ONLINE APPENDIX

Efficient consignment auctions*

Bing Liu† Simon Loertscher‡ Leslie M. Marx§

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†Department of Economics, Stanford University, Stanford, CA 94305, USA, bingliu@stanford.edu.

‡Department of Economics, Level 4, FBE Building, 111 Barry Street, University of Melbourne, Victoria 3010, Australia, simonl@unimelb.edu.au.

§Duke University, 100 Fuqua Drive, Durham, NC 27708, USA, marx@duke.edu.

1. Discussion and background

1.1. Mechanism design background Given endowments $\mathbf{r}$, the VCG consignment auction is $\langle \mathbf{Q}, \mathbf{T}_{\mathbf{r}} \rangle$, where $\mathbf{Q} : [\underline{\theta}, \overline{\theta}]^n \to \mathbb{R}^n_+$ is the efficient allocation rule stated below and $\mathbf{T}_{\mathbf{r}} : [\underline{\theta}, \overline{\theta}]^n \to \mathbb{R}^n$ is the payment rule defined below.

The efficient allocation rule can be written as follows: given type vector $\boldsymbol{\theta} \in [\underline{\theta}, \overline{\theta}]^n$ whose elements are ranked as $\theta_{(1)} > \theta_{(2)} > \cdots > \theta_{(n)}$ (to reduce notation, we ignore ties, which occur with zero probability), the ex post allocation to agent i with type and ranking $\theta_i = \theta_{(\ell)}$ is

$$Q_i(\boldsymbol{\theta}) \equiv \min\{k_i, \max\{0,\ 1 - \sum_{j=1}^{\ell-1} k_j\}\}.$$

That is, in the efficient allocation, the quantity allocated to agent i is the maximum quantity less than or equal to its capacity k_i such that all agents with greater types are fully served prior to allocating any units to agent i. Because $\mathbf{Q}$ is the efficient allocation rule, for all θ_i',

$$\sum_{j \in \mathcal{N}} \theta_j Q_j(\theta_i', \boldsymbol{\theta}_{-i}) \leq \sum_{j \in \mathcal{N}} \theta_j Q_j(\theta_i, \boldsymbol{\theta}_{-i}). \tag{OA.1}$$

The interim expected quantity for agent i is the defined as

$$q_i(\theta_i) \equiv \mathbb{E}_{\boldsymbol{\theta}_{-i}} \left[Q_i(\boldsymbol{\theta}) \right].$$

Noting that q_i is continuous in θ_i (using the assumption that F_i has support $[\underline{\theta}, \overline{\theta}]$ and positive density on the interior of the support), $q_i(\underline{\theta}) = 0$, and $q_i(\overline{\theta}) = k_i$ (using $k_i \leq 1$), one can define type $\hat{\theta}_i \in [\underline{\theta}, \overline{\theta}]$ for agent i by

$$\hat{\theta}_i \equiv \min \left\{ \theta \mid q_i(\theta) \geq r_i \right\},$$

which in our setup amounts to

$$q_i(\hat{\theta}_i) = r_i.$$

Given this definition of $\hat{\theta}_i$, the VCG payment rule for agent i is defined as

$$T_{\mathbf{r},i}(\boldsymbol{\theta}) \equiv \sum_{j \in \mathcal{N} \setminus \{i\}} \theta_j \left(Q_j(\hat{\theta}_i, \boldsymbol{\theta}_{-i}) - Q_j(\boldsymbol{\theta}) \right) + \hat{\theta}_i \left(Q_i(\hat{\theta}_i, \boldsymbol{\theta}_{-i}) - r_i \right),$$

and the expected payoff of agent i with type θ_i from reporting type θ_i' is

$$\begin{aligned} U_i(\theta_i, \theta_i') &\equiv \theta_i q_i(\theta_i') - \mathbb{E}_{\boldsymbol{\theta}_{-i}}\left[T_{\mathbf{r},i}(\theta_i', \boldsymbol{\theta}_{-i})\right] \\ &= \theta_i q_i(\theta_i') - \mathbb{E}_{\boldsymbol{\theta}_{-i}}\Big[\sum_{j \in \mathcal{N} \setminus \{i\}} \theta_j \left(Q_j(\hat{\theta}_i, \boldsymbol{\theta}_{-i}) - Q_j(\theta_i', \boldsymbol{\theta}_{-i}) \right) \Big] - \hat{\theta}_i q_i(\hat{\theta}_i) + \hat{\theta}_i r_i. \end{aligned}$$

To establish interim individual rationality, we need to show that for all i and all θ_i,

$$U_i(\theta_i, \theta_i) \geq \theta_i r_i$$

holds. To see that this is the case, notice that we have

$$\begin{aligned} U_i(\theta_i, \theta_i) &= \theta_i q_i(\theta_i) - \mathbb{E}_{\boldsymbol{\theta}_{-i}}\Big[\sum_{j \in \mathcal{N} \setminus \{i\}} \theta_j \left(Q_j(\hat{\theta}_i, \boldsymbol{\theta}_{-i}) - Q_j(\theta_i, \boldsymbol{\theta}_{-i}) \right) \Big] - \hat{\theta}_i q_i(\hat{\theta}_i) + \hat{\theta}_i r_i \\ &= \theta_i q_i(\hat{\theta}_i) - \hat{\theta}_i q_i(\hat{\theta}_i) + \hat{\theta}_i r_i - \mathbb{E}_{\boldsymbol{\theta}_{-i}}\Big[\sum_{j \in \mathcal{N}} \theta_j \left(Q_j(\hat{\theta}_i, \boldsymbol{\theta}_{-i}) - Q_j(\theta_i, \boldsymbol{\theta}_{-i}) \right) \Big] \\ &\geq \theta_i q_i(\hat{\theta}_i) - \hat{\theta}_i q_i(\hat{\theta}_i) + \hat{\theta}_i r_i \\ &= \theta_i r_i, \end{aligned}$$

where the first equality uses the definition of U_i, the second equality extends the summation to include agent i and simplifies, the inequality uses (OA.1), and the final equality uses $q_i(\hat{\theta}_i) = r_i$. Because $U_i(\theta_i, \theta_i) \geq \theta_i r_i$, which is agent i's outside option, we conclude that interim individual rationality is satisfied. Further, because $U_i(\hat{\theta}_i, \hat{\theta}_i) = \hat{\theta}_i r_i$, interim individual rationality is satisfied with equality for agent i with type $\hat{\theta}_i$.

By the *payoff equivalence theorem* (see, e.g., Myerson, 1981; Krishna, 2010; Börgers, 2015), given any incentive compatible mechanism, the interim expected payment rule for each agent i, $\mathbb{E}_{\boldsymbol{\theta}_{-i}}[T_{\mathbf{r},i}(\boldsymbol{\theta})]$ is pinned down, up to a constant, by the allocation rule. Fixing the allocation rule at the efficient allocation rule and noting that $\langle \mathbf{Q}, \mathbf{T_r} \rangle$ satisfies interim individual rationality with equality for agent i with type $\hat{\theta}_i$, it follows that the constants in the VCG allocation rule cannot be increased without violating interim individual rationality. Thus, the VCG mechanism maximizes expected revenue to the designer subject to ex post efficiency, incentive compatibility, and interim individual rationality.

By the payoff equivalence theorem and the definition of $\hat{\theta}_i$ and letting individual rationality bind for type $\hat{\theta}_i$,

$$t_{\mathbf{r},i}(\theta_i) = q_i(\theta_i)\theta_i - \int_{\hat{\theta}_i(r_i)}^{\theta_i} q_i(x)dx - \hat{\theta}_i(r_i)r_i. \tag{OA.2}$$

1.2. Ex post budget balance with an AGV mechanism As we now discuss, one can also consider Arrow-d'Aspremont-Gérard-Varet (AGV) spot market mechanisms, which generate zero budget surplus for all type realizations.[1] Thus, the assumption that the spot market mechanism must run no deficit in expectation can be replaced by an assumption of no deficit ex post. As shown by Börgers and Norman (2009), the assumption of no deficit in expectation is without loss of generality because for any Bayesian incentive compatible mechanism $\langle \mathbf{Q}, \mathbf{M} \rangle$ that is such that $\sum_{i\in\mathcal{N}} \mathbb{E}_{\theta_i}[m_i(\theta_i)] = \kappa$ holds for some $\kappa \in \mathbb{R}$, there is Bayesian incentive compatible mechanism $\langle \mathbf{Q}, \tilde{\mathbf{M}} \rangle$ with the same allocation rule and the same interim expected payments m_i whose revenue is κ ex post, i.e., that satisfies $\sum_{i\in\mathcal{N}} \tilde{M}_i(\boldsymbol{\theta}) = \kappa$ for all $\boldsymbol{\theta}$. Letting $\tilde{M}_i(\boldsymbol{\theta}) = m_i(\theta_i) - \sum_{j\neq i} m_j(\theta_j)/(n-1) + c_i$ with $c_i = (\mathbb{E}_{\theta_i}[m_i(\theta_i)] - \kappa)/(n-1)$, it further follows that if the mechanism $\langle \mathbf{Q}, \mathbf{M} \rangle$ satisfies interim individual rationality, then so does $\langle \mathbf{Q}, \tilde{\mathbf{M}} \rangle$.

We provide additional details in what follows.

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[1]See Krishna (2010, chapter 5), with references to Arrow (1979) and d'Aspremont and Gérard-Varet (1979), d'Aspremont and Gérard-Varet (1979b). See also Krishna and Perry (2000) and d'Aspremont et al. (2004).

Recall that we use $\mathbf{Q}$ to denote the efficient allocation rule. Denote the expected utility of agents other than agent i, given agent i's report of θ_i, plus the outside option of agent's i's worst-off type by

$$v_i(\theta_i) \equiv \sum_{j \neq i} \mathbb{E}_{\boldsymbol{\theta}_{-i}} \left[Q_j(\theta_i, \boldsymbol{\theta}_{-i}) \theta_j \right] + r_i \hat{\theta}_i$$

and define

$$h_i(\boldsymbol{\theta}_{-i}) \equiv \frac{1}{n-1} \sum_{j \neq i} v_j(\theta_j)$$

and

$$d_i \equiv \mathbb{E}_{\boldsymbol{\theta}_{-i}}[h_i(\boldsymbol{\theta}_{-i})] - \mathbb{E}_{\boldsymbol{\theta}_{-i}}[W(\underline{\theta}, \boldsymbol{\theta}_{-i})].$$

Proposition OA.1. *If* $\mathbf{r}$ *is such that*

$$\mathbb{E}_{\boldsymbol{\theta}} \left[\sum_{i \in \mathcal{N}} \left(W(\underline{\theta}, \boldsymbol{\theta}_{-i}) - \sum_{j \in \mathcal{N} \setminus \{i\}} Q_j(\boldsymbol{\theta}) \theta_j \right) \right] \geq \sum_{i \in \mathcal{N}} r_i \mathbb{E}_{\theta_i}[\theta_i], \tag{OA.3}$$

then the mechanism $\langle \mathbf{Q}, \overline{\mathbf{T}} \rangle$*, where for* $i \in \mathcal{N} \setminus \{1\}$*,*

$$\overline{T}_i(\boldsymbol{\theta}) \equiv h_i(\boldsymbol{\theta}_{-i}) - v_i(\theta_i) - d_i$$

and

$$\overline{T}_1(\boldsymbol{\theta}) \equiv h_1(\boldsymbol{\theta}_{-1}) - v_1(\theta_1) + \sum_{j \in \mathcal{N} \setminus \{1\}} d_j,$$

is ex post efficient, Bayesian incentive compatible, interim individual rational, and ex post budget balanced.

Proof. Recall that the AGV mechanism, $\langle \mathbf{Q}, \mathbf{T}^A \rangle$, where $T_i^A(\boldsymbol{\theta}) \equiv h_i(\boldsymbol{\theta}_{-i}) - v_i(\theta_i)$, satisfies Bayesian incentive compatibility and ex post budget balance. Because $\overline{\mathbf{T}}$ is defined to be the AGV payment rule adjusted by a constant, it follows that $\langle \mathbf{Q}, \overline{\mathbf{T}} \rangle$ is Bayesian incentive compatible. Further, because $\sum_{i \in \mathcal{N}} \overline{T}_i(\boldsymbol{\theta}) = \sum_{i \in \mathcal{N}} T_i^A(\boldsymbol{\theta}) = 0$, $\langle \mathbf{Q}, \overline{\mathbf{T}} \rangle$ satisfies ex post budget balance. It remains to show that $\langle \mathbf{Q}, \overline{\mathbf{T}} \rangle$ satisfies interim individual rationality.

Defining $u_i^T(\theta_i) \equiv \theta_i q_i(\theta_i) - \mathbb{E}_{\boldsymbol{\theta}_{-i}}[T_i(\boldsymbol{\theta})] - r_i\theta_i$, we need to show that for all $i \in \mathcal{N}$, $u_i^{\overline{T}}(\theta_i) \geq 0$. By the definition of $\hat{\theta}_i$ as the worst-off type for agent i, for all $\theta_i \in [\underline{\theta}, \overline{\theta}]$, $u_i^T(\theta_i) \geq u_i^T(\hat{\theta}_i)$, so it is sufficient to show that $u_i^{\overline{T}}(\hat{\theta}_i) \geq 0$.

Define

$$T_i^*(\boldsymbol{\theta}) \equiv W(\underline{\theta}, \boldsymbol{\theta}_{-i}) - \sum_{j \in \mathcal{N}\setminus\{i\}} Q_j(\boldsymbol{\theta})\theta_j - r_i\theta_i,$$

and note that (OA.3) implies that

$$\mathbb{E}_{\boldsymbol{\theta}}\left[\sum_{i \in \mathcal{N}} T_i^*(\boldsymbol{\theta})\right] \geq 0. \tag{OA.4}$$

Further, note that

$$u_i^{T^*}(\theta_i) = \theta_i q_i(\theta_i) - \mathbb{E}_{\boldsymbol{\theta}_{-i}}[T_i^*(\boldsymbol{\theta})] - r_i\theta_i,$$

so

$$u_i^{T^*}(\underline{\theta}) = 0. \tag{OA.5}$$

Further,

$$u_i^{T^*}(\theta_i) = \theta_i q_i(\theta_i) - \mathbb{E}_{\boldsymbol{\theta}_{-i}}[W(\underline{\theta}, \boldsymbol{\theta}_{-i}) - \sum_{j \in \mathcal{N}\setminus\{i\}} Q_j(\boldsymbol{\theta})\theta_j],$$

so, noting that by the definition of W, $\sum_{j \in \mathcal{N}} \frac{\partial Q_j(\boldsymbol{\theta})}{\partial \theta_i}\theta_j = 0$, which we can rewrite as

$$\sum_{j \in \mathcal{N}\setminus\{i\}} \frac{\partial Q_j(\boldsymbol{\theta})}{\partial \theta_i}\theta_j = -\theta_i q_i'(\theta_i),$$

we have

$$u_i^{T^*\prime}(\theta_i) = q_i(\theta_i) + \theta_i q_i'(\theta_i) + \mathbb{E}_{\boldsymbol{\theta}_{-i}}[\sum_{j \in \mathcal{N}\setminus\{i\}} \frac{\partial Q_j(\boldsymbol{\theta})}{\partial \theta_i}\theta_j] = q_i(\theta_i) \geq 0. \tag{OA.6}$$

Combining (OA.5) and (OA.6), for all $i \in \mathcal{N}$ and $\theta_i \in [\underline{\theta}, \overline{\theta}]$,

$$u_i^{T^*}(\theta_i) \geq 0. \tag{OA.7}$$

Turning to the constant d_i, note that

$$d_i = \mathbb{E}_{\boldsymbol{\theta}_{-i}}[h_i(\boldsymbol{\theta}_{-i})] - \mathbb{E}_{\boldsymbol{\theta}_{-i}}[W(\underline{\theta}, \boldsymbol{\theta}_{-i})] = \mathbb{E}_{\boldsymbol{\theta}}\left[T_i^A(\underline{\theta}, \boldsymbol{\theta}_{-i})\right] - \mathbb{E}_{\boldsymbol{\theta}}\left[T_i^*(\underline{\theta}, \boldsymbol{\theta}_{-i})\right],$$

which, using (OA.4) and $\sum_{i\in\mathcal{N}} T_i^A(\boldsymbol{\theta}) = 0$ implies that

$$\sum_{i\in\mathcal{N}} d_i \leq 0. \tag{OA.8}$$

Thus, for $i \neq 1$, we have

$$\begin{aligned} u_i^{\overline{T}}(\hat{\theta}_i) &= \hat{\theta}_i q_i(\hat{\theta}_i) - \mathbb{E}_{\boldsymbol{\theta}_{-i}}[\overline{T}_i(\hat{\theta}_i, \boldsymbol{\theta}_{-i})] - r_i\hat{\theta}_i \\ &= \hat{\theta}_i q_i(\hat{\theta}_i) + d_i - \mathbb{E}_{\boldsymbol{\theta}_{-i}}[T_i^A(\hat{\theta}_i, \boldsymbol{\theta}_{-i})] - r_i\theta_i \\ &= u_i^{T^*}(\hat{\theta}_i) \\ &\geq 0, \end{aligned}$$

where the inequality uses (OA.7) and

$$\begin{aligned} u_1^{\overline{T}}(\hat{\theta}_i) &= \hat{\theta}_1 q_1(\hat{\theta}_1) - \mathbb{E}_{\boldsymbol{\theta}_{-1}}[\overline{T}_1(\hat{\theta}_1, \boldsymbol{\theta}_{-1})] - r_1\hat{\theta}_1 \\ &= \hat{\theta}_i q_1(\hat{\theta}_i) - \sum_{j\in\mathcal{N}\setminus\{1\}} d_j - \mathbb{E}_{\boldsymbol{\theta}_{-1}}[T_1^A(\hat{\theta}_1, \boldsymbol{\theta}_{-1})] - r_1\hat{\theta}_1 \\ &\geq \hat{\theta}_i q_1(\hat{\theta}_i) + d_1 - \mathbb{E}_{\boldsymbol{\theta}_{-1}}[T_1^A(\hat{\theta}_1, \boldsymbol{\theta}_{-1})] - r_1\hat{\theta}_1 \\ &= u_1^{T^*}(\hat{\theta}_1) \\ &\geq 0, \end{aligned}$$

where the first inequality uses (OA.8) and the second inequality uses (OA.7). ■

1.3. Illustration of optimal overendowments As an illustration of the contrast between the interior solution $\mathbf{r}^*$ and overendowments $\mathbf{r}^o$, consider a setup with $n = 3$, $k_2 = k_3 = 2/3$, and $F_i(\theta) = \theta$ for all $i \in \{1, 2, 3\}$. Let the value of k_1 go from 0 to 2/3. For this range of k_1, Proposition 6 shows that $\mathbf{r}^o = (1, 0, 0)$ is a corner solution. Figure OA.1 plots the ex ante consignment auction revenue from the interior solution $\mathbf{r}^*$ and from the solution $(1, 0, 0)$, for $k_1 \in [0, 2/3]$. For sufficiently small k_1, the corner solution generates higher ex ante revenue. However, when k_1 is sufficiently large, even if $r_i > k_i$ is permitted, we have $\Pi_{\mathbf{r}^*} > \Pi_{\mathbf{r}^o}$.

Expected consignment auction revenue

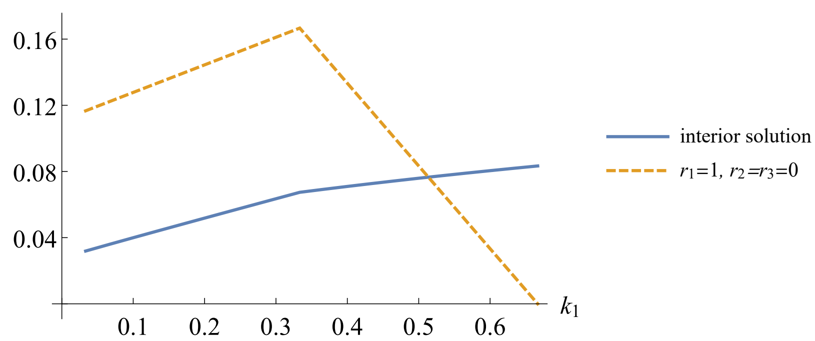

Figure OA.1: Ex ante consignment auction revenue as a function of k_1 under the interior solution for $\mathbf{r}$ and under the corner solution in which agent 1's endowment is equal to the entire supply. Assumes that $n = 3$, $k_2 = k_3 = \frac{2}{3}$, k_1 varies between 0 and $\frac{2}{3}$ as indicated, and costs are uniformly distributed on $[0, 1]$.

In the illustration of Figure OA.1, the ex ante revenue generated by the interior solution increases in k_1 but the rate of increase drops once k_1 is greater than 1/3. When k_1 increases, the expected consignment auction endowments to agents other than agent 1 decrease, which implies a lower common worst-off type and a higher revenue. The increase is relatively steep when $k_1 < 1/3$ because then agents expect to receive a positive quantity in the consignment auction even when they have the lowest type. The same explanation applies to the kink at $k_1 = 1/3$ in the expected surplus generated by the corner solution. For the corner solution, an increase in k_1 increases the outside option of agent 1 and, when $k_1 > 1/3$, *reduces* the consignment auction revenue.

As intuition, because agent i^* values anything beyond its capacity k_i at zero, giving the entire supply to agent i^* is analogous to creating a one-sided allocation problem for the consignment auction where the "supplier," agent i^*, is chosen to be the agent that trades "the least" in the consignment auction. Of course, the interior solution $\mathbf{r}^*$ continues to characterize a local maximum, and whether the interior or corner solution is the global maximum depends on the details of the setup.

2. Data

2.1. Calibration summary tables In this appendix, we first provide in Table OA.1 the summary statistics for the data that we use, which were assembled by Fowlie and Perloff (2013). Table OA.2 then provides the maximum demand and the average endowment for each facility in our selection as well as the calibrated values a_i for the power distribution, i.e., $F_i(\theta) = \theta^{a_i}$, and the endowments r_i^* for the power distribution, the distribution $F_i(\theta) = 1-(1-\theta)^{a_i}$, and the Beta distribution. It concludes with a formal comparison of the similarity of the distribution of the calibration endowments $\mathbf{r}^*$ for each these three specifications.

Table OA.2 shows for each facility i: the maximum demand k_i,[2] the average empirical endowment across periods, the calibrated distribution parameter a_i, and the optimal endowment r_i^* based on the power distribution. To evaluate the robustness of the results to other distributions, Table OA.2 also shows, for each facility i, the optimal endowments calculated based on the alternative distributions of $F_i(\theta_i) = 1 - (1 - \theta_i)^{a_i}$ and the symmetric beta distribution $Beta(a_i, a_i)$. The distribution parameters a_i for the two alternative distributions are calibrated to maximize (4). To compare the optimal endowments under the different distributions, we make use of the Kullback-Leibler (KL) distance, which is a common measure of the dissimilarities between two unit mass (probability) distributions. The KL distance between the endowments under the power distribution, i.e., $F(\theta) = \theta^{a_i}$, relative to the Beta distribution and to the distribution $F_i(\theta) = 1 - (1 - \theta)^{a_i}$, are small (0.0091 and 2.4747×10^{-5}, respectively). In terms of model fitting, the power distributions used in the

[2]For all facilities in the same group, that is for all $i \in g$, we have $k_i = k_g$, $F_i = F_g$ (which is equivalent to $a_i = a_g$), and hence $r_i^* = r_g^*$.

main model yield the lowest sum of squared errors, 8.477×10^{-5}, while the sum of squared errors for $F_i(\theta) = 1 - (1-\theta)^{a_i}$ is 1.150×10^{-4}. For the Beta distribution, it is 0.0121.

Table OA.1: Summary of statistics for a selection of facilities in the RECLAIM data assembled by Fowlie and Perloff (2013). Allocations (which correspond to endowments in the paper) and emissions (which correspond to quantities consumer in the paper) are measured in 1000 tons of NOx.

Facility name	Facility ID	Maximum allocation	Minimum allocation	Mean allocation	Stdev of allocation	Maximum emission	Minimum emission	Mean emission	Stdev of emission
RECOT, INC.	346	40.5	32.2	33.8	2.91	12.8	10.6	11.5	0.656
LA CO., INTERNAL SER	550	28.6	22.7	23.9	2.04	65.8	21.8	32.2	12.2
SUPERIOR INDUSTRIES	2083	8.96	7.15	7.51	0.634	24.8	8.47	15.8	4.36
MCP FOODS INC	2825	2.21	1.77	1.86	0.157	2.54	1.52	2.3	0.297
TABC, INC	3968	48.4	36.1	39.5	4.39	20.5	7.73	13.5	4.32
TEXTRON AEROSPACE FA	4451	2.9	2.32	2.43	0.205	2.17	1.37	1.72	0.24
GAINEY CERAMICS INC	5814	9.65	7.7	8.09	0.683	5.64	3.45	4.47	0.71
SO CAL GAS CO	5973	49	39.1	41	3.47	83.1	27.9	48.6	15.8
ALL AMERICAN ASPHALT	5998	2.32	1.85	1.94	0.164	2.13	1.08	1.58	0.307
PRAXAIR INC	7416	12.9	10.3	10.8	0.912	6.05	0.471	1.33	1.7
OWENS-BROCKWAY GLASS	7427	215	172	180	15.3	187	57.6	100	54.4
QUEMETCO INC	8547	39.5	31.5	33.1	2.8	37.4	19.5	26.7	7.21
SO CAL GAS CO/PLAYA	8582	44	35.1	36.9	3.12	44.4	10.7	27.1	10.8
SEMPRA ENERGY SOLUTI	9217	3.44	2.75	2.89	0.243	4.82	0.295	1.61	1.41
BREA CANYON OIL CO I	10340	8.44	6.73	7.07	0.597	0	0	0	0
SO CAL GAS CO	11119	1.49	1.19	1.25	0.105	2.93	1.03	1.67	0.566
ARMSTRONG WORLD INDU	12155	1.86	1.49	1.56	0.132	2.54	2.1	2.29	0.165
MISSION CLAY PRODUCT	12372	5.79	4.62	4.85	0.41	4.84	4.29	4.67	0.19
SO CAL GAS CO	14926	4.71	3.76	3.95	0.334	0.473	0.077	0.224	0.137
TECHALLOY CO., INC.	14944	5.64	4.5	4.73	0.399	0.419	0.219	0.284	0.0548
HIGGINS BRICK CO	15164	30	23.9	25.1	2.12	17.5	3.27	8.07	5.42
EXXONMOBIL OIL CORP	17344	2.88	2.29	2.41	0.203	2.15	0.131	1.33	0.696
WESTERN METAL DECORA	17956	1.15	0.918	0.965	0.0813	2.51	1.78	2.07	0.257
NORTHROP GRUMMAN COR	18294	6.04	4.82	5.06	0.427	11.8	7.23	9.49	1.45
US GYPSUM CO	18695	10.5	8.34	8.76	0.74	13.2	3.43	7.04	3.02
BLUE DIAMOND MATERIA	19390	3.97	3.16	3.32	0.281	10.7	4.76	6.33	1.68
ALPHA BETA CO./RALPH	21290	2.99	2.38	2.5	0.212	1.91	1.53	1.66	0.12
SGL TECHNIC INC, POL	37603	0.846	0.672	0.707	0.0605	4.99	3.37	4.23	0.55
BENTLEY MILLS INC	40034	6.82	5.44	5.72	0.483	5.53	4.12	4.77	0.428
HAYES LEMMERZ INTER	45953	14.9	11.9	12.5	1.05	28	0	11.4	9.68
OLS ENERGY-CHINO C/O	47781	40.3	32.2	33.8	2.85	36.6	23.1	30	3.57
WHEELABRATOR NORWALK	51620	50.2	40.1	42.1	3.55	63.7	29.8	42.2	9.71
REXAM PLC, REXAM BEV	52517	18.1	14.4	15.1	1.3	12.9	9.19	10.7	1.23
TREND OFFSET PRINTIN	53729	1.13	0.9	0.947	0.081	5.74	4.08	4.87	0.648
PACIFIC CONTINENTAL	59618	3.11	2.47	2.6	0.223	5.91	3.39	4.71	0.932
LA CITY, HARBOR DEPT	61962	5.16	4.11	4.32	0.365	1.82	0.888	1.43	0.317
DARLING INTERNATIONA	63180	7.7	6.14	6.45	0.544	9.52	4.5	7.66	1.76
U.S. DYEING & FINISH	83738	4.06	3.23	3.4	0.291	7.74	3.17	5.12	1.32
CARGILL INC	94930	1.45	1.16	1.21	0.104	1.45	0.907	1.1	0.143
CHROMA SYSTEMS PARTN	95212	3.87	3.09	3.24	0.274	3.52	1.86	2.66	0.549
THE TERMO COMPANY	97081	1.44	1.15	1.2	0.102	0.999	0.0245	0.605	0.348
VERTIS, INC	115130	0.927	0.738	0.775	0.0662	6.21	4.65	5.58	0.461
METAL COATERS OF CAL	115563	1.63	1.3	1.36	0.116	11	6.09	7.17	1.4
HITCO CARBON COMPOSI	800066	8.27	6.6	6.93	0.585	8.39	6.66	7.23	0.544
BOEING SATELLITE SYS	800067	6.38	5.09	5.35	0.451	2.78	1.61	2.09	0.358
LA CITY, DWP HAYNES	800074	307	245	257	21.7	335	80.5	137	74.2
EXXONMOBIL OIL CORPO	800089	696	558	586	48.2	857	622	763	72.3
MOBIL OIL CORP, NEWH	800094	3.31	2.64	2.77	0.234	3.3	0.391	1.82	1.03
SO CAL GAS CO (EIS U	800127	67.3	53.6	56.4	4.77	29	0.09	4.55	9.25
SO CAL GAS CO (EIS U	800128	69.5	55.5	58.3	4.92	215	134	179	28.8
US GOVT, AF DEPT, MA	800150	12	9.6	10.1	0.852	6.25	3.96	5.17	0.833
PASADENA CITY, DWP (	800168	47.9	38.1	40.1	3.42	53.7	4.02	26.3	16.4
LA CITY, DWP HARBOR	800170	32.6	26	27.3	2.3	96.3	20.1	57.6	26.7
PARAMOUNT PETR CORP	800183	113	90	94.5	7.98	84.6	52.9	68.6	9.88
THUMS LONG BEACH	800330	4.16	3.31	3.48	0.294	2.41	0.195	0.936	0.711
CALIFORNIA AIR NATIO	800344	1.08	0.864	0.908	0.0767	0.412	0.162	0.262	0.0821

Table OA.2: Calibration for a selection of facilities in the RECLAIM data assembled by Fowlie and Perloff (2013).

Facility name	maximum demand (k_i)	Average endowment	Power Distribution Parameter (a_i)	r_i^* ($F_i(\theta_i) = \theta_i^{a_i}$)	r_i^* ($F_i(\theta_i) = 1-(1-\theta_i)^{a_i}$)	r_i^* (Beta)
RECOT, INC.	0.011035	0.0404	0.676	0.00534	0.00532	0.00554
LA CO., INTERNAL SER	0.040117	0.0286	0.522	0.0192	0.0192	0.0201
SUPERIOR INDUSTRIES	0.014137	0.00898	0.836	0.00686	0.00683	0.0071
MCP FOODS INC	0.011035	0.00222	0.676	0.00534	0.00532	0.00554
TABC, INC	0.014137	0.0471	0.836	0.00686	0.00683	0.0071
TEXTRON AEROSPACE FA	0.011035	0.00291	0.676	0.00534	0.00532	0.00554
GAINEY CERAMICS INC	0.011035	0.00967	0.676	0.00534	0.00532	0.00554
SO CAL GAS CO	0.041669	0.0491	0.552	0.02	0.02	0.0209
ALL AMERICAN ASPHALT	0.011035	0.00233	0.676	0.00534	0.00532	0.00554
PRAXAIR INC	0.011035	0.0129	0.676	0.00534	0.00532	0.00554
OWENS-BROCKWAY GLASS	0.12977	0.216	0.545	0.0606	0.0604	0.0652
QUEMETCO INC	0.040117	0.0396	0.522	0.0192	0.0192	0.0201
SO CAL GAS CO/PLAYA	0.040117	0.0441	0.522	0.0192	0.0192	0.0201
SEMPRA ENERGY SOLUTI	0.011035	0.00345	0.676	0.00534	0.00532	0.00554
BREA CANYON OIL CO I	0	0.00846	1.5	0	0	0
SO CAL GAS CO	0.011035	0.00149	0.676	0.00534	0.00532	0.00554
ARMSTRONG WORLD INDU	0.011035	0.00187	0.676	0.00534	0.00532	0.00554
MISSION CLAY PRODUCT	0.011035	0.0058	0.676	0.00534	0.00532	0.00554
SO CAL GAS CO	0.011035	0.00473	0.676	0.00534	0.00532	0.00554
TECHALLOY CO., INC.	0.011035	0.00565	0.676	0.00534	0.00532	0.00554
HIGGINS BRICK CO	0.011035	0.03	0.676	0.00534	0.00532	0.00554
EXXONMOBIL OIL CORP	0.011035	0.00288	0.676	0.00534	0.00532	0.00554
WESTERN METAL DECORA	0.011035	0.00115	0.676	0.00534	0.00532	0.00554
NORTHROP GRUMMAN COR	0.011035	0.00605	0.676	0.00534	0.00532	0.00554
US GYPSUM CO	0.011035	0.0105	0.676	0.00534	0.00532	0.00554
BLUE DIAMOND MATERIA	0.011035	0.00398	0.676	0.00534	0.00532	0.00554
ALPHA BETA CO./RALPH	0.011035	0.00299	0.676	0.00534	0.00532	0.00554
SGL TECHNIC INC, POL	0.011035	0.000845	0.676	0.00534	0.00532	0.00554
BENTLEY MILLS INC	0.011035	0.00684	0.676	0.00534	0.00532	0.00554
HAYES LEMMERZ INTER	0.011035	0.0149	0.676	0.00534	0.00532	0.00554
OLS ENERGY-CHINO C/O	0.040117	0.0404	0.522	0.0192	0.0192	0.0201
WHEELABRATOR NORWALK	0.041669	0.0503	0.552	0.02	0.02	0.0209
REXAM PLC, REXAM BEV	0.011035	0.0181	0.676	0.00534	0.00532	0.00554
TREND OFFSET PRINTIN	0.011035	0.00113	0.676	0.00534	0.00532	0.00554
PACIFIC CONTINENTAL	0.011035	0.00311	0.676	0.00534	0.00532	0.00554
LA CITY, HARBOR DEPT	0.011035	0.00517	0.676	0.00534	0.00532	0.00554
DARLING INTERNATIONA	0.011035	0.00771	0.676	0.00534	0.00532	0.00554
U.S. DYEING & FINISH	0.011035	0.00406	0.676	0.00534	0.00532	0.00554
CARGILL INC	0.011035	0.00145	0.676	0.00534	0.00532	0.00554
CHROMA SYSTEMS PARTN	0.011035	0.00388	0.676	0.00534	0.00532	0.00554
THE TERMO COMPANY	0.011035	0.00144	0.676	0.00534	0.00532	0.00554
VERTIS, INC	0.011035	0.000927	0.676	0.00534	0.00532	0.00554
METAL COATERS OF CAL	0.011035	0.00163	0.676	0.00534	0.00532	0.00554
HITCO CARBON COMPOSI	0.011035	0.00829	0.676	0.00534	0.00532	0.00554
BOEING SATELLITE SYS	0.011035	0.00639	0.676	0.00534	0.00532	0.00554
LA CITY, DWP HAYNES	0.16093	0.307	0.546	0.0739	0.0737	0.0808
EXXONMOBIL OIL CORPO	0.45238	0.701	0.827	0.283	0.286	0.231
MOBIL OIL CORP, NEWH	0.011035	0.00332	0.676	0.00534	0.00532	0.00554
SO CAL GAS CO (EIS U	0.011035	0.0674	0.676	0.00534	0.00532	0.00554
SO CAL GAS CO (EIS U	0.11825	0.0697	1.25	0.0614	0.0611	0.0594
US GOVT, AF DEPT, MA	0.011035	0.0121	0.676	0.00534	0.00532	0.00554
PASADENA CITY, DWP (	0.040117	0.0479	0.522	0.0192	0.0192	0.0201
LA CITY, DWP HARBOR	0.080722	0.0326	0.404	0.0377	0.0376	0.0405
PARAMOUNT PETR CORP	0.036518	0.113	10.2	0.0185	0.0184	0.0183
THUMS LONG BEACH	0.011035	0.00416	0.676	0.00534	0.00532	0.00554
CALIFORNIA AIR NATIO	0.011035	0.00109	0.676	0.00534	0.00532	0.00554
Artifical Fac	0.25348	0.029	0.185	0.102	0.1	0.127

Notes: The underlying data are from Fowlie and Perloff (2013). The distribution parameters are calibrated to the model, and r_i^* is the estimated value of the optimal endowment for facility i.

2.2. Grouping of facilities In this section, we give justification on the grouping of the facilities. For each facility i, we use the 90th percentile of its normalized empirical emission overtime (which we abbreviatedly refer to as the maximum emission) as an approximation of its maximum normalized capacity k_i, and the average normalized empirical emission $\frac{1}{T} \max_{t \in T} \tilde{Q}_{i,t}$ (which we abbreviatedly refer to as the average emission), as an approximation of its ex ante expected normalized emission $\overline{r}_i$ that is determined by its maximum normalized capacity k_i and its distribution F_i. Hence a facility's maximum emission and average emission together reflect its maximum demand k_i and distribution F_i.

Figure OA.2 is a scatter plot of each facility's maximum emission and average emission which demonstrate a strong positive correlation. Figure OA.3 shows a histogram of all the facilities' maximum emissions and suggests that the facilities belong to 12 groups.[3]

[3]There is one facility with zero maximum emission. We model that facility as a separate group.

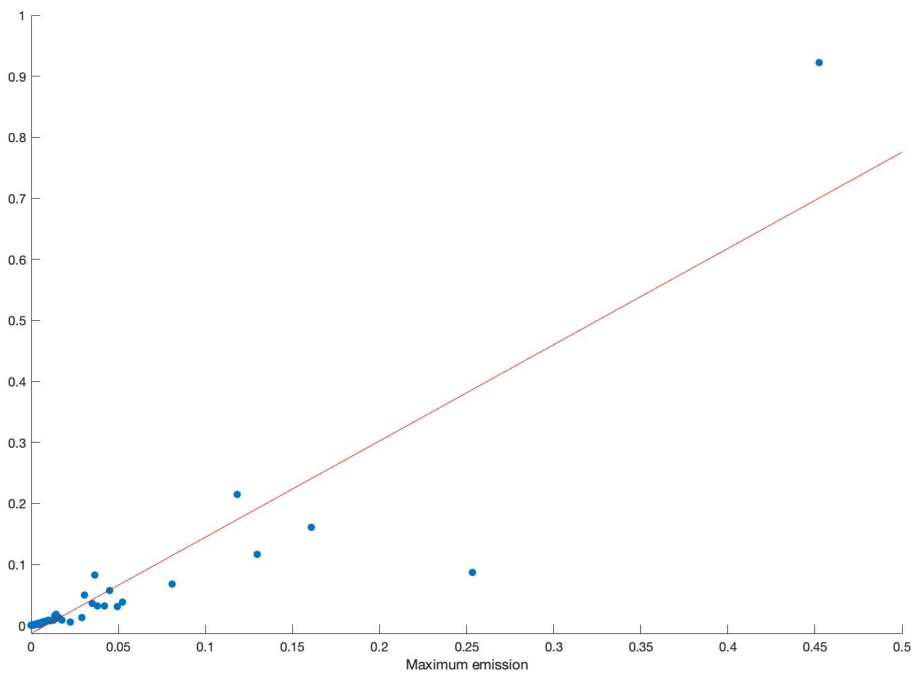

Figure OA.2: A scatter plot of each facility's maximum emission (vertical axis) and average emission (horizontal axis).

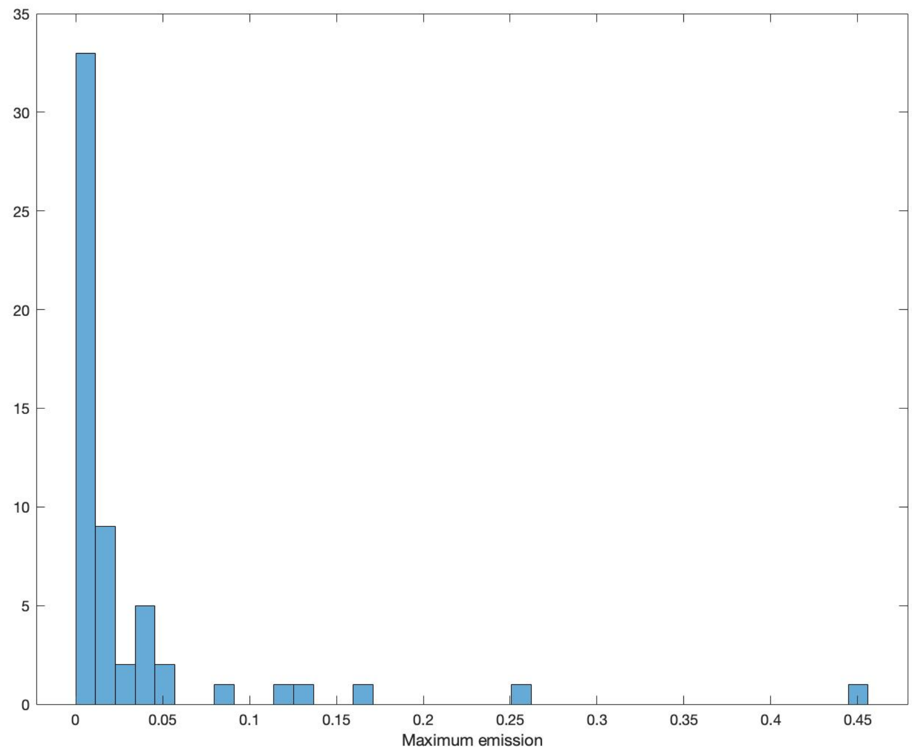

Figure OA.3: Histogram on the facilities' maximum emission, with one bar for each of the groups, except for the one facility with zero maximum emission.

3. Comparative statics We provide additional comparative statics results for the optimal endowments for the cases with strength-ordered agents, strength-ordered groups, and the α-hybrid auction.

3.1. Properties of optimal endowments with strength-ordered agents In Section 3.2 of the paper, we considered the question of how the optimal endowments $\mathbf{r}^*$ vary with other parameters. By focusing here on the special case with two agents, we can shed additional light on how optimal endowments vary with maximum demands and type distributions.

We begin with the following lemma:

Lemma OA.2. *For the case of two agents, the worst-off type $\hat{\theta}$ does not depend on the maximum demands and solves $F_1(\hat{\theta}) + F_2(\hat{\theta}) = 1$.*

Proof. First, for $i \in \{1, 2\}$, let $\overline{k}_i = \min\{k_i, 1\}$ and $\underline{k}_i = \min\{\overline{k}_i, 1 - \overline{k}_j\}$, so that $\overline{k}_i$ ($\underline{k}_i$) is the allocation to agent i if agent i has the higher (lower) realized type. Then we can write $q_i(\theta) = \overline{k}_i F_j(\theta) + \underline{k}_i(1 - F_j(\theta))$. Summing over $i \in \{1, 2\}$ and equating total allocation to total supply, $\hat{\theta}$ solves $1 = \overline{k}_i F_j(\hat{\theta}) + \underline{k}_i(1 - F_j(\hat{\theta})) + \overline{k}_j F_i(\hat{\theta}) + \underline{k}_j(1 - F_i(\hat{\theta}))$. Note that $1 - \underline{k}_i = \max\{1 - \overline{k}_i, \overline{k}_j\} = \overline{k}_j$ by the assumption that $k_1 + k_2 > 1$, so we have $\underline{k}_i = 1 - \overline{k}_j$, and we get

$$
\begin{aligned}
1 &= \overline{k}_i F_j(\hat{\theta}) + (1 - \overline{k}_j)(1 - F_j(\hat{\theta})) + \overline{k}_j F_i(\hat{\theta}) + (1 - \overline{k}_i)(1 - F_i(\hat{\theta})), \\
&= (\overline{k}_i + \overline{k}_j - 1)(F_i(\hat{\theta}) + F_j(\hat{\theta})) + (1 - \overline{k}_i) + (1 - \overline{k}_j),
\end{aligned}
$$

which we can rewrite as $(\overline{k}_i + \overline{k}_j - 1) = (\overline{k}_i + \overline{k}_j - 1)(F_i(\hat{\theta}) + F_j(\hat{\theta}))$. Dividing both sides by $(\overline{k}_i + \overline{k}_j - 1)$ yields $F_1(\hat{\theta}) + F_2(\hat{\theta}) = 1$ and completes the proof. ∎

Lemma OA.2 says that the common worst-off type implied by $\mathbf{r}^*$ in the two-agent problem is invariant to the maximum demands.

In the following, we denote by $\mathbf{r}^*(\mathbf{k})$ the optimal endowments given maximum demands $\mathbf{k}$. Using Lemma OA.2, we get:

Proposition OA.2. *For the case of two strength-ordered agents, the optimal endowment of the stronger agent: (i) increases with proportional increases to maximum demands, i.e., $r_1^*(\beta\mathbf{k})$ increases in $\beta > 0$ for all $\beta \leq 1/k_1$; (ii) increases with the maximum demand of the stronger agent and decreases with the maximal demand of the weaker agent, i.e., $r_1^*(\mathbf{k})$ is increasing in k_1 and decreasing in k_2 for $\mathbf{k}$ in the feasible range, $k_2 \leq k_1 \leq 1 < k_1 + k_2$.*

Proof. By Lemma OA.2, the common worst-off type implied by $\mathbf{r}^*$ solves $F_1(\hat{\theta}) + F_2(\hat{\theta}) = 1$. Because F_1 stochastically dominates F_2, we have $F_2(\hat{\theta}) > \frac{1}{2}$. Using the result of Lemma 1 that $q_1(\hat{\theta}) = r_1^*$, we have for all $\beta > 0$ with $\beta(k_1+k_2) \leq 1$, $r_1^*(\beta\mathbf{k}) = \beta k_1 F_2(\hat{\theta}) + (1-\beta k_2)(1-F_2(\hat{\theta}))$. Taking the derivative with respect to β, we have $\frac{\partial r_1^*(\beta\mathbf{k})}{\partial\beta} = k_1 F_2(\hat{\theta}) - k_2(1 - F_2(\hat{\theta}))$, which is positive because $k_1 > k_2$ and $F_2(\hat{\theta}) > \frac{1}{2}$. This completes the proof of part (i). Parts (ii) and (iii) follow by the invariance of $\hat{\theta}$ to maximum demands given by Lemma OA.2, writing out $r_1^*(\mathbf{k}) = k_1 F_2(\hat{\theta}) + (1 - k_2)(1 - F_2(\hat{\theta}))$, and taking derivatives. ■

Proposition OA.2 says that when both agents' maximum demands increase or, equivalently, total supply shrinks, the difference grows between the larger optimal endowment for the stronger agent and the smaller optimal endowment of the weaker agent. Further, Proposition OA.2 implies that optimal endowments respond to changes in the relative sizes of two agents as one might expect, with an increase in the size difference between firms resulting in an increase in the gap between the optimal endowments.

3.2. Properties of optimal endowments for strength-ordered groups In the setting with strength-ordered groups, in some cases, an agent may switch from one group to another, such as in the emission permit context when an energy provider changes from relying on fossil fuel to solar power. To study the effect of such transformations on the optimal endowments, suppose that there are two groups of agents, G_1 and G_2, where G_1 is stronger than G_2, and suppose that one agent changes its distribution and capacity so that it moves from G_1 to G_2. More specifically, let $m_1 \in \{0, \ldots, n\}$ be the number of agents in group G_1 and $m_2 \equiv n - m_1$ be the number of agents in group G_2. For $m_1 \in \{0, \ldots, n\}$, let $r_{G_i}^*(m_1)$ be the optimal endowments for an agent in group G_i when there are m_1 agents in group G_1.

Proposition OA.3. *In the setup with two strength-ordered groups, moving an agent from the weaker group to the stronger group increases the switching agent's optimal endowment; that is, for any $m_1 \in \{0, \ldots, n-1\}$, we have $r^*_{G_1}(m_1+1) > r^*_{G_2}(m_1)$.*

Proof. If $m_1 = 0$, then all agents are in group 2, so we have $r^*_{G_2}(0) = \frac{1}{n}$. For $m_1 = 1$, we have $r^*_{G_1}(1) > r^*_{G_2}(1)$, which implies that $nr^*_{G_1}(1) > 1$, so for $m_1 = 0$,

$$r^*_{G_1}(m_1+1) = r^*_{G_1}(1) > \frac{1}{n} = r^*_{G_2}(0) = r^*_{G_2}(m_1).$$

In addition, for all $m_1 \in \{1, \ldots, n-1\}$, $r^*_{G_1}(m_1) > r^*_{G_2}(m_1)$ and $r^*_{G_1}(m_1+1) > r^*_{G_2}(m_1+1)$, which imply that $nr^*_{G_2}(m_1) < 1$ and $nr^*_{G_1}(m_1+1) \geq 1$, so for $m_1 \in \{1, \ldots, n-1\}$, we have

$$r^*_{G_1}(m_1+1) \geq \frac{1}{n} > r^*_{G_2}(m_1).$$

Thus, for all $m_1 \in \{0, \ldots, n-1\}$, $r^*_{G_1}(m_1+1) > r^*_{G_2}(m_1)$. ■

On the one hand, Proposition 2 in the body of the paper suggests that the addition of "stronger" agents tends to reduce each existing agent's optimal endowment. On the other hand, Proposition OA.3 shows that for the case of two groups, when an agent transforms from a weak agent to a strong agent, its optimal endowment increases. Beyond two groups, how the two opposite effects interplay depends on the type distributions, scale of the maximum demands, and the group compositions.

3.3. Properties of optimal endowments in the α-hybrid auction Related to Section 3.3 in the paper, another interesting but somewhat elusive question is whether $\mathbf{r}^*(\alpha)$ becomes more or less dispersed as α increases. To see why the question is difficult to address with any degree of generality, consider the case with two agents, for which we know that, when $\alpha = 1$, $\hat{\theta}$ does not vary with k_1 and k_2. Summing up $q_i(\hat{\theta}) = k_i F_j(\hat{\theta}) + (1-k_j)(1-F_j(\hat{\theta}))$, equating the sum to α, and rearranging yields

$$1-(F_1(\hat{\theta}(\alpha))+F_2(\hat{\theta}(\alpha)))=\frac{1-\alpha}{k_1+k_2-1}.$$

Hence, $\hat{\theta}(\alpha)$ also increases in k_1 and k_2 and satisfies $\hat{\theta}(1)=\hat{\theta}$, where $\hat{\theta}$ is the value of the identical worst-off type at $\alpha=1$, which satisfies $F_1(\hat{\theta})+F_2(\hat{\theta})=1$.

Under additional assumptions, we then obtain the following result regarding the ratio of optimal endowments:

Proposition OA.4. *For the case of two agents, if $F_1(\theta)<F_2(\theta)$ for all $\theta\in(\underline{\theta},\overline{\theta})$, then $\lim_{\alpha\to 0}\frac{r_1^*(\alpha)}{r_2^*(\alpha)}>1$ and $\frac{r_1^*(\alpha)}{r_2^*(\alpha)}$ increases as α increases in the neighborhood of $\alpha=0$; if, in addition, $k_1\geq k_2$ and the distributions satisfy the monotone likelihood ratio property, i.e., $\frac{f_2(\theta)}{f_1(\theta)}$ decreases in θ, then $\frac{r_1^*(\alpha)}{r_2^*(\alpha)}$ is either increasing or hump-shaped in α.*

Proof. Notice first that $r_i^*(\alpha)=q_i(\hat{\theta}(\alpha))$, so $\frac{r_1^*(\alpha)}{r_2^*(\alpha)}=\frac{q_1(\hat{\theta}(\alpha))}{q_2(\hat{\theta}(\alpha))}$. At $\alpha=0$, we have $r_1^*(0)=r_2^*(0)$, but using L'Hôpital's rule and $q_i'(\theta)=f_j(\theta)(k_1+k_2-1)$ for $j\neq i$ and assuming $f_1(\underline{\theta})>0$, we have

$$\lim_{\alpha\to 1}\frac{r_1^*(\alpha)}{r_2^*(\alpha)}=\frac{f_2(\underline{\theta})}{f_1(\underline{\theta})},$$

which is greater than 1 because F_1 first-order stochastically dominates F_2. Moreover,

$$\left[\frac{q_1(\theta)}{q_2(\theta)}\right]'=\frac{k_1+k_2-1}{q_2(\theta)}\left[f_2(\theta)-f_1(\theta)\frac{q_1(\theta)}{q_2(\theta)}\right],$$

which is 0 at $\theta=\underline{\theta}$. The sign of the second derivative with respect to θ at $\theta=\underline{\theta}$ is the same as the sign of the derivative of $f_2q_2-f_1q_1$, where we drop arguments for convenience. This derivative is $[f_2q_2-f_1q_1]'=f_2'q_2-f_1'q_1+(k_1+k_2-1)[f_2-f_1]$. Because $q_1(\underline{\theta})=q_2(\underline{\theta})=0$ and $f_2(\underline{\theta})>f_1(\underline{\theta})$, the second derivative is positive in a small enough neighborhood of $\underline{\theta}$. Thus, the ratio $r_1^*(\alpha)/r_2^*(\alpha)$ increases as α increases in the neighborhood of $\underline{\theta}$.

To establish the result that $r_1^*(\alpha)/r_2^*(\alpha)$ is either increasing or hump-shaped in α, it suffices to show that $\frac{q_1(\theta)}{q_2(\theta)}$ is quasiconcave on $[\underline{\theta}, \overline{\theta}]$. To see this, rewrite $\left[\frac{q_1(\theta)}{q_2(\theta)}\right]'$ as

$$\left[\frac{q_1(\theta)}{q_2(\theta)}\right]' = \frac{k_1 + k_2 - 1}{q_2(\theta)} f_1(\theta) \left[\frac{f_2(\theta)}{f_1(\theta)} - \frac{q_1(\theta)}{q_2(\theta)}\right].$$

This is 0 if and only if $\left[\frac{f_2(\theta)}{f_1(\theta)} - \frac{q_1(\theta)}{q_2(\theta)}\right] = 0$. Because f_2/f_1 is decreasing and $\left[\frac{q_1(\theta)}{q_2(\theta)}\right]' = 0$, it follows that the second derivative is negative wherever $\left[\frac{q_1(\theta)}{q_2(\theta)}\right]' = 0$, establishing quasiconcavity.

If the monotone likelihood ratio property holds and $k_1 \geq k_2$, observe that $\left[\frac{q_1(\theta)}{q_2(\theta)}\right]' \geq 0$ is equivalent to $\frac{f_2(\theta)}{f_1(\theta)} \geq \frac{q_1(\theta)}{q_2(\theta)}$. Thus, for θ small, the ratio $\frac{q_1(\theta)}{q_2(\theta)}$ is increasing in θ. Because for θ sufficiently large $\frac{f_2(\theta)}{f_1(\theta)} < 1$, it follows that there is a $\theta_0 \in (\underline{\theta}, \overline{\theta})$ such that $\frac{f_2(\theta_0)}{f_1(\theta_0)} = \frac{q_1(\theta_0)}{q_2(\theta_0)}$ holds. By quasiconcavity, this is a maximum, and θ_0 is unique. Whether or not $r_1^*(\alpha)/r_2^*(\alpha)$ is monotone in α depends on whether $\theta_0 > \hat{\theta}$. That is, $r_1^*(\alpha)/r_2^*(\alpha)$ increases in α for all $\alpha \in [0, 1]$ if and only if $\theta_0 > \hat{\theta}$. ■

As Proposition OA.4 shows under certain conditions, the dispersion in $\mathbf{r}^*(\alpha)$ initially increases and then may decrease as the share of the supply that is endowed to the agents in stage one increases. In this case, the optimal endowments are most symmetric when the designer endows agents with either all or none of the supply, rather than an intermediate amount.

4. Consumer welfare As is customary in the auctions and related market design literature, our focus has been on allocative efficiency among the participating firms. That said, it is of course possible, and perhaps likely, that participating firms' profits do not capture all the social surplus that the use of emission permits generates. In this case, allocating efficiently among the participating firms is not necessarily the same as maximizing social surplus.

To address this issue, we now briefly derive conditions under which, in the presence of downstream consumers, maximizing the participating firms' surplus is equivalent to maximizing social surplus. Then we sketch how these conditions could be relaxed in future research, and what new questions this would raise.

Assume now that each firm i participating in the consignment auction serves a downstream market with a mass of k_i consumers. Let p_i be the unit price for consumers in market i, and let c_i be firm i's constant marginal cost of serving consumers in its market. Assuming that p_i is fixed, for example, due to regulatory constraints or long-term contracts, which is, for example, the case in many electricity markets, and that c_i is the realization of a random variable, the profit per consumer served is $\theta_i = p_i - c_i$. Letting v_i be the gross utility of a consumer in market i, consumer surplus per consumer served in market i is simply $v_i - p_i \equiv \gamma_i$. Thus, if $q_i \leq k_i$ consumers are served in market i, then the social surplus created in that market is $(\theta_i + \gamma_i)q_i$. It follows that maximizing social surplus across all markets is equivalent to maximizing firms' profits if $\gamma_i = \gamma$ for all $i \in \mathcal{N}$. In this case, a focus on the surplus of the regulated facilities is appropriate when the regulator has a social surplus standard.

Alternatively, as in Loertscher and Marx (2022); Loertscher and Marx (2022b), one may assume that the firms are monopolies in downstream consumer markets in which they are free to set the prices and that the input serves to increase the quality of the good downstream and thereby consumers' willingness to pay. If consumer surplus and firm profits vary in the same proportion across markets, then allocating efficiently among the firms is, again, equivalent to maximizing social surplus that accounts for consumer surplus.

Of course, the result in Proposition 7 that firms' incentives to invest are aligned with the social planner's if the market operates efficiently will not extend to the model with downstream consumers because firms do not internalize the full benefits of their investments on consumers. However, that effect is not related to whether or not there is a consignment auction.

More generally, consumer surplus may vary across markets, that is, $\gamma_i \neq \gamma_j$, and possibly with a firm's profit margin θ_i. With richer information about the downstream markets, one can accommodate generalizations like these by, for example, allowing the firms to exhibit decreasing marginal values, representing downward sloping consumer demand. Set asides and bid credits are instruments often used in auction design to promote outcomes that are deemed inadequately achieved through the course of bidding alone (see e.g. Milgrom, 2004). These instruments could, of course, also be used in the second stage of a consignment auction. An interesting question that their use would raise is what endowments would permit the second-stage reallocation mechanism to run without a deficit.

5. Proof of Proposition 6 As a preliminary step to proving Proposition 6, we first prove the following lemma:

Lemma OA.3. *For any $i \in \mathcal{N}$, $1 - \sum_{j \in \mathcal{N} \setminus \{i\}} q_j(\underline{\theta}) > k_i$.*

Proof. Using our assumption that $\sum_{i \in \mathcal{N}} k_i > 1$, for all $i \in \mathcal{N}$, we have

$$q_i(\underline{\theta}) = \max\{0, 1 - \sum_{j \in \mathcal{N} \setminus \{i\}} k_j\} < k_i. \tag{OA.9}$$

Given $i \in \mathcal{N}$, if for all $j \in \mathcal{N} \setminus \{i\}$, $q_j(\underline{\theta}) = 0$, we are done. If not, choose any $j \in \mathcal{N} \setminus \{i\}$ such that $q_j(\underline{\theta}) = 1 - \sum_{\ell \in \mathcal{N} \setminus \{j\}} k_\ell > 0$. Then,

$$\begin{aligned} 1 - \sum_{\ell \in \mathcal{N} \setminus \{i\}} q_\ell(\underline{\theta}) &= 1 - q_j(\underline{\theta}) - \sum_{\ell \in \mathcal{N} \setminus \{i,j\}} q_\ell(\underline{\theta}) \\ &= 1 - (1 - \sum_{\ell \in \mathcal{N} \setminus \{j\}} k_\ell) - \sum_{\ell \in \mathcal{N} \setminus \{i,j\}} q_\ell(\underline{\theta}) \\ &> 1 - (1 - \sum_{\ell \in \mathcal{N} \setminus \{j\}} k_\ell) - \sum_{\ell \in \mathcal{N} \setminus \{i,j\}} k_\ell \\ &= k_i, \end{aligned}$$

where the inequality uses (OA.9). □

We continue the proof by formulating the constrained optimization problem. Define agent i's interim expected payoff, not including the agent's outside option, by

$$u_i(\theta_i, \mathbf{r}) \equiv q_i(\theta_i)\theta_i - t_{i,\mathbf{r}}(\theta_i).$$

For agents with $r_i \geq k_i$, the interim expected payoff of its worst-off type is $u_i(\hat{\theta}_i, \mathbf{r}) = \hat{\theta}_i k_i$. Denote this set of agents by $\mathcal{N}_P$. For all $i \in \mathcal{N}_P$, $q_i(\hat{\theta}_i) = k_i$ and by the continuity of the density functions, we have $\hat{\theta}_i = \overline{\theta}$. Define $\mathcal{N}_P^c \equiv \mathcal{N} \backslash \mathcal{N}_P$. Let $\Delta_P(\mathbf{k}) \equiv \{\mathbf{r} : \sum_{i \in \mathcal{N}} r_i = 1, \forall i \in \mathcal{N}_p^c \ \max\{0, 1 - \sum_{j \in \mathcal{N} \backslash \{i\}} k_j\} \leq r_i \leq k_i, \forall i \in \mathcal{N}_p \ r_i \geq k_i\}$ be the set of feasible endowments. The constrained optimization problem is to

$$\max_{\mathbf{r} \in \Delta_P(\mathbf{k})} \sum_{i \in \mathcal{N}_P^c} \left[\int_{\underline{\theta}}^{\hat{\theta}_i(r_i)} q_i(x) dx - \hat{\theta}_i(r_i) r_i \right] + \sum_{i \in \mathcal{N}_P} \left[\int_{\underline{\theta}}^{\overline{\theta}} q_i(x) dx - \overline{\theta} k_i \right].$$

The associated Lagrangian is

$$\begin{aligned} L(\mathbf{r}, \mu, \boldsymbol{\lambda}^o, \boldsymbol{\lambda}^N, \boldsymbol{\lambda}^P) = & \sum_{i \in \mathcal{N}_P^c} \left[\int_{\underline{\theta}}^{\hat{\theta}_i(r_i)} q_i(x) dx - \hat{\theta}_i(r_i) r_i \right] + \sum_{i \in \mathcal{N}_P} \left[\int_{\underline{\theta}}^{\overline{\theta}} q_i(x) dx - \overline{\theta} k_i \right] \\ & + \mu(\sum_{i \in \mathcal{N}} r_i - 1) + \sum_{i \in \mathcal{N}_P^c} \lambda_i^o (r_i - \max\{0, 1 - \sum_{j \in \mathcal{N} \backslash \{i\}} k_j\}) \\ & - \sum_{i \in \mathcal{N}_P^c} \lambda_i^N (r_i - k_i) + \sum_{i \in \mathcal{N}_P} \lambda_i^P (r_i - k_i). \end{aligned}$$

An analysis similar to that of Case 3 in the proof of Proposition 1 shows that solution only exists when $\lambda_i^N = 0$ for all $i \in \mathcal{N}_P^c$. Stationarity implies that for all $i \in \mathcal{N}_P$, $\mu = -\lambda_i^P$. We rewrite the Lagrangian as

$$\begin{aligned} L(\mathbf{r}, \mu, \boldsymbol{\lambda}^o, \boldsymbol{\lambda}^N) = & \sum_{i \in \mathcal{N}_P^c} \left[\int_{\underline{\theta}}^{\hat{\theta}_i(r_i)} q_i(x) dx - \hat{\theta}_i(r_i) r_i \right] + \sum_{i \in \mathcal{N}_P} \left[\int_{\underline{\theta}}^{\overline{\theta}} q_i(x) dx - \overline{\theta} k_i \right] \\ & + \mu(\sum_{i \in \mathcal{N}} r_i - 1) + \sum_{i \in \mathcal{N}_P^c} \lambda_i^o (r_i - \max\{0, 1 - \sum_{j \in \mathcal{N} \backslash \{i\}} k_j\}) \\ & - \mu \sum_{i \in \mathcal{N}_P} (r_i - k_i). \end{aligned}$$

The Karush–Kuhn–Tucker conditions are as follows:

(a) (stationarity) for all $i \in \mathcal{N}$, $\frac{\partial L}{\partial r_i} = 0$, i.e., for all $i \in \mathcal{N}_P^c$,

$$\hat{\theta}_i(r_i) = \mu + \lambda_i^o;$$

(b) (complementary slackness) for all $i \in \mathcal{N}_P^c$,

$$\lambda_i^o(r_i - \max\{0, 1 - \sum_{j \in \mathcal{N} \setminus \{i\}} k_j\}) = 0$$

and for all $i \in \mathcal{N}_P$, $\mu(r_i - k_i) = 0$;

(c) (primal feasibility) $\mathbf{r} \in \Delta_P(\mathbf{k})$;

(d) (dual feasibility) for all $i \in \mathcal{N}_P^c$, $\lambda_i^o \geq 0$, and for all $i \in \mathcal{N}_P$, $\mu = -\lambda_i^P \leq 0$.

To complete the proof, we consider two exhaustive cases depending on the sign of λ_i^o.

Case 1. For all i, $\lambda_i^o = 0$. The definition of the worst-off types $\hat{\theta}_i$, stationarity and primal feasibility require the existence of $\mu \in [\underline{\theta}, \overline{\theta}]$ and $(r_i)_{i \in \mathcal{N}_P}$ such that

$$\sum_{i \in \mathcal{N}_P^c} q_i(\mu) + \sum_{i \in \mathcal{N}_P} r_i = 1.$$

If $\mathcal{N}_P = \emptyset$, then the solution to this case is the interior solution $\mathbf{r}^*$. If $\mathcal{N}_P \neq \emptyset$, then complementary slackness implies that $\mu = 0$. The problem is feasible if $\underline{\theta} = 0$ and for all $i \in \mathcal{N}_P^c$, $r_i = q_i(0)$. If $\underline{\theta} > 0$, then we can expand the type space by making $q_i(x) = 0$ for all $i \in \mathcal{N}$ and $x \in [0, \underline{\theta})$. The problem becomes to

$$\max_{\mathcal{N}_P \in \mathcal{P}(\mathcal{N})} \sum_{i \in \mathcal{N}_P} \left[\int_0^{\overline{\theta}} q_i(x)dx - \overline{\theta} k_i \right].$$

Because $\int_{\underline{\theta}}^{\overline{\theta}} q_i(x)dx - \overline{\theta} k_i < 0$, the solution is to choose $\mathcal{N}_P = \{i^*\}$ where

$$i^* \in \arg\min_{i \in \mathcal{N}} \int_{\underline{\theta}}^{\overline{\theta}} (k_i - q_i(x))dx.$$

Let $\mathbf{r}^o$ be such that for all $j \neq i^*$, $r_j^o \in [0, q_j(\underline{\theta})]$ and $r_{i^*}^o = 1 - \sum_{j \in \mathcal{N} \setminus \{i^*\}} r_j^o$. $\mathbf{r}^o$ is feasible since $r_{i^*} \geq 1 - \sum_{j \in \mathcal{N} \setminus \{i^*\}} q_j(\underline{\theta}) > k_{i^*}$ by Lemma OA.3. $\mathbf{r}^o$ is optimal since $\mathcal{N}_P = \{i^*\}$.

Case 2. For some $i \in \mathcal{N}$, $\lambda_i^o > 0$. Let $\mathcal{N}_o \equiv \{i \in \mathcal{N} \mid \lambda_i^o > 0\}$ and let $\mathcal{N}_o^c \equiv \mathcal{N}_P^c \setminus \mathcal{N}_o$. Then stationarity implies that for all $i \in \mathcal{N}_o$, $\hat{\theta}_i(r_i) = \mu + \lambda_i^o$, and for all $i \in \mathcal{N}_o^c$, $\hat{\theta}_i(r_i) = \mu$. By the result demonstrated in Online Appendix Section 1.1 that $q_i(\hat{\theta}_i) = \min\{r_i, k_i\}$ and complementary slackness, for all $i \in \mathcal{N}_o$,

$$q_i(\mu + \lambda_i^o) = \max\{0, 1 - \sum_{j \in \mathcal{N} \setminus \{i\}} k_j\},$$

for all $i \in \mathcal{N}_o^c$, $q_i(\mu) = r_i$. Primal feasibility implies

$$\sum_{i \in \mathcal{N}_o} q_i(\mu + \lambda_i^o) + \sum_{i \in \mathcal{N}_o^c} q_i(\mu) + \sum_{i \in \mathcal{N}_P} r_i = 1$$

If $\mathcal{N}_P = \emptyset$, then the solution to this case is the interior solution $\mathbf{r}^*$ by the same argument in Case 2 in the proof of Proposition 1. If $\mathcal{N}_P \neq \emptyset$, then complementary slackness implies that $\mu = 0$, and the solution has the same form as in Case 1 of this proof. ■

References for the Online Appendix

ARROW, K. (1979): "The Property Rights Doctrine and Demand Revelation under Incomplete Information," in *Economics and Human Welfare*, ed. by M. Boskin, New York: Academic Press, 23–39.

BÖRGERS, T. (2015): *An Introduction to the Theory of Mechanism Design*, Oxford University Press.

BÖRGERS, T. AND P. NORMAN (2009): "A Note on Budget Balance Under Interim Participation Constraints: The Case of Independent Types," *Economic Theory*, 39, 477–489.

D'ASPREMONT, C., J. CRÉMER, AND L.-A. GÉRARD-VARET (2004): "Balanced Bayesian Mechanisms," *Journal of Economic Theory*, 115, 385–396.

D'ASPREMONT, C. AND L.-A. GÉRARD-VARET (1979a): "Incentives and Incomplete Information," *Journal of Public Economics*, 11, 25–45.

——— (1979b): "On Bayesian Incentive Compatible Mechanisms," in *Aggregation and Revelation of Preferences*, ed. by J.-J. Laffont, North-Holland, 269–288.

KRISHNA, V. (2010): *Auction Theory*, Elsevier, second ed.

KRISHNA, V. AND M. PERRY (2000): "Efficient Mechanism Design," Working Paper, Penn State University.

LOERTSCHER, S. AND L. M. MARX (2022a): "Incomplete Information Bargaining with Applications to Mergers, Investment, and Vertical Integration," *American Economic Review*, 112, 616–649.

——— (2022b): "Vertical Integration with Incomplete Information," University of Melbourne.

MILGROM, P. (2004): *Putting Auction Theory to Work*, Cambridge University Press.

MYERSON, R. B. (1981): "Optimal Auction Design," *Math. of Op. Research*, 6, 58–73.