# Mergers, remedies, and incomplete information*

Simon Loertscher† Leslie M. Marx‡

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**Abstract**

We study an incomplete information model in which ex post efficiency is impossible unless the market for an asset includes firms that have the potential to act as either buyers or sellers of the asset depending on type realizations, characterize the set of efficiency-permitting asset ownership structures, and show that horizontal (vertical) mergers never make efficiency possible (impossible). Mergers that are neither horizontal nor vertical can have either effect. The analysis provides a foundation and guidance for divestitures and a test for whether post merger, with or without divestitures, ex post efficiency is possible. It shows that pure asset transfers are typically profitable bilaterally even when reducing social surplus, rationalizing roles for antitrust vigilance and a focus on bilateral transactions. Methodologically, the paper extends asset market models—generalized partnership models—to multi-dimensional types, and decreasing marginal values, which are empirically relevant. For these settings, it establishes possibility and concavity results.

**Keywords**: horizontal, vertical and conglomerate mergers, raising rivals' costs, Triple-IO
**JEL Classification**: D44, D82, L41

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†Department of Economics, Level 4, FBE Building, 111 Barry Street, University of Melbourne, Victoria 3010, Australia. Email: simonl@unimelb.edu.au.

‡Duke University, 100 Fuqua Drive, Durham, NC 27708, USA: Email: marx@duke.edu.

# 1 Introduction

The effects of and incentives for integration—horizontal, vertical, or otherwise—are of long-standing interest in economics, taking center stage in organizational economics, industrial organization, and antitrust. While non-horizontal integration has traditionally been viewed favorably by competition authorities, a combination of recent merger cases, controversies related to big tech, and empirical and theoretical advances have brought forth more skeptical perspectives regarding their effects on social and consumer surplus.[1] Recent empirical research find mixed effects of non-horizontal integration.[2] Competition authorities and ultimately courts faced with a proposed merger can either approve or block it, or approve it subject to so-called "remedies," including possibly structural remedies that require the merged entity to divest some of its joint assets to a third party. A core question, thus, is what decision should be taken.

In this paper, we provide a framework based on incomplete information that permits answers to the questions of which mergers can be approved, which should be blocked, and which ones can be approved subject to structural remedies. Our model distinguishes between buyers, sellers, and traders. Buyers (sellers) are firms that, under efficiency, never sell (buy), while traders sometimes buy and sometimes sell, depending on the circumstances. Because of incomplete information, the model also allows us to distinguish between pure *asset transfers*, whereby asset ownership changes without any commercially relevant (private) information being transferred, and *full integration*, in which case both ownership and information are integrated. While full integration reduces the number of firms by one, asset transfers do not affect the number of firms and therefore capture the essence of divestitures.

We model the market as an incentive compatible, individually rational mechanism that allocates ex post efficiently, provided that this is possible without running a deficit.[3] Incomplete information takes the form of firms having independent private values that are drawn

[1] *Vertical Merger Guidelines* were released by the U.S. DOJ and FTC in 2020 to replace the 1984 *Non-Horizontal Merger Guidelines*; however, in September of 2021, the FTC rescinded its support for those guidelines (FTC 2021). For analyses specifically related to big tech, see, for example, Baker et al. (2019); Kang and Muir (2022); and various policy reports (ACCC, 2019; Crémer et al., 2019; Furman et al., 2019; and Stigler Center, 2019). In models with complete information, the possibility of anticompetitive vertical integration arises from a "raising rivals' costs" theory of vertical foreclosure (Salop and Scheffman, 1983; Salinger, 1988; Hart and Tirole, 1990; Ordover et al., 1990; Riordan, 1998). Luco and Marshall (2020), Loertscher and Marx (2022), Chen and Rey (2023), and Choné et al. (forth.) provide recent empirical and theoretical analyses of non-horizontal integration in which the elimination of a double markup, which is a classic vertical merger defense based on complete information models with contracts restricted to linear prices (Cournot, 1838; Spengler, 1950), does not imply procompetitive effects.

[2] See Luco and Marshall (2020) document anticompetitive effects of integration in the soda industry while Chen et al. (2024) provide evidence of procompetitive effects in the movie industry.

[3] The market is also modeled as a mechanism in Loertscher and Marx (2022).

from continuous, commonly known distributions with positive densities on an identical support. Each firm is endowed with a nonnegative asset ownership share that is no more than its maximum demand, and the total asset endowment is less than the total demand at a price of zero so that there is scarcity. Asset transfers or full integration take place prior to the realization of private information.[4] After the firms' private values are realized, the market mechanism allocates the essential input among firms. We assume that, absent full integration, a firm's type is a one-dimensional random variable and that full integration creates a firm with decreasing marginal values and multi-dimensional type.[5]

Full integration by two buyers or two sellers corresponds to a *horizontal* merger, while full integration by a buyer and a seller constitutes a *vertical* merger. We show that horizontal mergers never make ex post efficiency possible, and vertical mergers never make it impossible. Viewing firms on the same side of the market as substitutes and firms on opposite sides as complements, this is naturally interpreted as the incomplete-information analogue to the results of Cournot (1838) that mergers between producers of substitutes (complements) decrease (increase) efficiency. Full integration involving a trader is neither a horizontal nor a vertical merger and can, as we show, have either effect. Referring to these as *conglomerate* mergers, we show that conglomerate mergers that "corner the market," for example by putting all ownership into the hands of the merged entity, never make ex post efficiency possible. In contrast, conglomerate mergers with a sufficiently strong vertical component do so.[6] More generally, given knowledge of the environment parameterized by the distributions,

[4]Although the model is static, asset ownership can be thought of as a long-term contract over an essential input such as trucks, emission permits, or long-term labor contracts, and private information can be viewed as pertaining to day-to-day business, with the asset allocation that follows the realization of private information being interpreted as the issuance of short-term leases. Online Appendix C.1 formalizes this in a multi-period model.

[5]The property of decreasing marginal values is empirically relevant. For example, it is consistent with the observation in Southern California's RECLAIM program that all firms are always allocated some permits (see e.g., Fowlie and Perloff, 2013; Liu et al., forth.). With constant marginal values, firms with low draws would typically be allocated zero permits.

[6]Viewing the mergers analyzed by Luco and Marshall (2020) and Chen et al. (2024) as conglomerate mergers that differ in their degree of "verticalness" also provides an explanation for the contrasting effects these papers document (see footnote 2). Luco and Marshall (2020) propose the "Edgeworth-Salinger" effect (Edgeworth, 1925; Salinger, 1991) as an alternative and at-face-value natural explanation for their empirical finding of price-increasing effects of integration by soda producers with downstream bottlers. According to this effect, a tax decrease or the elimination of a double markup can induce a multi-product monopoly to increase the prices on *all* of its products. However, the problem with the Edgeworth-Salinger effect is that it occurs rather rarely. See Armstrong and Vickers (2023), who provide a systematic analysis of necessary and sufficient conditions for Edgeworth's taxation paradox, which is at the heart of the Edgeworth-Salinger effect. Even though the Edgeworth paradox has a certain genericity, the conditions for it to obtain are demanding. For example, no symmetric demand system can give rise to it. This suggests that Hotelling's assessment that "[t]here is no basis known at present for denying that Edgeworth's phenomenon may pertain to a large proportion of ordinary situations, or for affirming that it is, in his language, a mere 'curiosum"' (Hotelling, 1932, p. 583) still applies today.

ownership structure, and maximum demands, it is always possible to derive the ex post efficiency-permitting set of ownership structures, which as we show is nonempty and convex. Importantly, this set can be used to gauge whether there are divestitures that permit ex post efficiency if, absent such remedies, ex post efficiency is not possible.

For the case with one-dimensional types, the second-best mechanism when ex post efficiency is not possible is known and can be used to compute firms' expected payoffs. Under this assumption, we show that there is a role for sustained antitrust vigilance because, typically, bilateral transfers of asset ownership are profitable even if they reduce social surplus. Therefore, bilateral transfers are a natural focus for antitrust—there would typically be privately profitable bilateral transactions even following an agreement by all firms on an ownership structure that permitted ex post efficiency. With ex ante identical firms, more symmetric ownership structures are better because they directly increase expected social surplus and because, indirectly, they permit better divestitures post merger.[7]

In extensions, we incorporate consumers surplus and provide conditions under which ex post efficiency in the market that we consider also maximizes consumer surplus, allow for investments, and discuss nonidentical supports and possible avenues for incorporating second-best mechanisms with full integration.

The paper contributes to the emerging literature on incomplete information industrial organization, such as Backus et al. (2019), Backus et al. (2020), Backus et al. (forth.), Larsen (2021), Larsen et al. (2021), Larsen and Zhang (2022), Byrne et al. (2022), Loertscher and Marx (2019, 2022), Choné et al. (forth.), Kang and Muir (2022), Barkley et al. (2024), and Condorelli and Padilla (2024), and the mechanism design literature, particularly the strand on partnership models that was initiated by Cramton et al. (1987), with subsequent contributions by, among many others, Che (2006), Figueroa and Skreta (2012), Lu and Robert (2001), Loertscher and Wasser (2019), and Liu et al. (forth.).[8] In particular, Backus et al. (2019) and Backus et al. (2020) provide empirical evidence that is consistent with incomplete information and difficult to reconcile with complete information bargaining models. Larsen (2021), Larsen et al. (2021) and Larsen and Zhang (2022) develop empirical methods to estimate types distributions and perform counterfactual analyses using incomplete information

[7]This result resonates with and provides a formalization of the notion that more symmetry—"leveling the playing field"—is, somehow, desirable, which is widely held in antitrust. Antitrust practitioners have long viewed symmetry, or at least the absence of its counterparts of "undue concentration" or "dominant firms," as procompetitive (see, e.g., Lanzillotti, 1961; Turner, 1969; Williamson, 1972).

[8]Because the design problem inevitably fails the condition that Myerson (1981) called regularity, the paper also contributes to a recent upsurge of interest in nonregular mechanism design problems such as Condorelli (2012), Dworczak ⓡ al. (2021), Loertscher and Muir (2022), and Akbarpour ⓡ al. (forth.), where nonregularity arises from assumptions on the distributions from which the agents draw their types or differences in the agents' utility functions. In contrast, in our setting, the nonregularity of the second-best mechanism design problem derives from the market structure.

bargaining models while Loertscher and Marx (2019, 2022) show that, with incomplete information, the widely used but typically vague notions of buyer power and countervailing power are well defined and derive from the primitives of the model. For one-dimensional private information, we combine the incomplete information bargaining approach of Williams (1987) and Loertscher and Marx (2022) with a generalized partnership model with heterogeneous distributions and maximum demands.[9] Among other things, this allows us to derive the social surplus maximizing policy for mergers that are pure transfers of asset ownership. For both one-dimensional and multi-dimensional private information, we characterize the set of ownership structures that permits ex post efficiency. As mentioned, this set is nonempty and convex.[10] Methodologically, it is the multi-dimensional generalization that allows us to make statements about whether horizontal, vertical, and conglomerate mergers make ex post efficiency (im)possible. As the existing literature has confined attention to settings with the property that, post-merger, an integrated firm's private information remains essentially one-dimensional, this is a substantial step forward.[11] Moreover, as mentioned in footnote 5, the ability to capture decreasing marginal values afforded by having multi-dimensional private information is empirically relevant. The setup with long-term ownership structures that affect the efficiency of the day-to-day market also constitutes an incomplete information formalization of the effects of forward markets on the operation of spot markets studied by, for example, Green and Newbery (1992), Allaz and Vila (1993), and Ito and Reguant (2016).

The remainder of the paper is structured as follows. In Section 2, we use a reduced-form model to illustrate the social surplus maximizing competition policy. In Section 3, we introduce the incomplete information model that provides a microfoundation for the reduced-form model, and we derive results pertaining to firms' private incentives for asset ownership transfers. In Section 4, we analyze pure asset ownership transfers and derive the social surplus maximizing merger policy. In Section 5, we consider the effects of full integration that combines the assets and private information of two firms. Section 6 contains extensions and discussion, and Section 7 concludes the paper.

[9]See also Loertscher and Marx (2024), whose analyses further generalize incomplete information bargaining among partners to allow for nonidentical supports, which, among other things, captures new aspects of the notions of upstream and downstream.

[10]Most of the partnership literature has studied problems with one-dimensional type distributions. Two exceptions are Yenmez (2015) and Agastya and Birulin (2018) who study, respectively, settings with interdependent values and settings with outside options that are independent from the value of continuing the partnership. The former shows an impossibility result, and the latter's emphasis is on impossibility of ex post efficiency as well.

[11]For example, Loertscher and Marx (2019) and Choné et al. (forth.) study one-to-many settings in which an integrated firm always only contracts with one party. The same is true for the analysis of vertical integration in Loertscher and Marx (2022). Horizontal mergers in that paper are tractable if, say, the two merging suppliers each have enough capacity to serve the entire market pre merger, so that post merger all that matters is the lower of their two cost draws, which is a one-dimensional random variable.

## 2 Reduced form

We first use a reduced-form model to illustrate the social surplus maximizing merger policy. This model provides both a rationale and guidance for divestitures and conditions for when a merger can be permitted with remedies and when instead it should be blocked. The subsequent section will deliver microfoundations for this simple yet illustrative model.footnoteThe model with one-dimensional private information and asset transfers below provides a precise micro-foundation for the reduced form. With full integration and multi-dimensional types, the gist of the results illustrated here carry over.

There is a set $\mathcal{N}$ of $n \geq 2$ firms. Each firm $i \in \mathcal{N}$ has some asset ownership $r_i \geq 0$, with total asset ownership normalized to 1, that is, $\sum_{i=1}^{n} r_i = 1$. Accordingly, the set of feasible ownership structures is the simplex

$$\Delta \equiv \Big\{ \mathbf{r} \in [0,1]^n \mid \sum_{i=1}^{n} r_i = 1 \Big\}.$$

We denote by $SS(\mathbf{r})$ the social surplus that results from ownership structure $\mathbf{r} \in \Delta$, which here is assumed to be a continuous function. Let $\mathcal{R}(\mathbf{r})$ be the set of all ownership structures that generate social surplus of at least $SS(\mathbf{r})$, that is,

$$\mathcal{R}(\mathbf{r}) \equiv \{\mathbf{r}' \in \Delta \mid SS(\mathbf{r}') \geq SS(\mathbf{r})\}.$$

Denote by $\mathcal{R}^e$ the set of all ownership structures that maximize $SS(\mathbf{r})$, that is, $\mathcal{R}^e = \arg\max_{\mathbf{r}' \in \Delta} SS(\mathbf{r}')$.

The sale of assets from firm $i$ with $r_i > 0$ to firm $j$ is modeled as a reduction $x \in (0, r_i]$ in firm $i$'s ownership and an increase by $x$ in firm $j$'s ownership, so that post-transaction the ownership structure consists of $r_i - x$ for firm $i$, of $r_j + x$ for firm $j$, and leaves ownership unchanged for all other firms. We consider the role of an antitrust authority that can exert control over asset ownership by approving or blocking a proposed sale of assets or by approving changes in asset ownership conditional on divestitures, in which case the acquiree must sell some of its newly acquired assets to a third party. Such divestitures are sometimes referred to as structural remedies.[12] Here we focus on identifying the set of divestitures that would be acceptable to a competition authority without regard for whether they would be acceptable to the parties involved. Given $\mathbf{r}$ and a transaction shifting $x \leq r_i$ from firm $i$ to firm $j$, the feasible divestitures are

$$\mathcal{D}_{i,j}(\mathbf{r}, x) \equiv \{\mathbf{r}' \in \Delta \mid r'_i = r_i - x,\ r'_j \in [r_j, r_j + x],\ \forall \ell \in \mathcal{N} \backslash \{i,j\},\ r'_\ell \geq r_\ell\}.$$

Put differently, divestitures can remove any amount less than or equal to $x$ from firm $j$ and

[12]Structural remedies are in contrast to behavioral remedies that restrict conduct rather than ownership.

shift it, or parts of it, to any other firm except for firm $i$, whose ownership remains $r_i - x$ before and after the divestiture.

We consider a competition authority that observes pre-transaction asset ownership and the social surplus level sets, $\mathcal{R}(\mathbf{r})$, and is notified of a transaction that proposes to shift a specified amount of asset ownership from one firm to another. We then have the following social surplus maximizing policy:

**Proposition 1.** *Let $\mathbf{r}^b$ and $\mathbf{r}^a$ be asset ownership before and after, respectively, a transaction that shifts asset ownership from firm $i$ to firm $j$. Then the social surplus maximizing policy*

- *allows the transaction if $\mathbf{r}^a \in \mathcal{R}^e$;*
- *allows it with divestiture if $\mathcal{D}_{i,j}(\mathbf{r}^b, r_i^b - r_i^a) \cap \mathcal{R}(\mathbf{r}^b) \neq \varnothing$;*
- *blocks it otherwise.*

Based on Proposition 1, we say that given initial ownership $\mathbf{r}^b$, ownership $\mathbf{r}'$ results from *optimal divestitures* following the sale of assets $x$ from firm $i$ to firm $j$ if we have $\mathbf{r}' \in \arg\max_{\mathbf{r} \in \mathcal{D}_{i,j}(\mathbf{r}^b, x)} SS(\mathbf{r})$.

For the next result, we assume that $SS(\mathbf{r})$ is *Schur concave*, which is to say that if $\mathbf{r}'$ majorizes $\mathbf{r}$, then $SS(\mathbf{r}') \leq SS(\mathbf{r})$, with strict inequality unless $\mathbf{r}' \in \mathcal{R}^e$.[13] Under this condition, for which we provide foundations in Section 4, Proposition 2 brings to light a novel benefit of symmetry—an increase in pre-transaction symmetry not only causes social surplus to be higher pre-transaction (as a direct consequence of Schur concavity), but also post-transaction following an optimal divestiture.

**Proposition 2.** *Assume that $SS(\mathbf{r})$ is Schur concave. Given $\tilde{\mathbf{r}}$ that majorizes $\mathbf{r}$, with $\tilde{r}_i = r_i = s > 0$, if $\tilde{\mathbf{r}}'$ and $\mathbf{r}'$ result from optimal divestitures after firm $i$ sells assets $\sigma \in (0, s]$ to another firm, then $\tilde{\mathbf{r}}'$ majorizes $\mathbf{r}'$ and*

$$SS(\tilde{\mathbf{r}}') \leq SS(\mathbf{r}'),$$

*with a strict inequality unless $\tilde{\mathbf{r}}' \in \mathcal{R}^e$.*

*Proof.* The proof follows from the straightforward, albeit tedious, application of the definition of majorization to show that $\tilde{\mathbf{r}}'$ majorizes $\mathbf{r}'$. We relegate details to Online Appendix B.1.

We summarize the key takeaway of Proposition 2 in the following corollary:

[13] The vector $\mathbf{r}'$ *majorizes* $\mathbf{r}$ if for all $j \in \{1, \ldots, n\}$, $\sum_{i=1}^{j} r'_{(i)} \geq \sum_{i=1}^{j} r_{(i)}$, where $r_{(i)}$ is the $i$-th largest element of $\mathbf{r}$, with a strict inequality for some $j$ and equality for $j = n$ (Marshall et al., 2011, p. 80). As shown by Marshall et al. (2011, Lemma B.1), if $\mathbf{r}'$ majorizes $\mathbf{r}$, then $\mathbf{r}$ can be obtained from $\mathbf{r}'$ by a finite number of $T$-transforms: given vector $(x_1, \ldots, x_n)$, a $T$-transform of $\mathbf{x}$ is a vector with two coordinates $x_j$ and $x_k$ replaced by $\lambda x_j + (1 - \lambda) x_k$ and $\lambda x_k + (1 - \lambda) x_j$ for some $\lambda \in (0, 1)$ (Marshall et al., 2011, p. 32). This maintains the majorization relation at each step.

**Corollary 1.** *If $SS(\mathbf{r})$ is Schur concave, then increased symmetry increases expected social surplus directly and makes divestiture-based remedies more effective.*

While this reduced-form analysis highlights key insights from this paper, it does not answer the question of whether firms' incentives to transact asset ownership align with the objective of a social surplus maximizing planner or whether there is a basis for sustained antitrust vigilance.

# 3 Model

We now provide a microfoundation for the reduced-form model.

## 3.1 Setup

At the outset, that is, before any transactions of ownership or integration take place, each firm $i \in \mathcal{N}$ has asset ownership $r_i \geq 0$ and maximum demand for assets $k_i \geq r_i$ with $k_i > 0$.[14] We assume that for all $i \in \mathcal{N}$, $\sum_{j \in \mathcal{N} \setminus \{i\}} k_j \geq 1$. Together with $k_i > 0$, this implies that $\sum_{i \in \mathcal{N}} k_i > 1$, that is, there is scarcity. The set of admissible ownership structures is thus

$$\Delta_{\mathbf{k}} \equiv \left\{ \mathbf{r} \in \times_{i \in \mathcal{N}} [0, k_i] \mid \textstyle\sum_{i \in \mathcal{N}} r_i = 1 \right\}.$$

We distinguish between the cases in which a firm's private information is *one-dimensional* and in which it is *multi-dimensional*. If firm $i$ is characterized by multi-dimensional private information with dimensionality $h_i \geq 2$, then that means that it is described by a vector of maximum demands $(k_i^1, \ldots, k_i^{h_i})$ and a vector of distributions $(F_i^1, \ldots, F_i^{h_i})$, where to fix ideas $(k_i^j, F_i^j)$ may be thought of as describing a downstream market $j$ that firm $i$ has exclusive access to serve. As usual, the differentiation between markets may be in geographic or product space. While we assume that firms do not compete downstream, foreclosure effects can still arise in the sense that shifts in asset ownership between two firms can impose externalities on firms not involved in the transaction, including reducing a nontransacting firm's expected allocation. For each $j \in \{1, \ldots, h_i\}$, the constant marginal value $\theta_i^j$ is an independent draw from the distribution $F_i^j$ with support $[0, 1]$ and positive density $f_i^j$. This implies that firm $i$, if multi-dimensional, has decreasing marginal values and a maximum demand of $k_i = \sum_{j=1}^{h_i} k_i^j$. For example, if $h_i = 2$ and $k_i^1 = k_i^2 = k$, then given a realization $\boldsymbol{\theta}_i = (\theta_i^1, \theta_i^2)$, firm $i$'s willingness to pay for the first $k$ units is $\max\{\theta_i^1, \theta_i^2\}$ and for the second $k$ units is $\min\{\theta_i^1, \theta_i^2\}$. We use $H \equiv \sum_{i \in \mathcal{N}} h_i$ to denote the total number of types. In the one-dimensional case, that is, if $h_i = 1$, we simply write $k_i, F_i$, and $f_i$, that is, we drop the

[14] If, contrary to our assumption, $k_i < r_i$, then firm $i$ is has no value for $r_i - k_i$ units, which it is willing to sell for free. It is without loss to assume, as we do, that $k_i > 0$, for otherwise, the assumption that $k_i \geq r_i$ would imply that $r_i = k_i = 0$, and so we could just eliminate firm $i$ from the market.

superscript 1. Distributions, maximum demands, and ownership are common knowledge, whereas a firm's realized types are its private information.

Given type realizations $\boldsymbol{\theta}$, the *ex post efficient* allocation (or, equivalently, ex post efficient consumption of the asset's services) is based on the ranking of the firms' types and denoted $\mathbf{Q}^e(\boldsymbol{\theta})$. For a given vector of types $\boldsymbol{\theta}$, we have

$$\mathbf{Q}^e(\boldsymbol{\theta}) \in \arg\max_{\mathbf{Q}} \sum_{i=1}^{n} \sum_{j=1}^{h_i} Q_i^j \theta_i^j \quad \text{subject to} \quad \sum_{i=1}^{n} \sum_{j=1}^{h_i} Q_i^j \leq 1 \text{ and } Q_i^j \in [0, k_i^j].$$

While $\mathbf{Q}_i^e(\boldsymbol{\theta})$ is a vector of length $h_i$, the total quantity allocated to firm $i$ is $Q_i^e(\boldsymbol{\theta}) \equiv \sum_{j=1}^{h_i} Q_i^{e,j}(\boldsymbol{\theta})$, and firm $i$'s interim expected total allocation is $q_i^e(\boldsymbol{\theta}_i) \equiv \mathbb{E}_{\boldsymbol{\theta}_{-i}}[Q_i^e(\boldsymbol{\theta})]$. The assumption that, for all $i \in \mathcal{N}$, $\sum_{j \neq i} k_j \geq 1$, implies that $q_i^e(\mathbf{0}) = 0$.

When ex post efficiency is possible, the market mechanism's budget surplus under binding individual rationality may be positive. In that case, we assume that the budget surplus is divided among the firms according to fixed shares: for $i \in \mathcal{N}$, we let $\eta_i \in [0, 1]$ denote firm $i$'s share of any budget surplus, where $\sum_{i \in \mathcal{N}} \eta_i = 1$.

We distinguish between pure *asset ownership transfers*, which refer to transfers of ownership that do not affect the set of firms or the private information held by any firm, and *full integration*. Formally, in an asset ownership transfer, firm $i$ sells to firm $j$ the amount $x \in [0, \min\{r_i, k_j - r_j\}]$, so that post transaction the ownership shares are $r_i - x$ and $r_j + x$ for firms $i$ and $j$ and remain the same as before the transaction for all other firms. Asset ownership transfers are analyzed in Section 4. In contrast, full integration of firms $i$ and $j$, which we analyze in Section 5, results in the two firms both combining their asset ownership and integrating their private information. Consequently, full integration reduces the number of firms by 1 and creates an integrated firm with a higher dimension of private information. For example, full integration of firms $i$ and $j$ with $h_i = h_j = 1$, results in an integrated firm with asset ownership $r_{i,j} = r_i + r_j$ and dimensionality $h_{i,j} = 2$, where the firm has maximum demand $k_i$ associated with the type drawn from $F_i$ and maximum demand $k_j$ associated with the type drawn from $F_j$. Of course, a multi-dimensional firm can always be thought of and microfounded as having emerged from the full integration of one-dimensional firms.

Throughout, we assume that decisions to transfer asset ownership or to fully integrate occur at the ex ante stage, that is, before the realization of any private information. This is also the stage at which we evaluate welfare. This reflects a view that private information pertains to short-term, "day-to-day" transactions, while mergers are long-term decisions and welfare and profits are naturally evaluated "on average."[15]

[15]See Online Appendix C.1 for a formalization of this view in a multi-period model in which the allocation $\mathbf{Q}$ in any given period represents short-term leases and the asset ownership $\mathbf{r}$ is fixed for all periods after transactions in period 0.

The timeline is as follows. First, given $\mathcal{N}$ and $\mathbf{r}$, two firms may engage in asset ownership transfers or full integration. This proposed transaction is then evaluated by antitrust authorities, including the possibility that divestitures are imposed and implemented. Second, private information is realized. Third, the day-to-day market operates and determines which firm consumes how many units of the services provided by the assets, and payments and firms' payoffs are realized.

The firms' asset ownership and maximum demands for assets determine whether they are buyers, sellers, or simply traders that have the potential to act as a buyer or a seller, depending on type realizations.[16] Firm $i$ is a *buyer* if it has no assets of its own to sell, $r_i = 0$; firm $i$ is a *seller* if it has no demand for additional assets, $k_i = r_i$; and otherwise firm $i$ is said to be a *trader*. Thus, for any trader $i$, we have $0 < r_i < k_i$.

## 3.2 Market mechanism

As do Loertscher and Marx (2022) and Liu et al. (forth.), we model the market as a mechanism operated by a (fictitious) market maker or designer that, as a function of firms' types $\boldsymbol{\theta} \in [0,1]^H$, chooses a feasible allocation $\mathbf{Q}$ and determines firms' payments $\mathbf{M} = (M_i)_{i\in\mathcal{N}} \in \mathbb{R}^n$, where $M_i$ is the payment from firm $i$ to the market maker. The idea of the market being organized by a fictitious entity has a long tradition in industrial organization, and economics more generally, starting with Cournot's (1938) auctioneer, who sets the market-clearing price given firms' quantity choices, and subsequently adapted by Walras (1874) to the general equilibrium setting. Because the firms are privately informed about their types, the mechanism can make $\mathbf{Q}$ and $\mathbf{M}$ depend on the realized $\boldsymbol{\theta}$ only if it induces the firms to reveal that information to the market maker, which is why, with private information, the natural extension of a Cournot-Walras auctioneer is that of a market maker who uses a mechanism. Specifically, we stipulate that this mechanism is *direct* in that it asks every agent to report its type and *incentive compatible* (IC)—makes reporting types truthful in either a Bayes Nash (BIC) or a dominant strategy (DIC) equilibrium—and *interim individually rational* (IR)—knowing its type and expected allocation and payment, every firm is weakly better off participating in the mechanism than walking away. Both IC and IR will be more formally defined below. A direct IC, IR mechanism is said to be *ex post efficient* if, for every $\boldsymbol{\theta} \in [0,1]^H$, it uses an ex post efficient allocation rule $\mathbf{Q}^e(\boldsymbol{\theta})$.

Ex post efficiency is said to be *possible* if there exists an ex post efficient, IC, IR mechanism that, in expectation, does not run a deficit, that is, $\mathbb{E}_{\boldsymbol{\theta}}[\sum_{i\in\mathcal{N}} M_i(\boldsymbol{\theta})] \geq 0$. If no such

[16] Lu and Robert (2001) refer to firms in this last category as "ex ante unidentified traders." Settings with traders are called *asset markets* in Loertscher and Marx (2020, 2023) and Delacrétaz et al. (2022). Liu et al. (forth.) document the empirical relevance of traders in the context of emission permit markets, where many emitters are net buyers of emission permits in some periods and net sellers in other periods.

mechanism exists, then ex post efficiency is said to be *impossible*. The no-deficit constraint simply means that there is no outside source pouring money into the market to grease its wheels.[17] Our primary focus when dealing with multi-dimensional types will be on whether or not ex post efficiency is possible.[18] This question has been the focus of much of the partnership literature with one-dimensional types upon which our model builds and expands.

For the setting with one-dimensional types, which we study in the remainder of this section and in the next section, we are able to characterize firms' incentives to transfer asset ownership ex ante and the social surplus maximizing policy for asset ownership transfers.

With one-dimensional private information, the allocation rule of the market maker's direct mechanism $\langle \mathbf{Q}, \mathbf{M} \rangle$ is a mapping $\mathbf{Q} : [0,1]^n \to \mathbb{R}^n$ satisfying $Q_i(\boldsymbol{\theta}) \in [0, k_i]$ and $\sum_{i \in \mathcal{N}} Q_i(\boldsymbol{\theta}) \leq 1$, and a payment rule $\mathbf{M} : [0,1]^n \to \mathbb{R}^n$, where for reports $\boldsymbol{\theta}$, $Q_i(\boldsymbol{\theta})$ specifies the quantity allocated to firm $i$ and $M_i(\boldsymbol{\theta})$ specifies the payment from firm $i$ to the market maker.[19] Here and in the next section, we focus on IR mechanisms that satisfy BIC and no deficit in expectation.[20] Firm $i$'s outside option from not participating is $\theta_i r_i$.

For a fixed mechanism $\langle \mathbf{Q}, \mathbf{M} \rangle$, we denote firm $i$'s interim expected allocation and payments, respectively, by

$$q_i(\theta_i) \equiv \mathbb{E}_{\boldsymbol{\theta}_{-i}}[Q_i(\theta_i, \boldsymbol{\theta}_{-i})] \quad \text{and} \quad m_i(\theta_i) \equiv \mathbb{E}_{\boldsymbol{\theta}_{-i}}[M_i(\theta_i, \boldsymbol{\theta}_{-i})].$$

The interim expected net payoff of firm $i$ from participating in the mechanism when its type is $\theta$ and when it reports its type truthfully, with net meaning net of the outside option $r_i\theta$, is denoted by

$$u_i(\theta) \equiv \theta(q_i(\theta) - r_i) - m_i(\theta).$$

The direct mechanism $\langle \mathbf{Q}, \mathbf{M} \rangle$ is BIC if for all $i \in \mathcal{N}$ and all $\theta, \theta' \in [0,1]$,

$$u_i(\theta) \geq \theta(q_i(\theta') - r_i) - m_i(\theta'), \tag{1}$$

[17] One could account for additional market frictions by replacing the no-deficit constraint by a constraint that market maker revenue be no less than some $\kappa \geq 0$. The effect is to shrink the set of ownership structures such that ex post efficiency is possible, potentially to the empty set, and to change the right side in inequality (3) to $\kappa$.

[18] The exceptions pertain to cases in which a firm $i$ with dimensionality $h_i \geq 2$ is characterized by maximum demands $k_i^j \geq 1$ for all $j \in \{1, \ldots, h_i\}$, in which case all that matters to determine the firm's willingness to pay is the highest of its $h_i$ draws, whose distribution is $\times_{j=1}^{h_i} F_i^j$, which is one-dimensional. It is precisely this property of effective one-dimensionality that is exploited in the prior literature; see, e.g., Choné et al. (forth.) and Loertscher and Marx (2019, 2022).

[19] By the Revelation Principle, a focus on direct mechanisms is without loss of generality. The constraint $Q_i \in [0, k_i]$ is for convenience and can be dropped by replacing $Q_i(\boldsymbol{\theta})$ with $\min\{k_i, Q_i(\boldsymbol{\theta})\}$ in firm $i$'s payoff.

[20] In our independent private values setting, any BIC, IR mechanism can be implemented as a dominant strategy incentive compatible (DIC), IR mechanism because of the monotonicity of the allocation rule implied by incentive compatibility. In this sense, the nature of IC is immaterial. Further, as shown in footnote 21, the focus on no deficit in expectation is also without loss of generality within the class of mechanisms that satisfy IR.

which implies that $u_i$ is convex. The mechanism $\langle \mathbf{Q}, \mathbf{M} \rangle$ is IR, if for all $i \in \mathcal{N}$ and all $\theta \in [0,1]$,

$$u_i(\theta) \geq 0. \tag{2}$$

Type $\hat{\theta}_i \in [0,1]$ is called a *worst-off type* of firm $i$ if $u_i(\theta) \geq u_i(\hat{\theta}_i)$ for all $\theta \in [0,1]$.

The *no-deficit constraint* is satisfied if:[21]

$$\sum_{i \in \mathcal{N}} \mathbb{E}_{\theta_i} \left[ m_i(\theta_i) \right] \geq 0. \tag{3}$$

By the standard characterization (see, e.g., Myerson, 1981), BIC holds if and only if:[22]

$$q_i \text{ is nondecreasing.} \tag{4}$$

By the envelope theorem (Milgrom and Segal, 2002, Corollary 1), $u_i'(\theta) = q_i(\theta) - r_i$ wherever $u_i$ is differentiable,[23] so that for all $\theta, \theta' \in [0,1]$,

$$u_i(\theta) = u_i(\theta') + \int_{\theta'}^{\theta} (q_i(y) - r_i) dy. \tag{5}$$

The definition of $u_i(\theta)$ and equation (5) imply that for all $\theta, \theta' \in [0,1]$,

$$m_i(\theta) = \theta(q_i(\theta) - r_i) - \int_{\theta'}^{\theta} (q_i(y) - r_i) dy - u_i(\theta'). \tag{6}$$

The market maker's mechanism maximizes the equally weighted sum of the firms' ex ante expected payoffs subject to IC, IR, and no deficit:

$$\max_{\mathbf{Q}, \mathbf{M}} \mathbb{E}_{\boldsymbol{\theta}} \Big[ \sum_{i \in \mathcal{N}} \big( \theta_i Q_i(\boldsymbol{\theta}) - M_i(\boldsymbol{\theta}) \big) \Big] \text{ subject to (1)–(3).} \tag{7}$$

The ex ante expected social surplus is then

$$SS(\mathbf{r}) = \sum_{i \in \mathcal{N}} \mathbb{E}_{\theta_i} [\theta_i q_i(\theta_i) - m_i(\theta_i)],$$

where $q_i$ and $m_i$ are the interim expected allocation and payment rules induced by the mechanism that solves (7).

[21] Condition (3) only requires no deficit in expectation, but this is without loss of generality because for any BIC mechanism $\langle \mathbf{Q}, \mathbf{M} \rangle$ that is such that $\sum_{i \in \mathcal{N}} \mathbb{E}_{\theta_i} [m_i(\theta_i)] = \kappa$ holds for some $\kappa \in \mathbb{R}$, there is a BIC mechanism $\langle \mathbf{Q}, \tilde{\mathbf{M}} \rangle$ with the same allocation rule and the same interim expected payments $m_i$ whose revenue is $\kappa$ ex post, i.e., that satisfies $\sum_{i \in \mathcal{N}} \tilde{M}_i(\boldsymbol{\theta}) = \kappa$ for all $\boldsymbol{\theta}$: let $\tilde{M}_i(\boldsymbol{\theta}) = m_i(\theta_i) - \sum_{j \neq i} m_j(\theta_j)/(n-1) + c_i$ with $c_i = (\mathbb{E}_{\theta_i}[m_i(\theta_i)] - \kappa)/(n-1)$. It further follows that if the mechanism $\langle \mathbf{Q}, \mathbf{M} \rangle$ satisfies IR, then so does $\langle \mathbf{Q}, \tilde{\mathbf{M}} \rangle$. For more on equivalences of this form, see Börgers and Norman (2009).

[22] To see that IC implies that $q_i$ is nondecreasing, consider two types $\theta, \theta' \in [\underline{\theta}_i, \overline{\theta}_i]$. IC for type $\theta$ and $\theta'$ requires, respectively, $q_i(\theta)\theta - m_i(\theta) \geq q_i(\theta')\theta - m_i(\theta')$ and $q_i(\theta)\theta' - m_i(\theta) \leq q_i(\theta')\theta' - m_i(\theta')$. Subtracting the latter from the former implies that $q_i(\theta)(\theta - \theta') \geq q_i(\theta')(\theta - \theta')$, which is equivalent to $q_i$ being nondecreasing. For more background on mechanism design, see Börgers (2015) and Krishna (2010).

[23] IC implies that $u_i$ is a maximum of a family of affine functions, which implies that $u_i$ is convex and so absolutely continuous and differentiable almost everywhere in the interior of its domain.

This setup generalizes prior literature in that we allow $k_i < 1$, with the implication that there may be multiple buyers, in contrast to a partnership dissolution setup in which all assets move to a single firm.[24]

For the case of one-dimensional private information, we define virtual value and virtual cost functions:[25]

$$\Psi_i^S(\theta) \equiv \theta + \frac{F_i(\theta)}{f_i(\theta)} \text{ and } \Psi_i^B(\theta) \equiv \theta - \frac{1 - F_i(\theta)}{f_i(\theta)},$$

and we define a firm's overall virtual type function given critical type $x$, as

$$\Psi_i(\theta, x) \equiv \begin{cases} \Psi_i^S(\theta) & \text{if } \theta \in [0, x), \\ \Psi_i^B(\theta) & \text{if } \theta \in [x, 1], \end{cases} \tag{8}$$

which is nonmonotone for $x \in (0, 1)$. This impacts the analysis of second-best mechanisms.

Using the definition in (8), we obtain the following lemma, which is a slight generalization of Cramton et al. (1987, Lemma 2) to allow for heterogeneous distributions and maximum demands:

**Lemma 1.** *Given an IC, IR mechanism $\langle \mathbf{Q}, \mathbf{M} \rangle$ and one-dimensional private information, the set of worst-off types for firm $i$ is the set of $\hat{\theta}_i$ such that $q_i(\hat{\theta}_i) = r_i$ if any such $\hat{\theta}_i$ exists and otherwise is the unique $\hat{\theta}_i$ such that $q_i(\theta_i) < r_i$ for all $\theta_i < \hat{\theta}_i$ and $q_i(\theta_i) > r_i$ for all $\theta_i > \hat{\theta}_i$, and firm $i$'s expected payment to the market maker is*

$$\mathbb{E}_{\theta_i}[m_i(\theta_i)] = \mathbb{E}_{\theta_i}\Big[\Psi_i(\theta_i, \hat{\theta}_i) q_i(\theta_i)\Big] - \hat{\theta}_i r_i - u_i(\hat{\theta}_i), \tag{9}$$

*where $u_i(\hat{\theta}_i) \geq 0$ (and $u_i(\hat{\theta}_i) = 0$ if IR binds for firm $i$'s worst-off type).*

*Proof.* See Online Appendix B.2.

Worst-off types for the case of multi-dimensional private information are characterized in Lemma 2 below.

[24] The literature typically assumes $k_i = 1$, including Myerson and Satterthwaite (1983); Cramton et al. (1987); Che (2006); Figueroa and Skreta (2012); Makowski and Mezzetti (1993); Loertscher and Wasser (2019). Other research, including Gresik and Satterthwaite (1989); Lu and Robert (2001); Liu et al. (forth.), relaxes the assumption that $k_i = 1$. Williams (1987) and Loertscher and Marx (2022) introduce bargaining weights but assume extremal ownership $r_i \in \{0, k_i\}$, so that each firm is either a buyer or a seller.

[25] The virtual value function $\Psi_i^B$ captures the marginal revenue associated with firm $i$. To see this, consider a seller with cost $c$ that makes a take-it-or-leave-it price offer $p$ to firm $i$. The seller's problem is $\max_{p \in [0,1]}(1 - F_i(p))(p - c)$. The first-order condition is $-f_i(p)(\Psi_i^B(p) - c) = 0$, which by the standard "marginal revenue equals marginal cost" condition means that $\Psi_i^B(p)$ is the marginal revenue associated with $i$'s demand. An analogous argument shows that $\Psi_i^S$ captures the marginal cost associated with $F_i$.

# 4 Asset ownership transfers

In this section, we characterize the set of ex post efficiency permitting ownership structures for the model with one-dimensional types, and then examine firms' incentives for pure asset ownership transfers.

## 4.1 Ex post efficiency permitting ownership structures

For $\mathbf{r} \in \Delta_{\mathbf{k}}$, denote by $\Pi^e(\mathbf{r})$ the maximum expected revenue to the market maker of an ex post efficient, IC, IR mechanism before any revenue is given back to the agents:

$$\Pi^e(\mathbf{r}) \equiv \mathbb{E}_{\boldsymbol{\theta}}[\sum_{i \in \mathcal{N}} (\Psi_i(\theta_i, \hat{\theta}_i^e) Q_i^e(\boldsymbol{\theta}) - \hat{\theta}_i^e r_i)],$$

where $\hat{\theta}_i^e$ satisfies $q_i^e(\hat{\theta}_i^e) = r_i$. (Because the allocation rule is fixed at the ex post efficient allocation rule, maximum expected revenue means that each firm's IR constraint is satisfied with equality at its worst-off type.) Consequently, ex post efficiency is possible if and only if $\Pi^e(\mathbf{r}) \geq 0$. We will show that $\Pi^e(\mathbf{r})$ is concave and maximized at a $\mathbf{r}^*$ such that each firm has the same worst-off type, at which point it is positive. Using the result of Lemma 1 that firm $i$'s interim expected allocation is equal to $r_i$ at its worst-off type, it follows that equalized worst-off types must be interior, and so at ownership $\mathbf{r}^*$, all firms are traders. Moreover, reflecting impossibility results for settings with buyers and sellers in the tradition of Vickrey (1961) and Myerson and Satterthwaite (1983), ex post efficiency is not possible without traders.[26]

**Proposition 3.** *Ex post efficiency is not possible if there are no traders; for all* $\mathbf{r}$ *in a nonempty convex set* $\mathcal{R}^e$*, ex post efficiency is possible, including when all firms are traders with ownership* $\mathbf{r}^*$*; and* $SS(\mathbf{r})$ *is concave, including being constant for* $\mathbf{r} \in \mathcal{R}^e$ *and strictly concave otherwise.*

*Proof.* See Appendix A.

To develop intuition for the concavity of $SS(\mathbf{r})$, note that a mechanism that is a convex combination of the social surplus maximizing mechanisms for two different ownership vectors is itself a feasible mechanism that satisfies IC, IR, and no deficit when the ownership vector is the same convex combination of the two different ownership vectors. But then the allocation rule can be adjusted in favor of one that generates weakly more social surplus, and strictly more outside of $\mathcal{R}^e$, which explains why $SS(\mathbf{r})$ is concave and strictly so outside of $\mathcal{R}^e$.

[26] For generalizations, see, for example, Williams (1999), Segal and Whinston (2016), or Delacrétaz et al. (2019).

## 4.2 Social surplus maximizing merger policy

By Proposition 3, transactions that move ownership towards $\mathcal{R}^e$ increase expected social surplus. This allows us to consider the potential positive and negative effects of changes in asset ownership and the question of when and whether divestitures can play a role in remedying negative effects. For now, we assume a social surplus standard, with social surplus defined as the sum of the firms' payoffs.[27]

Because the incomplete information model with one-dimensional types satisfies all of the conditions used in Proposition 1, the social surplus maximizing policy to approve pure asset ownership transfers, block them, or approve them subject to divestitures is as described in that proposition. Further, because every symmetric, concave function is Schur concave (Marshall et al., 2011, Proposition C.2), it follows that $SS(\mathbf{r})$ is Schur concave if all firms are ex ante identical, that is, if $F_i = F$ and $k_i = k$ for all $i \in \mathcal{N}$. Consequently, all firms being ex ante identical provides a foundation for Proposition 2 and Corollary 1. In particular, with symmetric firms, greater symmetry in asset ownership increases social surplus directly and also makes divestiture-based remedies more effective in the sense used in Proposition 2.

Figure 1 illustrates the ex post efficiency permitting set $\mathcal{R}^e$ and $\mathbf{r}^*$ for $n = 3$ and different assumptions on distributions. As shown in panel (a) and observed by Cramton et al. (1987), when firms are symmetric, $\mathbf{r}^*$ is symmetric, and ex post efficiency may be achievable even when $r_i = 0$ for any $i \in \mathcal{N}$. In this case, Corollary 1 implies that any shifts in the ownership structure towards symmetry are beneficial for social surplus. In contrast, when firms are not symmetric, $\mathbf{r}^*$ is not symmetric either and achieving ex post efficiency may require that a particular firm (firm 1 in the case of Figure 1(b)) has positive asset ownership.

This analysis provides the foundation for the guidance that divestitures should, if possible, be designed to secure ownership structures in $\mathcal{R}^e$. But to the extent that unmodeled transactions costs or market frictions are present, a competition authority might have a preference for ownership structures that are not just an element of $\mathcal{R}^e$, but that are robust to such unmodeled costs as best possible. This can be a achieved with the ownership structure $\mathbf{r}^*$, which maximizes expected revenue under ex post efficiency and binding IR for the firms' worst-off types.[28] This raises the question of how $\mathbf{r}^*$ varies with the size and strength of firms in the market. We address this question in Appendix C.3. As we show there, all else equal, firms with larger maximum demands or stronger distributions according to first-order

[27]The framework can be applied to analyze transactions under a consumer surplus standard under the extension to downstream consumers in Section 6.1.

[28]Following the interpretation mentioned in the introduction of asset ownership as long-term labor contracts, the social surplus benefit of symmetric ownership for symmetric firms is consistent with policies such as the Union of European Football Association's Financial Fair Play Regulations, which restrict European football clubs from consistently operating at a loss and thereby promote symmetric team rosters.

(a) Acquisition by firm 3 of firm 2's assets (b) Acquisition by firm 3 of firm 1's assets

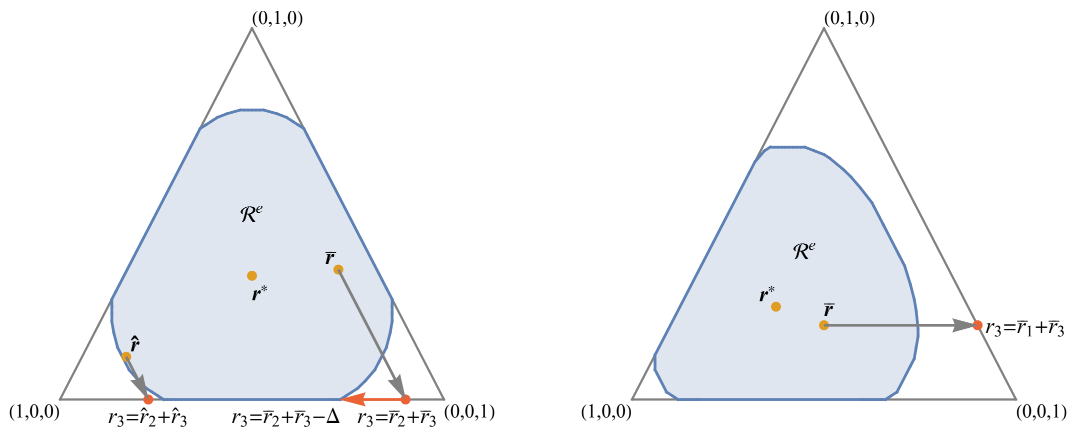


Figure 1: Ex post efficiency permitting set $\mathcal{R}^e$ (blue region), with example asset ownership transfers indicated by gray arrows. In panel (a), starting from $\bar{\mathbf{r}}$ there exists an ex post efficiency restoring divestiture of assets to firm 1, shown by the red arrow, but starting from $\hat{\mathbf{r}}$, no such divestiture exists. In panel (b), no divestiture to firm 2 restores ex post efficiency. Assumes $n = 3$ and $k_i = 1$ for all $i$. Panel (a) assumes uniformly distributed types for all firms, and panel (b) assumes that $F_1(x) = x^3$ and $F_2(x) = F_3(x) = x$, so that firm 1 is stronger than firms 2 and 3 in the sense of first-order stochastic dominance.

stochastic dominance have larger asset ownership in $\mathbf{r}^*$.

## 4.3 Incentives for asset ownership transfers

We now show that bilateral transactions of assets that occur at the ex ante stage are quite generally profitable even if they are socially harmful. In particular, an acquisition of assets that does not disrupt the efficiency of the market is profitable for the acquiring firm if it improves the firm's outside option and so improves its expected payoff from participation in the mechanism. Further, when $n > 2$, transactions that disrupt the efficiency of the market can also be profitable for the transacting firms.

We call an ownership structure $\mathbf{r}$ *stable* if there are no mutually beneficial pairwise transactions. We then have the following contrasting results, the first applying to the case of two firms and the second to the case of more than two firms:

**Proposition 4.** *Given $n = 2$, mutually beneficial transactions of assets exist if and only if $\mathbf{r} \notin \mathcal{R}^e$. Put differently, $\mathbf{r}$ is stable if and only if it is ex post efficiency permitting.*

*Proof.* If $\Pi^e(\mathbf{r}) < 0$, then ex post efficiency is not achieved under $\mathbf{r}$. The two firms can increase their joint payoff through, for example, a transaction that shifts the ownership structure to $\mathbf{r}^*$, where $\Pi^e(\mathbf{r}^*) \geq 0$, so ex post efficiency is achieved (and the ex post efficient surplus is divided between the two firms). If $\Pi^e(\mathbf{r}) \geq 0$, then ex post efficiency is achieved prior to any transaction (and the associated surplus is divided between the two firms), so no further increases in joint surplus are possible. This completes the proof. ■

Proposition 4 implies that with only two firms, one expects mutually beneficial asset transactions to allow ex post efficiency to be achieved. Accordingly, with two firms, a laissez-faire competition policy maximizes social surplus because any profitable asset ownership transfer increases social surplus.

In contrast, with more than two firms, a pair of firms may have a mutually beneficial transaction that reduces social surplus, and so imposes negative externalities on nontrading firms:

**Proposition 5.** *Given $n \geq 3$ and at least two firms that are traders: (i) if $\Pi^e(\mathbf{r}) > 0$, then a weakly (strictly if there are traders $i$ and $j$ with $\eta_i + \eta_j < 1$) mutually beneficial pairwise asset ownership transfer exists; and (ii) if $\Pi^e(\mathbf{r}) = 0$ and $F_i = F$ for all $i \in \mathcal{N}$, then there exists a strictly mutually beneficial pairwise asset ownership transfer that results in ownership structure $\mathbf{r}'$ with $\Pi^e(\mathbf{r}') < 0$.*

*Proof.* See Appendix A for a sketch of the proof and Online Appendix B.4 for details.

Proposition 5 states that ex post efficiency permitting market structures are unstable if there are at least two traders that jointly extract less than the full budget surplus. Negating this and noting that (i) if there are no traders, then the market structure cannot be ex post efficiency permitting, and (ii) if there are more than two traders, then there necessarily exists a pair of traders that jointly extract less than the full budget surplus, we have the following implication: Ex post efficiency permitting market structures are stable only if there is exactly one trader or if there are exactly two traders that jointly extract the full budget surplus. Further, with $F_i = F$ and more than two firms, ownership structures on the boundary of the ex post efficiency permitting region, i.e., $\mathbf{r}$ such that $\Pi^e(\mathbf{r}) = 0$, are also not stable if there are two traders because then those traders have an incentive for transactions that harm rivals and reduce social surplus below the ex post efficient level. The reason that two traders are required for this result is that the kind of transaction that can be profitable even though it moves away from ex post efficiency is one that shifts assets to the firm with the weakly higher worst-off type. If there are two traders, then this is always possible. In contrast, it is not possible if one firm is a seller because the seller will have the higher worst-off type, but no demand for additional assets; and it is not possible if one firm is a buyer because the buyer will have the lower worst-off type, but no assets to trade.

To summarize, we have:

**Corollary 2.** *For two firms, the set of stable ownership structures coincides with the ex post efficiency permitting set, but for more than two firms, ex post efficiency is not possible for* *any stable ownership structure with at least two traders when all firms have positive surplus shares (for all $i \in \mathcal{N}$, $\eta_i > 0$).*

Because traders have an incentive for asset ownership transfers that reduce the efficiency of the market below the ex post efficient level and because such transactions harm rivals, concerns related to raising rivals' costs can be tied to the overall efficiency effects of shifts of assets among traders.[29] The contrast in results between the case with two firms and the one with more than two firms is stark. It has a precursor in the complete information literature and the debate on the Coase Theorem, with Aivazian and Callen (1981) arguing that with more than two agents, the emptiness of the core may render efficient bargaining impossible, and Coase (1981) countering that his argument (Coase, 1960) was based on the case with two agents. With that in mind, Corollary 2 provides an incomplete information formalization of these opposing views and forces. Because bargaining externalities arise when asset transactions at the ex ante stage are profitable for the firms involved but are socially harmful, divestiture policies in our setting directly relate the theory of *raising rivals' costs* in industrial organization.[30]

Of course, if all $n$ firms negotiated simultaneously, they would find ex post efficiency permitting arrangements. But our analysis of bilateral asset transactions then also implies that these would not be immune to bilateral deviations. In that sense, this analysis also provides an explanation for why asset transactions are typically bilateral transactions.

While Proposition 3 shows that having traders can contribute to the efficient functioning of the market for asset usage (traders are necessary for ex post efficiency and having all firms be traders with appropriate ownership is sufficient for ex post efficiency), Proposition 5 and Corollary 2 show that traders can be the source of problems at the ex ante stage by having profitable bilateral transactions that reduce social surplus, making them of particular concern for competition authorities. Thus, a tension exists: traders are potentially problematic at the ex ante stage, but they improve market functioning at the ex post stage. Our results can be viewed as good news for a focus on structural rather than behavioral remedies by

[29]This result contrasts with the finding of Farrell and Shapiro (1990) that in a Cournot setup, a profitable reallocation of capital that reduces welfare benefits the rivals, which increase their output in response to the contraction in total output by the transacting firms. Thus, they do not get a "raising rivals' costs" effect because the welfare reduction is borne entirely by the transacting firms. In contrast, in our setup, the reduction in the efficiency of the market affects all firms. See Podwol and Raskovich (2021) for a model of vertical mergers with inputs purchased by auction, with application to the CVS-Aetna merger investigation.

[30]The "raising rivals' costs" theory of harm argues that following a vertical merger, the integrated firm will charge more to external buyers for the inputs that it controls. The profitability of such a strategy usually relies on diversion of downstream customers to the integrated firm (see, e.g., Salop and Scheffman, 1983, 1987; Ordover et al., 1990). Raising rivals' costs theories have played a prominent, and sometimes controversial, role in antitrust practice (see, e.g., Coate and Kleit, 1990; Salop, 2017). It is notable that raising rivals' costs effects that arise in our setting do not rely on diversion.

competition authorities because they suggest that structural remedies alone can be valuable and effective.

# 5 Full integration

We now turn to the analysis of full integration. As full integration even between two one-dimensional firms creates a multi-dimensional firm, we first derive the set of ex post efficiency permitting ownership structures. Then we analyze the effects of mergers under full integration on the market's ability to allocate ex post efficiently.

## 5.1 Possibility of ex post efficiency with multi-dimensional types

To derive the set of ex post efficiency permitting ownership structures when some or all firms have multi-dimensional types and decreasing marginal values, we construct a revenue-maximizing efficient mechanism subject to IC and IR constraints, and we characterize the conditions under which that mechanism satisfies the no deficit constraint.

For the model with multi-dimensional types, we require the mechanism to be *dominant strategy incentive compatible (DIC)*. Formally, a mechanism $\langle \mathbf{Q}, \mathbf{M} \rangle$ satisfies DIC if for all $i \in \mathcal{N}$, $\boldsymbol{\theta}_i$, $\boldsymbol{\theta}'_i$, and $\boldsymbol{\theta}_{-i}$, we have $\mathbf{Q}_i(\boldsymbol{\theta}) \cdot \boldsymbol{\theta}_i - M_i(\boldsymbol{\theta}) \geq \mathbf{Q}_i(\boldsymbol{\theta}'_i, \boldsymbol{\theta}_{-i}) \cdot \boldsymbol{\theta}_i - M_i(\boldsymbol{\theta}'_i, \boldsymbol{\theta}_{-i})$. This focus contrasts with the preceding analysis insofar as, there, the nature of incentive compatibility was not material (see footnote 20). Of course, it is still the case that any DIC mechanism is BIC, but with multi-dimensional types, it is an open question whether there is also an implication arrow that points the other way. The upshot of focusing on ex post efficient DIC mechanisms is that it implies, by Holmström's (1979) theorem, that the set of admissible mechanisms is the set of Groves' schemes (Groves, 1973). That is, at reported type profile $\boldsymbol{\theta}$, the payment from any firm $i$ to the market maker takes the form

$$M_i^G(\boldsymbol{\theta}) \equiv K_i(\boldsymbol{\theta}_{-i}) - \left(W(\boldsymbol{\theta}) - Q_i^e(\boldsymbol{\theta}) \cdot \boldsymbol{\theta}_i\right),$$

where $W(\boldsymbol{\theta})$ is social surplus under ex post efficiency at $\boldsymbol{\theta}$ and $K_i(\boldsymbol{\theta}_{-i})$ is a constant that is, independent of firm $i$'s reported type. The search for expected revenue maximizing, ex post efficient mechanisms that satisfy DIC and IR thus reduces to finding the constants $K_i(\boldsymbol{\theta}_{-i})$ for $i \in \mathcal{N}$ that satisfy IR and maximize expected revenue.

Using Lemma 1 and the fact that $q_i^e(\theta)$ is strictly increasing with $q_i^e(0) = 0$ and $q_i^e(1) = k_i$, if firm $i$ has a one-dimensional type, then its worst-off type under ex post efficiency, $\hat{\theta}_i^e$, is unique and determined by the condition $q_i^e(\hat{\theta}_i^e) = r_i$. Consequently, for one-dimensional firms, the worst-off types and IR pin down the constant $K_i(\boldsymbol{\theta}_{-i})$ as $K_i(\boldsymbol{\theta}_{-i}) = W(\hat{\theta}_i^e, \boldsymbol{\theta}_{-i}) - r_i\hat{\theta}_i^e$, where $W(\hat{\theta}_i^e, \boldsymbol{\theta}_{-i})$ is social surplus under ex post efficiency when firm $i$'s type is $\hat{\theta}_i$ and the types of all firms other than firm $i$ are $\boldsymbol{\theta}_{-i}$. In contrast, for a multi-dimensional firm $i$, there are typically multiple type vectors $\boldsymbol{\theta}_i$ that satisfy the condition for a worst-off type that $q_i^e(\boldsymbol{\theta}_i) = r_i$. This implies that the market maker (or analyst) faces the nontrivial problem of determining which of those worst-off types maximizes revenue. As we now show, constant worst-off types are revenue maximizing:

**Lemma 2.** *Constant worst-off types maximize expected revenue under binding IR for firms' worst-off types, and firm i's constant worst-off type is uniquely defined by* $\hat{\boldsymbol{\theta}}_i^e = (\hat{\theta}_i^e, \ldots, \hat{\theta}_i^e)$ *such that* $q_i^e(\hat{\boldsymbol{\theta}}^e{}_i) = r_i$.

*Proof.* See Appendix A.

The proof of Lemma 2 shows that firm $i$'s interim expected payoff under the ex post efficient allocation, denoted by $u_i^e(\boldsymbol{\theta}_i)$, is convex and that, using envelope theorem results of Milgrom and Segal (2002),[31] $u_i^e(\boldsymbol{\theta}_i)$ can be written in terms of a path integral with respect to a path connecting an arbitrary $\boldsymbol{\theta}_i'$ and $\boldsymbol{\theta}_i$, which allows its gradient to be written as a function of $(k_i^1, \ldots, k_i^{h_i})$, $r_i$, and $q_i^e(\boldsymbol{\theta}_i)$, where that gradient has all components equal to 0 if and only if $q_i^e(\boldsymbol{\theta}_i) = r_i$ for all $i \in \mathcal{N}$. This establishes that, analogous to the case of one-dimensional types, any worst-off type vector $\boldsymbol{\theta}_i^w$ for firm $i$ satisfies $q_i^e(\boldsymbol{\theta}_i^w) = r_i$. We then show that firm $i$'s expected payment is maximized when its worst-off type is a constant vector.

In light of Lemma 2, the revenue maximizing choice of the constant $K_i(\boldsymbol{\theta}_{-i})$ for a firm $i$ whose type is multi-dimensional is

$$K_i(\boldsymbol{\theta}_{-i}) = W(\hat{\boldsymbol{\theta}}_i^e, \boldsymbol{\theta}_{-i}) - r_i \hat{\theta}_i^e,$$

where $\hat{\boldsymbol{\theta}}_i^e$ is the constant worst-off type defined in Lemma 2. Note that $r_i \hat{\theta}_i^e$ is the value of $i$'s outside option when its type is $\hat{\boldsymbol{\theta}}_i^e$ because that type has constant marginal values. Consequently, the expected revenue maximizing, ex post efficient, DIC, IR mechanism has, for $i \in \mathcal{N}$, the payment rule

$$M_i^{VCG}(\boldsymbol{\theta}) \equiv W(\hat{\boldsymbol{\theta}}_i^e, \boldsymbol{\theta}_{-i}) - W(\boldsymbol{\theta}) + \mathbf{Q}_i^e(\boldsymbol{\theta}) \cdot \boldsymbol{\theta}_i - r_i \hat{\theta}_i^e, \tag{10}$$

where $\hat{\boldsymbol{\theta}}_i^e$ is, in slight abuse of notation, a scalar if $i$'s type is one-dimensional and consists of the constant worst-off types given by Lemma 2 otherwise. The acronym $VCG$ in (10) stands for Vickrey (1961), Clarke (1971), and Groves (1973), who first analyzed mechanisms of this form.

With the mechanism $\langle \mathbf{Q}^e, \mathbf{M}^{VCG} \rangle$ in hand, we can now define the set $\mathcal{R}^e$ of ex post efficiency permitting asset ownership vectors. As shown below, like in the case with one-

[31] Specifically, we apply Milgrom and Segal's Corollary 1 under their adaptation to the case of multi-dimensional types and relaxing an assumption through an application of the Monotone Selection Theorem of Milgrom and Shannon (1994) (see Milgrom and Segal, 2002, footnote 10).

dimensional types, $\mathcal{R}^e$ is nonempty, convex, and includes the ownership vector $\mathbf{r}^*$ that equalizes all firms' worst-off types.[32]

**Proposition 6.** *There exists* $\mathbf{r}^* \in \Delta_{\mathbf{k}}$ *and* $\hat{\theta}^* \in (0,1)$ *such that each firm with a one-dimensional type has worst-off type* $\hat{\theta}^*$ *and each firm with a multi-dimensional type has worst-off type* $(\hat{\theta}^*, \ldots, \hat{\theta}^*)$*; moreover, the set of ex post efficiency permitting ownership vectors,* $\mathcal{R}^e$*, is nonempty, convex, and contains* $\mathbf{r}^*$*.*

*Proof.* See Appendix A.

An implication of Proposition 6 is that our analysis for asset ownership transfers regarding which mergers should be approved or blocked and the potential role for social surplus restoring divestitures carries over to transactions that involve the full integration of firms, with one difference and one qualification. The difference is that with full integration, the set of ex post efficiency permitting ownership structures varies with the transaction, whereas for pure asset ownership transfers, it is fixed. Consequently, divestitures that restore ex post efficiency must be calculated with regard to the post-transaction ex post efficiency permitting set. But because divestitures only involve a change in asset ownership, that set is the same before and after divestiture. The qualification is that with full integration that induces a nondegenerate multi-dimensional firm, the analysis can only be applied to mergers such that ex post efficiency is possible before or after the transaction (and any divestitures) because with multi-dimensional types the second-best mechanism for asset ownership structures that do not permit ex post efficiency is not known. In contrast, with pure asset ownership transfers and one-dimensional types, the second-best analysis applies to any asset ownership structure.

## 5.2 Horizontal and vertical mergers

It is useful to distinguish different categories of full integration. We refer to the full integration of two buyers or of two sellers as a *horizontal merger* and to the full integration of one buyer and one seller as a *vertical merger.* Mergers that are neither strictly horizontal nor strictly vertical are referred to as *conglomerate mergers* and are analyzed in the next subsection.

[32] The result that the ownership $\mathbf{r}$ that equalizes the firms' worst-off types also permits ex post efficiency is the driving force behind the possibility result of Cramton et al. (1987) for $F_i = F$ and $k_i = 1$ for all $i$. Che (2006) provides the first generalization to nonidentical distributions while maintaining $k_i = 1$, while Liu et al. (forth.) show that this insight extends to $k_i \neq k_j$. Segal and Whinston (2011) derive a status quo, which can be thought of as some $\mathbf{r}$ in our context, that permits ex post efficiency for a more general allocation problem, which when applied to the present problem is not necessarily $\mathbf{r}^*$. See also Schweizer (2006) for a fixed-point based possibility result. Analyzing a partnership model with nonidentical distributions and $k_i = 1$ in which the market maker maximizes a weighted sum of social surplus and revenue, Loertscher and Wasser (2019) show that, with private values, the objective function remains concave in $\mathbf{r}$ and that the optimal $\mathbf{r}$ maintains the property that firms' worst-off types are equal if possible.

We say that a merger *makes ex post efficiency possible* if ex post efficiency is possible after the merger whereas it was not before. Likewise, we say that a merger *makes ex post efficiency impossible* if ex post efficiency was possible before the merger but not after it. A merger is *socially desirable* if it can make ex post efficiency possible and never makes it impossible. Likewise, a merger is *socially undesirable* if it can make ex post efficiency impossible and never makes it possible.

**Theorem 1.** *Horizontal mergers are socially undesirable, whereas vertical mergers are socially desirable.*

*Proof.* See Appendix A.

The first part of Theorem 1 resonates with the result for one-dimensional private information of Loertscher and Marx (2022) that, in a two-sided setting, horizontal mergers weakly reduce social surplus, and the result of of Loertscher and Marx (2019) that a horizontal merger in a procurement setting with one buyer and multiple sellers harms the buyer.[33] The vertical merger result in Theorem 1 highlights an important difference to the second-best analysis afforded by one-dimensional types. To illustrate, consider a setting with, before integration, one buyer and $n-1$ sellers where $n \geq 2$. Then ex post efficiency is not possible prior to a vertical merger of the buyer with one seller, and it remains impossible post merger unless $n = 2$. As discussed in Section 6.3, which extends Loertscher and Marx (2022, Proposition 7), with nonoverlapping supports there exist cases in which a vertical merger reduces social surplus. For example, if there is one buyer with one-unit demand and type support $[1, 2]$ and two sellers, each with one-unit supply and type support $[0, 1]$, then ex post efficiency is possible pre-merger, but following vertical integration, the vertically integrated firm acts as a buyer vis à vis the outside seller with type distributed on $[0, 1]$, and so ex post efficiency is not possible—a vertical merger in this case creates a Myerson-Satterthwaite problem.

## 5.3 Assessing conglomerate mergers

We now turn to mergers that are neither purely horizontal nor purely vertical, which, as mentioned, we refer to as conglomerate mergers. For example, the merger of two traders would be a conglomerate merger, as would the merger of a trader with a buyer or a seller. Of course, a trader that has positive but very low asset ownership is "close," in a sense, to being a buyer, and a trader that has asset ownership less than but close to its maximum demand resembles a seller. Together with Theorem 1, this observation suggests that some

[33] In Loertscher and Marx (2019), a horizontal merger is neutral for social surplus if the buyer uses an efficient procurement mechanism because the buyer's value is commonly known, but it increases the expected payment that the buyer has to make.

conglomerate mergers are socially desirable, while others are socially undesirable. This intuition is indeed correct as stated in the following proposition, which follows from the proof of Proposition 8 below:

**Proposition 7.** *Some conglomerate mergers are socially desirable while others are socially undesirable.*

Proposition 7 raises the questions which conglomerate mergers are socially desirable and which ones are not, and to what extent and how harm from an otherwise undesirable conglomerate merger can be offset by divestitures. In what follows, we address these questions.

Say that a merger between a buyer (seller) and another firm $j$ with ownership $r_j$ has a strong vertical component if $r_j$ is close to $k_j$ (close to 0) and a strong horizontal component if $r_j$ is close to 0 (close to $k_j$). We first show that the intuition that a conglomerate merger is socially desirable if it has a sufficiently strong vertical component and socially undesirable if it has a sufficiently strong horizontal component is correct.

**Proposition 8.** *Assume $n \geq 3$. If $r_1 = 0$, then there exist $\underline{r}_2 \in (0, k_2)$ and $\overline{r}_2 \in (0, k_2)$ such that the full integration of firms 1 and 2 is socially undesirable if $r_2 \in [0, \underline{r}_2)$ and socially desirable if $r_2 \in (\overline{r}_2, k_2]$. Conversely, if $r_1 = k_1$, then there exist $\hat{\underline{r}}_2 \in (0, k_2)$ and $\hat{\overline{r}}_2 \in (0, k_2)$ such that the full integration of firms 1 and 2 is socially desirable if $r_2 \in [0, \hat{\underline{r}}_2)$ and socially undesirable if $r_2 \in (\hat{\overline{r}}_2, k_2]$.*

*Proof.* If $r_2 = 0$, then the merger is horizontal and so, by Theorem 1, is socially undesirable. By continuity of the market maker's revenue, this also holds for $r_2$ positive but sufficiently close to zero. If $r_2 = k_2$, then the merger is vertical and so, by Theorem 1, is socially desirable. By continuity, this holds for $r_2$ less than but sufficiently close to $k_2$. The argument that proves the converse is essentially the same and based on Theorem 1 and continuity. ■

Proposition 8 provides nuanced guidance by relating the effects to a conglomerate merger to how strongly vertical or horizontal the merger is. In contrast, our next result provides sufficient condition for a conglomerate to be socially undesirable.

We say that a merger *corners the market* if after the merger all remaining firms are either buyers or sellers. All remaining firms are buyers if the merger involves the only two firms with positive ownership, and all remaining firms are sellers if the merger involves the only firm(s) whose net demand exceeds their ownership. (For the merger to be a conglomerate merger, at least one of the merging firms needs to have positive ownership that is less than its maximum demand.) A merger that corners the market leads to two-sided setting with either a single or a single seller, i.e. the merged entity. Consequently, the following result follows from existing impossibility results for two-sided settings with multi-dimensional types:[34]

**Corollary 3.** *A conglomerate merger that corners the market is socially undesirable if ex post efficiency is possible before the merger.*

Corollary 3 relates to Loertscher and Marx (2022, Proposition 7) on integration rendering ex post efficiency impossible if it results in a two-sided market, that is, a setting with only buyers and sellers whose distributions have identical supports, and no traders. Because ex post efficiency is possible if all firms have the same worst-off type, it is also the case that a conglomerate merger is socially desirable if ex post efficiency is not possible pre merger and if the worst-off types of all firms are the same post merger. Further, because the market maker's revenue is continuous, this result extends to conglomerate mergers that make the worst-off types of all firms sufficiently close to being the same.

Of course, whether a merger equalizes all firms worst-off types is, arguably, harder to observe in data typically available in the merger review process than whether it corners the market. However, given knowledge of $F_i$, $k_i$, and $r_i$ for all $i \in \mathcal{N}$, it is always possible to compute the ex post efficiency permitting regions pre and post merger and therefore to assess whether a conglomerate merger is socially desirable or socially undesirable and whether there are divestitures that make ex post efficiency possible post merger if it is not possible without divestitures, and if so, what these remedies are.

As an illustration, consider the case of $n = 3$ with uniformly distributed types and a merger of firms 1 and 2. In a partnership setup with $k_i = 1$ for all $i \in \mathcal{N}$, the merged entity applies all its assets to the larger of its two type draws. As a result, as mentioned in footnote 18, the merged entity behaves as a firm with a one-dimensional type that is drawn from the distribution of the maximum of independent draws from the two pre-merger distributions.

Figure 2 illustrates that for some pre-integration asset ownership, full integration decreases expected social surplus ($\mathbf{r}$ in the blue shaded region but not in the orange shaded region), while for other pre-integration asset ownership, full integration increases expected social surplus ($\mathbf{r}$ in the orange shaded region but not in the blue shaded region). The possibility of social surplus increasing mergers is perhaps surprising—it does not arise in a one-sided or two-sided settings (Loertscher and Marx, 2019, 2022),[35] but rather relies on the presence of traders that buy for some type realizations and sell for others. For intuition, consider the example of the partnership setting shown in Figure 2(b) with $\mathbf{r} = (0, 0.8, 0.2)$.

[34] For example, Loertscher and Mezzetti (2019) show that the Walrasian price gap is a lower bound for the deficit the market maker incurs on every unit that is traded under ex post efficiency.

[35] For example, in a two-sided setting with one buyer and multiple sellers, where $\mathbf{r} = (0, 1/(n-1), \dots, 1/(n-1))$ and $\mathbf{k} = (1, 1/(n-1), \dots, 1/(n-1))$, a merger of sellers merely reduces competition among sellers, and a merger of the buyer and a seller both reduces competition among sellers and reduces the buyer's willingness to pay for outside units, both with negative consequences for social surplus.

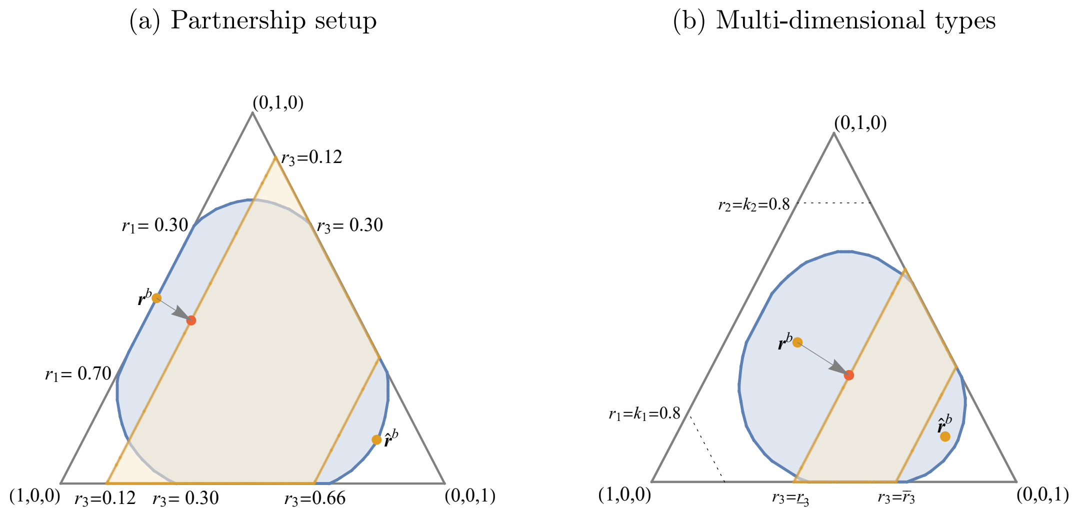


Figure 2: Effects of the full integration of firms 1 and 2 on the ex post efficiency permitting set, including the pre-merger (blue) ex post efficiency permitting set and the post-merger (orange) set of vectors $(r_1, r_2, r_3)$ such that ex post efficiency is possible following the full integration of firms 1 and 2. Panel (a) assumes $k_1 = k_2 = k_3 = 1$. Panel (b) assumes that $k_1 = k_2 = 0.8$ and $k_3 = 1$. In both panels, given pre-merger ownership $\mathbf{r}^b$, the merger of firms 1 and 2 reduces expected social surplus, but can be remedied through a divestiture of assets to firm 3; however, no divestiture can restore pre-merger expected social surplus when pre-merger ownership is $\hat{\mathbf{r}}^b$. In both panels, pre-merger types are one-dimensional and uniformly distributed.

In this case, ex post efficiency is not possible because, in effect, the ownership structure is too asymmetric relative to the firms' symmetric productivities. However, the integration of firms 1 and 2 improves their distribution, allowing the new firm to, in effect, "grow into" its large ownership of 0.8. In this way, full integration can increase social surplus because it better aligns relatively large ownership with a relatively strong distribution (see Proposition C.1 in Online Appendix C.3 for a formalization of this).

In the case displayed in Figure 2(b), the merger combines the two identical firms whose maximum demands are smaller than the outside firm's, giving the merger a substantial horizontal aspect. As shown in the figure, the set of ownership structures such that full integration makes ex post efficiency impossible (blue but not orange) is relatively large, while the set of ownership structures such that the reverse occurs (orange but not blue) is very small. While, consistent with Proposition 7, we see the possibility of both beneficial and harmful conglomerate mergers, the largely pessimistic view for this particular setup with its substantial horizontal aspect is supported by the result of Theorem 1 that horizontal mergers are socially undesirable.

Figure 2 also illustrates that, as was the case for transfers of asset ownership, harms associated with full integration can sometimes be remedied through divestitures.[36] This also illustrates the relevance of the three components of a competition authority's social surplus maximizing response to a merger described in Proposition 1.

**Screens and practical implementation.** The above analysis rests, evidently, on the assumption that $F_i$, $k_i$ and $r_i$ are known for all $i \in \mathcal{N}$. But the procedure can, of course, also be applied if some of this information needs to be estimated with data typically available in a merger review as illustrated in Appendix D.[37] Consequently, the procedure illustrated in Figure 2 can, in principle, serve as a screen for merger effects and as guidance for divestitures to eliminate merger harm.

# 6 Extensions and discussion

In the first three subsections below, we show how the model can be extended to accommodate downstream consumers and consumer surplus considerations, investment, and nonidentical supports of the firms' type distributions, respectively. Then in the fourth subsection, we discuss paths toward second-best mechanisms with full integration that consist of transforming multi-dimensional distributions into one-dimensional distributions that are equivalent in the sense of inducing the same ex post efficiency permitting set.

## 6.1 Downstream consumers and consumer surplus

We now extend the model by adding downstream consumers and provide conditions such that, taking as given that firms have market power vis à vis downstream consumers, social surplus maximization as described above is equivalent to the maximization of consumer surplus. Under these conditions, the prescriptions above are appropriate regardless of whether the focus is on social surplus or consumer surplus.

To accommodate downstream consumers and consumer surplus considerations, assume that asset usage is an input that improves the quality of a product that firms sell in individual downstream markets. Let $P_i(y)$ be the willingness to pay of a typical consumer in market $i$ for the $y$-th unit of quality 1 and assume that the willingness to pay for the $y$-th unit of quality $q \in [0, k_i]$ is $qP_i(y)$. (This extends straightforwardly to the case of multi-dimensional,

[36]For example, focusing on panel (a), when $r_3$ is less than 0.12 while $r_1$ and $r_2$ are similar (blue region in the middle, left part of the simplex), full integration of firms 1 and 2 reduces social surplus, but, in that case, the harm can be remedied through a divestiture of assets to firm 3. In contrast, when $r_3$ is greater than 0.66 while $r_1$ and $r_2$ have similar asset ownership (blue region in the lower, right corner of the simplex), the harm from full integration of firms 1 and 2 cannot be remedied through a divestiture of assets to firm 3.

[37]As shown there, under the parameterization that $F_i(\theta) = 1-(1-\theta)^{\alpha_i}$ with $\alpha_i > 0$, distributions can be calibrated using the firms' markets shares and one firm's margin. Maximum demands can be derived from historical market data while asset ownership will typically be observable. With these values in hand, merger effects (and divestiture effects) can be estimated as illustrated in Appendix D.

decreasing marginal values.) Assume further that the marginal cost of production of the downstream product is zero, the private information of each firm $i$ pertains to the mass $\sigma_i > 0$ of identical consumers in its downstream market, and the inverse demand function $P_i(y)$ is decreasing. A maximizer of $yP_i(y)$ over $y$ is denoted by $y_i^*$. For a given realization $\sigma_i$ of the mass of consumers in market $i$, the firm's marginal willingness to pay for quality $q$ is $\theta_i \equiv \sigma_i y_i^* P_i(y_i^*)$.

Given input quality $q \in [0, k_i]$, in equilibrium per-capita consumer surplus in market $i$ is $q(\int_0^{y_i^*} P_i(y)dy - y_i^* P_i(y_i^*))$, per-capita profit is $q y_i^* P_i(y_i^*)$, and, dividing consumer surplus by firm profit, the pass-through rate of firm profit to consumer surplus in market $i$ is

$$\gamma_i \equiv \frac{1}{y_i^* P_i(y_i^*)} \int_0^{y_i^*} P_i(y)dy - 1.$$

If all downstream markets have the same consumer surplus pass-through, that is, if $\gamma_i = \gamma$ for all $i \in \mathcal{N}$, then social surplus maximization, with social surplus narrowly defined as firms' profits, is the appropriate objective even for a planner that also accounts for downstream consumer surplus, including the case of a planner that only values consumer surplus.

Of course, the condition that $\gamma_i = \gamma$ for all $i \in \mathcal{N}$ will not be universally met. For example, if some firm $j$ only serves an export market and the antitrust authority does not account for consumer surplus in other countries, then $\gamma_j = 0$ will hold, and the authority would like the market (and the market structure) to discriminate against firm $j$.

## 6.2 Investment

Investment incentives feature prominently in concurrent competition policy debates and merger cases as well as in organizational economics. For example, investment incentives were at center stage in the Dow-Dupont merger in the United States or the EC's GE-Almstrom merger decision.[38] As is reasonably well known in the mechanism design literature, with incomplete information, efficient markets imply efficient investments quite generally.[39] Because the same does not appear to be the case for practitioners and scholars working in antitrust, we now provide a short, yet general, analysis of investment incentives with incomplete information.

Consider the model that allows for firms to have multi-dimensional types. Each firm $i$ has a set of possible investments $\mathcal{I}_i$ with each investment $I_i \in \mathcal{I}_i$ for firm $i$ being associated with

[38] See the U.S. DOJ's Competitive Impact Statement in the Dow-Dupont merger (https://www.justice.gov/atr/case-document/file/973951/download, pp. 2, 10, 15, 16) or the EC's statement in Annex I, paragraph 32 (https://ec.europa.eu/competition/mergers/cases/decisions/m7278_6808_3.pdf). For additional examples, see, e.g., Gilbert and Sunshine (1995) and Katz and Shelanski (2017).

[39] See, for example, Milgrom (2004) for the second-price auction and Krähmer and Strausz (2007), Loertscher and Marx (2022), and Liu et al. (forth.) for more general setups with one-dimensional types.

a known cost $C_i(I_i)$. Investments affect the firms' type distributions without affecting their supports, so that, when accounting for investments, we now write $f_i(\theta, I_i)$ for the density of $i$'s type $\theta$ given investment $I_i$ if firm $i$'s type is one-dimensional. After the investment $I_i$ of multi-dimensional firm $i$, the density its $j$-th type $\theta_i^j$ is denoted $f_i^j(\theta_i^j, I_i)$. For a multi-dimensional firm $i$, let $f_i(\boldsymbol{\theta}_i, I_i) = \times_{j=1}^{h_i} f_i^j(\theta_i^j, I_i)$ denote its joint density.

Letting $f(\boldsymbol{\theta}, \mathbf{I}) \equiv \times_{i \in \mathcal{N}} f_i(\boldsymbol{\theta}_i, I_i)$, where in slight abuse of notation $\boldsymbol{\theta}_i$ is a scalar if firm $i$'s type is one-dimensional, expected social surplus under ex post efficiency is

$$SS^{\text{invest}}(\mathbf{I}) = \int_{[0,1]^H} W(\boldsymbol{\theta}) f(\boldsymbol{\theta}, \mathbf{I}) d\boldsymbol{\theta} - \sum_{i \in \mathcal{N}} C_i(I_i),$$

where, as may be recalled, $H$ is the number of types. We make the weak assumption that there exists a feasible vector of investments $\bar{\mathbf{I}}$ that maximizes $SS^{\text{invest}}(\mathbf{I})$.

Consider now the following investment game. In stage one, all firms simultaneously choose their investments, which are neither observable nor contractible. If a vector of investments $\mathbf{I}$ is a pure strategy Nash equilibrium outcome of the game with market structure $\mathbf{r}$, then the market operates with expected revenue computed according to $f(\boldsymbol{\theta}, \mathbf{I})$. In this setup, if the market is ex post efficient given investments $\bar{\mathbf{I}}$, then the investment game has a Nash equilibrium in which each firm $i \in \mathcal{N}$ invests $\bar{I}_i$.[40] The intuition is simple. Under the VCG mechanism, every firm is the residual claimant to the social surplus generated by its type and therefore in expectation of the social value of its investment. Thus, the individual incentives are perfectly aligned with the social planner's objective.

Moreover, there exists an asset ownership structure, denoted $\mathbf{r}^*_{\bar{\mathbf{I}}}$, that permits ex post efficiency and induces the investments $\bar{\mathbf{I}}$ as a Nash equilibrium outcome of the two-stage game. Specifically, for each $i$, let $q_i^e(\boldsymbol{\theta}; \mathbf{I}_{-i}) = \mathbb{E}_{\boldsymbol{\theta}_{-i}}[\mathbf{Q}_i^e(\boldsymbol{\theta}, \boldsymbol{\theta}_{-i})]$ be firm $i$'s interim expected allocation under the ex post efficient allocation rule given investments $\mathbf{I}_{-i}$ by the other firms, with the expectation taken with respect to the density $f_{-i}(\boldsymbol{\theta}_{-i}, \mathbf{I}_{-i}) \equiv \times_{j \neq i} f_j(\boldsymbol{\theta}_j, I_j)$, where in abuse of notation $\mathbf{Q}_i^e$ (and $\boldsymbol{\theta}$ and $\hat{\boldsymbol{\theta}}^e$) are scalars if $i$'s type is one-dimensional. Then there is a unique $\hat{\boldsymbol{\theta}}^e(\mathbf{I}) \in (0,1)$ such that $\sum_{i \in \mathcal{N}} q_i^e(\hat{\boldsymbol{\theta}}^e(\mathbf{I}), \mathbf{I}_{-i}) = 1$. Defining $\mathbf{r}^*_{\mathbf{I}} \equiv (q_1^e(\hat{\boldsymbol{\theta}}^e(\mathbf{I}); \mathbf{I}_{-1}), \ldots, q_n^e(\hat{\boldsymbol{\theta}}^e(\mathbf{I}); \mathbf{I}_{-n}))$, the result is an implication of Proposition 6.

Because efficient markets induce efficient investments, policies that make markets efficient are desirable with and without consideration for investment incentives. Put differently, given an efficient market, if there are concerns that investment incentives will be adversely affected by a change in asset ownership, the concern must derive from the change causing the market to no longer operate efficiently.

[40] This generalizes the insights of Krähmer and Strausz (2007), who assume that the cost functions are differentiable, and of Liu et al. (forth.), who like Krähmer and Strausz, assume one-dimensional types. Krähmer and Strausz show further that if firm $i$'s investment affects firm $j$'s distribution, then the VCG mechanism does not induce efficient investments.

### 6.3 Integration in a setting with nonidentical supports

Up to here, we have assumed that all distributions had identical supports of $[0, 1]$. We now drop this restriction. Beyond generality, the immediate purposes of this extension are that it allows us to nest settings with nonidentical supports studied in Loertscher and Marx (2022) and to generalize the analysis beyond the one there. Among other things, and in contrast to Theorem 1 above, vertical integration can be socially undesirable with nonidentical supports. While this basic insight was present in Loertscher and Marx (2022), the analysis there was restricted to settings with either only one buyer or only one seller before vertical integration because that paper did not tackle challenges associated with having firms whose types are multi-dimensional.[41]

To relate our results to those in that paper, we assume that firms are divided into a set $\mathcal{N}_U$ of $N_U \geq 1$ "upstream" sellers and a set $\mathcal{N}_D$ of $N_D \geq 1$ "downstream" buyers. Prior to integration, all firms are assumed to have one-dimensional types and maximum demand $k$, where for upstream seller $i$, $r_i = k$, and for downstream buyer $i$, $r_i = 0$. The support of the upstream sellers' type distributions is $[0, 1]$, while the support of the downstream buyers' type distributions is $[\underline{\theta}, 1 + \underline{\theta}]$ with $\underline{\theta} \geq 0$. This setting nests the case of identical supports studied thus far by setting $\underline{\theta} = 0$, nonidentical but overlapping supports, which corresponds to $\underline{\theta} \in (0, 1)$, and nonoverlapping supports, $\underline{\theta} \geq 1$. Detailed background for the following and a discussion of the different configurations is provided in Online Appendix C.2.

If $\underline{\theta} < 1$ and $N_U = N_D = 1$, then ex post efficiency is not possible before vertical integration because of the impossibility result of Myerson and Satterthwaite (1983), but becomes possible with vertical integration. Thus, in this case, a vertical merger is socially desirable even away from identical supports.

Now turn attention to the case of $\underline{\theta} \geq 1$. Then for any $N_U$ and $N_D$, ex post efficiency is possible without vertical integration—a second-price auction to shorten the long side of the market (if any) with reserve $p \in [1, \underline{\theta}]$ induces the ex post efficient allocation in dominant strategies without running a deficit or violating IR. Consequently, vertical integration cannot increase social surplus. Whether it is socially harmful depends on the specifics of the environment. If $N_U = N_D$ prior to integration, then ex post efficiency remains possible after vertical integration because in the market after vertical integration, the integrated firm can without loss be assumed to be self-sufficient and make in house in order to satisfy its demand,

[41] It is true that a merger between a buyer and seller in the setting of Loertscher and Marx (2022) creates a firm that has a two-dimensional type. However, because of the one-to-many setting before integration, the trading position of the integrated firm is predetermined—either it will trade as a buyer, which happens if there was only one buyer before integration, in which case the minimum draw from the two distributions is the relevant statistic, or it will trade as a seller, in which case it is the distribution of the maximum draw that matters.

leaving a market with $N_U - 1$ upstream and $N_D - 1$ downstream firms. As just seen, ex post efficiency is possible in this case. Interestingly, the innocuous nature of vertical integration is not necessarily monotone in the number of upstream firms. That is, vertical integration can make ex post efficiency impossible when $N_U > N_D$. For $N_D = 1$, this observation was made in Loertscher and Marx (2022). Effectively, the vertically integrated firm becomes a buyer on the input market where there are $N_U - 1 \geq 1$ independent suppliers, in which case, by a generalized impossibility theorem in the spirit of Myerson and Satterthwaite (1983), ex post efficiency is not possible. As shown in Online Appendix C.2, the potential for socially harmful vertical integration when $\underline{\theta} \geq 1$ extends to settings with $N_D > 1$ before integration. However, any nonintegrating downstream firms always buy under ex post efficiency, and therefore always pay $\underline{\theta}$ for every unit they obtain, so for $N_D > 1$, increases in $\underline{\theta}$ translate into increases in revenue and thereby mitigate any negative effects of vertical integration.

## 6.4 Toward second-best mechanisms with full integration

As mentioned, for the model with multi-dimensional types, the second-best mechanism is not known, and there appears to be limited hope that it will be any time soon. Even without having the second-best mechanism for multi-dimensional types, some results regarding firms' incentives to engage in full integration can still be obtained. For example, if $\mathbf{r} \in \mathcal{R}^e$, then any vertical merger and any conglomerate merger that results in ex post efficiency still being possible will be profitable for the merging firms.[42] The same would not be true for a horizontal merger because two buyers would still have a zero outside option after the merger and two sellers would still have the same outside option after the merger. These incentive effects reinforce concerns regarding horizontal mergers and resonate with the notion that vertical mergers tend to be benign.

In pursuit of addressing the second-best, a natural path forward is to transform the multi-dimensional random variable characterizing the merged firm into a one-dimensional one that is, in a sense to be defined, equivalent. The set $\mathcal{R}^e$ that characterizes ownership structures that permit ex post efficiency after full integration by two one-dimensional firms may be useful in that regard because a one-dimensional distribution that induces the same set $\mathcal{R}^e$, with all other firms the same, would be equivalent in a meaningful sense.

Consider the case with $k_i < 1$ for one or both of the merging firms prior to the merger so that after the merger, the merged firm's type is nondegenerately multidimensional (in contrast, the merged firm's type is effectively one-dimensional if the merging firms have maximum demands of 1—see footnote 18). In this case, with additional assumptions, one

[42]To see this, note that the allocations are the same before and after integration and the merged firm's outside option increases relative to the sum of merging firms' pre-merger outside options because the merged firm can optimize to take advantage of whichever type is larger.

can still derive a one-dimensional type distribution that induces the same set $\mathcal{R}^e$. As an illustration, consider the case of $k_1 = k_2 < 1 = k_3$ and uniformly distributed types. Following a merger of firms 1 and 2, the merged firm's type is two-dimensional, with associated maximum demands of $k_1$ and $k_2$, respectively, and its asset ownership is $r_1 + r_2$. The post-merger set of ex post efficiency permitting ownership vectors has the form $\mathcal{R}^e = \{(1-r_3, r_3) \mid r_3 \in [\underline{r}_3, \overline{r}_3]\}$, where $0 < \underline{r}_3 < \overline{r}_3 < 1$. As shown in Online Appendix C.4, one can construct a density $\tilde{f}$ such that this same ex post efficiency permitting set $\mathcal{R}^e$ obtains when the merged entity has a one-dimensional type drawn from a distribution with density $\tilde{f}$. This density can then provide a basis for estimating effects outside $\mathcal{R}^e$ based on the (known) second-best mechanism for the case in which the merged firm has a one-dimensional type.

# 7 Conclusion

This paper studies an incomplete information model in which traders—firms that as a function of type realizations buy or sell—are necessary for ex post efficiency. For identical supports, horizontal (vertical) mergers never make ex post efficiency possible (impossible), while some conglomerate mergers, that is, mergers that are neither horizontal nor vertical, make it possible while others foil it. The analysis provides both a rationale and guidance for divestitures that can eliminate harm from mergers. Because the analysis is based on a mechanism design approach, the effects identified derive from the primitives of the model and do not rest on contractual restrictions. Full integration, defined as a merger that combines the merging firms' assets and their private information, creates a multi-dimensional firm with decreasing marginal values. The behavior of the model with multi-dimensional types is remarkably similar to the one with one-dimensional types under ex post efficiency. For the model with one-dimensional types and identical supports, we show that, quite generally, firms have private incentives for bilateral asset ownership transfers ex ante even when these induce the market to operate inefficiently, suggesting a role for sustained antitrust vigilance and that bilateral transactions are the natural focus of attention. We show that conglomerate mergers with a sufficiently strong horizontal (vertical) component never make ex post efficiency possible (impossible). This insight plausibly extends beyond the confines of our model. As a concrete case in point, consider full integration between two airlines. This would typically be a conglomerate merger because on some routes the two firms' services are complements, while on others their services are substitutes. The more important are the latter, the heavier is the weight on the horizontal aspect of the merger. Of the many avenues for future research to explore, a particularly promising one is to allow for downstream externalities by assuming that allocating inputs to some firm affects the profits that other firms generate.

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# A Appendix: Proofs

*Proof of Proposition* 3. Liu et al. (forth.) establish that (see Online Appendix B.3 for details): (i) the mapping $\Pi^e(\mathbf{r})$ is strictly concave in $\mathbf{r}$, which implies that $\mathcal{R}^e$ is convex;[43] and (ii) the unique ownership structure that maximizes $\Pi^e(\mathbf{r})$, denoted $\mathbf{r}^*$, is such that all firms have the same worst-off types. Further, the proof of Proposition 6, which is stated for multi-dimensional types but also encompasses one-dimensional types, shows that $\Pi^e(\mathbf{r}^*)$ is positive, implying that $\mathbf{r}^* \in \mathcal{R}^e$. It remains to show that $SS(\mathbf{r})$ is concave and strictly concave outside of $\mathcal{R}^e$. For any $\mathbf{r} \in \Delta_{\mathbf{k}}$, let $\langle \mathbf{Q_r}, \mathbf{M_r} \rangle$ denote the expected social surplus maximizing mechanism, subject to IC, IR, and no deficit. Let $\mathbf{r}, \mathbf{r}' \in \Delta_{\mathbf{k}}$ and $\mu \in [0,1]$ be given. Because $\langle \mathbf{Q_r}, \mathbf{M_r} \rangle$ and $\langle \mathbf{Q_{r'}}, \mathbf{M_{r'}} \rangle$ satisfy IC, IR, and no deficit, when the ownership structure is $\mu\mathbf{r}+(1-\mu)\mathbf{r}'$, the mechanism $\langle \hat{\mathbf{Q}}, \hat{\mathbf{M}} \rangle$ such that $\hat{q}_i(\theta_i) = \mu q_{\mathbf{r},i}(\theta_i)+(1-\mu)q_{\mathbf{r}',i}(\theta_i)$ and $\hat{m}_i(\theta_i) = \mu m_{\mathbf{r},i}(\theta_i)+(1-\mu)m_{\mathbf{r}',i}(\theta_i)$ also satisfies these constraints. Total expected social surplus from $\langle \hat{\mathbf{Q}}, \hat{\mathbf{M}} \rangle$ is $\mu SS(\mathbf{r}) + (1-\mu)SS(\mathbf{r}')$, but $\langle \mathbf{Q}_{\mu\mathbf{r}+(1-\mu)\mathbf{r}'}, \mathbf{M}_{\mu\mathbf{r}+(1-\mu)\mathbf{r}'} \rangle$ maximizes expected social surplus subject to the constraints, so $\mu SS(\mathbf{r}) + (1-\mu)SS(\mathbf{r}') \leq SS(\mu\mathbf{r}+(1-\mu)\mathbf{r}')$, which implies that $SS(\mathbf{r})$ is concave. For $\mathbf{r} \in \mathcal{R}^e$, the market mechanism with the efficient allocation rule satisfies the no-deficit constraint. If $\mathbf{r} \notin \mathcal{R}^e$, then the no-deficit constraint cannot be satisfied with the efficient mechanism, implying that $SS(\mathbf{r}) < SS^e \equiv SS(\mathbf{r}^*)$. Thus, $SS(\mathbf{r})$ is strictly concave for $\mathbf{r} \notin \mathcal{R}^e$. This completes the proof of Proposition 3. ■

*Sketch of the proof of Proposition* 5. We begin with a lemma. For the purposes of the lemma, given an IC mechanism $\langle \mathbf{Q}, \mathbf{M} \rangle$ and ownership structure $\mathbf{r}$, we let $\hat{\theta}_i^{\mathbf{Q}}(r_i)$ denote the minimum worst-off type of firm $i$ with assets $r_i$ under allocation rule $\mathbf{Q}$, and let $\Pi^{\mathbf{Q}}(\mathbf{r})$ denote the expected budget surplus under allocation rule $\mathbf{Q}$ and ownership structure $\mathbf{r}$ when IR constraints are satisfied with equality for the worst-off types.

**Lemma A.1.** *Given an IC mechanism* $\langle \mathbf{Q}, \mathbf{M} \rangle$ *and ownership structure* $\mathbf{r}$ *with* $\hat{\theta}_i^{\mathbf{Q}}(r_i) \geq$

[43] $\Pi^e(\mathbf{r})$ is strictly concave in $\mathbf{r}$, but $\mathcal{R}^e$ is only convex (and not necessarily strictly convex) because $\mathcal{R}^e \equiv \{\mathbf{r} \mid \Pi^e(\mathbf{r}) \geq 0\} \cap \Delta_{\mathbf{k}}$. So, $\mathcal{R}^e$ is not strictly convex where it intersects with the boundary of $\Delta_{R,\mathbf{k}}$.

$\hat{\theta}_j^{\mathbf{Q}}(r_j)$, *if* $\mathbf{r}'$ *is derived from* $\mathbf{r}$ *by a* $T$*-transform that shifts assets to firm* $i$ *from firm* $j$*, then*

$$\Pi^{\mathbf{Q}}(\mathbf{r}') - \Pi^{\mathbf{Q}}(\mathbf{r}) < 0.$$

*Proof of Lemma* A.1. Take $\mathbf{Q}$ as given. Let $q_i(\theta_i)$ be firm $i$'s interim expected allocation when its type is $\theta_i$. Let $u_i^{\mathbf{Q}}(\theta_i, r_i)$ be firm $i$'s interim expected payoff net of its outside option when its type is $\theta_i$, its resource ownership is $r_i$, and the payment rule is such that IR is satisfied with equality for worst-off types. Assume that $\hat{\theta}_i^{\mathbf{Q}}(r_i) \geq \hat{\theta}_j^{\mathbf{Q}}(r_j)$. Using Lemma 1, we have

$$\begin{aligned}
\Pi^{\mathbf{Q}}(\mathbf{r}) &= \sum_{i\in\mathcal{N}} \Big(\mathbb{E}_{\theta_i}\Big[\Psi_i(\theta_i, \hat{\theta}_i^{\mathbf{Q}}(r_i))q_i(\theta_i)\Big] - r_i\hat{\theta}_i^{\mathbf{Q}}(r_i) - u_i^{\mathbf{Q}}(\hat{\theta}_i^{\mathbf{Q}}(r_i), r_i)\Big) \\
&= \sum_{i\in\mathcal{N}} \Big(\int_0^{\hat{\theta}_i^{\mathbf{Q}}(r_i)} \Psi_i^S(\theta)q_i(\theta)dF_i(\theta) + \int_{\hat{\theta}_i^{\mathbf{Q}}(r_i)}^1 \Psi_i^B(\theta)q_i(\theta)dF_i(\theta) - r_i\hat{\theta}_i^{\mathbf{Q}}(r_i) - u_i^{\mathbf{Q}}(\hat{\theta}_i^{\mathbf{Q}}(r_i), r_i)\Big).
\end{aligned}$$

Differentiating the second line above with respect to $r_i$, the three terms involving $\frac{d\hat{\theta}_i^{\mathbf{Q}}(r_i)}{dr_i}$ cancel, $u_i^{\mathbf{Q}}(\hat{\theta}_i^{\mathbf{Q}}(r_i), r_i)$ is constant at zero as $r_i$ changes, and we are left with $\frac{\partial\Pi^{\mathbf{Q}}(\mathbf{r})}{\partial r_i} = -\hat{\theta}_i^{\mathbf{Q}}(r_i)$. It follows from the Monotone Selection Theorem of Milgrom and Shannon (1994) that $\hat{\theta}_i^{\mathbf{Q}}(r_i)$ is increasing in $r_i$, and so all second partial derivatives are negative. Further, all cross-partial derivatives are zero. Thus, $\Pi^{\mathbf{Q}}$ is strictly concave. Because $\Pi^{\mathbf{Q}}$ is strictly concave and $\nabla\Pi^{\mathbf{Q}}(\mathbf{r}) = (-\hat{\theta}_1^{\mathbf{Q}}, \ldots, -\hat{\theta}_n^{\mathbf{Q}})$, where we drop the dependence of $\hat{\theta}_i^{\mathbf{Q}}$ on $r_i$, it follows that for $\mathbf{r}'$ derived from $\mathbf{r}$ by shifting amount $\Delta > 0$ from firm $j$ to firm $i$, we have

$$\begin{aligned}
\Pi^{\mathbf{Q}}(\mathbf{r}') - \Pi^{\mathbf{Q}}(\mathbf{r}) &= \Pi^{\mathbf{Q}}(r_i + \Delta, r_j - \Delta, \mathbf{r}_{-i,j}) - \Pi^{\mathbf{Q}}(r_i, r_j, \mathbf{r}_{-i,j}) \\
&< (\Delta, -\Delta, \mathbf{0}_{-i,j}) \cdot (-\hat{\theta}_1^{\mathbf{Q}}, \ldots, -\hat{\theta}_n^{\mathbf{Q}}) = -\Delta(\hat{\theta}_i^{\mathbf{Q}} - \hat{\theta}_j^{\mathbf{Q}}) \leq 0,
\end{aligned}$$

where the final inequality uses the assumption that $\hat{\theta}_i^{\mathbf{Q}} \geq \hat{\theta}_j^{\mathbf{Q}}$. which completes the proof. □

The remainder of the proof of Proposition 5 proceeds in two parts:

*Part* (i). Suppose that $\Pi^e(\mathbf{r}) > 0$ and there exist two traders indexed by 1 and 2 with $\eta_1 + \eta_2 \leq 1$. By virtue of the firms being traders, $0 < r_1 < k_1$ and $0 < r_2 < k_2$. Without loss of generality, we can assume that $\hat{\theta}_2^e(r_2) \leq \hat{\theta}_1^e(r_1)$. Because $r_1 < k_1$ and $0 < r_2$, there exists $\Delta > 0$ sufficiently small that the ownership vector $\tilde{\mathbf{r}}(\Delta)$ defined by $\tilde{r}_1(\Delta) \equiv r_1 + \Delta$, $\tilde{r}_2(\Delta) \equiv r_2 - \Delta$, and $\tilde{\mathbf{r}}_{-\{1,2\}}(\Delta) \equiv \mathbf{r}_{-\{1,2\}}$ is a feasible ownership vector (i.e., $r_1 + \Delta \leq k_1$ and $0 \leq r_2 - \Delta$). Further, using the continuity of $\Pi^e$ and the assumption that $\Pi^e(\mathbf{r}) > 0$, there exists $\Delta > 0$ sufficiently small that $\Pi^e(\tilde{\mathbf{r}}(\Delta)) > 0$. Taking $\Delta$ to satisfy these conditions, ex post efficiency is achieved under both $\mathbf{r}$ and $\tilde{\mathbf{r}}(\Delta)$, and by Lemma A.1,

$$\Pi^e(\tilde{\mathbf{r}}(\Delta)) < \Pi^e(\mathbf{r}).$$

It then follows (details are provided in Online Appendix B.4) that the change in the joint expected gross surplus of firms 1 and 2 from a change in ownership structure from $\mathbf{r}$ to $\tilde{\mathbf{r}}(\Delta)$ is $(1-\eta_1-\eta_2)\left(\Pi^e(\mathbf{r})-\Pi^e(\tilde{\mathbf{r}}(\Delta))\right)\geq 0$. The inequality is strict if $\eta_1+\eta_2<1$.

*Part* (ii). Assume, as in the statement of the proposition, that $n\in\{3,4,\dots\}$, $\Pi^e(\mathbf{r})=0$, and firms 1 and 2 are traders. Without loss of generality, assume that $\hat{\theta}^e_1(r_1)\geq\hat{\theta}^e_2(r_2)$. Define ownership structure $\tilde{\mathbf{r}}(\Delta)$ by $\tilde{r}_1(\Delta)\equiv r_1+\Delta$, $\tilde{r}_2(\Delta)\equiv r_2-\Delta$, and $\tilde{\mathbf{r}}_{-\{1,2\}}(\Delta)\equiv\mathbf{r}_{-\{1,2\}}$, which is feasible for $\Delta\in[0,\min\{k_1-r_1,r_2\}]$, which is a nonempty interval because firms 1 and 2 are traders. Because we are considering a shift from the firm with the weakly lower worst-off type to the firm with the weakly higher worst-off type, by Lemma A.1, for all $\Delta>0$ in the feasible range, we have $\Pi^e(\tilde{\mathbf{r}}(\Delta))<\Pi^e(\mathbf{r})=0$. The expected gross payoff of firm $i$ with type $\theta_i$ is $\theta_i q_i(\theta_i)-m_i(\theta_i)$, and by Lemma 1, $\mathbb{E}_{\theta_i}[m_i(\theta_i)]=\mathbb{E}_{\theta_i}[\Psi_{i,0}(\theta_i,\hat{\theta}_i)q_i(\theta_i)]-u_i(\hat{\theta}_i)-\hat{\theta}_i r_i$, where binding IR for the firms' worst-off types implies that $u_i(\hat{\theta}_i)=0$. One can then show that the expected gross payoff of firm $i$, $\tilde{u}_i(\Delta)$, satisfies for $\Delta>0$ sufficiently small, $\sum_{i\in\{1,2\}}\tilde{u}_i(\Delta)>\sum_{i\in\{1,2\}}\tilde{u}_i(0)$, which establishes that the envisioned transaction between firms 1 and 2 is strictly mutually beneficial. Using $\Pi^e(\tilde{\mathbf{r}}(\Delta))<\Pi^e(\mathbf{r})=0$, such transactions result in $\Pi^e(\tilde{\mathbf{r}}(\Delta))<0$, completing the proof. Full details are in Online Appendix B.4. □

*Proof of Lemma* 2. Define $v_i:[0,1]^{h_i}\times[0,1]\to\mathbb{R}$ such that $v_i(\boldsymbol{\theta}_i,x)$ is firm $i$'s willingness to pay when its type is $\boldsymbol{\theta}_i$ and its allocation is $x$, i.e.,

$$v_i(\boldsymbol{\theta}_i,x)\equiv\max\nolimits_{\mathbf{Q}_i\text{ s.t. }Q_i^j\in[0,k_i^j],\ \sum_{j=1}^{h_i}Q_i^j\leq x}\mathbf{Q}_i\cdot\boldsymbol{\theta}_i.$$

Given ex post efficient, DIC mechanism $\langle\mathbf{Q}^e,\mathbf{M}\rangle$, let $u_i^e(\boldsymbol{\theta}_i)$ denote firm $i$'s interim expected payoff net of its outside option:

$$u_i^e(\boldsymbol{\theta}_i)\equiv\mathbb{E}_{\boldsymbol{\theta}_{-i}}[v_i(\boldsymbol{\theta}_i,Q_i^e(\boldsymbol{\theta}))-v_i(\boldsymbol{\theta}_i,r_i)-M_i(\boldsymbol{\theta})].$$

**Step 1.** As a first step in the proof, we show that for firm $i$ with multi-dimensional type, $u_i^e$ is convex with gradient that exists almost everywhere.

Consider firm $i$ with $h_i$-dimensional type and type space $[0,1]^{h_i}$. Because the type space, which is the finite product of unit intervals, is smoothly connected, the focus on Groves schemes is without loss of generality (Holmström, 1979), and because we are interested in revenue-maximizing, efficient, DIC mechanisms that respect IR, we consider a version of the VCG mechanism. Define the VCG transfer for firm $i$ at type profile $\boldsymbol{\theta}$ given an arbitrary reference type $\boldsymbol{\theta}_i^c$ by (to conserve on notation here, we ignore the constant term required to ensure that IR is satisfied):

$$M_i(\boldsymbol{\theta})\equiv\sum_{j\in\mathcal{N}\setminus\{i\}}\mathbf{Q}_j^e(\boldsymbol{\theta}_{-i},\boldsymbol{\theta}_i^c)\cdot\boldsymbol{\theta}_j+\mathbf{Q}_i^e(\boldsymbol{\theta}_{-i},\boldsymbol{\theta}_i^c)\cdot\boldsymbol{\theta}_i^c-\sum_{j\in\mathcal{N}\setminus\{i\}}\mathbf{Q}_j^e(\boldsymbol{\theta}_{-i},\boldsymbol{\theta}_i)\cdot\boldsymbol{\theta}_j.\qquad\text{(A.1)}$$

To see that this mechanism endows the agents with dominant strategies to report types truthfully, note that at type profile $\boldsymbol{\theta}$, $i$'s payoff is

$$\mathbf{Q}_i^e(\boldsymbol{\theta}) \cdot \boldsymbol{\theta}_i - M_i(\boldsymbol{\theta}) = \sum_{j \in \mathcal{N}} \mathbf{Q}_j^e(\boldsymbol{\theta}_{-i}, \boldsymbol{\theta}_i) \cdot \boldsymbol{\theta}_j - \sum_{j \in \mathcal{N} \setminus \{i\}} \mathbf{Q}_j^e(\boldsymbol{\theta}_{-i}, \boldsymbol{\theta}_i^c) \cdot \boldsymbol{\theta}_j - \mathbf{Q}_i^e(\boldsymbol{\theta}_{-i}, \boldsymbol{\theta}_i^c) \cdot \boldsymbol{\theta}_i^c,$$

where first term on the right side is maximized social surplus at type profile $\boldsymbol{\theta}$, so by changing the report to induce a different allocation that expression cannot be increased. Because the other terms on the right side are independent of $\boldsymbol{\theta}_i$, it follows that the mechanism endows the agents with dominant strategies.

Because $\langle \mathbf{Q}^e, \mathbf{M} \rangle$ is then also Bayesian IC, it follows that for all $\boldsymbol{\theta}_i, \boldsymbol{\theta}_i' \in [0,1]^{h_i}$,

$$u_i^e(\boldsymbol{\theta}_i) \geq \mathbb{E}_{\boldsymbol{\theta}_{-i}}[\mathbf{Q}_i^e(\boldsymbol{\theta}_i', \boldsymbol{\theta}_{-i}) \cdot \boldsymbol{\theta}_i] - v_i(\boldsymbol{\theta}_i, r_i) - \mathbb{E}_{\boldsymbol{\theta}_{-i}}[M_i(\boldsymbol{\theta}_i', \boldsymbol{\theta}_{-i})].$$

This implies that $u_i^e$ is the maximum of a family of affine functions, which implies that it is convex and so absolutely continuous and differentiable almost everywhere in the interior of its domain.

**Step 2.** We now show that, letting $\tilde{\theta}_i^j$ be the $j$-th largest element of $\boldsymbol{\theta}_i$ and $\tilde{k}_i^j$ be the maximum demand associated with the $j$-th largest element of $\boldsymbol{\theta}_i$, we have

$$\nabla u_i^e(\tilde{\boldsymbol{\theta}}_i) = (\tilde{k}_i^1, \ldots, \tilde{k}_i^\ell, q_i^e(\boldsymbol{\theta}_i) - \sum_{j=1}^{\ell} \tilde{k}_i^j, 0, \ldots, 0) - (\tilde{k}_i^1, \ldots, \tilde{k}_i^{\ell'}, r_i - \sum_{j=1}^{\ell'} \tilde{k}_i^j, 0, \ldots, 0), \quad \text{(A.2)}$$

where $\ell$ is such that $0 < q_i^e(\boldsymbol{\theta}_i) - \sum_{j=1}^{\ell} \tilde{k}_i^j \leq \tilde{k}_i^{\ell+1}$ and $\ell'$ is such that $0 < r_i - \sum_{j=1}^{\ell'} \tilde{k}_i^j \leq \tilde{k}_i^{\ell'+1}$.

Given type $\boldsymbol{\theta}_i$, allocation $z$, and transfer $y$, firm $i$'s payoff net of its outside option is $\phi_i(\boldsymbol{\theta}_i, z, y) \equiv v_i(\boldsymbol{\theta}_i, z) - y - v_i(\boldsymbol{\theta}_i, r_i)$, which is differentiable and absolutely continuous in $\boldsymbol{\theta}_i$ for all $(z, y) \in [0,1] \times \mathbb{R}$. Further, $v_i$ has strictly increasing differences in $(\boldsymbol{\theta}_i, z)$—intuitively, increases in types make the quantity more valuable and increases in the quantity make higher types more valuable.

Denote the set of outcomes that are accessible to agent $i$ in the VCG mechanism by $X_i \equiv \{\mathbb{E}_{\boldsymbol{\theta}_{-i}}[(Q_i^e(\boldsymbol{\theta}_{-i}, \boldsymbol{\theta}_i), M_i(\boldsymbol{\theta}_{-i}, \boldsymbol{\theta}_i))] \mid \boldsymbol{\theta}_i \in [0,1]^{h_i}\}$, and let

$$X_i^*(\boldsymbol{\theta}_i) \equiv \{(z, y) \in X_i \mid \phi_i(\boldsymbol{\theta}_i, z, y) = \sup_{(z', y') \in X_i} \phi_i(\boldsymbol{\theta}_i, z', y')\}.$$

Then the Monotone Selection Theorem of Milgrom and Shannon (1994) implies that for any $(z^*(\boldsymbol{\theta}_i), y^*(\boldsymbol{\theta}_i)) \in X_i^*(\boldsymbol{\theta}_i)$, $z^*(\boldsymbol{\theta}_i)$ is nondecreasing in $\boldsymbol{\theta}_i$ and $\frac{\partial v_i(\boldsymbol{\theta}_i, z)}{\partial \theta_i^j}$ is nondecreasing in $z$. Therefore, for all $\boldsymbol{\theta}_i' \in [0,1]^{h_i}$,

$$\frac{\partial \phi(\boldsymbol{\theta}_i, z^*(\boldsymbol{\theta}_i'), y^*(\boldsymbol{\theta}_i'))}{\partial \theta_i^j} = \frac{\partial v_i(\boldsymbol{\theta}_i, z^*(\boldsymbol{\theta}_i'))}{\partial \theta_i^j} - \frac{\partial v_i(\boldsymbol{\theta}_i, r_i)}{\partial \theta_i^j}$$

is bounded below by $\frac{\partial v_i(\boldsymbol{\theta}_i, z^*(\mathbf{0}))}{\partial \theta_i^j} - \frac{\partial v_i(\boldsymbol{\theta}_i, r_i)}{\partial \theta_i^j}$ and bounded above by $\frac{\partial v_i(\boldsymbol{\theta}_i, z^*(\mathbf{1}))}{\partial \theta_i^j} - \frac{\partial v_i(\boldsymbol{\theta}_i, r_i)}{\partial \theta_i^j}$, which implies that $\frac{\partial \phi(\boldsymbol{\theta}_i, z, y)}{\partial \theta_i^j}$ is uniformly bounded on $(\boldsymbol{\theta}_i, z, y) \in [0,1]^{h_i} \times X^*([0,1]^{h_i})$.

Further, because $[0,1]^{h_i}$ is smoothly connected, given any $\boldsymbol{\theta}_i', \boldsymbol{\theta}_i \in [0,1]^{h_i}$, there is a path $\mathcal{C}$ in $[0,1]^{h_i}$ described by a continuously differentiable function $\boldsymbol{\tau} : [0,1] \to [0,1]^{h_i}$ such that $\boldsymbol{\tau}(0) = \boldsymbol{\theta}_i'$ and $\boldsymbol{\tau}(1) = \boldsymbol{\theta}_i$. Because $\phi(\boldsymbol{\theta}_i, z, y)$ is differentiable in $\boldsymbol{\theta}_i \in [0,1]^{h_i}$ and $\frac{\partial \phi(\boldsymbol{\theta}_i, z, y)}{\partial \theta_i^j}$ is uniformly bounded on $(\boldsymbol{\theta}_i, z, y) \in [0,1]^{h_i} \times X^*([0,1]^{h_i})$, the function $\hat{\phi} : [0,1] \times [0,1] \times \mathbb{R} \to \mathbb{R}$ defined by $\hat{\phi}(a, z, y) \equiv \phi(\boldsymbol{\tau}(a), z, y)$ satisfies the assumptions of Milgrom and Segal's (2002) Theorem 2 and Corollary 1 (as modified in their footnote 10). It then follows that we can express $u_i^e(\boldsymbol{\theta}_i) = \phi_i(\boldsymbol{\theta}_i, z^*(\boldsymbol{\theta}_i), y^*(\boldsymbol{\theta}_i))$ in terms of the path integral with respect to $\boldsymbol{\tau}$ as follows:

$$u_i^e(\boldsymbol{\theta}_i) = u_i^e(\boldsymbol{\theta}_i') + \int_{\mathcal{C}} \Big( \times_{j=1}^{h_i} \frac{\partial \mathbb{E}_{\boldsymbol{\theta}_{-i}}[v_i(\boldsymbol{\tau}, Q_i^e(\boldsymbol{\theta}_{-i}, \boldsymbol{\tau}))]}{\partial \theta_i^j} - \times_{j=1}^{h_i} \frac{\partial v_i(\boldsymbol{\tau}, r_i)}{\partial \theta_i^j} \Big) d\boldsymbol{\tau}.$$

This implies that

$$\frac{\partial u_i^e(\boldsymbol{\theta}_i)}{\partial \theta_i^j} = \frac{\partial \mathbb{E}_{\boldsymbol{\theta}_{-i}}[v_i(\boldsymbol{\theta}_i, Q_i^e(\boldsymbol{\theta})]}{\partial \theta_i^j} - \frac{\partial v_i(\boldsymbol{\theta}_i, r_i)}{\partial \theta_i^j},$$

which, using the implication of the definition of $v_i$ that

$$v_i(\boldsymbol{\theta}_i, z) = \sum_{j=1}^{\ell} \tilde{\theta}_i^j \tilde{k}_i^j + \tilde{\theta}_i^{\ell+1}(z - \sum_{j=1}^{\ell} \tilde{k}_i^j),$$

where $\ell$ satisfies $0 < z - \sum_{j=1}^{\ell} k_i^j \le \tilde{k}_i^{\ell+1}$, gives us the result that (A.2) holds.

**Step 3.** Using the result that $u_i^e$ is convex, it follows that at any worst-off type $\boldsymbol{\theta}_i^w$ for firm $i$, we have $\nabla u_i^e(\boldsymbol{\theta}_i^w) = (0, \ldots, 0)$, which, using the expression for $\nabla u_i^e$ from (A.2), implies that $q_i^e(\boldsymbol{\theta}_i^w) = r_i$. Because $q_i^e(\boldsymbol{\theta}_i)$ is increasing in $\boldsymbol{\theta}_i$, the second-order condition is also satisfied.

We now show that the constant worst-off type $\hat{\boldsymbol{\theta}}_i = (\hat{\theta}_i, \ldots, \hat{\theta}_i)$ such that $q_i^e(\hat{\boldsymbol{\theta}}_i) = r_i$ is revenue maximizing. First note that because $q_i^e(\mathbf{0}) < \min\{1, \sum_{j=1}^{h_i} k_i^j\} = q_i^e(\mathbf{1})$ and because $q_i^e$ is continuously increasing, such a type $\hat{\boldsymbol{\theta}}_i$ exists and is unique. Second, consider the payment of firm $i$ under a VCG mechanism defined with respect to some, at this point arbitrary, reference type $\boldsymbol{\theta}_i^c$. That is, firm $i$'s payment at type profile $\boldsymbol{\theta}$ is

$$M_i^{VCG}(\boldsymbol{\theta}_i^c, \boldsymbol{\theta}) \equiv W(\boldsymbol{\theta}_i^c, \boldsymbol{\theta}_{-i}) - W(\boldsymbol{\theta}) + \mathbf{Q}_i^e(\boldsymbol{\theta}) \cdot \boldsymbol{\theta}_i - v_i(\boldsymbol{\theta}_i^c, r_i), \tag{A.3}$$

where $v_i(\boldsymbol{\theta}_i^c, r_i)$ is the outside option for firm $i$'s reference type. This implies that firm $i$'s expected payment is $m_i^{VCG}(\boldsymbol{\theta}_i^c) \equiv \mathbb{E}_{\boldsymbol{\theta}}[M_i^{VCG}(\boldsymbol{\theta}_i^c, \boldsymbol{\theta})]$. By the envelope theorem and (A.2), the gradient of $m_i^{VCG}(\boldsymbol{\theta}_i^c)$ with respect to $\boldsymbol{\theta}_i^c$ is

$$(\tilde{k}_i^1, \ldots, \tilde{k}_i^\ell, q_i^e(\boldsymbol{\theta}_i) - \sum_{j=1}^{\ell} \tilde{k}_i^j, 0, \ldots, 0) - (\tilde{k}_i^1, \ldots, \tilde{k}_i^h, r_i - \sum_{j=1}^{h} \tilde{k}_i^j, 0, \ldots, 0).$$

This derivative is nonnegative for all $\boldsymbol{\theta}_i^c$ such that $q_i^e(\boldsymbol{\theta}_i^c) \ge r_i$, which is the set of types for firm $i$ that are of interest from here on, and it is strictly positive if $q_i^e(\boldsymbol{\theta}_i^c) > r_i$. Because

$q_i^e(\cdot)$ is strictly increasing, that is, $q_i^e(\boldsymbol{\theta}_i) > q_i^e(\boldsymbol{\theta}'_i)$ for any $\boldsymbol{\theta}_i$ and $\boldsymbol{\theta}'_i$ satisfying $\theta_i^j \geq \theta_i^{j\prime}$ for all $j \in \{1, \ldots, h_i\}$ with at least one strict inequality, it follows that among all types $\boldsymbol{\theta}_i^c$ satisfying $q_i^e(\boldsymbol{\theta}_i^c) \geq r_i$ and $\max \boldsymbol{\theta}_i^c = \theta^1$, revenue is maximized by the vector of constant types $(\theta^1, \ldots, \theta^1)$. Consequently, the reference type that maximizes revenue while respecting firm $i$'s IR constraint is the unique constant type $\hat{\boldsymbol{\theta}}_i$ satisfying $q_i^e(\hat{\boldsymbol{\theta}}_i) = r_i$. ■

*Proof of Proposition* 6. Firm $i$'s VCG payment given constant worst-off type $\hat{\boldsymbol{\theta}}_i$ is given by (10). The derivative of firm $i$'s payment with respect to $r_i$ is $-\hat{\theta}_i$ and the second derivative is $-\frac{d\hat{\theta}_i}{dr_i} \leq 0$, with cross-derivatives that are zero. Thus, the expected revenue under ex post efficiency is concave in $\mathbf{r}$, implying that $\mathcal{R}^e$ is convex.

Assume that all firms have the same constant worst-off type $\hat{\theta}^*$. That is, firms with a one-dimensional type have worst-off type $\hat{\theta}^*$ and firm $i$ with a multi-dimensional type has worst-off type $\hat{\boldsymbol{\theta}}_i^* = (\hat{\theta}^*, \ldots, \hat{\theta}^*)$. A simple revealed preference argument establishes that

$$W(\boldsymbol{\theta}) - W(\hat{\boldsymbol{\theta}}_i^*, \boldsymbol{\theta}_{-i}) \leq (\boldsymbol{\theta}_i - \hat{\boldsymbol{\theta}}_i^*) \cdot \mathbf{Q}_i^e(\boldsymbol{\theta}) = \boldsymbol{\theta}_i \cdot \mathbf{Q}_i^e(\boldsymbol{\theta}) - \hat{\theta}^* Q_i^e(\boldsymbol{\theta}), \qquad \text{(A.4)}$$

because as firm $i$'s type changes from $\boldsymbol{\theta}_i$ to $\hat{\boldsymbol{\theta}}_i^*$, the planner could keep the allocation fixed at $\mathbf{Q}^e(\boldsymbol{\theta})$. Optimizing, it means that it can do weakly better, and strictly better for a positive measure of types given that we assume positive densities on $(0, 1)$. (Inequality (A.4) is the multi-unit generalization of (A.6) in Liu et al. (forth.).) Because $\sum_{i \in \mathcal{N}} Q_i^e(\boldsymbol{\theta}) = 1$, summing up both sides in (A.4) yields $\sum_{i \in \mathcal{N}} (W(\boldsymbol{\theta}) - W(\hat{\boldsymbol{\theta}}_i^*, \boldsymbol{\theta}_{-i})) \leq W(\boldsymbol{\theta}) - \hat{\theta}^*$, which is equivalent to

$$0 \leq \sum_{i \in \mathcal{N}} W(\hat{\boldsymbol{\theta}}_i^*, \boldsymbol{\theta}_{-i}) - (n-1) W(\boldsymbol{\theta}) - \hat{\theta}^*. \qquad \text{(A.5)}$$

Using (10), firm $i$'s VCG transfer given worst-off type $\hat{\boldsymbol{\theta}}_i^*$ is

$$M_i^{VCG}(\boldsymbol{\theta}) \equiv W(\hat{\boldsymbol{\theta}}_i^*, \boldsymbol{\theta}_{-i}) - W(\boldsymbol{\theta}) + \mathbf{Q}_i^e(\boldsymbol{\theta}) \cdot \boldsymbol{\theta}_i - r_i \hat{\theta}^*. \qquad \text{(A.6)}$$

Summing these transfers across all firms and using $\sum_{i \in \mathcal{N}} r_i = 1$, we have

$$\sum_{i \in \mathcal{N}} M_i^{VCG}(\boldsymbol{\theta}) = \sum_{i \in \mathcal{N}} W(\hat{\boldsymbol{\theta}}_i^*, \boldsymbol{\theta}_{-i}) - (n-1) W(\boldsymbol{\theta}) - \hat{\theta}^* \geq 0,$$

where the inequality uses (A.5). Thus, the revenue of the VCG mechanism in which all agents have the same worst-off type $\hat{\theta}^*$ is never negative, and the inequality is strict for a positive measure of types because (A.4) is strict for a positive measure of types, implying that the revenue is positive in expectation. Moreover, it satisfies IR for each firm $i$ because firm $i$'s interim expected payoff net of its outside option for its worst-off type is

$$\mathbb{E}_{\boldsymbol{\theta}_{-i}} \left[ \hat{\boldsymbol{\theta}}_i^* \cdot \mathbf{Q}_i^e(\hat{\boldsymbol{\theta}}_i^*, \boldsymbol{\theta}_{-i}) - M_i^{VCG}(\hat{\boldsymbol{\theta}}_i^*, \boldsymbol{\theta}_{-i}) \right] - r_i \hat{\theta}^* = 0,$$

where the equality uses (A.6), and, by the definition of the worst-off type, firm $i$'s interim expected payoff net of its outside option for any other type is weakly greater.

It remains only to establish that there exists $\mathbf{r}^* \in \Delta_{\mathbf{k}}$ such that equalized worst-off types exist. Thus, we now state and prove the following lemma:

**Lemma A.2.** *There exists $\mathbf{r}^* \in \Delta_{\mathbf{k}}$ and $\hat{\theta}^* \in (0,1)$ such that for all $i$ with one-dimensional types, $r_i^* = q_i^e(\hat{\theta}^*)$, and for all $i$ with multi-dimensional types, $r_i^* = q_i^e(\hat{\boldsymbol{\theta}}_i^*)$, where $\hat{\boldsymbol{\theta}}_i^* \equiv (\hat{\theta}^*, \ldots, \hat{\theta}^*) \in [0,1]^{h_i}$.*

*Proof of Lemma* A.2. We decompose the multi-dimensional firms to create a setup with the same number of firms as the total number of types, $\sum_{i\in\mathcal{N}} h_i$. For $i \in \mathcal{N}$, we let $\tilde{\mathcal{N}}_i$ be the indices of the extended set of firms derived from firm $i$'s types, with $|\tilde{\mathcal{N}}_i| = h_i$, $\tilde{\mathcal{N}}_i \cap \tilde{\mathcal{N}}_j = \varnothing$ for $i \neq j$, and $\tilde{\mathcal{N}} = \cup_{i\in\mathcal{N}}\tilde{\mathcal{N}}_i$. Further, define $\tilde{\mathcal{N}}_{-i} \equiv \cup_{j\in\mathcal{N}\setminus\{i\}}\tilde{\mathcal{N}}_j$ to be the set of extended firms derived from firms other than firm $i$. For each $i \in \tilde{\mathcal{N}}$, let $\tilde{k}_i$ denote the associated maximum demand and $\tilde{F}_i$ the associated distribution for extended firm $i$. For each of our actual firms $i \in \mathcal{N}$, we define the interim expected allocation under ex post efficiency when firm $i$ has the constant type $(\theta_i, \ldots, \theta_i)$ by

$$\tilde{q}_i^e(\theta_i) = \sum_{\mathcal{A}\subset\tilde{\mathcal{N}}_{-i}} \max\{0, \min\{k_i, 1 - \sum_{j\in\mathcal{A}} \tilde{k}_j\}\} \prod_{j\in\mathcal{A}}(1 - \tilde{F}_j(\theta_i)) \prod_{j\in\tilde{\mathcal{N}}_{-i}\setminus\mathcal{A}} \tilde{F}_j(\theta_i),$$

which is continuous and increasing in $\theta_i$ on $[0,1]$ and satisfies $\tilde{q}_i^e(\theta_i) \in [0, k_i]$. Then by an argument analogous to that of Lemma B.5, we have

$$\sum_{i\in\mathcal{N}} \tilde{q}_i^e(0) < 1. \tag{A.7}$$

We now define a real-valued function on $[0,1]$, denoted $g$, that will allow us to identify a common worst-off type $g(t) \equiv \sum_{i\in\mathcal{N}} \tilde{q}_i^e(t)$. Using (A.7), we have $g(0) = 0$, and using the assumption of excess demand, we have $g(1) > 1$. Further, $g(t)$ is continuously increasing. Thus, there exists a unique $\hat{\theta}^* \in (0,1)$ such that $g(\hat{\theta}^*) = 1$. Given $\hat{\theta}^*$, define for $i \in \mathcal{N}$, $r_i^* \equiv g_i(\hat{\theta}^*)$, which satisfies $\mathbf{r}^* \in \Delta_{\mathbf{k}}$, which completes the proof of Lemma A.2. □

Together, these results complete the proof of Proposition 6. ■

*Proof of Theorem* 1. In what follows, Lemma A.3 proves the result for a merger of buyers, Lemma A.4 applies to a merger of sellers, and Lemma A.5 applies to a vertical merger. In these lemmas, we use the notation $v_i(x, \boldsymbol{\theta}_i)$ to denote firm $i$'s willingness to pay for quantity $x$ when its type is $\boldsymbol{\theta}_i$, i.e., $v_i(x, \boldsymbol{\theta}_i) \equiv \max_{\mathbf{Q}_i \text{ s.t. } Q_i^\ell \in [0,k_i^\ell],\ \sum_{\ell=1}^{h_i} Q_i^\ell \leq x} \mathbf{Q}_i \cdot \boldsymbol{\theta}_i$. Analogously, we use $v_{i,j}(\boldsymbol{\theta}_i, \boldsymbol{\theta}_j, x)$ to denote the willingness to pay of the integrated firm that combines firms $i$ and $j$ for quantity $x$, $v_{i,j}(x, \boldsymbol{\theta}_i, \boldsymbol{\theta}_j) \equiv \max_{\mathbf{Q}_i, \mathbf{Q}_j \text{ s.t. } Q_i^\ell \in [0,k_i^\ell],\ Q_j^\ell \in [0,k_j^\ell],\ \sum_{\ell=1}^{h_i} Q_i^\ell + \sum_{\ell=1}^{h_j} Q_j^\ell \leq x} \mathbf{Q}_i \cdot \boldsymbol{\theta}_i + \mathbf{Q}_j \cdot \boldsymbol{\theta}_j$.

**Lemma A.3.** *A merger of two buyers decreases expected revenue under ex post efficiency.*

*Proof of Lemma* A.3. Consider full integration between two buyers, $b_1$ and $b_2$ with $r_{b_1} = r_{b_2} = 0$. Let $\boldsymbol{\theta}_{b_1}$, $\boldsymbol{\theta}_{b_2}$, and $\boldsymbol{\theta}_o$ be the vectors of types for buyer $b_1$, buyer $b_2$, and all the other firms, respectively. Because the VCG payments of the nonintegrating firms are not affected by the integration, we can focus on the change in payments made by the integrating firms. It is sufficient to show that the sum of the VCG payments of buyers $b_1$ and $b_2$ is greater than or equal to (and strictly greater for a positive measure set of type realizations) the VCG payment of the firm formed through the integration of buyers $b_1$ and $b_2$. Thus, it is sufficient to show that the following inequality holds (and strictly for a positive measure set of type realizations), where the left side is the sum of the VCG payments of buyers $b_1$ and $b_2$, and the right side is the VCG payment of the firm formed through the integration of buyers $b_1$ and $b_2$:

$$\begin{aligned} & W(\mathbf{0},\boldsymbol{\theta}_{b_2},\boldsymbol{\theta}_o) - W(\boldsymbol{\theta}_{b_1},\boldsymbol{\theta}_{b_2},\boldsymbol{\theta}_o) + v_{b_1}(Q^e_{b_1}(\boldsymbol{\theta}_{b_1},\boldsymbol{\theta}_{b_2},\boldsymbol{\theta}_o),\boldsymbol{\theta}_{b_1}) \\ & + W(\boldsymbol{\theta}_{b_1},\mathbf{0},\boldsymbol{\theta}_o) - W(\boldsymbol{\theta}_{b_1},\boldsymbol{\theta}_{b_2},\boldsymbol{\theta}_o) + v_{b_2}(Q^e_{b_2}(\boldsymbol{\theta}_{b_1},\boldsymbol{\theta}_{b_2},\boldsymbol{\theta}_o),\boldsymbol{\theta}_{b_2}) \\ \geq\ & W(\mathbf{0},\mathbf{0},\boldsymbol{\theta}_o) - W(\boldsymbol{\theta}_{b_1},\boldsymbol{\theta}_{b_2},\boldsymbol{\theta}_o) + v_{b_1,b_2}(Q^e_{b_1}(\boldsymbol{\theta}_{b_1},\boldsymbol{\theta}_{b_2},\boldsymbol{\theta}_o) + Q^e_{b_2}(\boldsymbol{\theta}_{b_1},\boldsymbol{\theta}_{b_2},\boldsymbol{\theta}_o),\boldsymbol{\theta}_{b_1},\boldsymbol{\theta}_{b_2}). \end{aligned}$$

Noting that $v_{b_1,b_2}(Q^e_{b_1} + Q^e_{b_2},\boldsymbol{\theta}_{b_1},\boldsymbol{\theta}_{b_2}) = v_{b_1}(Q^e_{b_1},\boldsymbol{\theta}_{b_1}) + v_{b_2}(Q^e_{b_2},\boldsymbol{\theta}_{b_2})$ (where we drop the argument $(\boldsymbol{\theta}_{b_1},\boldsymbol{\theta}_{b_2},\boldsymbol{\theta}_o)$ on the allocations $Q^e_{b_1}$ and $Q^e_{b_2}$), we can rewrite this as

$$W(\mathbf{0},\boldsymbol{\theta}_{b_2},\boldsymbol{\theta}_o) + W(\boldsymbol{\theta}_{b_1},\mathbf{0},\boldsymbol{\theta}_o) \geq W(\boldsymbol{\theta}_{b_1},\boldsymbol{\theta}_{b_2},\boldsymbol{\theta}_o) + W(\mathbf{0},\mathbf{0},\boldsymbol{\theta}_o). \tag{A.8}$$

Let type vector $\boldsymbol{\theta} = (\boldsymbol{\theta}_{b_1},\boldsymbol{\theta}_{b_2},\boldsymbol{\theta}_o)$ be given. Let $(\theta_{(1)},...,\theta_{(h)})$ be the ranked list of types from largest to smallest and (with some abuse of notation) let $k_{(j)}$ be the maximum demand corresponding to the $j$-th highest type. Letting $\ell$ be such that $\sum_{i=1}^{\ell-1} k_{(i)} < 1 \leq \sum_{i=1}^{\ell} k_{(i)}$, it follows that under ex post efficiency, the types $\boldsymbol{\theta}^{top} \equiv (\theta_{(1)},...,\theta_{(\ell-1)})$ are served up to their maximum demands, with the remainder of the supply, $k^r \equiv 1 - \sum_{i=1}^{\ell-1} k_{(i)}$, going to type $\theta_{(\ell)}$, i.e.,

$$W(\boldsymbol{\theta}_{b_1},\boldsymbol{\theta}_{b_2},\boldsymbol{\theta}_o) = \sum_{i=1}^{\ell-1} \theta_{(i)} k_{(i)} + \theta_{(\ell)} k^r. \tag{A.9}$$

For $i \in \{1,\ldots,\ell\}$, we let $I(\theta_{(i)})$ denote the firm associated with $\theta_{(i)}$. Further, for $j \in \{b_1,b_2,o\}$, define $k_j^{top} \equiv \sum_{i=1}^{\ell-1} k_{(i)} \cdot 1_{I(\theta_{(i)})=j}$ and $k_j^r \equiv k^r \cdot 1_{I(\theta_{(\ell)})=j}$, where, by definition

$$k_{b_1}^{top} + k_{b_2}^{top} + k_o^{top} + k_{b_1}^r + k_{b_2}^r + k_o^r = 1. \tag{A.10}$$

With some abuse of notation, for $i \in \{1,2\}$, we define $\tilde{v}_{b_i,o}\left(x,\boldsymbol{\theta}_{b_i}\backslash\boldsymbol{\theta}^{top},\boldsymbol{\theta}_o\backslash\boldsymbol{\theta}^{top}\right)$ to be the joint willingness to pay of buyer $b_i$ and all firms other than buyer $b_1$ and buyer $b_2$ for quantity $x$ when buyer $b_i$'s type is $\tilde{\boldsymbol{\theta}}_{b_i}$ given by $\tilde{\theta}^j_{b_i} = 0$ if $\theta^j_{b_i}$ is in the top $\ell - 1$ types and $\tilde{\theta}^j_{b_i} = \theta^j_{b_i}$ otherwise, and analogously for the other firms' types, and adjusting the maximum demand for type $\theta_{(\ell)}$ to be $k_{(\ell)} - k^r$.

We can now write

$$
\begin{aligned}
& W(\mathbf{0},\boldsymbol{\theta}_{b_2},\boldsymbol{\theta}_o)+W(\boldsymbol{\theta}_{b_1},\mathbf{0},\boldsymbol{\theta}_o) \\
= \; & \textstyle\sum_{i=1}^{\ell-1}\theta_{(i)}k_{(i)}\left(1_{I(\theta_{(i)})=b_2}+1_{I(\theta_{(i)})=o}\right)+\theta_{(\ell)}(k_{b_2}^r+k_o^r)+\tilde{v}_{b_2,o}\left(1-k_{b_2}^{top}-k_o^{top}-k_{b_2}^r-k_o^r,\boldsymbol{\theta}_{b_2}\backslash\boldsymbol{\theta}^{top},\boldsymbol{\theta}_o\backslash\boldsymbol{\theta}^{top}\right) \\
& +\textstyle\sum_{i=1}^{\ell-1}\theta_{(i)}k_{(i)}\left(1_{I(\theta_{(i)})=b_1}+1_{I(\theta_{(i)})=o}\right)+\theta_{(\ell)}(k_{b_1}^r+k_o^r)+\tilde{v}_{b_1,o}\left(1-k_{b_1}^{top}-k_o^{top}-k_{b_1}^r-k_o^r,\boldsymbol{\theta}_{b_1}\backslash\boldsymbol{\theta}^{top},\boldsymbol{\theta}_o\backslash\boldsymbol{\theta}^{top}\right) \\
= \; & W(\boldsymbol{\theta}_{b_1},\boldsymbol{\theta}_{b_2},\boldsymbol{\theta}_o)+\textstyle\sum_{i=1}^{\ell-1}\theta_{(i)}k_{(i)}1_{I(\theta_{(i)})=o}+\theta_{(\ell)}k_o^r \\
& +\tilde{v}_{b_2,o}\left(1-k_{b_2}^{top}-k_o^{top}-k_{b_2}^r-k_o^r,\boldsymbol{\theta}_{b_2}\backslash\boldsymbol{\theta}^{top},\boldsymbol{\theta}_o\backslash\boldsymbol{\theta}^{top}\right)+\tilde{v}_{b_1,o}\left(1-k_{b_1}^{top}-k_o^{top}-k_{b_1}^r-k_o^r,\boldsymbol{\theta}_{b_1}\backslash\boldsymbol{\theta}^{top},\boldsymbol{\theta}_o\backslash\boldsymbol{\theta}^{top}\right) \\
\geq \; & W(\boldsymbol{\theta}_{b_1},\boldsymbol{\theta}_{b_2},\boldsymbol{\theta}_o)+\textstyle\sum_{i=1}^{\ell-1}\theta_{(i)}k_{(i)}1_{I(\theta_{(i)})=o}+\theta_{(\ell)}k_o^r \\
& +\tilde{v}_{b_2,o}\left(1-k_{b_2}^{top}-k_o^{top}-k_{b_2}^r-k_o^r,\mathbf{0},\boldsymbol{\theta}_o\backslash\boldsymbol{\theta}^{top}\right)+\tilde{v}_{b_1,o}\left(1-k_{b_1}^{top}-k_o^{top}-k_{b_1}^r-k_o^r,\mathbf{0},\boldsymbol{\theta}_o\backslash\boldsymbol{\theta}^{top}\right) \\
= \; & W(\boldsymbol{\theta}_{b_1},\boldsymbol{\theta}_{b_2},\boldsymbol{\theta}_o)+v_o(2-k_{b_1}^{top}-k_{b_2}^{top}-k_o^{top}-k_{b_1}^r-k_{b_2}^r-k_o^r,\boldsymbol{\theta}_o) \\
= \; & W(\boldsymbol{\theta}_{b_1},\boldsymbol{\theta}_{b_2},\boldsymbol{\theta}_o)+v_o(1,\boldsymbol{\theta}_o) \\
= \; & W(\boldsymbol{\theta}_{b_1},\boldsymbol{\theta}_{b_2},\boldsymbol{\theta}_o)+W(\mathbf{0},\mathbf{0},\boldsymbol{\theta}_o),
\end{aligned}
$$

where the second equality uses (A.9), the first inequality follows because we reduce the types in the $\tilde{v}$ expressions for $b_1$ and $b_2$ to zero, the third equality adds up the units being allocated to outside types, and the second-to-last equality uses (A.10). This establishes (A.8), and with a strict inequality for a positive measure of type realizations, completing the proof. □

**Lemma A.4.** *A merger of two sellers decreases expected revenue under ex post efficiency.*

*Proof of Lemma* A.4. Consider full integration between two sellers, $s_1$ and $s_2$ with $r_{s_1}=k_{s_1}$ and $r_{s_2}=k_{s_2}$. Let $\boldsymbol{\theta}_{s_1}$, $\boldsymbol{\theta}_{s_2}$, and $\boldsymbol{\theta}_o$ be the vectors of types for seller $s_1$, seller $s_2$, and all the other firms, respectively. As in the proof of Lemma A.3, the VCG payments of the nonintegrating firms are not affected by the integration, so we can focus on the change in payments made by the integrating firms. It is sufficient to show that the following inequality holds (and strictly for a positive measure set of types):

$$
\begin{aligned}
& W(\mathbf{0},\boldsymbol{\theta}_{s_2},\boldsymbol{\theta}_o)-W(\boldsymbol{\theta}_{s_1},\boldsymbol{\theta}_{s_2},\boldsymbol{\theta}_o)+v_{s_1}(Q_{s_1}^e(\boldsymbol{\theta}_{s_1},\boldsymbol{\theta}_{s_2},\boldsymbol{\theta}_o),\boldsymbol{\theta}_{s_1})-r_{s_1} \\
& +W(\boldsymbol{\theta}_{s_1},\mathbf{0},\boldsymbol{\theta}_o)-W(\boldsymbol{\theta}_{s_1},\boldsymbol{\theta}_{s_2},\boldsymbol{\theta}_o)+v_{s_2}(Q_{s_2}^e(\boldsymbol{\theta}_{s_1},\boldsymbol{\theta}_{s_2},\boldsymbol{\theta}_o),\boldsymbol{\theta}_{s_2})-r_{s_2} \\
\geq \; & W(\mathbf{0},\mathbf{0},\boldsymbol{\theta}_o)-W(\boldsymbol{\theta}_{s_1},\boldsymbol{\theta}_{s_2},\boldsymbol{\theta}_o)+V_{s_1,s_2}(Q_{s_1}^e(\boldsymbol{\theta}_{s_1},\boldsymbol{\theta}_{s_2},\boldsymbol{\theta}_o)+Q_{s_2}^e(\boldsymbol{\theta}_{s_1},\boldsymbol{\theta}_{s_2},\boldsymbol{\theta}_o),\boldsymbol{\theta}_{s_1},\boldsymbol{\theta}_{s_2})-r_{s_1}-r_{s_2}.
\end{aligned}
$$

Because the merging sellers' outside options drop out of this expression, the problem is identical to that in Lemma A.3, but with "$s$" replacing "$b$", and so the result of Lemma A.3 applies, thereby completing the proof. □

**Lemma A.5.** *A merger of a buyer and a seller increases expected revenue under ex post efficiency.*

*Proof.* Consider full integration between a buyer $b$ with $r_b=0$ and a seller $s$ with $r_s=k_s\leq 1$. Let $\boldsymbol{\theta}_b$, $\boldsymbol{\theta}_s$, and $\boldsymbol{\theta}_o$ be the vectors of types for buyer $b$, seller $s$, and all the other firms, respectively. Let $v_{all}(x, \boldsymbol{\theta}_s, \boldsymbol{\theta}_b, \boldsymbol{\theta}_o)$ be the joint willingness to pay of all firms for quantity $x$ units and have types $\boldsymbol{\theta}_s$, $\boldsymbol{\theta}_b$, and $\boldsymbol{\theta}_o$. Because the VCG payments of the nonintegrating firms are not affected by the integration of firms $s$ and $b$, we can focus on the change in payments made by the integrating firms. We show that the following inequality holds (and strictly for a positive measure set of type realizations), where the left side is the sum of the VCG payments of buyer $b$ and seller $s$, and the right side is the VCG payment of the firm formed through the integration of $b$ and $s$, where that firm's worst-off type is denoted $(\omega, \dots, \omega)$:

$$\begin{aligned}
& W(\mathbf{1}, \boldsymbol{\theta}_b, \boldsymbol{\theta}_o) - W(\boldsymbol{\theta}_s, \boldsymbol{\theta}_b, \boldsymbol{\theta}_o) + v_s(Q_s^e(\boldsymbol{\theta}_s, \boldsymbol{\theta}_b, \boldsymbol{\theta}_o), \boldsymbol{\theta}_s) - r_s \\
& + W(\boldsymbol{\theta}_s, \mathbf{0}, \boldsymbol{\theta}_o) - W(\boldsymbol{\theta}_s, \boldsymbol{\theta}_b, \boldsymbol{\theta}_o) + v_b(Q_b^e(\boldsymbol{\theta}_s, \boldsymbol{\theta}_b, \boldsymbol{\theta}_o), \boldsymbol{\theta}_b) \\
\leq \; & W(\boldsymbol{\omega}, \boldsymbol{\omega}, \boldsymbol{\theta}_o) - W(\boldsymbol{\theta}_s, \boldsymbol{\theta}_b, \boldsymbol{\theta}_o) + v_{s,b}(Q_b^e(\boldsymbol{\theta}_s, \boldsymbol{\theta}_b, \boldsymbol{\theta}_o) + Q_s^e(\boldsymbol{\theta}_s, \boldsymbol{\theta}_b, \boldsymbol{\theta}_o), \boldsymbol{\theta}_s, \boldsymbol{\theta}_b) - \omega r_s.
\end{aligned}$$

Noting that $k_s = r_s$ and that $v_{s,b}(Q_b^e + Q_s^e, \boldsymbol{\theta}_s, \boldsymbol{\theta}_b) = v_s(Q_s^e, \boldsymbol{\theta}_s) + v_b(Q_b^e, \boldsymbol{\theta}_b)$ (where we drop the argument $(\boldsymbol{\theta}_s, \boldsymbol{\theta}_b, \boldsymbol{\theta}_o)$ on the allocations $Q_s^e$ and $Q_b^e$), we can rewrite this as

$$W(\mathbf{1}, \boldsymbol{\theta}_b, \boldsymbol{\theta}_o) - W(\boldsymbol{\omega}, \boldsymbol{\omega}, \boldsymbol{\theta}_o) - (1-\omega)k_s \leq W(\boldsymbol{\theta}_s, \boldsymbol{\theta}_b, \boldsymbol{\theta}_o) - W(\boldsymbol{\theta}_s, \mathbf{0}, \boldsymbol{\theta}_o). \tag{A.11}$$

By the definition of the ex post efficient allocation and since $Q_s^e(\mathbf{1}, \boldsymbol{\theta}_b, \boldsymbol{\theta}_o) = k_s$, we have

$$\begin{aligned}
W(\boldsymbol{\omega}, \boldsymbol{\omega}, \boldsymbol{\theta}_o) &\geq \omega Q_s^e(\mathbf{1}, \boldsymbol{\theta}_b, \boldsymbol{\theta}_o) + v_{b,o}(1 - Q_s^e(\mathbf{1}, \boldsymbol{\theta}_b, \boldsymbol{\theta}_o), \boldsymbol{\omega}, \boldsymbol{\theta}_o) \qquad \text{(A.12)} \\
&= \omega k_s + v_{b,o}(1 - k_s, \boldsymbol{\omega}, \boldsymbol{\theta}_o),
\end{aligned}$$

with a strict inequality for a positive measure set of types realizations. It then follows that

$$\begin{aligned}
& W(\mathbf{1}, \boldsymbol{\theta}_b, \boldsymbol{\theta}_o) - W(\boldsymbol{\omega}, \boldsymbol{\omega}, \boldsymbol{\theta}_o) - (1-\omega)k_s \\
\leq \; & k_s + v_{b,o}(1 - k_s, \boldsymbol{\theta}_b, \boldsymbol{\theta}_o) - \omega k_s - v_{b,o}(1 - k_s, \boldsymbol{\omega}, \boldsymbol{\theta}_o) - (1-\omega)k_s \\
\leq \; & k_s + v_{b,o}(1 - k_s, \boldsymbol{\theta}_b, \boldsymbol{\theta}_o) - \omega k_s - v_o(1 - k_s, \boldsymbol{\theta}_o) - (1-\omega)k_s \\
= \; & v_{b,o}(1 - k_s, \boldsymbol{\theta}_b, \boldsymbol{\theta}_o) - v_o(1 - k_s, \boldsymbol{\theta}_o) \\
\leq \; & v_{b,o}(1 - k_s, \boldsymbol{\theta}_b, \boldsymbol{\theta}_o) - v_o(1 - k_s, \boldsymbol{\theta}_o) + v_{all}(k_s, \boldsymbol{\theta}_s, \boldsymbol{\theta}_b, \boldsymbol{\theta}_o) - v_{s,o}(k_s, \boldsymbol{\theta}_s, \boldsymbol{\theta}_o) \\
= \; & W(\boldsymbol{\theta}_s, \boldsymbol{\theta}_b, \boldsymbol{\theta}_o) - W(\boldsymbol{\theta}_s, \mathbf{0}, \boldsymbol{\theta}_o),
\end{aligned}$$

where the first inequality uses (A.12). This establishes (A.11), including with a strict inequality for a positive measure set of type realizations. Thus, a vertical merger increases expected revenue under ex post efficiency and binding IR for firms' worst-off types. □

Together, Lemmas A.3–A.5 complete the proof of Theorem 1. ■