) + \int_k^{\overline{k}} \hat{q}^S(k')dk' - k\hat{q}^S(k)\\ &= \hat{m}^S(\overline{k}) - \hat{q}^S(\overline{k})\overline{k} + \int_k^{\overline{k}} \hat{q}^S(k')dk' - k\hat{q}^S(k).\end{aligned}$$

If the worst-valuing type never trades, and so by individual rationality pays zero, then we have

$$\hat{m}_i(v_i) = \hat{q}_i(v_i)v_i - \int_{\underline{v}}^{v_i} \hat{q}_i(t_i)dt_i \tag{16}$$

and

$$\hat{m}^S(k) = \int_k^{\overline{k}} \hat{q}^S(k')dk' - k\hat{q}^S(k). \tag{17}$$

The seller's expected revenue from buyer is then

$$\begin{aligned}\sum_{i=1}^n E_{\mathbf{v}}[\hat{m}_i(v_i)] &= \sum_{i=1}^n \int_{\underline{v}}^{\overline{v}} \hat{m}_i(v_i)f_i(v_i)dv_i\\ &= \sum_{i=1}^n \int_{\underline{v}}^{\overline{v}} \left(\hat{q}_i(v_i)v_i - \int_{\underline{v}}^{v_i} \hat{q}_i(t_i)dt_i\right)f_i(v_i)dv_i\\ &= \sum_{i=1}^n \left(\int_{\underline{v}}^{\overline{v}} \hat{q}_i(v_i)v_i f_i(v_i)dv_i - \int_{\underline{v}}^{\overline{v}}\int_{t_i}^{\overline{v}} \hat{q}_i(t_i)f_i(v_i)dv_i dt_i\right)\\ &= \sum_{i=1}^n \left(\int_{\underline{v}}^{\overline{v}} \hat{q}_i(v_i)v_i f_i(v_i)dv_i - \int_{\underline{v}}^{\overline{v}} \hat{q}_i(t_i)(1-F_i(t_i))dt_i\right)\\ &= \sum_{i=1}^n \int_{\underline{v}}^{\overline{v}} \hat{q}_i(v_i)\left(v_i - \frac{1-F_i(v_i)}{f_i(v_i)}\right)f_i(v_i)dv_i\\ &= \sum_{i=1}^n \int_{\underline{v}}^{\overline{v}} \hat{q}_i(v_i)\Phi_i(v_i)f_i(v_i)dv_i\\ &= E_{\mathbf{v}}\left[\sum_{i=1}^n \hat{q}_i(v_i)\Phi_i(v_i)\right],\end{aligned}$$

where the first equality uses the definition of the expectation, the second uses (16), the third switches the order of integration, the fourth integrates, the fifth collects terms, the sixth uses the definition of the virtual value Φ_i, and the last equality uses the definition of the expectation.

Applying similar steps, and using (17), one can show that

$$E_k\left[\hat{m}^S(k)\right] = E_k\left[\Gamma(k)\hat{q}^S(k)\right].$$

Thus, we have the result that

$$\Pi = E_{\mathbf{v},k}\left[\sum_{i=1}^{n} \Phi_i(v_i) q_i(\boldsymbol{v}, k) - \Gamma(k)\hat{q}^S(k)\right].$$

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