Online Appendix

to accompany

“Mergers, remedies, and incomplete information”

by

Simon Loertscher and Leslie M. Marx

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This Online Appendix contains the following material (sections are labeled starting with “B” to distinguish these appendices from Appendix A in the paper). In Section B, we present proofs omitted from the body of the paper and the full proofs for those that were only sketched in the body of the paper. In Section C, we provide extensions and discussion. In Section D, we illustrate based on the Republic-Santek transaction how the divestiture policies that we discuss can be implemented in practice using market data that is typically available in a merger review process.

B Details of sketched and omitted proofs

B.1 Proof of Proposition 2

Proof of Proposition 2. The social surplus results follow from the Schur-concavity of $SS(\mathbf{r})$ once we have shown that $\tilde{\mathbf{r}}'$ majorizes $\mathbf{r}'$. We prove this through the use of three lemmas, which we state and prove below. Consider a vector $\mathbf{r} \in \Delta$ with one element $r_\ell \geq \sigma > 0$. Then the amount σ can be spread among the $n-1$ elements of the vector $\mathbf{r}_{-\ell}$. Let $\overline{\mathbf{r}}(\ell, \sigma)$ be the most symmetric vector that is obtained by this procedure, that is, the most symmetric among vectors that are obtained by distributing σ among the elements in $\mathbf{r}_{-\ell}$ and replacing r_ℓ with $r_\ell - \sigma$. We show that given $\tilde{\mathbf{r}}, \mathbf{r} \in \Delta$ with $\tilde{r}_{\tilde{\ell}} = r_\ell \geq \sigma > 0$, if $\tilde{\mathbf{r}}$ majorizes $\mathbf{r}$, then $\overline{\tilde{\mathbf{r}}}(\tilde{\ell}, \sigma)$ majorizes $\overline{\mathbf{r}}(\ell, \sigma)$. We begin in Lemma B.1 by characterizing $\overline{\mathbf{r}}(\ell, \sigma)$. As a matter of notation, given vector $\mathbf{x}$, we use $x_{(i)}$ to denote the i-th highest element of $\mathbf{x}$. We define $x_{(0)} \equiv \infty$ and use $\mathbf{x}^\downarrow$ to denote $(x_{(1)}, \ldots, x_{(|\mathbf{x}|)})$.

Lemma B.1. *Given* $\mathbf{r} \in \Delta$ *with* $r_\ell \geq \sigma > 0$, *we have* $\overline{\mathbf{r}}(\ell, \sigma) = (\mathbf{r}'_{-\ell}, r_\ell - \sigma)$, *where* ${\mathbf{r}'_{-\ell}}^\downarrow = (r_{-\ell(1)}, \ldots, r_{-\ell(j)}, x, \ldots, x)$ *for* $j \in \{0, \ldots, n-1\}$ *and* $x \in [r_{-\ell(j+1)}, r_{-\ell(j)}]$ *such that* $\sum_{i=j+1}^{n-1}(x - r_{-\ell(i)}) = \sigma$.

Proof of Lemma B.1. Take as given $\mathbf{r} \in \Delta$ with $r_\ell \geq \sigma > 0$, and let $\mathbf{r}'$ be constructed from $\mathbf{r}$ as in the statement of the lemma. Thus, as in the statement of the lemma, there exists $j \in \{0, \ldots, n-1\}$ such that for $i \in \{0, \ldots, j\}$, $r'_{-\ell(i)} = r_{-\ell(i)}$, and for $i \in \{j+1, \ldots, n-1\}$,

$r'_{-\ell(i)} = x$. Take an arbitrary vector $\hat{\mathbf{r}} \in \Delta$ with $\hat{r}_{\hat{\ell}} = r_\ell - \sigma$ for some $\hat{\ell} \in \mathcal{N}$, $\hat{\mathbf{r}}_{-\hat{\ell}} \neq \mathbf{r}'_{-\ell}$, and $\hat{\mathbf{r}}_{-\hat{\ell}} = \mathbf{r}_{-\ell} + (\varepsilon_1, \ldots, \varepsilon_{n-1})$, where $\varepsilon_i \geq 0$ and $\sum_{i=1}^{n-1} \varepsilon_i = \sigma$. We show that $\hat{\mathbf{r}}$ majorizes $\mathbf{r}'$, which then completes the proof.

We first show that $\hat{\mathbf{r}}_{-\hat{\ell}}$ majorizes $\mathbf{r}'_{-\ell}$. If $j = 0$, then $\mathbf{r}'_{-\ell} = (x, \ldots, x)$, which is majorized by any other n-dimensional vector whose elements sum to the same amount, thus including $\hat{\mathbf{r}}_{-\hat{\ell}}$. So assume that $j \geq 1$. By the construction of $\hat{\mathbf{r}}_{-\hat{\ell}}$, we have for all $i \in \{1, \ldots, j\}$, $\hat{r}_{-\hat{\ell}(i)} \geq r_{-\ell(i)} = r'_{-\ell(i)} \geq x$, which implies that for all $h \in \{1, \ldots, j\}$,

$$\sum_{i=1}^{h} \hat{r}_{-\hat{\ell}(i)} \geq \sum_{i=1}^{h} r'_{-\ell(i)}.$$

Let $\hat{\jmath}$ be the largest index such that $\hat{r}_{-\hat{\ell}(\hat{\jmath})} \geq x$. From the argument above, we know that $\hat{\jmath} \geq j$, and we know that for $h \in \{1, \ldots, \hat{\jmath}\}$,

$$\sum_{i=1}^{h} \hat{r}_{-\hat{\ell}(i)} \geq \sum_{i=1}^{h} r'_{-\ell(i)}, \tag{B.1}$$

where (B.1) holds with a strict inequality for at least one $h \in \{1, \ldots, \hat{\jmath}\}$: to see this, note that if (B.1) is not strict for $h = j$, then for all $i \in \{1, \ldots, j\}$, $\hat{r}_{-\hat{\ell}(i)} = r'_{-\ell(i)}$, and it follows from $\hat{\mathbf{r}}_{-\hat{\ell}} \neq \mathbf{r}'_{-\ell}$ and the fact that both $\hat{\mathbf{r}}_{-\hat{\ell}}$ and $\mathbf{r}'_{-\ell}$ sum to the same amount that $\hat{r}_{-\hat{\ell}(j+1)} > r'_{-\hat{\ell}(j+1)} = x$, which implies that $\hat{\jmath} \geq j + 1$ and that (B.1) is strict for $h = j + 1$.

Because $\hat{\mathbf{r}}_{-\hat{\ell}}$ and $\mathbf{r}'_{-\ell}$ sum to the same amount, we have $\sum_{i=1}^{n-1} \hat{r}_{-\hat{\ell}(i)} = \sum_{i=1}^{n-1} r'_{-\ell(i)}$, which we can write as

$$\sum_{i=1}^{j} (\hat{r}_{-\hat{\ell}(i)} - r'_{-\ell(i)}) + \sum_{i=j+1}^{\hat{\jmath}} (\hat{r}_{-\hat{\ell}(i)} - x) = \sum_{i=\hat{\jmath}+1}^{n-1} (x - \hat{r}_{-\hat{\ell}(i)}), \tag{B.2}$$

where the summation from $j + 1$ to $\hat{\jmath}$ is defined to be zero if $\hat{\jmath} = j$. Using this, for $h \in \{\hat{\jmath} + 1, \ldots, n - 1\}$, we have

$$\begin{aligned}
\sum_{i=1}^{h} \hat{r}_{-\hat{\ell}(i)} - \sum_{i=1}^{h} r'_{-\ell(i)} &= \sum_{i=1}^{j} (\hat{r}_{-\hat{\ell}(i)} - r'_{-\ell(i)}) + \sum_{i=j+1}^{\hat{\jmath}} (\hat{r}_{-\hat{\ell}(i)} - x) + \sum_{i=\hat{\jmath}+1}^{h} (\hat{r}_{-\hat{\ell}(i)} - x) \\
&= \sum_{i=\hat{\jmath}+1}^{n-1} (x - \hat{r}_{-\hat{\ell}(i)}) + \sum_{i=\hat{\jmath}+1}^{h} (\hat{r}_{-\hat{\ell}(i)} - x) \\
&= (n - 1 - h)x - \sum_{i=h+1}^{n-1} \hat{r}_{-\hat{\ell}(i)} \\
&\geq 0,
\end{aligned}$$

where the first equality uses $r'_{-\ell(i)} = x$ for $i > j$, the second equality uses (B.2), the third equality rearranges, and the inequality uses $h \geq \hat{\jmath} + 1$, which implies that for $i \in \{h +$

$1, \ldots, n-1\}$, $\hat{r}_{-\hat{\ell}(i)} < x$. Combining this with (B.1), we conclude that $\hat{\mathbf{r}}_{-\hat{\ell}}$ majorizes $\mathbf{r}'_{-\ell}$.

We are left to show that $\mathbf{r}' = (\mathbf{r}'_{-\ell}, r_\ell - \sigma)$. Because $\hat{\mathbf{r}}$ was arbitrary, this is equivalent to showing that $\mathbf{r}'$ is majorized by $\hat{\mathbf{r}} = (\hat{\mathbf{r}}_{-\hat{\ell}}, r_\ell - \sigma)$. Let ρ be the rank of $r_\ell - \sigma$ in $\mathbf{r}'$, i.e., $r_\ell - \sigma = \mathbf{r}'_{(\rho)}$, and let $\hat{\rho}$ be the rank of $r_\ell - \sigma$ in $\hat{\mathbf{r}}$, i.e., $\hat{r}_{\hat{\ell}} = r_\ell - \sigma = \hat{\mathbf{r}}_{(\hat{\rho})}$ (breaking ties in favor of larger ρ and $\hat{\rho}$, respectively). For $h \in \{1, \ldots, \min\{\hat{\rho}-1, \rho-1\}\}$ and for $h \in \{\max\{\hat{\rho}, \rho\}, \ldots, n\}$,

$$\sum_{i=1}^h \hat{r}_{(i)} - \sum_{i=1}^h r'_{(i)} = \sum_{i=1}^h \hat{r}_{-\hat{\ell}(i)} - \sum_{i=1}^h r'_{-\ell(i)} \geq 0,$$

where the inequality uses that $\hat{\mathbf{r}}_{-\hat{\ell}}$ majorizes $\mathbf{r}'_{-\ell}$. It remains to show that $\sum_{i=1}^h \hat{r}_{(i)} - \sum_{i=1}^h r'_{(i)} \geq 0$ for $\hat{\rho} < \rho$ and $h \in \{\hat{\rho}, \ldots, \rho-1\}$ and for $\rho < \hat{\rho}$ and $h \in \{\rho-1, \ldots, \hat{\rho}\}$. If $\hat{\rho} < \rho$, then for $h \in \{\hat{\rho}, \ldots, \rho-1\}$,

$$\begin{aligned}\sum_{i=1}^h \hat{r}_{(i)} - \sum_{i=1}^h r'_{(i)} &= \sum_{i=1}^{h-1} \hat{r}_{-\hat{\ell}(i)} + r_\ell - \sigma - \sum_{i=1}^{h} r'_{-\ell(i)} \geq \sum_{i=1}^{h-1} \hat{r}_{-\hat{\ell}(i)} + \hat{r}_{-\hat{\ell}(h)} - \sum_{i=1}^{h} r'_{-\ell(i)} \\ &= \sum_{i=1}^{h} \hat{r}_{-\hat{\ell}(i)} - \sum_{i=1}^{h} r'_{-\ell(i)} \geq 0,\end{aligned}$$

where the first inequality uses that for $i \geq \hat{\rho}$, $\hat{r}_{-\hat{\ell}(i)} \leq \hat{r}_{(i)} \leq \hat{r}_{(\hat{\rho})} = r_\ell - \sigma$, and the second inequality uses that $\hat{\mathbf{r}}_{-\hat{\ell}}$ majorizes $\mathbf{r}'_{-\ell}$. If $\rho < \hat{\rho}$, then for $h \in \{\rho, \ldots, \hat{\rho}-1\}$,

$$\begin{aligned}\sum_{i=1}^h \hat{r}_{(i)} - \sum_{i=1}^h r'_{(i)} &= \sum_{i=1}^{h} \hat{r}_{-\hat{\ell}(i)} - \sum_{i=1}^{h-1} r'_{-\ell(i)} - (r_\ell - \sigma) \\ &= \sum_{i=1}^{h-1} \hat{r}_{-\hat{\ell}(i)} - \sum_{i=1}^{h-1} r'_{-\ell(i)} + \hat{r}_{-\hat{\ell}(h)} - (r_\ell - \sigma) \geq \sum_{i=1}^{h-1} \hat{r}_{-\hat{\ell}(i)} - \sum_{i=1}^{h-1} r'_{-\ell(i)} \geq 0,\end{aligned}$$

where the first inequality uses that for $i < \hat{\rho}$, $\hat{r}_{-\hat{\ell}(i)} = \hat{r}_{(i)} \geq \hat{r}_{(\hat{\rho})} = r_\ell - \sigma$, and the second inequality uses that $\hat{\mathbf{r}}_{-\hat{\ell}}$ majorizes $\mathbf{r}'_{-\ell}$. Thus, we conclude that $\hat{\mathbf{r}}$ majorizes $\mathbf{r}'$, which completes the proof. □

Next, in Lemma B.2, we show that majorization extends to subvectors when a common element is removed:

Lemma B.2. *If $\tilde{\mathbf{r}}$ majorizes $\mathbf{r}$ and $\tilde{r}_{\tilde{\ell}} = r_\ell$, then $\tilde{\mathbf{r}}_{-\tilde{\ell}}$ majorizes $\mathbf{r}_{-\ell}$.*

Proof of Lemma B.2. Let $\tilde{j}$ and j be the rank of $\tilde{r}_{\tilde{\ell}}$ and r_ℓ, respectively, breaking ties in favor of larger $\tilde{j}$ and j. If $j \leq \tilde{j}$, then the result follows because then $\sum_{i=1}^h \tilde{r}_{-\hat{\ell}(i)} - \sum_{i=1}^h r_{-\ell(i)}$ is either the same as $\sum_{i=1}^h \tilde{r}_{(i)} - \sum_{i=1}^h r_{(i)}$, and so nonnegative (and positive for at least one h), or even greater by the amount $r_{(j)} - r_{(j+1)} \geq 0$. So, suppose that $j > \tilde{j}$. It is sufficient to check that for $h \in \{\tilde{j}+1, \ldots, j-1\}$ we have $\sum_{i=1}^h \tilde{r}_{-\hat{\ell}(i)} \geq \sum_{i=1}^h r_{-\ell(i)}$. By the definition of j, we have $r_{(i)} \geq r_\ell$ for $i < j$, and by the definition of $\tilde{j}$, we have $\tilde{r}_{(i)} \leq \tilde{r}_{\tilde{\ell}}$ for $i \geq \tilde{j}+1$. Thus, for $h \in \{\tilde{j}, \ldots, j-1\}$, we have

$$r_{-\ell(h)} = r_{(h)} \geq r_\ell = \tilde{r}_{\tilde{\ell}} \geq \tilde{r}_{(h)} \geq \tilde{r}_{-\tilde{\ell}(h)} \tag{B.3}$$

and

$$\begin{aligned} \sum_{i=1}^{h} \tilde{r}_{-\tilde{\ell}(i)} &= \sum_{i=1}^{h} \tilde{r}_{(i)} - \tilde{r}_{\tilde{\ell}} + \tilde{r}_{-\tilde{\ell}(h)} \geq \sum_{i=1}^{h} r_{(i)} - r_\ell + \tilde{r}_{-\tilde{\ell}(h)} \\ &= \sum_{i=1}^{h} r_{-\ell(i)} + r_\ell - r_{-\ell(h)} - r_\ell + \tilde{r}_{-\tilde{\ell}(h)} = \sum_{i=1}^{h} r_{-\ell(i)} - r_{-\ell(h)} + \tilde{r}_{-\tilde{\ell}(h)} \geq \sum_{i=1}^{h} r_{-\ell(i)}, \end{aligned}$$

where the first inequality uses that $\tilde{\mathbf{r}}$ majorizes $\mathbf{r}$ and the inequality second uses (B.3). $\square$

Finally, in Lemma B.3, we show use Lemmas B.1 and B.2 to show that majorization extends to optimally divested ownership structures:

Lemma B.3. *Given $\tilde{\mathbf{r}}, \mathbf{r} \in \Delta_R^{n-1}$ with $\tilde{r}_{\tilde{\ell}} = r_\ell$ and $\sigma \in (0, r_\ell]$, if $\tilde{\mathbf{r}}$ majorizes $\mathbf{r}$, then $\bar{\tilde{\mathbf{r}}}(\ell, \sigma)$ majorizes $\bar{\mathbf{r}}(\ell, \sigma)$.*

Proof of Lemma B.3. Let $\tilde{j}$ and $\tilde{x}$ be the parameters of the optimal divestiture given $\tilde{\mathbf{r}}$, $\tilde{\ell}$, and σ, and let j and x be the parameters of the optimal divestiture given $\mathbf{r}$, ℓ, and σ as derived in Lemma B.1. That is, $\tilde{x} \in [\tilde{r}_{-\hat{\ell}(\tilde{j}+1)}, \tilde{r}_{-\hat{\ell}(\tilde{j})}]$ and $x \in [r_{-\ell(j+1)}, r_{-\ell(j)}]$ and

$$\sum_{i=\tilde{j}+1}^{n-1} (\tilde{x} - \tilde{r}_{-\tilde{\ell}(i)}) = \sum_{i=j+1}^{n-1} (x - r_{-\ell(i)}) = \sigma. \tag{B.4}$$

Assume that $\tilde{\mathbf{r}}$ majorizes $\mathbf{r}$. It follows from Lemma B.2 that $\tilde{\mathbf{r}}_{-\tilde{\ell}}$ majorizes $\mathbf{r}_{-\ell}$.

Let $\bar{\tilde{\mathbf{r}}}(\ell, \sigma) \equiv (\tilde{\mathbf{r}}'_{-\tilde{\ell}}, r_{\tilde{\ell}} - \sigma)$ and $\bar{\mathbf{r}}(\ell, \sigma) \equiv (\mathbf{r}'_{-\ell}, r_\ell - \sigma)$ and suppose that $\tilde{\mathbf{r}}'_{-\tilde{\ell}}$ does not majorize $\mathbf{r}'_{-\ell}$. Then there exists a smallest $\hat{h} \in \{1, \ldots, n-1\}$ such that $\sum_{i=1}^{\hat{h}} (\tilde{r}'_{-\tilde{\ell}(i)} - r'_{-\ell(i)}) < 0$. Given that $\tilde{\mathbf{r}}_{-\tilde{\ell}}$ majorizes $\mathbf{r}_{-\ell}$ and that $\tilde{r}'_{-\tilde{\ell}(i)}$ and $r'_{-\ell(i)}$ coincide with $\tilde{r}_{-\tilde{\ell}(i)}$ and $r_{-\ell(i)}$ for $i \leq \min\{j, \tilde{j}\}$ and that both $\tilde{\mathbf{r}}'_{-\tilde{\ell}}$ and $\mathbf{r}'_{-\ell}$ are constant for $i > \max\{j, \tilde{j}\}$, it must be that $\hat{h} \in (\min\{j, \tilde{j}\}, \max\{j, \tilde{j}\}]$.

Case 1. $\tilde{j} < \hat{h} \leq j$. In this case, we can rewrite (B.4) as

$$\sum_{i=\tilde{j}+1}^{j} (\tilde{x} - \tilde{r}_{-\tilde{\ell}(i)}) + \sum_{i=j+1}^{n-1} (\tilde{x} - \tilde{r}_{-\tilde{\ell}(i)}) = \sum_{i=j+1}^{n-1} (x - r_{-\ell(i)})$$

or, using $\tilde{x} \in [\tilde{r}_{-\hat{\ell}(\tilde{j}+1)}, \tilde{r}_{-\hat{\ell}(\tilde{j})}]$,

$$\sum_{i=\tilde{j}+1}^{j} \underbrace{(\tilde{x} - \tilde{r}_{-\tilde{\ell}(i)})}_{\text{positive}} + (n-1-j)(\tilde{x} - x) = \sum_{i=j+1}^{n-1} (\tilde{r}_{-\tilde{\ell}(i)} - r_{-\ell(i)}) \leq 0,$$

which implies that $\tilde{x} < x$. But then

$$\begin{aligned}\sum_{i=1}^{n-1}\left(r'_{-\ell(i)} - \tilde{r}'_{-\tilde{\ell}(i)}\right) &> \sum_{i=\hat{h}+1}^{n-1}\left(r'_{-\ell(i)} - \tilde{r}'_{-\tilde{\ell}(i)}\right) = \sum_{i=\hat{h}+1}^{j}\left(r'_{-\ell(i)} - \tilde{x}\right) + \sum_{i=j+1}^{n-1}(x - \tilde{x}) \\ &\geq \sum_{i=\hat{h}+1}^{j}(x - \tilde{x}) + \sum_{i=j+1}^{n-1}(x - \tilde{x}) \geq 0,\end{aligned}$$

where the first inequality uses $\sum_{i=1}^{\hat{h}}(\tilde{r}'_{-\tilde{\ell}(i)} - r'_{-\ell(i)}) < 0$, the second inequality uses $x < r'_{-\ell(i)}$ for $i \leq j$, and the third inequality uses $\tilde{x} < x$. This violates the summing up condition, which requires that $\sum_{i=1}^{n-1} r'_{-\ell(i)} = \sum_{i=1}^{n-1} \tilde{r}'_{-\ell(i)}$, giving us a contradiction.

Case 2. $j < \hat{h} \leq \tilde{j}$. For this case, we know that at $\hat{h}$-th highest element, $\mathbf{r}'_{-\ell}$ is already equal to x, but $\tilde{\mathbf{r}}'_{-\tilde{\ell}}$ is still equal to $\tilde{\mathbf{r}}_{-\ell}$. By the definition of $\hat{h}$, it must be that the change from $\sum_{i=1}^{\hat{h}-1} r'_{-\ell(i)}$ to $\sum_{i=1}^{\hat{h}} r'_{-\ell(i)}$, which is equal to x, is larger than the change from $\sum_{i=1}^{\hat{h}-1} \tilde{r}'_{-\tilde{\ell}(i)}$ to $\sum_{i=1}^{\hat{h}} \tilde{r}'_{-\tilde{\ell}(i)}$. Thus, we have $x > \tilde{r}'_{-\tilde{\ell}(\hat{h})}$, which means that $x > \tilde{r}'_{-\tilde{\ell}(\hat{h})} \geq \cdots \geq \tilde{r}'_{-\tilde{\ell}(n-1)}$, which means that $\sum_{i=\hat{h}+1}^{n-1} r'_{-\ell(i)} > \sum_{i=\hat{h}+1}^{n-1} \tilde{r}'_{-\tilde{\ell}(i)}$, so we have

$$\sum_{i=1}^{n-1} r'_{-\ell(i)} = \sum_{i=1}^{\hat{h}} r'_{-\ell(i)} + \sum_{i=\hat{h}+1}^{n-1} r'_{-\ell(i)} > \sum_{i=1}^{\hat{h}} \tilde{r}'_{-\tilde{\ell}(i)} + \sum_{i=\hat{h}+1}^{n-1} \tilde{r}'_{-\tilde{\ell}(i)} = \sum_{i=1}^{n-1} \tilde{r}'_{-\ell(i)},$$

which contradicts the summing up condition, which requires that $\sum_{i=1}^{n-1} r'_{-\ell(i)} = \sum_{i=1}^{n-1} \tilde{r}'_{-\ell(i)}$.

Thus, we conclude that $\tilde{\mathbf{r}}'_{-\tilde{\ell}}$ majorizes $\mathbf{r}'_{-\ell}$, which completes the proof of Lemma B.3. □

Combining Lemmas B.1–B.3 completes the proof of Proposition 2. ■

B.2 Proof of Lemma 1

Proof of Lemma 1. As noted in Section 3, an implication of IC is that $u_i'(\theta) = q_i(\theta) - r_i$ wherever u_i is differentiable, which by IC is almost everywhere. Given this, the monotonicity of u_i implies the following characterization of the set of worst-off types for firm i, denoted by $\Omega_i \equiv \arg\min_{\theta_i \in [0,1]} u_i(\theta_i)$ (see also Cramton et al. (1987, Lemma 2) and Loertscher and Wasser (2019)):

$$\Omega_i = \begin{cases} \{\theta_i \in [0,1] \mid q_i(\theta_i) = r_i\} & \text{if } \exists \theta_i \in [0,1] \text{ s.t. } q_i(\theta_i) = r_i, \\ \{\theta_i \in [0,1] \mid q_i(z) < r_i \ \forall z < \theta_i \text{ and } q_i(z) > r_i \ \forall z > \theta_i\} & \text{otherwise.} \end{cases}$$

In the first case in which there exists $\theta_i \in [0,1]$ such that $q_i(\theta_i) = r_i$, the set Ω_i is a (possibly degenerate) interval, and in the second case, Ω_i is a singleton.

Taking the expression for $m_i(\theta)$ in (6), with θ' replaced by $\hat{\theta}_i$, we have $\int_0^1 m_i(\theta) dF_i(\theta) = \int_0^1 (q_i(\theta) - r_i)\theta dF_i(\theta) - \int_{\hat{\theta}_i}^1 \int_{\hat{\theta}_i}^{\theta} (q_i(x) - r_i) dx dF_i(\theta) + \int_0^{\hat{\theta}_i} \int_{\theta}^{\hat{\theta}_i} (q_i(x) - r_i) dx dF_i(\theta) - u_i(\hat{\theta}_i)$. Changing the order of integration in the double integrals and substituting the virtual type functions and noting that $\mathbb{E}_{\theta_i}[\Psi_i(\theta_i, \hat{\theta}_i)] = \hat{\theta}_i$ gives the result. ■

B.3 Additional detail for the proof of Proposition 3

Proof of Proposition 3. We begin with a lemma:

Lemma B.4. $\Pi^e(\mathbf{r})$ *is strictly concave in* $\mathbf{r}$*, and* $\mathcal{R}^e$ *is convex in* $\mathbf{r}$.[1]

Proof of Lemma B.4. The second part of the statement is an implication of the first, so we prove the first part. Using Lemma 1, we have

$$\Pi^e(\mathbf{r}) = \sum_{i \in \mathcal{N}} \left(\mathbb{E}_{\theta_i} \left[\Psi_i(\theta_i, \hat{\theta}_i^e(r_i)) q_i^e(\theta_i) \right] - r_i \hat{\theta}_i^e(r_i) \right), \tag{B.5}$$

where $\hat{\theta}_i^e(r_i)$ is firm i's worst-off type under ex post efficiency. This can be rewritten as

$$\Pi^e(\mathbf{r}) = \sum_{i \in \mathcal{N}} \left(\int_0^{\hat{\theta}_i^e(r_i)} \Psi_i^S(\theta) q_i^e(\theta) dF_i(\theta) + \int_{\hat{\theta}_i^e(r_i)}^1 \Psi_i^B(\theta) q_i^e(\theta) dF_i(\theta) - r_i \hat{\theta}_i^e(r_i) \right).$$

Differentiating with respect to r_i, the three terms involving $\frac{d\hat{\theta}_i^e}{dr_i}$ cancel, and we are left with $\frac{\partial \Pi^e(\mathbf{r})}{\partial r_i} = -\hat{\theta}_i^e(r_i)$. Because $\hat{\theta}_i^e(r_i)$ increases in r_i, all second partial derivatives are negative. Further, all cross-partial derivatives are zero. This completes the proof. □

Because $-\hat{\theta}_i^e(r_i)$ is the derivative of $\Pi^e(\mathbf{r})$ with respect to r_i and because $\Pi^e(\mathbf{r})$ is strictly concave, it follows that the unique ownership structure that maximizes $\Pi^e(\mathbf{r})$ subject to the constraint that $\sum_{i \in \mathcal{N}} r_i = 1$, denoted $\mathbf{r}^*$, is such that all firms have the same worst-off types.

Further, the proof of Proposition 6, which is stated for multi-dimensional types but also encompasses one-dimensional types, shows that $\Pi^e(\mathbf{r}^*)$ is positive, implying that $\mathbf{r}^* \in \mathcal{R}^e$. We now show that a $\mathbf{r}^*$ exists that induces equal worst-off types.

Lemma B.5. *There exists a unique ownership structure* $\mathbf{r}^* \in \Delta_{\mathbf{k}}$ *such that* $\hat{\theta}_i^e(r_i^*) = \hat{\theta}^e \in (0, 1)$ *for all* $i \in \mathcal{N}$.

Proof of Lemma B.5. Given Lemma 1, we need only show that there exists $\hat{\theta} \in (0, 1)$ such that $\mathbf{r}^* = (q_1^e(\hat{\theta}), \ldots, q_n^e(\hat{\theta})) \in \Delta_{\mathbf{k}}$. Define $\mathcal{N}_{-i} \equiv \mathcal{N} \backslash \{i\}$. By the definition of ex post efficiency, for all $i \in \mathcal{N}$ and $\theta \in [0, 1]$,

$$q_i^e(\theta_i) = \sum_{\mathcal{A} \subset \mathcal{N}_{-i}} \max\{0, \min\{k_i, 1 - \sum_{j \in \mathcal{A}} k_j\}\} \prod_{j \in \mathcal{N}_{-i} \backslash \mathcal{A}} F_j(\theta_i) \prod_{j \in \mathcal{A}} (1 - F_j(\theta_i)),$$

which is continuous and increasing in θ_i on $[0, 1]$. Under our maintained assumption that $\sum_{j \neq i} k_j \geq 1$, $q_i^e(0) = 1$, so we have $\sum_{i \in \mathcal{N}} q_i^e(0) < 1$. The assumption of excess demand

[1] $\Pi^e(\mathbf{r})$ is strictly concave in $\mathbf{r}$, but $\mathcal{R}^e$ is only convex (and not necessarily strictly convex) because $\mathcal{R}^e \equiv \{\mathbf{r} \mid \Pi^e(\mathbf{r}) \geq 0\} \cap \Delta_{\mathbf{k}}$. So, $\mathcal{R}^e$ is not strictly convex where it intersects with the boundary of $\Delta_{R,\mathbf{k}}$.

implies that $\sum_{i\in\mathcal{N}} q_i^e(1) > 1$, so we have

$$\sum_{i\in\mathcal{N}} q_i^e(0) < 1 < \sum_{i\in\mathcal{N}} q_i^e(1).$$

By the continuity and monotonicity of $q_i^e(\cdot)$ on $[0,1]$, there exists a unique $\hat{\theta} \in (0,1)$ such that $\sum_{i\in\mathcal{N}} q_i^e(\hat{\theta}) = 1$. Further, $q_i^e(\hat{\theta}) \in [0, k_i]$ for all θ. So, $(q_1^e(\hat{\theta}), \ldots, q_n^e(\hat{\theta})) \in \Delta_{\mathbf{k}}$. □

It remains to show that $SS(\mathbf{r})$ is concave and strictly concave outside of $\mathcal{R}^e$. For any $\mathbf{r} \in \Delta_{\mathbf{k}}$, let $\langle \mathbf{Q}_{\mathbf{r}}, \mathbf{M}_{\mathbf{r}} \rangle$ denote the expected social surplus maximizing mechanism, subject to IC, IR, and no deficit. Let $\mathbf{r}, \mathbf{r}' \in \Delta_{\mathbf{k}}$ and $\mu \in [0,1]$ be given. Because $\langle \mathbf{Q}_{\mathbf{r}}, \mathbf{M}_{\mathbf{r}} \rangle$ and $\langle \mathbf{Q}_{\mathbf{r}'}, \mathbf{M}_{\mathbf{r}'} \rangle$ satisfy IC, IR, and no deficit, when the ownership structure is $\mu\mathbf{r} + (1-\mu)\mathbf{r}'$, the mechanism $\langle \hat{\mathbf{Q}}, \hat{\mathbf{M}} \rangle$ such that $\hat{q}_i(\theta_i) = \mu q_{\mathbf{r},i}(\theta_i) + (1-\mu) q_{\mathbf{r}',i}(\theta_i)$ and $\hat{m}_i(\theta_i) = \mu m_{\mathbf{r},i}(\theta_i) + (1-\mu) m_{\mathbf{r}',i}(\theta_i)$ also satisfies these constraints. Total expected social surplus from $\langle \hat{\mathbf{Q}}, \hat{\mathbf{M}} \rangle$ is $\mu SS(\mathbf{r}) + (1-\mu) SS(\mathbf{r}')$, but $\langle \mathbf{Q}_{\mu\mathbf{r}+(1-\mu)\mathbf{r}'}, \mathbf{M}_{\mu\mathbf{r}+(1-\mu)\mathbf{r}'} \rangle$ maximizes expected social surplus subject to the constraints, so

$$\mu SS(\mathbf{r}) + (1-\mu) SS(\mathbf{r}') \leq SS(\mu\mathbf{r} + (1-\mu)\mathbf{r}'),$$

which implies that $SS(\mathbf{r})$ is concave.

For $\mathbf{r} \in \mathcal{R}^e$, the market mechanism with the efficient allocation rule satisfies the no-deficit constraint. If $\mathbf{r} \notin \mathcal{R}^e$, then the no-deficit constraint cannot be satisfied with the efficient mechanism, implying that $SS(\mathbf{r}) < SS^e \equiv SS(\mathbf{r}^*)$. Thus, $SS(\mathbf{r})$ is strictly concave for $\mathbf{r} \notin \mathcal{R}^e$.

This completes the proof of Proposition 3. ■

B.4 Proof of Proposition 5

Proof of Proposition 5. We begin with a lemma. For the purposes of the lemma, given an IC mechanism $\langle \mathbf{Q}, \mathbf{M} \rangle$ and ownership structure $\mathbf{r}$, we let $\hat{\theta}_i^{\mathbf{Q}}(r_i)$ denote the minimum worst-off type of firm i with assets r_i under allocation rule $\mathbf{Q}$, and let $\Pi^{\mathbf{Q}}(\mathbf{r})$ denote the expected budget surplus under allocation rule $\mathbf{Q}$ and ownership structure $\mathbf{r}$ when IR constraints are satisfied with equality for the worst-off types.

Lemma B.6. *Given an IC mechanism $\langle \mathbf{Q}, \mathbf{M} \rangle$ and ownership structure $\mathbf{r}$ with $\hat{\theta}_i^{\mathbf{Q}}(r_i) \geq \hat{\theta}_j^{\mathbf{Q}}(r_j)$, if $\mathbf{r}'$ is derived from $\mathbf{r}$ by a T-transform that shifts assets to firm i from firm j, then*

$$\Pi^{\mathbf{Q}}(\mathbf{r}') - \Pi^{\mathbf{Q}}(\mathbf{r}) < 0.$$

Proof of Lemma B.6. Take $\mathbf{Q}$ as given. Let $q_i(\theta_i)$ be firm i's interim expected allocation when its type is θ_i. Let $u_i^{\mathbf{Q}}(\theta_i, r_i)$ be firm i's interim expected payoff net of its outside option when its type is θ_i, its resource ownership is r_i, and the payment rule is such that IR is satisfied with equality for worst-off types. Assume that $\hat{\theta}_i^{\mathbf{Q}}(r_i) \geq \hat{\theta}_j^{\mathbf{Q}}(r_j)$. Using Lemma 1, we have

$$\begin{aligned}
\Pi^{\mathbf{Q}}(\mathbf{r}) &= \sum_{i\in\mathcal{N}} \Big(\mathbb{E}_{\theta_i}\Big[\Psi_i(\theta_i, \hat{\theta}_i^{\mathbf{Q}}(r_i))q_i(\theta_i)\Big] - r_i\hat{\theta}_i^{\mathbf{Q}}(r_i) - u_i^{\mathbf{Q}}(\hat{\theta}_i^{\mathbf{Q}}(r_i), r_i)\Big) \\
&= \sum_{i\in\mathcal{N}} \Big(\int_0^{\hat{\theta}_i^{\mathbf{Q}}(r_i)} \Psi_i^S(\theta)q_i(\theta)dF_i(\theta) + \int_{\hat{\theta}_i^{\mathbf{Q}}(r_i)}^1 \Psi_i^B(\theta)q_i(\theta)dF_i(\theta) - r_i\hat{\theta}_i^{\mathbf{Q}}(r_i) - u_i^{\mathbf{Q}}(\hat{\theta}_i^{\mathbf{Q}}(r_i), r_i)\Big) \\
&= \sum_{i\in\mathcal{N}} \Big(\int_0^1 \Psi_i^S(\theta)q_i(\theta)f_i(\theta)d\theta - u_i^{\mathbf{Q}}(1, r_i) - r_i\Big),
\end{aligned}$$

where the final equality uses equation (5), which implies that $u_i^{\mathbf{Q}}(1, r_i) + r_i = u_i^{\mathbf{Q}}(\hat{\theta}_i(r_i), r_i) + r_i\hat{\theta}_i^{\mathbf{Q}}(r_i) + \int_{\hat{\theta}_i^{\mathbf{Q}}(r_i)}^1 q_i(y)dy$. Differentiating the second line above with respect to r_i, the three terms involving $\frac{d\hat{\theta}_i^{\mathbf{Q}}(r_i)}{dr_i}$ cancel, $u_i^{\mathbf{Q}}(\hat{\theta}_i^{\mathbf{Q}}(r_i), r_i)$ is constant at zero as r_i changes, and we are left with $\frac{\partial \Pi^{\mathbf{Q}}(\mathbf{r})}{\partial r_i} = -\hat{\theta}_i^{\mathbf{Q}}(r_i)$. It follows from the Monotone Selection Theorem of Milgrom and Shannon (1994) that $\hat{\theta}_i^{\mathbf{Q}}(r_i)$ is increasing in r_i, and so all second partial derivatives are negative. Further, all cross-partial derivatives are zero. Thus, $\Pi^{\mathbf{Q}}$ is strictly concave (and the final line of the displayed expression above, which does not depend on $\hat{\theta}_i^{\mathbf{Q}}$, shows that this result holds irrespective of the selection of worst-off type from the set of worst-off types). Because $\Pi^{\mathbf{Q}}$ is strictly concave and $\nabla\Pi^{\mathbf{Q}}(\mathbf{r}) = (-\hat{\theta}_1^{\mathbf{Q}}, \ldots, -\hat{\theta}_n^{\mathbf{Q}})$, where we drop the dependence of $\hat{\theta}_i^{\mathbf{Q}}$ on r_i, it follows that for $\mathbf{r}'$ derived from $\mathbf{r}$ by shifing amount $\Delta > 0$ from firm j to firm i, we have

$$\begin{aligned}
\Pi^{\mathbf{Q}}(\mathbf{r}') - \Pi^{\mathbf{Q}}(\mathbf{r}) &= \Pi^{\mathbf{Q}}(r_i + \Delta, r_j - \Delta, \mathbf{r}_{-i,j}) - \Pi^{\mathbf{Q}}(r_i, r_j, \mathbf{r}_{-i,j}) \\
&< (\Delta, -\Delta, \mathbf{0}_{-i,j}) \cdot (-\hat{\theta}_1^{\mathbf{Q}}, \ldots, -\hat{\theta}_n^{\mathbf{Q}}) \\
&= -\Delta(\hat{\theta}_i^{\mathbf{Q}} - \hat{\theta}_j^{\mathbf{Q}}) \\
&\leq 0,
\end{aligned}$$

where the final inequality uses the assumption that $\hat{\theta}_i^{\mathbf{Q}} \geq \hat{\theta}_j^{\mathbf{Q}}$. which completes the proof. □

The remainder of the proof of Proposition 5 proceeds in two parts:

Part (i). Suppose that $\Pi^e(\mathbf{r}) > 0$ and there exist two traders indexed by 1 and 2 with $\eta_1 + \eta_2 \leq 1$. By virtue of the firms being traders, $0 < r_1 < k_1$ and $0 < r_2 < k_2$. Without loss of generality, we can assume that $\hat{\theta}_2^e(r_2) \leq \hat{\theta}_1^e(r_1)$. Because $r_1 < k_1$ and $0 < r_2$, there exists $\Delta > 0$ sufficiently small that the ownership vector $\tilde{\mathbf{r}}(\Delta)$ defined by $\tilde{r}_1(\Delta) \equiv r_1 + \Delta$, $\tilde{r}_2(\Delta) \equiv r_2 - \Delta$, and $\tilde{\mathbf{r}}_{-\{1,2\}}(\Delta) \equiv \mathbf{r}_{-\{1,2\}}$ is a feasible ownership vector (i.e., $r_1 + \Delta \leq k_1$ and $0 \leq r_2 - \Delta$). Further, using the continuity of Π^e and the assumption that $\Pi^e(\mathbf{r}) > 0$, there exists $\Delta > 0$ sufficiently small that $\Pi^e(\tilde{\mathbf{r}}(\Delta)) > 0$. Taking Δ to satisfy these conditions, ex post efficiency is achieved under both $\mathbf{r}$ and $\tilde{\mathbf{r}}(\Delta)$, and by Lemma B.6,

$$\Pi^e(\tilde{\mathbf{r}}(\Delta)) < \Pi^e(\mathbf{r}). \tag{B.6}$$

Defining $\hat{m}_i(r_i) \equiv \mathbb{E}_{\theta_i}\left[\Psi_{i,0}(\theta_i, \hat{\theta}_i^e(r_i))q_i^e(\theta_i)\right] - r_i\hat{\theta}_i^e(r_i)$ and noting that $\Pi^e(\mathbf{r}) = \sum_{i\in\mathcal{N}} \hat{m}_i(r_i)$, it follows that

$$\Pi^e(\mathbf{r}) - \Pi^e(\tilde{\mathbf{r}}(\Delta)) = \hat{m}_1(r_1) + \hat{m}_2(r_2) - \hat{m}_1(r_1 + \Delta) - \hat{m}_2(r_2 - \Delta). \tag{B.7}$$

Because firm i's expected net surplus under ex post efficiency is

$$u_i^e(\mathbf{r}) \equiv \mathbb{E}_{\theta_i}\left[\theta_i(q_i^e(\theta_i) - r_i)\right] - \hat{m}_i(r_i) + \eta_i\Pi^e(\mathbf{r}),$$

the change in the joint expected gross surplus of firms 1 and 2 from a change in ownership structure from $\mathbf{r}$ to $\tilde{\mathbf{r}}(\Delta)$ is

$$\begin{aligned}
& u_1^e(\tilde{\mathbf{r}}(\Delta)) + u_2^e(\tilde{\mathbf{r}}(\Delta)) - u_1^e(\mathbf{r}) - u_2^e(\mathbf{r}) + (\tilde{r}_1(\Delta) - r_1)\mathbb{E}_{\theta_1}[\theta_1] + (\tilde{r}_2(\Delta) - r_2)\mathbb{E}_{\theta_2}[\theta_2] \\
= \;& -\hat{m}_1(r_1 + \Delta) + \eta_1\Pi^e(\tilde{\mathbf{r}}(\Delta)) - \hat{m}_2(r_2 - \Delta) + \eta_2\Pi^e(\tilde{\mathbf{r}}(\Delta)) \\
& +\hat{m}_1(r_1) - \eta_1\Pi^e(\mathbf{r}) + \hat{m}_2(r_2) - \eta_2\Pi^e(\mathbf{r}) \\
= \;& (1 - \eta_1 - \eta_2)\left(\Pi^e(\mathbf{r}) - \Pi^e(\tilde{\mathbf{r}}(\Delta))\right) \\
\geq \;& 0,
\end{aligned}$$

where the first equality uses the definition of $u_i^e(\cdot)$, the second equality uses (B.7), and the inequality uses the assumption that $\eta_1 + \eta_2 \leq 1$ and (B.6). The inequality is strict if $\eta_1 + \eta_2 < 1$. Thus, the joint expected gross payoff of firms 1 and 2 increases (weakly if $\eta_1 + \eta_2 \leq 1$ and strictly if $\eta_1 + \eta_2 < 1$) as a result of shifting amount Δ of firm 2's assets to firm 1, which completes the proof of the first part of the proposition.

Part (ii). Assume, as in the statement of the proposition, that $n \in \{3, 4, \dots\}$, $\Pi^e(\mathbf{r}) = 0$, and firms 1 and 2 are traders. Without loss of generality, assume that $\hat{\theta}_1^e(r_1) \geq \hat{\theta}_2^e(r_2)$. Define ownership structure $\tilde{\mathbf{r}}(\Delta)$ by $\tilde{r}_1(\Delta) \equiv r_1 + \Delta$, $\tilde{r}_2(\Delta) \equiv r_2 - \Delta$, and $\tilde{\mathbf{r}}_{-\{1,2\}}(\Delta) \equiv \mathbf{r}_{-\{1,2\}}$, which is feasible for $\Delta \in [0, \min\{k_1 - r_1, r_2\}]$. This is a nonempty interval because firms 1 and 2 are traders. Because we are considering a shift from the firm with the weakly lower worst-off type to the firm with the weakly higher worst-off type, by Lemma B.6, for all $\Delta > 0$ in the feasible range, we have

$$\Pi^e(\tilde{\mathbf{r}}(\Delta)) < \Pi^e(\mathbf{r}) = 0. \tag{B.8}$$

Recall that given IC, IR market mechanism $\langle \mathbf{Q}, \mathbf{M} \rangle$, the expected gross payoff of firm i with type θ_i is $\theta_i q_i(\theta_i) - m_i(\theta_i)$, and by Lemma 1,

$$\mathbb{E}_{\theta_i}[m_i(\theta_i)] = \mathbb{E}_{\theta_i}[\Psi_{i,0}(\theta_i, \hat{\theta}_i)q_i(\theta_i)] - u_i(\hat{\theta}_i) - \hat{\theta}_i r_i, \tag{B.9}$$

where binding IR for the firms' worst-off types implies that $u_i(\hat{\theta}_i) = 0$.

Given $\Delta > 0$ and $\tilde{\mathbf{r}}(\Delta)$, the solution value of the Lagrangian associated with the (second-best) market mechanism $\langle \mathbf{Q}^*, \mathbf{M}^* \rangle$ is

$$\mathbb{E}_{\boldsymbol{\theta}}\Big[\sum_{i\in\mathcal{N}} \theta_i q_i^*(\theta_i; \hat{\boldsymbol{\theta}}^*(\Delta), \rho^*(\Delta))\Big] - (1-\rho^*(\Delta))\Pi^*(\tilde{\mathbf{r}}(\Delta); \hat{\boldsymbol{\theta}}^*(\Delta), \rho^*(\Delta)),$$

where $\hat{\boldsymbol{\theta}}^*(\Delta)$ and $\rho^*(\Delta)$ are, respectively, the solution values for the worst-off types and the Lagrange multiplier on the no-deficit constraint, $q_i^*(\theta_i; \hat{\boldsymbol{\theta}}^*(\Delta), \rho^*(\Delta))$ is the solution value for the interim expected allocation rule for firm i, and, letting $m_i^*(\theta_i; \tilde{\mathbf{r}}(\Delta), \hat{\boldsymbol{\theta}}^*(\Delta), \rho^*(\Delta))$ be the solution value for the interim expected payment rule for firm i, the solution value for the expected budget surplus is

$$\Pi^*(\tilde{\mathbf{r}}(\Delta), \hat{\boldsymbol{\theta}}^*(\Delta), \rho^*(\Delta)) \equiv \sum_{i\in\mathcal{N}} \mathbb{E}_{\theta_i}\Big[m_i^*(\theta_i; \tilde{\mathbf{r}}(\Delta), \hat{\boldsymbol{\theta}}^*(\Delta), \rho^*(\Delta))\Big].$$

Thus, given Δ, the expected gross payoff of firm i is

$$\tilde{u}_i(\Delta) \equiv \mathbb{E}_{\theta_i}\left[\left(\theta_i - \Psi_{i,0}(\theta_i, \hat{\theta}_i^*(\Delta))\right) q_i^*(\theta_i; \hat{\boldsymbol{\theta}}^*(\Delta), \rho^*(\Delta))\right] + \hat{\theta}_i^*(\Delta)\tilde{r}_i(\Delta).$$

Using this and (B.9), we have

$$\Pi^*(\tilde{\mathbf{r}}(\Delta), \hat{\boldsymbol{\theta}}^*(\Delta), \rho^*(\Delta)) = \sum_{i\in\mathcal{N}} \left(\mathbb{E}_{\theta_i}\Big[\Psi_{i,0}(\theta_i; \hat{\theta}_i^*(\Delta)) q_i^*(\theta_i; \hat{\boldsymbol{\theta}}^*(\Delta), \rho^*(\Delta))\Big] - \tilde{u}_i(\Delta) - \hat{\theta}_i^*(\Delta)\tilde{r}_i(\Delta)\right).$$

Given the assumption that $F_i = F$ for all $i \in \mathcal{N}$, the interim expected allocation rule for each firm i has the form illustrated in Figure B.1, and, correspondingly, we have

$$q_i^*(\theta_i; \hat{\boldsymbol{\theta}}^*(\Delta), \rho^*(\Delta)) = \begin{cases} r_i & \text{if } \underline{z}_i(\rho^*(\Delta)) \le \theta_i \le \overline{z}_i(\rho^*(\Delta)), \\ q_i^e(\theta_i) & \text{otherwise,} \end{cases} \tag{B.10}$$

where $[\underline{z}_i(\rho^*(\Delta)), \overline{z}_i(\rho^*(\Delta))]$ is the ironing range for firm i and for ρ sufficiently close to 1,

$$\underline{z}_i'(\rho) < 0 \quad \text{and} \quad \overline{z}_i'(\rho) > 0. \tag{B.11}$$

Further, $\hat{\theta}_i^*(\Delta) \in [\underline{z}_i(\rho^*(\Delta)), \overline{z}_i(\rho^*(\Delta))]$ and $r_i > q_i^e(\theta_i)$ for $\theta_i < \hat{\theta}_i^*(\Delta)$ and $r_i < q_i^e(\theta_i)$ for $\theta_i > \hat{\theta}_i^*(\Delta)$, as illustrated in Figure B.1.

To establish that the envisioned transaction between firms 1 and 2 is strictly mutually beneficial, we need to show that for $\Delta > 0$ sufficiently small,

$$\sum_{i\in\{1,2\}} \tilde{u}_i(\Delta) > \sum_{i\in\{1,2\}} \tilde{u}_i(0). \tag{B.12}$$

By (B.8), the first-best is not possible. Thus, under the second-best, the solution value for the mechanism's budget surplus, Π^*, satisfies

$$\Pi^*(\tilde{\mathbf{r}}(\Delta); \hat{\boldsymbol{\theta}}^*(\Delta), \rho^*(\Delta)) = 0. \tag{B.13}$$

(a) Virtual type functions (b) Interim expected allocation

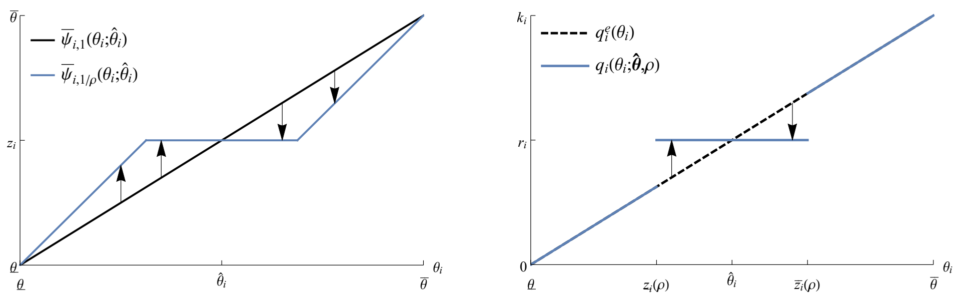

Figure B.1: Illustration of the effects of an increase in ρ above 1 on the ironed weighted virtual types and interim expected allocation. Assumes $\rho > 1$.

Further, because $\Pi^e(\tilde{\mathbf{r}}(0)) = \Pi^e(\mathbf{r}) = 0$, it follows that $\rho^*(0) = 1$ and that $\rho^*(\Delta)$ increases as Δ increases above 0, i.e.

$$\left.\frac{\partial \rho^*(\Delta)}{\partial \Delta}\right|_{\Delta \downarrow 0} \geq 0 \text{ and } \rho^*(\Delta) > \rho^*(0) \text{ for } \Delta > 0. \tag{B.14}$$

Using the definitions of $\tilde{u}_i(\Delta)$ and $\Pi^*(\tilde{\mathbf{r}}(\Delta); \hat{\boldsymbol{\theta}}^*(\Delta), \rho^*(\Delta))$, and dropping the argument Δ for $\hat{\boldsymbol{\theta}}^*(\Delta)$ and $\rho^*(\Delta)$, we can write

$$\begin{aligned}\sum_{i\in\{1,2\}} \tilde{u}_i(\Delta) &= \sum_{i\in\{1,2\}} \mathbb{E}_{\theta_i}\left[\theta_i q_i^*(\theta_i; \hat{\boldsymbol{\theta}}^*, \rho^*)\right] - \Pi^*(\tilde{\mathbf{r}}(\Delta); \hat{\boldsymbol{\theta}}^*, \rho^*) \\ &\quad + \sum_{j\in\mathcal{N}\setminus\{1,2\}} \left(\mathbb{E}_{\theta_j}\left[\Psi_{j,0}(\theta_j, \hat{\theta}_j^*) q_j^*(\theta_j; \hat{\boldsymbol{\theta}}^*, \rho^*)\right] - \hat{\theta}_j^* \tilde{r}_j(\Delta)\right) \qquad \text{(B.15)} \\ &= \sum_{i\in\{1,2\}} \mathbb{E}_{\theta_i}\left[\theta_i q_i^*(\theta_i; \hat{\boldsymbol{\theta}}^*, \rho^*)\right] + \sum_{j\in\mathcal{N}\setminus\{1,2\}} \left(\mathbb{E}_{\theta_j}\left[\Psi_{j,0}(\theta_j, \hat{\theta}_j^*) q_j^*(\theta_i; \hat{\boldsymbol{\theta}}^*, \rho^*)\right] - \hat{\theta}_j^* r_j\right),\end{aligned}$$

where the second equality uses (B.13) and $\tilde{r}_j(\Delta) = r_j$ for $j \in \mathcal{N}\setminus\{1,2\}$. Thus, the joint expected payoff of firms 1 and 2 is equal to their expected utility from consumption plus the expected payments by their rivals.

By construction of the virtual type functions and $q_i^*(\hat{\theta}_i^*; \hat{\boldsymbol{\theta}}^*, \rho^*) = r_i$, we have:[2]

$$\frac{\partial}{\partial \hat{\theta}_i}\left(\mathbb{E}_{\theta_i}\left[\Psi_{i,0}(\theta_i, \hat{\theta}_i^*) q_i^*(\theta_i; \hat{\boldsymbol{\theta}}^*, \rho^*)\right] - \hat{\theta}_i^* r_i\right) = \int_{\underline{\theta}}^{\overline{\theta}} \Psi_{i,0}(\theta_i, \hat{\theta}_i^*) \frac{\partial q_i^*(\theta_i; \hat{\boldsymbol{\theta}}^*, \rho^*)}{\partial \hat{\theta}_i} dF_i(\theta_i). \tag{B.16}$$

[2]To see this, note that:

$$\begin{aligned}&\frac{\partial}{\partial \hat{\theta}_i}\left(\mathbb{E}_{\theta_i}\left[\Psi_{i,0}(\theta_i, \hat{\theta}_i) q_i(\theta_i; \hat{\boldsymbol{\theta}}, \rho)\right] - \hat{\theta}_i r_i\right) = \frac{\partial}{\partial \hat{\theta}_i}\left(\int_{\underline{\theta}}^{\hat{\theta}_i} \Psi_{i,0}^S(\theta_i) q_i(\theta_i; \hat{\boldsymbol{\theta}}, \rho) dF_i(\theta_i) + \int_{\hat{\theta}_i}^{\overline{\theta}} \Psi_{i,0}^B(\theta_i) q_i(\theta_i; \hat{\boldsymbol{\theta}}, \rho) dF_i(\theta_i)\right) - r_i \\ &= (\Psi_{i,0}^S(\hat{\theta}_i) - \Psi_{i,0}^B(\hat{\theta}_i)) q_i(\hat{\theta}_i; \hat{\boldsymbol{\theta}}, \rho) f_i(\hat{\theta}_i) + \int_{\underline{\theta}}^{\hat{\theta}_i} \Psi_{i,0}^S(\theta_i) \frac{\partial q_i(\theta_i; \hat{\boldsymbol{\theta}}, \rho)}{\partial \hat{\theta}_i} dF_i(\theta_i) + \int_{\hat{\theta}_i}^{\overline{\theta}} \Psi_{i,0}^B(\theta_i) \frac{\partial q_i(\theta_i; \hat{\boldsymbol{\theta}}, \rho)}{\partial \hat{\theta}_i} dF_i(\theta_i) \\ &= q_i(\hat{\theta}_i; \hat{\boldsymbol{\theta}}, \rho) + \int_{\underline{\theta}}^{\overline{\theta}} \Psi_{i,0}(\theta_i, \hat{\theta}_i) \frac{\partial q_i(\theta_i; \hat{\boldsymbol{\theta}}, \rho)}{\partial \hat{\theta}_i} dF_i(\theta_i) - r_i = \int_{\underline{\theta}}^{\overline{\theta}} \Psi_{i,0}(\theta_i, \hat{\theta}_i) \frac{\partial q_i(\theta_i; \hat{\boldsymbol{\theta}}, \rho)}{\partial \hat{\theta}_i} dF_i(\theta_i).\end{aligned}$$

Using (B.16) and noting that $q_i^*(\theta_i;\hat{\boldsymbol{\theta}},\rho)$ is independent of $\hat{\boldsymbol{\theta}}$ when $\rho=1$, it then follows that the derivative of the right side of (B.15) with respect to $\hat{\boldsymbol{\theta}}$ is zero when evaluated at $\Delta=0$. Thus, when considering the effect of a marginal change in Δ on $\sum_{i\in\{1,2\}}\tilde{u}_i(\Delta)$ at $\Delta=0$, we need only consider effects that come through $\rho^*(\Delta)$. But note that, by the envelope theorem, the partial derivative of

$$\mathcal{L}(\rho)=\mathbb{E}_{\boldsymbol{\theta}}\left[\sum_{i\in\mathcal{N}}\theta_i q_i^*(\theta_i;\hat{\boldsymbol{\theta}}^*,\rho)\right]-(1-\rho)\Pi^*(\tilde{\mathbf{r}}(\Delta);\hat{\boldsymbol{\theta}}^*,\rho)+\sum_{i\in\mathcal{N}}\mu_i u_i(\hat{\theta}_i^*)$$

with respect to ρ evaluated at $\rho=\rho^*$ is 0, so we have

$$\frac{\partial}{\partial\rho}\sum_{i\in\mathcal{N}}\mathbb{E}_{\theta_i}\left[\theta_i q_i^*(\theta_i;\hat{\boldsymbol{\theta}}^*,\rho)\right]\Bigg|_{\rho=\rho^*}=0. \tag{B.17}$$

Thus, differentiating (B.15) with respect to Δ, we are left with only the effects that come through $\rho^*(\Delta)$ (as mentioned above), and we obtain, noting that $\hat{\theta}_j^*=\hat{\theta}_j^e$ at $\Delta=0$,

$$\begin{aligned}
\sum_{i\in\{1,2\}}\tilde{u}_i'(0) &= \rho^{*\prime}(0)\frac{\partial}{\partial\rho}\left(\sum_{i\in\{1,2\}}\mathbb{E}_{\theta_i}\left[\theta_i q_i^*(\theta_i;\hat{\boldsymbol{\theta}}^*,\rho)\right]+\sum_{j\in\mathcal{N}\setminus\{1,2\}}\mathbb{E}_{\theta_j}\left[\Psi_{j,0}(\theta_j,\hat{\theta}_j)q_j^*(\theta_j;\hat{\boldsymbol{\theta}}^*,\rho)\right]\right)\Bigg|_{\rho=\rho^*(0)}\\
&= \rho^{*\prime}(0)\frac{\partial}{\partial\rho}\left(\sum_{i\in\mathcal{N}}\mathbb{E}_{\theta_i}\left[\theta_i q_i^*(\theta_i;\hat{\boldsymbol{\theta}}^*,\rho)\right]+\sum_{j\in\mathcal{N}\setminus\{1,2\}}\mathbb{E}_{\theta_j}\left[\left(\Psi_{j,0}(\theta_j,\hat{\theta}_j)-\theta_j\right)q_j^*(\theta_j;\hat{\boldsymbol{\theta}}^*,\rho)\right]\right)\Bigg|_{\rho=\rho^*(0)}\\
&= \rho^{*\prime}(0)\frac{\partial}{\partial\rho}\left(\sum_{j\in\mathcal{N}\setminus\{1,2\}}\mathbb{E}_{\theta_j}\left[\left(\Psi_{j,0}(\theta_j,\hat{\theta}_j)-\theta_j\right)q_j^*(\theta_j;\hat{\boldsymbol{\theta}}^*,\rho)\right]\right)\Bigg|_{\rho=\rho^*(0)}\\
&= \rho^{*\prime}(0)\frac{\partial}{\partial\rho}\left(\sum_{j\in\mathcal{N}\setminus\{1,2\}}\left(\int_{\underline{\theta}}^{\hat{\theta}_j^*}F_j(\theta_j)q_j^*(\theta_j;\hat{\boldsymbol{\theta}}^*,\rho)d\theta_j-\int_{\hat{\theta}_j^*}^{\overline{\theta}}\left(1-F_j(\theta_j)\right)q_j^*(\theta_j;\hat{\boldsymbol{\theta}}^*,\rho)d\theta_j\right)\right)\Bigg|_{\rho=\rho^*(0)}\\
&= \rho^{*\prime}(0)\frac{\partial}{\partial\rho}\left(\sum_{j\in\mathcal{N}\setminus\{1,2\}}\left(\int_{\underline{\theta}}^{\underline{z}_j(\rho)}F_j(\theta_j)q_j^e(\theta_j)d\theta_j+\int_{\underline{z}_j(\rho)}^{\hat{\theta}_j^*}F_j(\theta_j)r_j d\theta_j\right.\right.\\
&\qquad\left.\left.-\int_{\hat{\theta}_j^*}^{\overline{z}_j(\rho)}\left(1-F_j(\theta_j)\right)r_j d\theta_j-\int_{\overline{z}_j(\rho)}^{\overline{\theta}}\left(1-F_j(\theta_j)\right)q_j^e(\theta_j)d\theta_j\right)\right)\Bigg|_{\rho=\rho^*(0)}\\
&= \rho^{*\prime}(0)\sum_{j\in\mathcal{N}\setminus\{1,2\}}\left(\underbrace{\underline{z}_j'(\rho^*(0))}_{\text{negative}}F_j(\underline{z}_j(\rho^*(0)))\underbrace{\left(q_j^e(\underline{z}_j(\rho^*(0)))-r_j\right)}_{\text{negative}}\right.\\
&\qquad\left.-\underbrace{\overline{z}_j'(\rho^*(0))}_{\text{positive}}\left(1-F_j(\overline{z}_j(\rho^*(0)))\right)\underbrace{\left(r_j-q_j^e(\overline{z}_j(\rho^*(0)))\right)}_{\text{negative}}\right)\\
&> 0,
\end{aligned}$$

where the first equality uses the definition of $\tilde{u}_i$, the second equality rearranges, the third equality uses (B.17), the fourth equality uses the definition of $\Psi_{i,0}$, the fifth equality uses (B.10), the sixth equality differentiates and rearranges, and the inequality uses the observations above, including (B.11) and (B.14), which hold strictly for $\Delta>0$ sufficiently small.

Thus, $\sum_{i\in\{1,2\}} \tilde{u}_i'(0) \geq 0$ and for $\Delta > 0$ sufficiently small $\sum_{i\in\{1,2\}} \tilde{u}_i(\Delta) > \sum_{i\in\{1,2\}} \tilde{u}_i(0)$, which implies that transactions between firms 1 and 2 of $\Delta > 0$ sufficiently small are mutually beneficial. By Lemma B.6, such transactions result in $\Pi^e(\tilde{\mathbf{r}}(\Delta)) < 0$, which completes the proof. ■

C Appendix: Extensions and discussion

C.1 Extension to multiple periods

Here, we define a T-period extension of the static model without discounting and with one-dimensional types. Fix an ownership structure $\mathbf{r}$. Within each period, types are realized (independently across firms and time), firms participate in the market to lease assets to or from other firms, and then firms realize payoffs associated with their total asset holdings (initial assets, minus assets leased to others, plus assets leased from others). At the end of the period, the leases expire and firms' asset holdings revert to the ownership structure $\mathbf{r}$.

If one assumes that a firm's participation decision in one period has no implications for whether it can participate in future periods, then the IR constraints in each period remain the same as in our static model. Thus, the ex post efficiency permitting set remains simply $\mathcal{R}^e$.

If, instead, a firm that chooses not to participate in period t cannot participate in any future period, then firm i's IR constraint is relaxed. Firm i participates in period t as long as $u_i(\theta_i) \geq -(T-t)\mathbb{E}_{\theta_i}[u(\theta_i)]$, rather than simply as long as $u_i(\theta_i) \geq 0$. Thus, the set of ownership structures that result in ex post efficiency in period $t < T$ is a superset of $\mathcal{R}^e$. However, only ownership structures in $\mathcal{R}^e$ permit ex post efficiency in every period. In this sense, the key features of the static setup extend to the multi-period model.

C.2 Extension to nonidentical supports

In this appendix, we provide additional details related to the extension to nonidentical supports discussed in Section 6.3. This extension also allows us to connect with prior results related to vertical integration.

As described in Section 6.3, we assume that firms are divided into a set $\mathcal{N}_U$ of $N_U \geq 1$ "upstream" sellers with support $[0,1]$ and a set $\mathcal{N}_D$ of $N_D \geq 1$ "downstream" buyers with support $[\underline{\theta}, 1+\underline{\theta}]$, where prior to integration, all firms have one-dimensional types and have maximum demands of k. Thus, each seller i has $r_i = k$ and each buyer i has $r_i = 0$, which implies that $k = 1/N_U$.

In this context, integration between an upstream seller s (i.e., $s \in \mathcal{N}_U$ with $r_s = k$) and a downstream buyer b (i.e., $b \in \mathcal{N}_D$ with $r_b = 0$) creates an integrated firm i that is a trader with $r_i = k$ and $k_i = 2k$, which is naturally thought of as a vertically integrated firm. Consequently, in what follows, we use the term *vertical integration* to refer to full integration between an upstream seller and a downstream buyer.

Case I. *If $N_U = 1$, $N_D = 1$, and $\underline{\theta} < 1$ (overlapping supports), then vertical integration permits ex post efficiency whereas prior to full integration ex post efficiency is not possible.*
In the case of one upstream seller and one downstream buyer and $\underline{\theta} < 1$, because the supports overlap, ex post efficiency is not possible by Myerson and Satterthwaite (1983). But, because full integration eliminates an agency problem within the firm, ex post efficiency is achieved following a merger of the buyer and seller. Thus, as observed in Loertscher and Marx (2022, Proposition 6), in this case vertical integration increases social surplus.

Case II. *If $N_U > 1$, $N_D = 1$, and $\underline{\theta} \geq 1$ (nonoverlapping supports), then vertical integration reduces social surplus because ex post efficiency is possible before but not after vertical integration.*
We obtain a contrasting result for the setting with one downstream buyer, multiple upstream sellers, and $\underline{\theta} \geq 1$. In the pre-integration market, ex post efficiency is possible—for example based on a second-price auction with reserve $p \in [1, \underline{\theta}]$. Integration between the downstream buyer and one upstream seller leaves us with $N_U - 1 \geq 1$ sellers and one integrated firm that has a two-dimensional type and k units of the asset. Ex post efficiency requires that k units be allocated to the integrated firm's buyer-side type, which is drawn from $[\underline{\theta}, 1 + \underline{\theta}]$, and for the remaining $1 - k$ units of supply to be allocated to the N_U largest seller types, whether that is the integrated firm's seller-side type or one of the independent sellers' types. In this setting, one can essentially remove the integrated buyer and the integrated firm's k units from the problem because that allocation occurs for all type realizations. What remains then are N_U entities all with the same support, where one (the seller side of the integrated firm) acts as a buyer and the remaining as sellers. This is a two-sided setting with a common support in which ex post efficiency is not possible (Delacrétaz et al., 2019). Thus, in this case, as in Loertscher and Marx (2022, Proposition 7), vertical integration decreases social surplus.

Case III. *If $N_U > 1$, $N_D = 1$, and $\underline{\theta} \in [0, 1)$ (overlapping supports), then the social surplus effects of vertical integration depend, in general, on the number of sellers.*
As in Loertscher and Marx (2022, Online Appendix F.2.B), with one buyer, multiple sellers, and overlapping supports, the social surplus effects of vertical integration depend, in general, on the number of sellers. We know from Williams (1999) and Makowski and Mezzetti (1993) that ex post efficiency is possible with nonidentical supports if N_U is large enough. Because vertical integration between buyer 1 and seller 1 creates an integrated firm that is a buyer

(because all other firms are sellers, the integrated firm can only act as a buyer) with willingness to pay for k units from the independent sellers of $\min\{v_1, c_1\}$ whose support is $[\underline{\theta}, 1]$, where v_1 (c_1) is the buyer's value (seller's cost). The results of Williams (1999) for this case imply that ex post efficiency is not possible. Hence, vertical integration is socially harmful whenever N_S and $\underline{\theta}$ are such that ex post efficiency is possible pre-integration. With identical supports, ex post efficiency is not possible with or without vertical integration if $N_U > 1$ (see, e.g., Williams, 1999). Further, with identical supports and uniformly distributed types, as the number of sellers grows large, the change in social surplus due to vertical integration is nonmonotone in the number of outside suppliers and, in the limit, approaches zero from below (see Loertscher and Marx (2022, Figure F.1(a))).

So far, these results have stayed within a "one-to-many" setting. The focus on one buyer and multiple sellers ensures that the post-integration firm can be viewed as effectively having a one-dimensional type because, in the absence of any other buyers or traders to sell to, the integrated firm can only act as a buyer vis à vis the other firms, so only its maximum willingness to pay for an external unit is relevant.

However, the methodology developed in this paper gives us the ability to go beyond the one-to-many cases and to consider many-to-many markets, thereby extending the results of Loertscher and Marx (2022) on vertical integration.

Case IV. *If $N_U = N_D$ and $\underline{\theta} \geq 1$ (nonoverlapping supports), then vertical integration does not affect social surplus because ex post efficiency is possible before and after vertical integration.*

With an equal number N of upstream sellers and downstream buyers and $\underline{\theta} \geq 1$, ex post efficiency is possible. For example, trade at a fixed price $p \in [1, \underline{\theta}]$ achieves ex post efficiency. Following the integration of one buyer and one seller, ex post efficiency continues to require that the buyer-side type of the integrated firm receives k units. We can think of the integrated firm's k units as allocated to its buyer-side type, leaving us with $N - 1$ downstream buyers with supports $[\underline{\theta}, 1 + \underline{\theta}]$ and maximum demands of k, and $k(N-1)$ units to be allocated to them from the upstream sellers with supports $[0, 1]$ (the seller-side of the integrated firm acts a buyer with support $[0, 1]$ and so is never allocated anything). Ex post efficiency can then be achieved in this residual market with a posted price $p \in [1, \underline{\theta}]$, with the result that vertical integration in this case is neutral for social surplus.

Case V. *If $N_U > N_D \geq 2$ and $\underline{\theta} \geq 1$ (nonoverlapping supports), then whether vertical integration increases or decreases social surplus depends on market details.*

With $N_U > N_D \geq 2$, and $\underline{\theta} \geq 1$, ex post efficiency is possible in the pre-integration market, for example based on a posted price of $p \in [1, \underline{\theta}]$. Following integration between an upstream seller and a downstream buyer, ex post efficiency requires that k units be allocated to the integrated firm's buyer type, and $k(N_D - 1)$ to the independent buyers. The remaining $k(N_U - N_D)$ units must then be allocated to the firms with the highest $N_U - N_D$ types among the $N_S - 1$ independent sellers and the integrated firm's seller type. A payment of $\underline{\theta}$ can be required from each of the independent buyers, which may or may not be sufficient to "grease the wheels" for the remaining transactions, which must occur among firms with a common support.

As an example, consider the case with $N_U = 3$ and $N_D = 2$. Let c_1 denote the type of the integrated seller and c_2 and c_3 denote the types of the independent sellers. Then ex post revenue based on VCG payments with binding IR for the firms' worst-off types, where the integrated firm's worst-off type is $(\underline{\theta}, 0)$, is $k(\underline{\theta} + \max\{c_2, c_3\} - 2\max\{c_1, c_2, c_3\})$. Thus, revenue is increasing in $\underline{\theta}$ and positive for all type realizations if $\underline{\theta} \geq 2$. For $\underline{\theta} \in [0, 2)$, the sign of expected revenue depends on the distributions of the sellers' types. For example, with $\underline{\theta} = 1$ and uniformly distributed types, expected revenue is negative, so ex post efficiency is not possible post-integration because the no-deficit constraint cannot be satisfied. However, if instead the sellers' costs are drawn from the distribution $G(c) = c^{1/4}$ (with expected value of $1/5$), then expected revenue is positive and so the no-deficit constraint is satisfied, giving us the result that ex post efficiency is possible. This establishes that in this case the effects of vertical integration can go either way and depend, in general, on $\underline{\theta}$ and the sellers' type distributions.

Figure C.1 illustrates that for $N_U > 2 = N_D$ and $\underline{\theta} = 1$, the expected revenue in the post-integration market under binding IR for the firms' worst-off type varies with the number of upstream sellers N_U. Using $(\mathbf{x})_{(j)}$ to denote the j-th highest element of $\mathbf{x}$ and assuming that $N_U > N_D$, ex post revenue as a function of N_U and N_D is given by

$$\frac{1}{N_U}\left((N_D - 1)\underline{\theta} + \sum_{j=1}^{N_U - N_D} (\mathbf{c}_{-1})_{(j)} - (N_U - 1)\sum_{j=1}^{N_U - N_D} (\mathbf{c})_{(j)} + \sum_{i=2}^{N_U}\sum_{j=1}^{N_U - N_D - 1} (\mathbf{c}_{-i})_{(j)}\right). \quad \text{(C.18)}$$

As N_U grows large, the sums of all but the j lowest order statistics for $j \in \{1, \ldots, N_D + 1\}$ approach $(N_U - j)\mu$, where μ is the expected cost of a seller. Thus, as N_U grows large, (C.18) approaches $\frac{\underline{\theta} - (N_D - 1)\mu}{N_U}$, which then approaches zero, and from above if $N_D = 2$, as illustrated in Figure C.1. Further, increasing the number of downstream firms has the effect of increasing the number of firms that pay $\underline{\theta}$, which for $\underline{\theta}$ sufficiently large increases expected revenue and makes vertical integration less harmful.

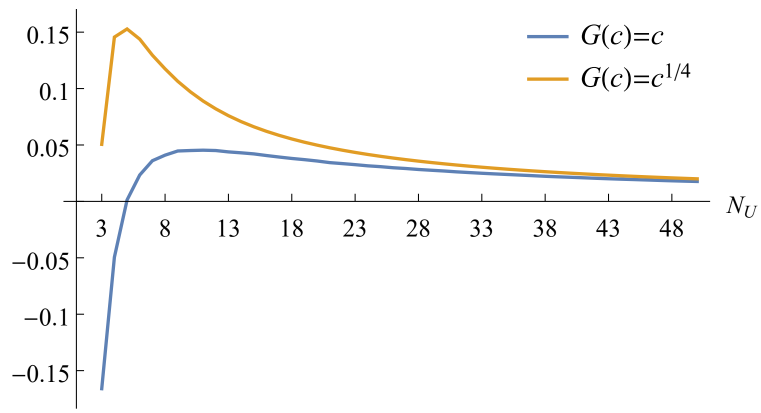

Figure C.1: Expected revenue following the integration of one upstream seller with one downstream buyer under the ex post efficient allocation and binding IR for firms' worst-off types. Assumes $N_U > 2 = N_D$, $\underline{\theta} = 1$, and common maximum demands of $k = 1/N_U$. Sellers' costs are drawn from distribution G on $[0, 1]$, and buyers' values are drawn from the uniform distribution on $[1, 2]$.

C.3 Optimal ownership and revenue under ex post efficiency

Much of the discussion above has focused on whether ex post efficiency is possible. This analysis provides the foundation for the guidance that divestitures should, if possible, be designed to secure ownership structures in $\mathcal{R}^e$. But to the extent that unmodeled transactions costs or market frictions are present, a competition authority might have a preference for ownership structures that are not just an element of $\mathcal{R}^e$, but that are robust to such unmodeled costs as best possible. This can be achieved with the ownership structure $\mathbf{r}^*$, which maximizes expected revenue under ex post efficiency and binding IR for the firms' worst-off types. This raises the question of how $\mathbf{r}^*$ varies with the size and strength of firms in the market. The possibility of differences in firms' maximum demands allows for differences in firm sizes, and differences in firms' distributions can be thought of as differences in productivity across the firms, and across types within a firm.

Consider firms i and j with the same dimensionality of their types, $h_i = h_j = h$, and maximum demand vectors $\mathbf{k}_i = (k_i^1, \ldots, k_i^h)$ and $\mathbf{k}_j = (k_j^1, \ldots, k_j^h)$. All else equal between firms i and j, $\mathbf{k}_i \geq \mathbf{k}_j$ and $\mathbf{k}_i \neq \mathbf{k}_j$ imply that $q_i^e(\boldsymbol{\theta}) > q_j^e(\boldsymbol{\theta})$ for $\boldsymbol{\theta} \in (0,1)^h$. Further, all else equal between firms i and j, if F_i^ℓ first-order stochastically dominates F_j^ℓ for all ℓ and $F_i^\ell \neq F_j^\ell$ for at least one ℓ, then $q_i^e(\boldsymbol{\theta}) > q_j^e(\boldsymbol{\theta})$ for $\boldsymbol{\theta} \in (0,1)^h$. Combining these observations, we have the following result:[3]

Proposition C.1. *All else equal, firms with larger maximum demands or stronger dis-*

[3]For a proof for the case of one-dimensional types, see Liu et al. (forth., Proposition 2). See also the related result of Che (2006) for one-dimensional types, $k_i = 1$ for all $i \in \mathcal{N}$, and distributions ranked by first-order stochastic dominance.

tributions according to first-order stochastic dominance have larger asset ownership in $\mathbf{r}^*$*: that is, given firms* i *and* j *with* h*-dimensional types: (i) assuming that* $F_i^\ell = F_j^\ell$ *for all* $\ell \in \{1, \ldots, h\}$*, if* $\mathbf{k}_i \geq \mathbf{k}_j$ *and* $\mathbf{k}_i \neq \mathbf{k}_j$*, then* $r_i^* > r_j^*$*; and (ii) assuming that* $\mathbf{k}_i = \mathbf{k}_j$*, if* F_i^ℓ *first-order stochastically dominates* F_j^ℓ *for all* $\ell \in \{1, \ldots, h\}$ *and* $F_i^\ell \neq F_j^\ell$ *for some* ℓ*, then* $r_i^* > r_j^*$*.*

Proposition C.1 offers guidance for a divestiture strategy that, in an abundance of caution, strives to identify an ownership structure that is maximally robust to unmodeled market frictions. In this case, the target asset ownership structure should be $\mathbf{r}^*$, which gives relatively more asset ownership to firms with relatively greater maximum demands and to firms with relatively greater value for their use. Further, if firms are ex ante identical, that is, have identical maximum demands and distributions, then this robustness criteria is met by having symmetric asset ownership. This provides a rationale for divestitures in markets with symmetric firms that promote symmetric asset ownership.

An open question of practical relevance remains for mergers such that ex post efficiency is possible neither before nor after the merger and such that a second-best analysis is not available because the multi-dimensionality of the integrated firm's type is nontrivial. Of course, if a post-merger divestiture exists that makes ex post efficiency possible, then a social surplus maximizing authority should approve the merger and require such a divestiture because the merger cum divestiture offers the opportunity to increase social surplus. So, the open issue pertains to the subset of mergers such that ex post efficiency is possible neither before nor after the merger, even with a divestiture, and when the second-best mechanism post merger is not known. A natural and feasible way of evaluating such a merger would be to compare expected revenue under ex post efficiency before and after the merger, which in either case is negative because ex post efficiency is impossible. If the merger increases that revenue, then a natural rule would be to approve the merger, and to otherwise block it.[4] This rule is in the spirit of the above discussion of divestiture strategies and of the paper more broadly insofar as larger revenue under ex post efficiency offers more "leeway" for the market to operate "well." It also has a (partial) foundation in some one-dimensional setups, where the second-best mechanism is known. For example, with ex ante identical firms, both social surplus and expected revenue under ex post efficiency are Schur-concave. Thus, if revenue under ex post efficiency increases because the ownership structure becomes more symmetric, then social surplus increases as well. Because majorization is a partial order, revenue can increase without the ownership structure becoming more symmetric, and reliability of the test would require social surplus to increase as well in such situations. While for models

[4]Of course, the rule could be augmented by also considering all possible divestitures post-merger and to evaluate expected revenue after both the merger and the divestitures.

with interior ownership assessing whether this is the case is challenging even numerically, it is clearly the case in a bilateral trade setting à la Myerson and Satterthwaite (1983) in which the buyer's support shifts upward.[5]

In what follows, we provide an illustration of its application beyond this case. Specifically, we provide an illustration in a partnership setup, which is essentially one-dimensional, of the rule of approving a merger if the merger increases revenue under ex post efficiency. Consider the case of three firms and suppose that pre-merger ownership is $\mathbf{r} = (0.85, 0.1, 0.05)$. Figure C.2, particularly panel (b) indicates that the iso-expected social surplus curve associated with $\mathbf{r} = (0.85, 0.1, 0.05)$ lies largely, if not entirely, to the right of the iso-ex post efficient revenue curve through that point. Thus, increases in ex post efficient revenue due to merger imply increases in expected social surplus.

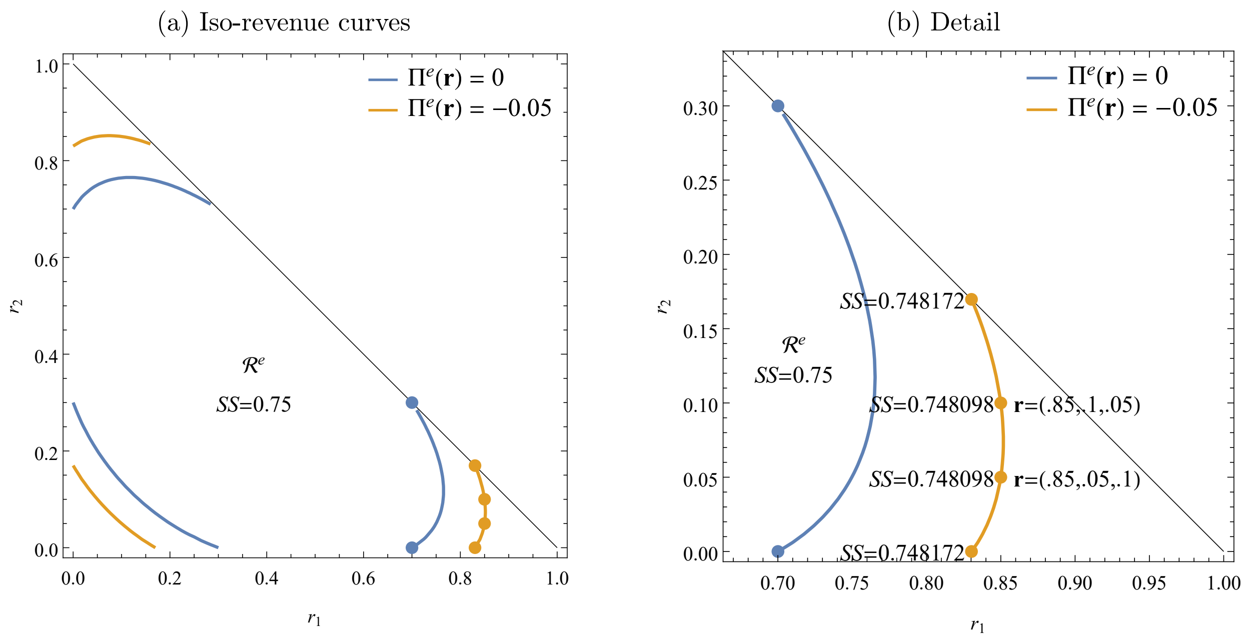

Figure C.2: Iso-ex post efficient revenue curves, i.e., constant $\Pi^e(\mathbf{r})$, and expected second-best social surplus. Assumes $n = 3$, $k_i = 1$, and uniformly distributed types. The iso-revenue curve for revenue -0.050976 includes $\mathbf{r} = (0.85, 0.05, 0.1)$ and $\mathbf{r} = (0.830275, 0.169725, 0)$, and the related vectors by symmetry. At $\mathbf{r} = (0.830275, 0.169725, 0)$, we have $\rho^* = 1.078037$ and the worst-off types are $\hat{\boldsymbol{\theta}} = (0.877443, 0.413357, 0)$. At $\mathbf{r} = (0.85, 0.05, 0.1)$, we have $\rho^* = 1.081305$ and $\hat{\boldsymbol{\theta}} = (0.886987, 0.258573, 0.318155)$.

[5] To be precise, assume firm 1 is a buyer whose type is drawn from a distribution with shifting support $[\underline{\theta}, \underline{\theta} + 1]$ and firm 2 is a seller whose type is drawn from a distribution with support $[0, 1]$. Ex post revenue under ex post efficiency and DIC is $\max\{\theta_2, \underline{\theta}\} - \min\{\theta_1, 1\}$, which increases in $\underline{\theta}$, and expected revenue increases because of that and because the probability that the seller is paid less than θ_1 increases as $\underline{\theta}$ increases. At the same time, social surplus under the second-best mechanism increases as $\underline{\theta}$ increases. That said, revenue under ex post efficiency is not a universally reliable indicator for the performance of the second-best mechanism; see, for example, Loertscher and Marx (2023, Propositions 4 and 5) and the discussion after Theorem 1 above.

C.4 Details for transforming a setting with multi-dimensional types into one with one-dimensional types

In this appendix, we provide details related to the discussion in Section 6.4 on how the two-dimensional types that arise due to the full integration of two firms might reasonably be transformed into a one-dimensional type.

Consider a setting with three pre-merger firms with one-dimensional types, ex post efficient allocation $\mathbf{Q}^e(\boldsymbol{\theta})$, and ex post welfare $W(\boldsymbol{\theta})$. We analyze the merger of firms 1 and 2 to create a post-merger firm with a two-dimensional type and asset ownership $r_{1,2} \equiv r_1 + r_2$. For this post-merger setting, the ex post efficient allocation for the merged firm is $Q^e_{1,2}(\boldsymbol{\theta}) \equiv Q^e_1(\boldsymbol{\theta}) + Q^e_2(\boldsymbol{\theta})$. The worst-off types (ω, ω) for the merged firm and $\hat{\theta}_3$ for firm 3 satisfy $q^e_{1,2}(\omega, \omega) = r_1 + r_2$ and $q^e_3(\hat{\theta}_3) = r_3$. Thus, the VCG revenue accounting for the merged entity's two-dimensional type is

$$\begin{aligned}\Pi^e(r_{1,2}, r_3) \;\;=\;\; & \mathbb{E}_{\boldsymbol{\theta}}[W(\omega,\omega,\theta_3) - W(\theta_1,\theta_2,\theta_3) + Q^e_1(\theta_1,\theta_2,\theta_3)\theta_1 + Q^e_2(\theta_1,\theta_2,\theta_3)\theta_2] - \omega r_{1,2} \\ & + \mathbb{E}_{\boldsymbol{\theta}}[W(\theta_1,\theta_2,\hat{\theta}_3) - W(\theta_1,\theta_2,\theta_3) + Q^e_3(\theta_1,\theta_2,\theta_3)\theta_3] - \hat{\theta}_3 r_3,\end{aligned}$$

where the expectations are taken with respect to the pre-merger distributions.

As we have shown, $\Pi^e(r_{1,2}, r_3)$ is concave and positive at a unique $\mathbf{r}^*$, which implies that we have unique cutoffs $\underline{r}_3$ and $\overline{r}_3$ such that $\Pi^e(1-\underline{r}_3, \underline{r}_3) = 0$ and $\Pi^e(1-\overline{r}_3, \overline{r}_3) = 0$, where $\Pi^e(1-r_3, r_3) > 0$ for all $r_3 \in (\underline{r}_3, \overline{r}_3)$. Thus, for the post-merger setting with two-dimensional types, $\mathcal{R}^e = \{(1-r_3, r_3) \mid r_3 \in [\underline{r}_3, \overline{r}_3]\}$.

We can then construct a density $\tilde{f}$ such that when the merged entity has a one-dimensional type drawn from a distribution with density $\tilde{f}$, along with asset ownership $r_{1,2}$ and maximum demand of $\tilde{k} \equiv \min\{1, k_1 + k_2\}$, then the ex post efficiency permitting set is once again $\mathcal{R}^e$. In the one-dimensional setup, $\tilde{Q}^e_{1,2}(\theta_{1,2}, \theta_3) \equiv \tilde{k} \cdot 1_{\theta_{1,2} > \theta_3} + (1-k_3) \cdot 1_{\theta_{1,2} < \theta_3}$ and $\tilde{Q}^e_3(\theta_{1,2}, \theta_3) \equiv k_3 \cdot 1_{\theta_{1,2} < \theta_3} + (1-\tilde{k}) \cdot 1_{\theta_{1,2} > \theta_3}$, and welfare is $\tilde{W}(\theta_{1,2}, \theta_3) \equiv \tilde{Q}^e_{1,2}(\theta_{1,2}, \theta_3)\theta_{12} + \tilde{Q}^e_3(\theta_{1,2}, \theta_3)\theta_3$. The merged entity's worst-off type is $\tilde{\omega}(r_{1,2})$ satisfying $\tilde{q}^e_{1,2}(\tilde{\omega}(r_{1,2})) = r_{1,2}$, which does not depend on $\tilde{f}$. The nonmerging firm has interim expected allocation rule $\tilde{q}^e_3(\theta_3; \tilde{f}) \equiv \int_0^1 \tilde{Q}^e_3(\theta_{1,2}, \theta_3)\tilde{f}(\theta_{1,2})d\theta_{1,2}$ and worst-off type $\tilde{\theta}_3(r_3; \tilde{f})$ defined by $\tilde{q}^e_3(\tilde{\theta}_3(r_3; \tilde{f}); \tilde{f}) = r_3$, where we explicitly note the dependence on $\tilde{f}$.

We parameterize the density $\tilde{f}$ as the piecewise uniform density

$$\tilde{f}(\theta) = (\tilde{f}_1, \ldots, \tilde{f}_\ell) \cdot (1_{\theta \in [0,1/\ell)}, 1_{\theta \in [1/\ell, 2/\ell)}, \ldots, 1_{\theta \in [(\ell-1)/\ell, 1]}),$$

where ℓ is a sufficiently large integer, $\tilde{f}_i > 0$, and $\sum_{i=1}^{\ell} \frac{1}{\ell}\tilde{f}_i = 1$.

The expected budget surplus under ex post efficiency with one-dimensional types, as a function of the firms' asset ownership as well as firm 3's worst-off type, is then

$$
\begin{aligned}
\tilde{\Pi}^e(r_{1,2}, r_3; \tilde{f}) \;\equiv\; & \int_0^1 \int_0^1 \Big(\tilde{W}(\tilde{\omega}(r_{1,2}), \theta_3) - 2\tilde{W}(\theta_{1,2}, \theta_3) + \tilde{Q}^e_{1,2}(\theta_{1,2}, \theta_3)\theta_{1,2} \\
& + \tilde{W}(\theta_{1,2}, \tilde{\theta}_3(r_3; \tilde{f})) + \tilde{Q}^e_3(\theta_{1,2}, \theta_3)\theta_3 \Big) \tilde{f}(\theta_{12}) f(\theta_3) d\theta_{12} d\theta_3 \\
& - \tilde{\omega}(r_{1,2}) r_{1,2} - \tilde{\theta}_3(r_3; \tilde{f}) r_3.
\end{aligned}
$$

One can then solve for $(\tilde{f}_1, \ldots; \tilde{f}_\ell)$, $\tilde{\theta}_3(\underline{r}_3; \tilde{f})$, and $\tilde{\theta}_3(\overline{r}_3; \tilde{f})$ such that we have, simultaneously, $\tilde{\Pi}^e(1 - \underline{r}_3, \underline{r}_3; \tilde{f}) = \tilde{\Pi}^e(1 - \overline{r}_3, \overline{r}_3; \tilde{f}) = 0$, $\tilde{q}^e_3(\tilde{\theta}_3(\underline{r}_3; \tilde{f}); \tilde{f}) = \underline{r}_3$, and $\tilde{q}^e_3(\tilde{\theta}_3(\overline{r}_3; \tilde{f}); \tilde{f}) = \overline{r}_3$. This gives us the density $\tilde{f}$ such that the ex post permitting set in the one-dimensional setting is, as in the two-dimensional setting, $\mathcal{R}^e = \{(1 - r_3, r_3) \mid r_3 \in [\underline{r}_3, \overline{r}_3]\}$, which is illustrated in Figure C.3.

(a) Ex post efficiency permitting sets

(b) Virtual types for transformed distribution

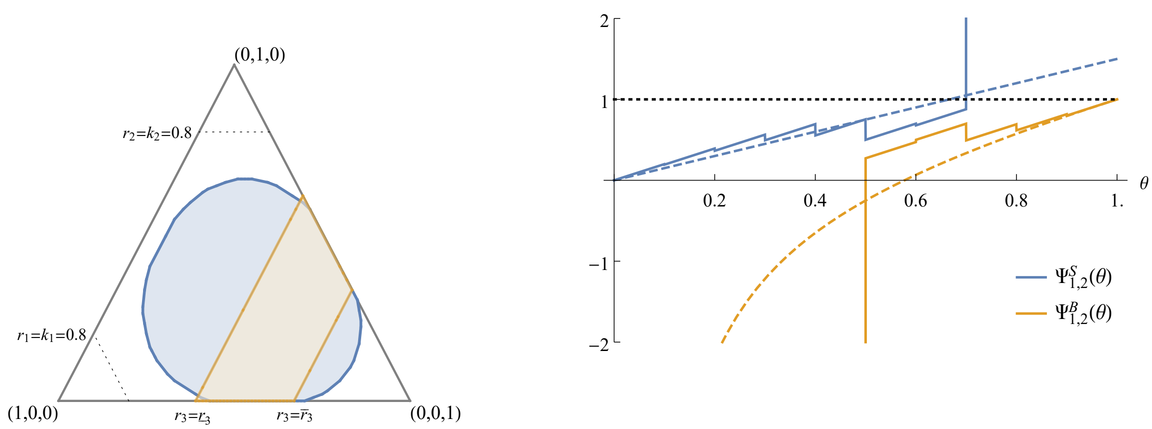

Figure C.3: Panel (a) shows the pre-merger (blue) and post-merger (orange) ex post efficiency permitting set based on two-dimensional types. Panel (b) shows the corresponding virtual types (solid lines) when the merged firm's type is transformed to be one-dimensional based on the piecewise uniform type distribution with 10 segments ($\ell = 10$). For comparison, dashed lines show the virtual type functions for the distribution for the maximum of two types. Assumes that the pre-merger types are uniformly distributed and that $k_1 = k_2 = 0.8$ and $k_3 = 1$.

D Application to the Republic-Santek transaction

A natural question is how the divestiture policies that we discuss can be implemented in practice. In a merger review context, the parties would need to provide details of their own holdings to the competition authority and assist the competition authority in understanding the nature of upstream and downstream competitive constraints. In addition, public filings and industry analyst reports provide relevant information for assessing market structure, maximum capacities, market shares, and parties' margins. This means that **r** can plausibly be treated as observable.

In this appendix, we show how the framework of this paper can be applied to market data that is typically available in a merger review process. As we now illustrate, historical market shares on the input market and maximum allocations can be used to estimate firms' expected allocations and maximum demands. Under parametric assumptions about firms' distributions, one can estimate these parameters to match the firms' historical market shares and to determine, for any given $\mathbf{r}$, how efficient the market operates. With these estimates in hand, one can then estimate sets like $\mathcal{R}^e$ and $\mathcal{R}(\mathbf{r})$, determine whether a proposed transaction is harmful, and what (if any) divestitures are capable of offsetting that harm or would even permit the first-best if the first-best was not possible prior to the transaction.

(a) First-best permitting ownership structures $\mathcal{R}^e$ along with $\mathbf{r}^b$ and $\mathbf{r}^*$

(b) Social surplus preserving ownership structures relative to $\mathbf{r}^b$, $\mathcal{R}(\mathbf{r}^b)$

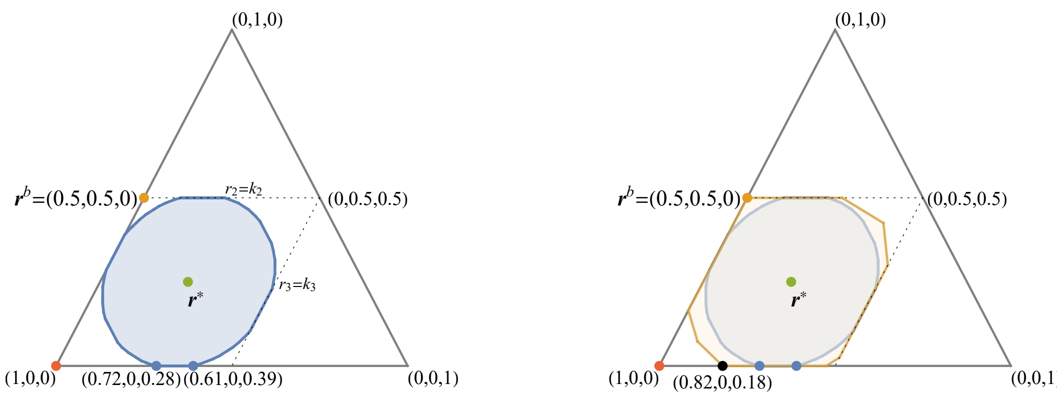

Figure D.1: First-best permitting region (blue) and social surplus preserving region $\mathcal{R}(\mathbf{r}^b)$ (yellow) for upstream market shares $\mathbf{r} \in \Delta_{\mathbf{k}}$ and $\mathbf{k} = (1, 1/2, 1/2)$ and distributions calibrated to the waste management market, which has $\mathbf{r}^b = (1/2, 1/2, 0)$. Specifically, $F_i(\theta) = 1 - (1-\theta)^{s_i}$, with $s_1 = 1$, $s_2 = 1.2$, and $s_3 = 0.8$.

In our illustration, there are three firms. The initial ownership structure is $\mathbf{r}^b = (1/2, 1/2, 0)$ and the vector of maximum demands is $\mathbf{k} = (1, 1/2, 0)$. The transaction consist of firm 2 selling its assets to firm 1, resulting in $\mathbf{r}^a = (1, 0, 0)$. Panel (a) in Figure D.1 displays $\mathbf{r}^b$, $\mathbf{r}^a$, and the estimated $\mathbf{r}^*$ and $\mathcal{R}^e$. As the figure shows, the first-best is not possible with or without the transaction. Panel (b) adds the estimated set $\mathcal{R}(\mathbf{r}^b)$ (in yellow) and divestitures that restore social surplus equal to $SS(\mathbf{r}^b)$ as well as divestitures that permit the first-best.

Our application is inspired by the 2021 transaction involving waste management companies Republic and Santek.[6] However, the specific data that we use are hypothetical.

[6]For background, see Loudermilk et al. (2023) and the U.S. DOJ's website on "U.S. and Plaintiff States v. Republic Services, Inc. and Santek Waste Services, LLC" (https://www.justice.gov/atr/case/us-and-state-alabama-v-republic-services-inc-and-santek-waste-services-llc).

Republic and Santek both held upstream waste disposal assets and also operated downstream waste collection businesses. In addition to using its own upstream assets, Republic also purchased upstream assets from Santek (and other firms). For Santek, in addition to consuming its own upstream assets, it also sold upstream assets to other firms, including to Republic and a third firm, Regional, that held no upstream assets of its own. Motivated by these facts, we model Republic as a trader, Santek, as a seller, and Regional as a buyer.

In general, one would expect the upstream ownership structure, $\mathbf{r}^b$, to be observable. The maximum demands $\mathbf{k}$ can be estimated using data on historical allocations. Specifically, assuming one has observations of input good market quantities q_i^t over $t \in \{1, \ldots, T\}$ periods that result from T independent instances of the market we studied, the maximum demands can be estimated by $\hat{k}_i = \max_t\{q_i^t\}$. This is a good approximation if T is large enough because, regardless of how efficient the market operates, every once in a while the type realizations will be such that firm i is allocated its maximum demand k_i. Likewise, firm i's expected or average quantity $\hat{q}_i$ can be estimated by $\hat{q}_i = \frac{\sum_t q_i^t}{T}$, and firm i's estimated market share generated in the input market will be $\hat{\varsigma}_i = \hat{q}_i/R$, where $R = \sum_{i \in \mathcal{N}} r_i$ is total supply.

For our illustration, we assume the following industry data:

Firm i	r_i^b	$\hat{k}_i$	Market share $\hat{\varsigma}_i$	Type
1. Republic	1/2	1	50%	trader
2. Santek	1/2	1/2	24%	seller
3. Regional	0	1/2	26%	buyer

To complete the specification of the market, we then need to estimate the firms' type distributions. To do so, we assume a class of parameterized distributions and calibrate the firms' parameters based on the information in the table above. Specifically, we model each firm i as having a type distribution of the form $F_i(\theta) \equiv 1-(1-\theta)^{s_i}$, where $s_i > 0$. See Waehrer and Perry (2003) for an axiomatic foundation for this basic structure.[7] The firms' market shares together with an additional identifying assumption, such as the margin for one of the firms, then determine the distributional parameters. For computational convenience, instead of using one firm's margin, we use as the identifying assumption that Republic has uniformly distributed types, i.e., $\hat{s}_{\text{Republic}} = 1$, which implies that the ironing parameter for Republic's ironed virtual type function has an analytic form. Proceeding in this way, we estimate the firms' distributional parameters as follows. Given $\mathbf{r}$, $\hat{\mathbf{k}}$, and arbitrary distributional

[7]Waehrer and Perry (2003) show that their three properties of no externalities, homogeneity, and constant returns are satisfied if and only if there exists a distribution function G with support $[0, 1]$ such that for all i and $c \in [0, 1)$, $F_i(c) = 1 - (1 - G(c))^{s_i}$ for $s_i > 0$.

parameters $\mathbf{s}$, one can solve numerically for ρ^* and Republic's worst-off type, $\hat{\theta}^*_{\text{Republic}}$ (because Santek is a seller, $\hat{\theta}^*_{\text{Santek}} = 1$, and because Regional is a buyer, $\hat{\theta}^*_{\text{Regional}} = 0$). Thereby one obtains the firms' interim expected allocation rules and the associated market shares. One can iterate over distributional parameters to calibrate to the market shares $\hat{\varsigma}$. The distributional parameters shown in the table below imply that $\rho^* = 1.09$ and $\hat{\theta}^*_{\text{Republic}} = 0.50$, which then imply the market shares $\hat{\varsigma}_i$ shown in the table above.[8] The table below displays the distributional parameters $\hat{s}_i$ obtained through this procedure:

Firm i	$\hat{s}_i$
1. Republic	1
2. Santek	1.2
3. Regional	0.8

With the distributional parameters in hand, we can calculate the ex post efficiency permitting region for ownership structures, $\mathcal{R}^e$, as well as the set $\mathcal{R}(\mathbf{r}^b)$, which contains all the ownership structures that generate a social surplus of at least $SS(\mathbf{r}^b)$. Both of these regions are illustrated in Figure D.1. As the figure shows, ex post efficiency is not possible under the industry's pre-transaction upstream ownership structure $\mathbf{r}^b = (1/2, 1/2, 0)$, and the upstream ownership structure that maximizes Π^e is $\mathbf{r}^* = (1/2, 1/4, 1/4)$.[9] The post-transaction markets structure is indicated as a red dot in Figure D.1 and given by $(1, 0, 0)$. The minimal divestiture that offsets the harm from that transaction, which requires Republic to divest 36% to Regional, is indicated by the black dot in Figure D.1(b). If it divests between 57% and 78%, ex post efficiency is achieved after the transaction with divestiture whereas it was not possible before the transaction.

Ultimately, the DOJ allowed Republic's acquisition of Santek subject to the divestiture of a number of Santek's assets to approved buyers.[10]

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[8] If there are traders that do not have uniformly distributed types, then one must also solve numerically for their ironing parameters. This applies, for example, for the problem of estimating $\mathcal{R}(\bar{\mathbf{r}})$, displayed in panel (b) of Figure D.1.

[9] In the efficient allocation, Santek and Regional are allocated their full maximum demand if and only if their type is greater than the type of Republic. Thus, given that Republic's type is uniformly distributed, we have $q^e_{\text{Santek}}(\theta) = q^e_{\text{Regional}}(\theta) = \theta$. For Republic, we have $q^e_{\text{Republic}}(\theta) = 2 - (1-\theta)^{0.8} - (1-\theta)^{1.2}$. The upstream ownership structure that equalizes the firms' worst-off types to be $\hat{\theta}$ is then $\mathbf{r}^*$ satisfying $r^*_{\text{Santek}} = r^*_{\text{Regional}} = \hat{\theta}$, $r^*_{\text{Republic}} = 2 - (1-\hat{\theta})^{0.8} - (1-\hat{\theta})^{1.2}$, and $r^*_{\text{Republic}} + r^*_{\text{Santek}} + r^*_{\text{Regional}} = 2$. Solving this, we get $\hat{\theta} = 0.502$, and so $r^*_{\text{Santek}} = r^*_{\text{Regional}} = 0.5024$ and $r^*_{\text{Republic}} = 0.9951$, which rounds to $\mathbf{r}^* = (1, 0.5, 0.5)$. Even though Santek and Republic's distributional parameters differ, their r^*_i's are the same.

[10] U.S. DOJ, "U.S. and Plaintiff States v. Republic Services, Inc. and Santek Waste Services, LLC" (https://www.justice.gov/atr/case/us-and-state-alabama-v-republic-services-inc-and-santek-waste-services-llc).

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