Online appendix to accompany "Asymptotically optimal prior-free clock auctions" by Loertscher and Marx

In this appendix, we describe two alternative clock auction formats and illustrate the rate of convergence of the optimal prior-free clock auction. In Section 1, we describe discriminatory clock auctions in a setting in which the designer can a priori place subsets of buyers and subsets of sellers into groups of symmetric agents while allowing for asymmetries across different groups. In Section 2, we describe quasi-clock auctions that implement the Bayesian optimal mechanism in a two-sided setup without violating privacy preservation for any trading agents other than the marginal pair. In Section 3, we discuss and illustrate rates of convergence.

1 Discriminatory clock auctions

Generalized Bayesian mechanism design setting

We now allow, without requiring, the possibility that agents have characteristics that are observable to the designer, so that the designer can a priori place subsets of agents into groups of symmetric agents while allowing for asymmetries across different groups. For example, traders of carbon emission permits might be identifiable as either power plants, cement manufacturers, or other manufacturers, with traders within a group being symmetric, but with the possibility of asymmetries across groups.

Let $\mathcal{N}$ and $\mathcal{M}$ denote the sets of buyers and sellers with cardinalities n and m. Let $\mathcal{Z}^B$ and $\mathcal{Z}^S$ be the sets of groups for buyers and sellers when private information pertains to both sides of the market. (When only one side is privately informed, there is, of course, no point distinguishing between groups on the side of the market without private information.) We refer to the setup in which $|\mathcal{Z}^B| = |\mathcal{Z}^S| = 1$ studied thus far as the *symmetric setup*. Let $n^b \geq 1$ be the number of buyers in buyer group b and let $m^s \geq 1$ be the number of sellers in group s, where $n = \sum_{b \in \mathcal{Z}^B} n^b$ and $m = \sum_{s \in \mathcal{Z}^S} m^s$. We assume that at least one buyer group and at least one seller group has 2 or more members. The group membership of each buyer and seller is common knowledge.

Each buyer in group b draws its value independently from the continuously differentiable distribution F^b with support $[\underline{v}^b, \overline{v}^b]$ and positive density f^b, and each seller in group s draws its cost independently from the continuously differentiable distribution G^s with support $[\underline{c}^s, \overline{c}^s]$ and positive density g^s. Each agent is privately informed about its type, but the types and distributions from which they are drawn are unknown to the mechanism designer and the agents. The designer only knows the group identity of each buyer and seller, that agents in the same group draw their types from the same distribution, that group-specific weighted virtual types given by

$$\Phi^b_\alpha(v) \equiv v - \alpha \frac{1 - F^b(v)}{f^b(v)} \quad \text{and} \quad \Gamma^s_\alpha(c) \equiv c + \alpha \frac{G^s(c)}{g^s(c)}$$

are increasing for each buyer and seller group and that the supports satisfy the no-full trade condition

$$\min_{s \in \mathcal{Z}^S} \{\overline{c}^s\} \geq \max_{b \in \mathcal{Z}^B} \overline{v}^b > \min_{s \in \mathcal{Z}^S} \{\underline{c}^s\} \geq \max_{b \in \mathcal{Z}^B} \{\underline{v}^b\},$$

which generalizes the no-full trade condition (1), that is, the assumption that $\overline{c} \geq \overline{v} > \underline{c} \geq \underline{v}$, to the setup with heterogeneous groups.

Under the stipulated assumptions, the allocation rule for the Bayesian optimal mechanism in the setup with two-sided private information can be described as follows: For a given realization of values and costs $(\mathbf{v}, \mathbf{c})$, rank all weighted virtual values in decreasing and all weighted virtual costs in increasing order, irrespective of group membership, and then have all those buyers and sellers trade who would trade in a Walrasian market if weighted virtual values and costs were true values and costs. That is, letting, for all $i \in \mathcal{N}$ in buyer group b and all $j \in \mathcal{M}$ in seller group s, $V_i \equiv \Phi^b_\alpha(v_i)$ and $C_j \equiv \Gamma^s_\alpha(c_j)$ and

$$\mathbf{V} \equiv (V_1, ..., V_n) \quad \text{and} \quad \mathbf{C} \equiv (C_1, ..., C_m),$$

the optimal quantity traded is given by the largest integer k satisfying $V_{(k)} \geq C_{[k]}$, where we use the usual conventions of setting $V_{(0)} = \infty = C_{[m+1]}$ and $C_{[0]} = -\infty = V_{(n+1)}$. In the dominant strategy implementation, trading buyers in group b pay p^b_k and trading sellers in group s receive p^s_k, where

$$p^b_k = \Phi^{b^{-1}}_\alpha \left(\max\{V_{(k+1)}, C_{[k]}\} \right) \quad \text{and} \quad p^s_k = \Gamma^{s^{-1}}_\alpha \left(\min\{C_{[k+1]}, V_{(k)}\} \right).$$

When private information pertains only to buyers, the optimal quantity is the largest index k such that $V_{(k)} \geq c_{[k]}$. In that case, in the dominant-strategy implementation, trading buyers in group b pay $p^b_k = \Phi^{b^{-1}}_\alpha \left(\max\{V_{(k+1)}, c_{[k]}\} \right)$. Analogously, when private information pertains only to sellers, the optimal quantity is the largest index k such that $v_{(k)} \geq C_{[k]}$, and in the dominant-strategy implementation, trading sellers in group s receive $p^s_k = \Gamma^{s^{-1}}_\alpha \left(\min\{C_{[k+1]}, v_{(k)}\} \right)$.

Discriminatory clock auction

In the generalization of the clock auction to the setup with heterogeneous groups of buyers and sellers, there are separate, but synchronized, clock prices for each buyer group and each seller group. Although buyers in different groups may pay different prices and sellers in different groups may receive different prices, the mechanism remains envy free within groups.

A *discriminatory clock auction* defines state transitions for state space $\hat{\Omega}$, defined below, based on buyer and seller functions $\hat{\phi} : \hat{\Omega} \to \mathbb{R}$ and $\hat{\gamma} : \hat{\Omega} \to \mathbb{R}$ and target function $\hat{\tau} : \hat{\Omega} \to \mathbb{R}$. Thus, we denote a discriminatory clock auction by $\widehat{\mathcal{C}}_{\hat{\phi},\hat{\gamma},\hat{\tau}}$. At $t \in \{0, 1, ...\}$, the state is $\hat{\boldsymbol{\omega}}_t = (z_t, \hat{\boldsymbol{\omega}}_t^B, \hat{\boldsymbol{\omega}}_t^S)$, where $z_t \in \{0, 1\}$ specifies whether the clock auction has ended ($z_t = 1$) or not ($z_t = 0$), $\hat{\boldsymbol{\omega}}_t^B = \times_{b \in \mathcal{Z}^B} \hat{\boldsymbol{\omega}}_t^b$, and $\hat{\boldsymbol{\omega}}_t^S = \times_{s \in \mathcal{Z}^S} \hat{\boldsymbol{\omega}}_t^s$, where $\hat{\boldsymbol{\omega}}_t^b = (\mathcal{N}^{A^b}, \mathbf{x}^b, p^b)$ and $\hat{\boldsymbol{\omega}}_t^s = (\mathcal{M}^{A^s}, \mathbf{x}^s, p^s)$ are group-specific buyer and seller states with components analogous to the symmetric case. We require that $\hat{\phi}$ is increasing in each p^b, that $\hat{\gamma}$ is increasing in each p^s, and that, as in the symmetric case, $\hat{\tau}(\hat{\boldsymbol{\omega}}_t) \in [\hat{\phi}(\hat{\boldsymbol{\omega}}_t), \hat{\gamma}(\hat{\boldsymbol{\omega}}_t)]$ whenever $\hat{\phi}(\hat{\boldsymbol{\omega}}_t) \le \hat{\gamma}(\hat{\boldsymbol{\omega}}_t)$. The state is initialized as in the symmetric case.

For $t \in \{0, 1, ...\}$, if $z_t = 0$, then $\hat{\boldsymbol{\omega}}_{t+1}$ is determined as follows:

If $\sum_{b \in \mathcal{Z}^B} n^{A^b} = \sum_{s \in \mathcal{Z}^S} m^{A^s}$: If $\sum_{b \in \mathcal{Z}^B} n^{A^b} = 0$ or $\hat{\phi}(\hat{\boldsymbol{\omega}}_t) \ge \hat{\gamma}(\hat{\boldsymbol{\omega}}_t)$, then $\hat{\boldsymbol{\omega}}_{t+1} = (1, \hat{\boldsymbol{\omega}}_t^B, \hat{\boldsymbol{\omega}}_t^S)$. Otherwise, proceed as follows (the order in which clock prices on either side of the market are moved is again immaterial): Increase the vector of buyer clock prices from $\boldsymbol{p}^B = (p^b)_{b \in \mathcal{Z}^B}$ by increasing the clock prices for the smallest number of buyer groups possible so as to increase $\hat{\phi}(\hat{\boldsymbol{\omega}}_t)$,[42] until either there is an exit by a group $\hat{b}$ buyer i at clock price vector $\hat{\mathbf{p}}^B$, in which case $\hat{\boldsymbol{\omega}}_{t+1}^{\hat{b}} = (\mathcal{N}^{A^{\hat{b}}} \backslash \{i\}, (\mathbf{x}^b, \hat{p}^{\hat{b}}), \hat{p}^{\hat{b}})$ and for $b \ne \hat{b}$, $\hat{\boldsymbol{\omega}}_{t+1}^b = (\mathcal{N}^{A^b}, \mathbf{x}^b, \hat{p}^b)$, or the buyer clock prices reach with no exit $\tilde{\mathbf{p}}^B$ such that $\hat{\phi}$ is equal to $\hat{\tau}(\hat{\boldsymbol{\omega}}_t)$ when it is evaluated at the state $\hat{\boldsymbol{\omega}}_t$ with $\boldsymbol{p}^B$ replaced by $\tilde{\mathbf{p}}^B$, in which case for all b, $\hat{\boldsymbol{\omega}}_{t+1}^b = (\mathcal{N}^{A^b}, \mathbf{x}^b, \tilde{p}^b)$. In analogous fashion, decrease the vector of seller clock prices and update the seller state. If both $\hat{\phi}$ and $\hat{\gamma}$, evaluated at the adjusted clock prices, reach the target $\hat{\tau}(\hat{\boldsymbol{\omega}}_t)$ with no exit, then $z_{t+1} = 1$; otherwise $z_{t+1} = 0$.

If $\sum_{b \in \mathcal{Z}^B} n^{A^b} > \sum_{s \in \mathcal{Z}^S} m^{A^s}$, increase only the buyer clock prices as above until there is an exit or all buyer clock prices reach $\overline{p} > \max_{b \in \mathcal{Z}^B} \overline{v}^b$, and similarly for sellers if $\sum_{b \in \mathcal{Z}^B} n^{A^b} < \sum_{s \in \mathcal{Z}^S} m^{A^s}$, with lower bound $\underline{p} < \min_{s \in \mathcal{Z}^S} \underline{c}^s$. The states transition analogously to the symmetric case.

When the auction ends, active buyers pay and active sellers receive their groups' clock prices.

Asymptotically optimal prior-free discriminatory clock auction

Focusing on the case of two-sided private information, we show how our asymptotically optimal prior-free clock auction can be generalized to account for differentiated groups of

[42] That is, $\hat{\phi}(\hat{\boldsymbol{\omega}}_t)$ depends, in general, on the B clock prices $(p^b)_{b \in \mathcal{Z}^B}$, but it may be invariant to some of those prices, for example if all members of a group have already exited. In addition, $\hat{\phi}(\hat{\boldsymbol{\omega}}_t)$ may be invariant to each individual price when others are held constant, for example if it is equal to the minimum of virtual type estimates across groups and two or more groups are tied for having the minimum estimated virtual type. We require that the vector of group-specific clock prices be increased in such a way that $\hat{\phi}(\hat{\boldsymbol{\omega}}_t)$ increases *and* such that for any group-specific clock price that is increased, $\hat{\phi}(\hat{\boldsymbol{\omega}}_t)$ would not increase if that clock price were not changed.

buyers and sellers. The generalization for one-sided private information follows along similar lines.

We define the prior-free discriminatory clock auction $\widehat{\mathcal{C}}_{\hat{\phi},\hat{\gamma},\hat{\tau}}$ that corresponds to the prior-free optimal clock auction $\mathcal{C}_{\phi,\gamma,\tau}$ defined by (5)–(9). For each buyer group b and seller group s, define σ_j^b and σ_j^s to be group-specific spacing estimates, analogous to the symmetric setup. If $\hat{\boldsymbol{\omega}}_t$ indicates, for each buyer group b, a clock price p^b and number of active bidders n^{A^b}, and if $\hat{\boldsymbol{\omega}}_t$ indicates, for each seller group s, a clock price p^s and number of active bidders m^{A^s}, then let

$$\hat{\phi}(\hat{\boldsymbol{\omega}}_t) = \min_{b\in\mathcal{Z}^B \text{ s.t. } n^{A^b}\geq 1} p^b - \chi_{\alpha,n^{A^b}+1}\sigma^b_{n^{A^b}+1} \text{ and } \hat{\gamma}(\hat{\boldsymbol{\omega}}_t) = \max_{s\in\mathcal{Z}^S \text{ s.t. } m^{A^s}\geq 1} p^s + \chi_{\alpha,m^{A^s}+1}\sigma^s_{m^{A^s}+1}.$$

The target function corresponding to (9) is $\hat{\tau}(\hat{\boldsymbol{\omega}}_t) = \min\left\{\hat{\gamma}(\hat{\boldsymbol{\omega}}_t), \max\left\{\hat{\phi}(\hat{\boldsymbol{\omega}}_t), \tilde{\delta}\right\}\right\}$, where $\tilde{\delta}$ satisfies

$$\sum\nolimits_{b\in\tilde{\mathcal{Z}}^B} \frac{1}{\tilde{\delta}-\left(p^b-\chi_{\alpha,n^{A^b}+1}\sigma^b_{n^{A^b}+1}\right)-(2-\alpha)\sigma^b_{n^{A^b}+1}} = \sum\nolimits_{s\in\tilde{\mathcal{Z}}^S} \frac{1}{\left(p^s+\chi_{\alpha,m^{A^s}+1}\sigma^s_{m^{A^s}+1}\right)-\tilde{\delta}-(2-\alpha)\sigma^s_{m^{A^s}+1}},$$

where $\tilde{\mathcal{Z}}^B$ is the set of buyer groups with $n^{A^b} \geq 1$ and $p^b - \chi_{\alpha,n^{A^b}+1}\sigma^b_{n^{A^b}+1} < \tilde{\delta}$ (so that the target function is only defined with respect to buyer groups that still have at least one active buyer and whose estimated virtual types are currently below the target) and $\tilde{\mathcal{Z}}^S$ is the set of seller groups satisfying $m^{A^s} \geq 1$ and $p^s + \chi_{\alpha,m^{A^s}+1}\sigma^s_{m^{A^s}+1} > \tilde{\delta}$.

One can show that the results on asymptotic optimality extend to heterogeneous groups. This combined with arguments analogous to the case of symmetric buyers and symmetric sellers establishes the following result.

Proposition OA.1 *In the setup with heterogeneous groups of buyers and sellers, there exists a prior-free discriminatory clock auction that is asymptotically optimal and sequentially consistent.*

2 Quasi-clock auctions

An implication of Proposition 1 in the body of the paper is that in two-sided settings the Bayesian optimal mechanism does not permit a clock implementation. The reason for this is that the thresholds for trading on either side of the market—for example, in the symmetric setting $\max\{v_{(k+1)}, \Phi_\alpha^{-1}(\Gamma_\alpha(c_{[k]}))\}$ for buyers and $\min\{c_{[k+1]}, \Gamma_\alpha^{-1}(\Phi_\alpha(v_{(k)}))\}$ for sellers—depend on information that is provided by an agent who optimally trades. A tradeoff thus arises in two-sided settings between the desirable properties of clock auctions and the benefits of Bayesian optimality.

We now briefly discuss how one could augment a clock auction and implement the Bayesian optimal mechanism without violating privacy preservation for any trading agents other than the marginal pair, that is, other than the buyer with value $v_{(k)}$ and the seller with cost $c_{[k]}$ when the optimal quantity traded is k. We refer to the augmented clock auction as a *quasi-clock auction*. To save space, we restrict the discussion to the setting with symmetric buyers and symmetric sellers. The generalization to heterogeneous groups of buyers and sellers is a straightforward extension. Just like the clock auction, a quasi-clock auction consists of two clocks. It proceeds similarly to the clock auction.

Assuming each agent stays active until the clock price equals its type, when the number of active agents on each side of the market is $k-1$ and the buyer and seller clocks stop at the prices $p^B = v_{(k)}$ and $p^S = c_{[k]}$, the buyers with values in the vector $\mathbf{v}_{(k+1)}$ and the sellers with costs in the vector $\mathbf{c}_{[k+1]}$ become inactive as in the clock auction. However, in contrast to the clock auction, the buyer and seller with types $v_{(k)}$ and $c_{[k]}$, who have just exited, may still trade. In particular, they still trade if $\Phi_\alpha(p^B) \geq \Gamma_\alpha(p^S)$, in which case the trading buyers pay $\Phi_\alpha^{-1}(\Gamma_\alpha(p^S))$ and the trading sellers receive $\Gamma_\alpha^{-1}(\Phi_\alpha(p^B))$, rather than the clock prices. If $\Phi_\alpha(p^B) < \Gamma_\alpha(p^S)$, then the quasi-clock auction proceeds until the earlier of the two events: the target prices are reached or an additional agent exits.

In the quasi-clock auction, all agents are price-takers at all times. Consequently, just like the clock auction, the quasi-clock auction endows agents with dominant strategies. It also preserves the privacy of all but at most one trading agent on each side of the market. However, by Li (2017, Theorem 3), it sacrifices the obviousness of the dominant strategies.

Because virtual types can be estimated analogously to the case of clock auctions, prior-free quasi-clock auctions can also be constructed that are asymptotically optimal, and sequentially consistent, and minimize mean square error among nearest neighbor estimators.

The alternative of a quasi-clock auction, which preserves the privacy of almost all trading agents, raises the question as to why one is concerned about privacy preservation. If privacy preservation is desired primarily to protect traders from hold-up by the designer as discussed, e.g., by Lucking-Reiley (2000), then quasi-clock auctions arguably do as good a job as clock auctions, provided buyers and sellers observe the clock prices on the other side of the market, whence they can infer the prices they face. Also, although to a slightly lesser extent than clock auctions, quasi-clock auctions protect the designer from criticism of "money left on the table" because only the marginal traders' values and costs are revealed. Because the difference between the revealed types and prices will typically be "small," quasi-clock auctions will also not perform much worse than clock auctions if the motivation for privacy preservation is post-auction hold-up, e.g., in the form of taxation. Therefore, if quasi-clock auctions do not appear appealing for practical purposes, this may have less to do with their limited ability to preserve privacy than with their failure to satisfy other desiderata such as the obviousness of the dominant strategies and the weak group strategy-proofness this implies.[43]

[43]Satterthwaite and Williams (2002) find that for uniform distributions, the efficiency loss of any incentive compatible, ex ante budget balanced mechanism is of the same order as the gain from trade of the marginal pair, suggesting that departing from clock auctions to always execute this marginal trade may not be worth

3 Rates of convergence

In this section, we provide simulation results illustrating the performance of our prior-free clock auction in the small. As described in Section 3.4 of the paper, an $r(n)$-nearest neighbor estimator has mean square error of order $(r(n)/n)^4 + 1/r(n)$, which is minimized when $r(n)$ is proportional to $n^{4/5}$ (Silverman, 1986, Chapters 3 and 5.2.2), in which case the approximate value of the mean integrated square error tends to zero at the rate $n^{-4/5}$ (Silverman, 1986, Chapter 3.7.2).

To illustrate performance in the small, we focus on an asymptotically optimal clock auction that is sequentially consistent and uses a virtual type estimator that achieves the minimum mean square error among nearest neighbor estimators for a designer placing weight $\alpha \in [0, 1]$ on revenue. Specifically, we analyze the clock auction defined as follows: For any state $\boldsymbol{\omega}$ with buyer clock price p^B, seller clock price p^S, and an equal number $j - 1$ of active buyers and sellers, the buyer and seller functions are

$$\phi(\boldsymbol{\omega}) = p^B - \chi_{\alpha,j}\sigma_j^v \text{ and } \gamma(\boldsymbol{\omega}) = p^S + \chi_{\alpha,j}\sigma_j^c, \tag{23}$$

where

$$\chi_{\alpha,j} \equiv \max\{0,\ \alpha(j-2) - (1-\alpha)\}. \tag{24}$$

and

$$\sigma_j^v \equiv \begin{cases} \frac{\hat{v}_{(j)} - \hat{v}_{(j+\min\{r(n),n-j\})}}{\min\{r(n),n-j\}}, & \text{if } j < n \\ \frac{1}{n+1}, & \text{otherwise} \end{cases} \text{ and } \sigma_j^c \equiv \begin{cases} \frac{\hat{c}_{[j+\min\{r(m),m-j\}]} - \hat{c}_{[j]}}{\min\{r(m),m-j\}}, & \text{if } j < m \\ \frac{1}{m+1}, & \text{otherwise,} \end{cases} \tag{25}$$

where $r(j) = j^{4/5}$.

To define the target function, we initialize the target at $\underline{c}$, i.e., until the state $\boldsymbol{\omega}$ reflects at least one exit on each side of the market, let $\tau(\boldsymbol{\omega}) = \underline{c}$.[44] Once there is at least one exit on each side of the market, if the state shows an equal number $j - 1$ of active buyers and sellers, then we define the target function as

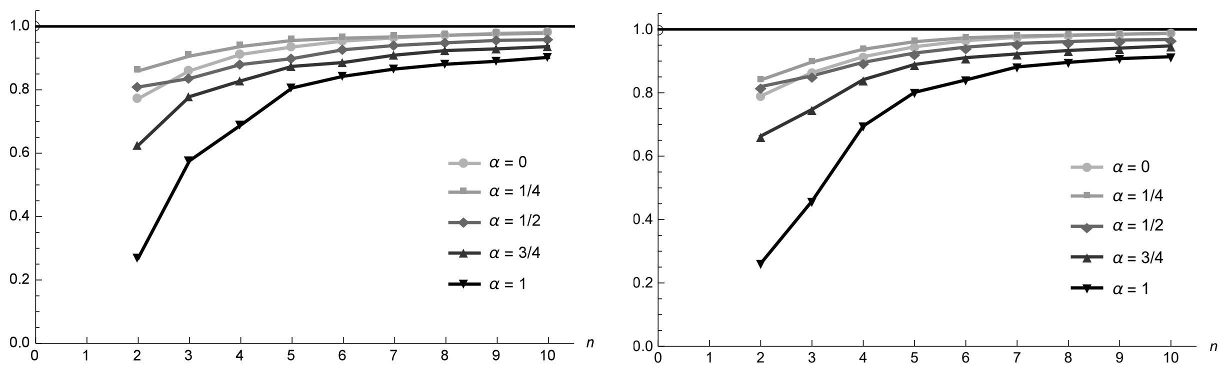
$$\tau(\boldsymbol{\omega}) = \min\left\{\gamma(\boldsymbol{\omega}),\ \max\left\{\phi(\boldsymbol{\omega}),\ \frac{\phi(\boldsymbol{\omega}) + \gamma(\boldsymbol{\omega})}{2} + \left(1 - \frac{\alpha}{2}\right)\left(\sigma_j^v - \sigma_j^c\right)\right\}\right\}. \tag{26}$$

As shown in Figure 5(a), this prior-free clock auction achieves over 80% of optimal expected revenue with only six buyer-seller pairs, and it achieves over 75% of optimal social surplus even when there are as few as 2 buyers and 2 sellers. To illustrate results for small

the cost.

[44]This initialization handicaps the mechanism because it means that there is no possibility that all buyers and sellers could trade. For the comparisons provided here, it seems appropriate to eliminate the possibility that an initial target, necessarily uninformed by any data from the mechanism, happens to permit full trade.

markets with α away from the extremes, Figure 5(a) also shows the performance of our prior-free clock auction relative to the optimal mechanism for various intermediate values of α. Generally speaking, the smaller is α, the smaller is the impact of estimation error, and so the better is the performance of the prior-free clock auction. However, the mechanism's use of the first buyer and seller exits for estimation is a greater disadvantage relative to the optimal mechanism, given α, when n and α are small. Thus, for small numbers of agents, the relative performance of the prior-free clock auction can be better for larger values of α. This is the case in Figure 5(a), where the line for $\alpha = 0$ dips below the lines for $\alpha = 1/4$ and $\alpha = 1/2$ when n is small. Figure 5(b) shows similar results for values and costs drawn from the Lognormal distribution.



(a) Prior-free performance relative to the α-optimal mechanism: Uniform distribution

(b) Prior-free performance relative to the α-optimal mechanism: Lognormal distribution

Figure 5: Ratio of the expected weighted objective in the prior-free clock auction $\mathcal{C}_{\phi,\gamma,\tau}$, with ϕ, γ, and τ given by (23)–(26) with $r(j) = j^{4/5}$, relative to the expected weighted objective in the α-optimal mechanism for various weights α on revenue. Panel (a): values and costs drawn from the Uniform distribution on $[0, 1]$. Panel (b): values and costs drawn from the Lognormal distribution derived from the Normal distribution with mean zero and standard deviation 1. Panels (a) and (b) are both based on Monte Carlo simulation (5000 auctions).

References for online appendix

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