Make or buy or sell*

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Abstract

We study an incomplete information model in which ex post efficiency is impossible without firms that sometimes make, buy, or sell, characterize the set of efficiency-permitting ownership structures, and show that horizontal (vertical) mergers never make efficiency possible (impossible). Conglomerate mergers, which are neither horizontal nor vertical, can have either effect. The analysis provides rationale and guidance for divestitures and shows that, with one-dimensional types and identical supports, equal bargaining weights are necessary for efficiency, and that pure asset transfers are typically profitable bilaterally even when reducing social surplus, rationalizing a role for antitrust vigilance and a focus on bilateral transactions.

Keywords: horizontal, vertical and conglomerate mergers, raising rivals' costs, bargaining power, Triple-IO

JEL Classification: D44, D82, L41

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1 Introduction

The effects of and incentives for integration—horizontal, vertical, or otherwise—are of long-standing interest in economics, taking center stage in organizational economics, industrial organization, and antitrust. While non-horizontal integration has traditionally been viewed favorably by competition authorities, a combination of recent merger cases, controversies related to big tech, and empirical and theoretical advances have brought forth more skeptical perspectives regarding their effects on social and consumer surplus.[1] Recent empirical research documents anticompetitive effects of vertical integration in the soda industry and procompetitive effects in the movie industry.[2] Understanding and disentangling the various effects requires a conceptual framework that is rich enough to capture these effects, yet remains simple and tractable.

In this paper, we provide an incomplete information model that allows us to distinguish between horizontal and vertical mergers and mergers that are neither, which are sometimes referred to as *conglomerate* mergers. We show that horizontal mergers are socially undesirable and vertical mergers are socially desirable insofar as the former never make the first-best (also known as ex post efficiency) possible, and the latter never make it impossible. In contrast, conglomerate mergers cannot be classified in this way and require, in general, further scrutiny. Conglomerate mergers that *corner* the market by putting all the assets into the hands of a single firm never permit ex post efficiency after the merger while conglomerate mergers that have a sufficiently strong horizontal (vertical) component do not make ex post efficiency possible (impossible). The analysis also delivers a rationale and guidance for divestitures—that is, requirements imposed on a merged entity to sell off some of its assets to a third party—to eliminate harm resulting from a merger.[3]

Specifically, we study a setup with independent private values that are drawn from continuous, commonly known distributions with positive densities on an identical support. Each

[1] *Vertical Merger Guidelines* were released by the U.S. DOJ and FTC in 2020 to replace the 1984 *Non-Horizontal Merger Guidelines*; however, in September of 2021, the FTC rescinded its support for those guidelines (FTC 2021). For analyses specifically related to big tech, see, for example, Baker et al. (2019); Kang and Muir (2022); and various policy reports (ACCC, 2019; Crémer et al., 2019; Furman et al., 2019; and Stigler Center, 2019). In models with complete information, the possibility of anticompetitive vertical integration arises from a "raising rivals' costs" theory of vertical foreclosure (Salop and Scheffman, 1983; Salinger, 1988; Hart and Tirole, 1990; Ordover et al., 1990; Riordan, 1998). Luco and Marshall (2020), Loertscher and Marx (2022), Chen and Rey (2023), and Choné et al. (forth.) provide recent empirical and theoretical analyses of non-horizontal integration in which the elimination of a double markup, which is a classic vertical merger defense based on complete information models with contracts restricted to linear prices (Cournot, 1838; Spengler, 1950), does not imply procompetitive effects.

[2] See Luco and Marshall (2020) and Chen et al. (2023), respectively.

[3] In the Online Appendix, we illustrate based on the Republic-Santek transaction how the divestiture policies that we discuss can be implemented in practice using market data that is typically available in a merger review process.

firm is endowed with a nonnegative asset ownership share that is no more than its maximum demand, and the total asset endowment is less than the total demand at a price of zero so that there is scarcity. Prior to the realization of private information, there can be transfers of asset ownership between firms or even the full integration of pairs of firms. Although the model is static, asset ownership can be thought of as a long-term contract over an essential input such as trucks, emission permits, or long-term labor contracts, and private information as pertaining to the day-to-day business, with the asset allocation that follows the realization of private information being interpreted as the issuance of short-term leases.[4] After the firms' private values are realized, the market allocates the essential input among firms based on incomplete information bargaining.

We distinguish two sorts of ex ante transactions. With pure *asset ownership transfers*, one firm's ownership is reduced by some amount and another firm's ownership is increased by that amount. From a modeling perspective, the important implication of pure asset ownership transfers is that the set of firms remains fixed and no commercially relevant information is affected by the transfer. Assuming that, at the outset, each firm has a one-dimensional type and constant marginal value up to its maximum demand, this means that each firm's private information remains one-dimensional after the transfer. This permits us to perform a wide range of informative analyses involving second-best mechanisms. Pure asset ownership transfers play an important role, for example, as part of divestiture-based structural remedies. In contrast, upon *full integration* between two firms, the number of firms is reduced by one and the integrated firm not only combines the asset ownership of the two standalone firms, but also has two-dimensional private information because it draws two types, and has decreasing marginal value for the asset.

Firms whose asset ownership is equal to their maximum demands are *sellers* because if they trade after the realization of private information, then only a sale can lead to gains from trade. Analogously, firms with zero ownership are *buyers*. In contrast, firms with positive ownership that is less than their maximum demand will, under ex post efficiency, sometimes buy, sometimes sell, and sometimes not trade at all. We refer to these firms as *traders*. If such a firm does not buy or sell, it consumes its endowment, which can be interpreted as in-house production, that is, as *make*. Full integration by two buyers or two sellers constitutes a *horizontal* merger, while full integration by a buyer and a seller is a *vertical* merger. Vertical mergers thus create traders. Mergers involving a trader are neither horizontal nor vertical and therefore categorized as *conglomerate* mergers. Given the realization of private information, the market is modeled as an incentive compatible, interim individually rational mechanism. We say that ex post efficiency is *possible* for a given asset ownership structure

[4]See Appendix B.1 for a formalization of a multi-period model.

if there exists such a mechanism that always allocates efficiently and does not run a deficit in expectation. Otherwise, ex post efficiency is said to be *impossible*.

We show that ex post efficiency is always possible for a nonempty convex set of ownership structures. This is a generalization of the possibility result of Cramton et al. (1987) to heterogeneous type distributions and maximum demands and, after full integration, to multidimensional types. Horizontal mergers eliminate an otherwise competing bid and therefore can never make ex post efficiency possible and sometimes make it impossible. Conversely, vertical mergers never eliminate competing bids and get rid of an agency problem within the firm.[5] Therefore, vertical mergers never make ex post efficiency impossible and sometimes make it possible. By construction, conglomerate mergers combine horizontal and vertical elements and their effects can, therefore, go either way. By continuity, conglomerate mergers that are almost horizontal are socially undesirable and conglomerate mergers that are sufficiently close to vertical are socially desirable. Any merger that creates a firm that corners the market by putting the entire asset ownership into its hands precludes ex post efficiency by a generalization of the impossibility theorem of Myerson and Satterthwaite (1983). For the case with one-dimensional types, we show that equalization of bargaining power, championed by Galbraith (1952, 1954) and criticized by Stigler (1954), is a necessary condition for ex post efficiency; that there is a role for sustained antitrust vigilance because, typically, bilateral transfers of asset ownership are profitable even if they reduce social surplus; and that with ex ante identical firms, more symmetric ownership structures are better because they directly increase expected social surplus and because, indirectly, they permit better divestitures post merger.[6] The result that bilateral transfers of asset ownership can be profitable even if they reduce social surplus also means that bilateral transfers are a natural focus—there would typically be privately profitable bilateral transactions even following an agreement by all firms on an ownership structure that permitted ex post efficiency.

In extensions, we incorporate consumers surplus and provide conditions under which ex post efficiency in the market we consider also maximizes consumer surplus, allow for investments, and discuss nonidentical supports and possible avenues for incorporating second-best mechanisms with full integration.

There are three strands of related literature. First, the paper relates to competition policy and, in particular, to non-horizontal mergers. Interest in these has recently been sparked by

[5] Riordan (1990) takes the elimination, or at least reduction, of private information within the integrated entity as a defining feature of vertical integration, while Hart (1995) begs to differ.

[6] This result resonates with and formalizes the notion that more symmetry—“leveling the playing field”—is, somehow, desirable, which is widely held in antitrust. Antitrust practitioners have long viewed symmetry, or at least the absence of its counterparts of “undue concentration” or “dominant firms,” as procompetitive (see, e.g., Lanzillotti, 1961; Turner, 1969; Williamson, 1972).

concerns related to big tech, theoretical and empirical advances, and controversial merger decisions (see footnote 1 for references and background). Conglomerate mergers offer scope for debate and further analysis because, even with identical supports, they can either decrease or increase social surplus. Even though in extensions with nonoverlapping supports our setting recovers and extends the finding of socially harmful vertical integration of Loertscher and Marx (2022), a key take-away from our analysis in this regard is that the concerns related to big tech and mergers involving multi-product firms stem primarily from these being conglomerate rather than vertical mergers.[7]

Second, the paper contributes to the emerging literature on incomplete information industrial organization, such as Backus et al. (2020), Backus et al. (forth.), Larsen (2021), Larsen et al. (2021), Larsen and Zhang (2022), Byrne et al. (2022), Loertscher and Marx (2019, 2022), Choné et al. (forth.), Kang and Muir (2022), and Barkley et al. (2023), and the mechanism design literature, particularly the strand on partnership models that was initiated by Cramton et al. (1987), with subsequent contributions by, among many others, Che (2006), Figueroa and Skreta (2012), Lu and Robert (2001), Loertscher and Wasser (2019), and Liu et al. (forth.).[8] For one-dimensional private information, we combine the incomplete information bargaining approach of Williams (1987) and Loertscher and Marx (2022) with a generalized partnership model with heterogeneous distributions and maximum demands. Among other things, this allows us to generalize the notion of social surplus increasing "countervailing power," popularized by Galbraith (1952, 1954) and viewed skeptically by Stigler (1954), and to conclude that, with identical supports, equalization of bargaining power is a necessary condition for ex post efficiency. With equal bargaining weights, this also allows us to derive the social surplus maximizing policy for mergers that are pure transfers of asset

[7]For example, an at face value natural explanation for the empirical finding of Luco and Marshall (2020) of price-increasing effects of integration by soda producers with downstream bottlers is what they refer to as the "Edgeworth-Salinger" effect (Edgeworth, 1925; Salinger, 1991) according to which a tax decrease (or the elimination of a double markup) can induce a multi-product monopoly to increase the prices on *all* of its products. However, to the extent that the soda producers were already vertically integrated prior to the acquisitions of bottlers in their study, these were, in the categorization of our paper, conglomerate rather than vertical mergers. Armstrong and Vickers (forth.) provide a systematic analysis of necessary and sufficient conditions for Edgeworth's taxation paradox, which is at the heart of the Edgeworth-Salinger effect. Even though the Edgeworth paradox has a certain genericity, the conditions for it to obtain are quite demanding. For example, no symmetric demand system can give rise to it. This suggests that Hotelling's assessment that "[t]here is no basis known at present for denying that Edgeworth's phenomenon may pertain to a large proportion of ordinary situations, or for affirming that it is, in his language, a mere 'curiosum"' (Hotelling, 1932, p. 583) still applies today.

[8]Because the design problem inevitably fails the condition that Myerson (1981) called regularity, the paper also contributes to a recent upsurge of interest in nonregular mechanism design problems such as Condorelli (2012), Dworczak ⓡ al. (2021), Loertscher and Muir (2022) and Akbarpour ⓡ al. (2020), where nonregularity arises from assumptions on the distributions from which the agents draw their types or differences in the agents' utility functions. In contrast, in our setting, the nonregularity of the second-best mechanism design problem derives from the market structure.

ownership. For both one-dimensional and multi-dimensional private information, we characterize the set of ownership structures that permits ex post efficiency. This set has the properties of being nonempty and convex.[9] While this set (and the set of firms) changes when two firms fully integrate, it is the multi-dimensional generalization that allows us to make statements about the social surplus effects of horizontal, vertical, and conglomerate mergers in general market settings. As the existing literature has confined attention to settings with the property that, post-merger, an integrated firm's private information remains essentially one-dimensional, this is a substantial step forward.[10] Moreover, the ability to capture decreasing marginal values afforded by having multi-dimensional private information is empirically relevant (Liu et al., forth.). The setup with long-term ownership structures that affect the efficiency of the day-to-day market also constitutes an incomplete information formalization of the effects of forward markets on the operation of spot markets studied by, for example, Green and Newbery (1992) and Allaz and Vila (1993).

Last, the paper relates to the literature on the theory of the firm and the "make-or-buy" decision by explicitly accounting for the "sale" that is the counterpart to the "buy." Because, with identical supports, ex post efficiency is impossible when each firm either always only buys or only sells when it trades, the endogeneity of the buy-or-sell decision is at the core of the paper. It also shares with the property rights literature in the tradition of Grossman and Hart (1986) and Hart and Moore (1990) that the asset ownership structure matters for efficiency. In contrast to that literature, in settings with incomplete information like ours, the underlying mechanics are almost the complete converse. The ownership structure affects how efficient bargaining is and, if the ownership structure permits ex post efficient trade, then it also provides the agents with incentives for socially optimal investments, whereas in the traditional property rights literature, bargaining is efficient by assumption and investment is plagued by hold up due to efficient complete information bargaining.

The remainder of the paper is structured as follows. In Section 2, we use a reduced-form model to derive the social surplus maximizing competition policy. In Section 3, we introduce the incomplete information model that provides a microfoundation for the reduced-form model, and we derive the results pertaining to countervailing power and firms' private

[9]Most of the partnership literature has studied problems with one-dimensional type distributions. Two exceptions are Yenmez (2015) and Agastya and Birulin (2018) who study, respectively, settings with interdependent values and settings with outside options that are independent from the value of continuing the partnership. The former shows an impossibility result, and the latter's emphasis is on impossibility of ex post efficiency as well.

[10]For example, Loertscher and Marx (2019) and Choné et al. (forth.) study one-to-many settings in which an integrated firm always only contracts with one party. The same is true for the analysis of vertical integration in Loertscher and Marx (2022). Horizontal mergers in that paper are tractable if, say, the two merging suppliers each have enough capacity to serve the entire market pre merger, so that post merger all that matters is the lower of their two cost draws, which is a one-dimensional random variable.

incentives for asset ownership transfers. In Section 4, we analyze pure asset ownership transfers and derive the social surplus maximizing merger policy. In Section 5, we consider the effects of full integration that combines the assets and private information of two firms. Section 6 contains extensions and discussion, and Section 7 concludes the paper.

2 Reduced form

We first use a reduced-form model for which we derive the social surplus maximizing merger policy. This model provides both a rationale and guidance for divestitures and conditions for when a merger can be permitted with remedies and when instead it should be blocked. The subsequent section will deliver microfoundations for this simple yet illustrative model.

There is a set $\mathcal{N}$ of $n \geq 2$ firms. Each firm $i \in \mathcal{N}$ has some asset ownership $r_i \geq 0$, with total asset ownership normalized to 1, that is, $\sum_{i=1}^{n} r_i = 1$. Accordingly, the set of feasible ownership structures is the simplex

$$\Delta \equiv \left\{ \mathbf{r} \in [0,1]^n \mid \sum_{i=1}^{n} r_i = 1 \right\}.$$

We denote by $SS(\mathbf{r})$ the social surplus that results from ownership structure $\mathbf{r} \in \Delta$, which here is assumed to be a continuous function. Let $\mathcal{R}(\mathbf{r})$ be the set of all ownership structures that generate social surplus of at least $SS(\mathbf{r})$, that is,

$$\mathcal{R}(\mathbf{r}) \equiv \{\mathbf{r}' \in \Delta \mid SS(\mathbf{r}') \geq SS(\mathbf{r})\}.$$

Denote by $\mathcal{R}^e$ the set of all ownership structures that maximize $SS(\mathbf{r})$, that is, $\mathcal{R}^e = \arg\max_{\mathbf{r}' \in \Delta} SS(\mathbf{r}')$.

The sale of assets from firm i with $r_i > 0$ to firm j is modeled as a reduction $x \in (0, r_i]$ in firm i's ownership and an increase by x in firm j's ownership, so that post-transaction the ownership structure consists of $r_i - x$ for firm i, of $r_j + x$ for firm j, and leaves ownership unchanged for all other firms. We consider the role of an antitrust authority that can exert control over asset ownership by approving or blocking a proposed sale of assets or by approving changes in asset ownership conditional on divestitures, in which case the acquiree must sell some of its newly acquired assets to a third party. Such divestitures are sometimes referred to as structural remedies.[11] Given $\mathbf{r}$ and a transaction shifting $x \leq r_i$ from firm i to firm j, the feasible divestitures are

$$\mathcal{D}_{i,j}(\mathbf{r}, x) \equiv \{\mathbf{r}' \in \Delta \mid r'_i = r_i - x,\ r'_j \in [r_j, r_j + x],\ \forall \ell \in \mathcal{N} \backslash \{i, j\},\ r'_\ell \geq r_\ell\}.$$

Put differently, divestitures can remove any amount less than or equal to x from firm j and

[11] Structural remedies are in contrast to behavioral remedies that restrict conduct rather than ownership.

shift it, or parts of it, to any other firm except for firm i, whose ownership remains $r_i - x$ before and after the divestiture.

Thus, we have the following social surplus maximizing policy:

(a) Acquisition by firm 3 of firm 2's assets (b) Acquisition by firm 3 of firm 1's assets

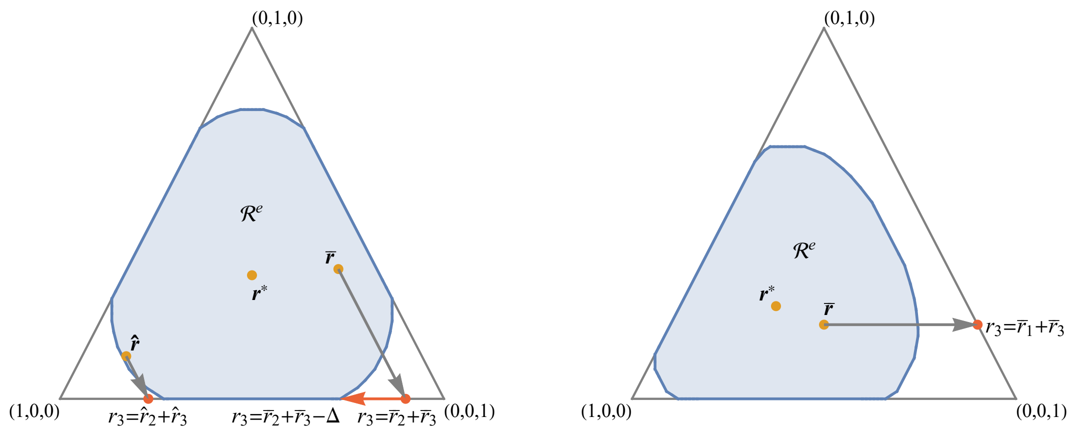

Figure 1: Ex post efficiency permitting set $\mathcal{R}^e$ (blue region), with asset ownership transfers indicated with gray arrows. In panel (a), starting from $\bar{\mathbf{r}}$ there exists an ex post efficiency restoring divestiture of assets to firm 1, shown by the red arrow, but starting from $\hat{\mathbf{r}}$, no such divestiture exists. In panel (b), no divestiture to firm 2 restores ex post efficiency. Points labeled $\mathbf{r}^*$ are defined and discussed in Section 4.1. Microfoundations for $\mathcal{R}^e$ are contained in Sections 3 and 4. In the model defined there, the panels correspond to the case with $n = 3$ and $k_i = 1$ for all i, where panel (a) assumes uniformly distributed types for all firms, and panel (b) assumes that $F_1(x) = x^3$ and $F_2(x) = F_3(x) = x$, so that firm 1 is stronger than firms 2 and 3 in the sense of first-order stochastic dominance.

Proposition 1. *Let $\mathbf{r}^b$ and $\mathbf{r}^a$ be asset ownership before and after, respectively, a transaction that shifts asset ownership from firm i to firm j. Then the social surplus maximizing policy*

- *allows the transaction if $\mathbf{r}^a \in \mathcal{R}^e$;*
- *allows it with divestiture if $\mathcal{D}_{i,j}(\mathbf{r}^b, r_i^b - r_i^a) \cap \mathcal{R}(\mathbf{r}^b) \neq \varnothing$;*
- *blocks it otherwise.*

Based on Proposition 1, we say that given initial ownership $\mathbf{r}^b$, ownership $\mathbf{r}'$ results from *optimal divestitures* following the sale of assets x from firm i to firm j if we have $\mathbf{r}' \in \arg\max_{\mathbf{r} \in \mathcal{D}_{i,j}(\mathbf{r}^b, x)} SS(\mathbf{r})$.

For the next result, we assume that $SS(\mathbf{r})$ is *Schur concave*, which is to say that if $\mathbf{r}'$ majorizes $\mathbf{r}$, then $SS(\mathbf{r}') \leq SS(\mathbf{r})$, with strict inequality unless $\mathbf{r}' \in \mathcal{R}^e$.[12] Under this condition, for which we provide foundations in Section 4, Proposition 2 brings to light a

[12]The vector $\mathbf{r}'$ *majorizes* $\mathbf{r}$ if for all $j \in \{1, \dots, n\}$, $\sum_{i=1}^{j} r'_{(i)} \geq \sum_{i=1}^{j} r_{(i)}$, where $r_{(i)}$ is the i-th largest element of $\mathbf{r}$, with a strict inequality for some j and equality for $j = n$ (Marshall et al., 2011, p. 80). As shown by Marshall et al. (2011, Lemma B.1), if $\mathbf{r}'$ majorizes $\mathbf{r}$, then $\mathbf{r}$ can be obtained from $\mathbf{r}'$ by a finite

novel benefit of symmetry—an increase in pre-transaction symmetry not only causes social surplus to be higher pre-transaction (as a direct consequence of Schur concavity), but also post-transaction following an optimal divestiture.

Proposition 2. *Assume that $SS(\mathbf{r})$ is Schur concave. Given $\tilde{\mathbf{r}}$ that majorizes $\mathbf{r}$, with $\tilde{r}_i = r_i = s > 0$, if $\tilde{\mathbf{r}}'$ and $\mathbf{r}'$ result from optimal divestitures after firm i sells assets $\sigma \in (0, s]$ to another firm, then $\tilde{\mathbf{r}}'$ majorizes $\mathbf{r}'$ and*

$$SS(\tilde{\mathbf{r}}') \leq SS(\mathbf{r}'),$$

with a strict inequality unless $\tilde{\mathbf{r}}' \in \mathcal{R}^e$.

Proof. See Appendix A.1.

We summarize the key takeaway of Proposition 2 in the following corollary:

Corollary 1. *If $SS(\mathbf{r})$ is Schur concave, then increased symmetry benefits expected social surplus directly and also makes divestiture-based remedies more effective.*

Of course, without a microfoundation, which the subsequent sections will provide, Proposition 2 and Corollary 1 are to a greater or lesser extent tautological.

While this reduced-form analysis highlights key insights from this paper, it does not answer the question of whether firms' incentives to transact asset ownership align with the objective of a social surplus maximizing planner or whether there is a basis for sustained antitrust vigilance. It also leaves open questions concerning bargaining power effects and, naturally, of what is a microfoundation for the reduced form. These are the questions we address in the following two sections.

3 Model

We now provide a microfoundation for the reduced-form model. After introducing the setup, we also derive a first set of results based on second-best mechanisms.

3.1 Setup

At the outset, that is, before any transactions of ownership or integration take place, each firm $i \in \mathcal{N}$ has asset ownership $r_i \geq 0$ and maximum demand for assets $k_i \geq r_i$ with $k_i > 0$.[13]

number of T-transforms: given vector $(x_1, \ldots, x_n)$, a T-transform of $\mathbf{x}$ is a vector with two coordinates x_j and x_k replaced by $\lambda x_j + (1-\lambda)x_k$ and $\lambda x_k + (1-\lambda)x_j$ for some $\lambda \in (0, 1)$ (Marshall et al., 2011, p. 32). This maintains the majorization relation at each step.

[13]If, contrary to our assumption, $k_i < r_i$, then firm i is has no value for $r_i - k_i$ units, which it is willing to sell for free. It is without loss to assume, as we do, that $k_i > 0$, for otherwise, the assumption that $k_i \geq r_i$ would imply that $r_i = k_i = 0$, and so we could just eliminate firm i from the market.

We assume that for all $i \in \mathcal{N}$, $\sum_{j \in \mathcal{N} \setminus \{i\}} k_j \geq 1$. Together with $k_i > 0$, this implies that $\sum_{i \in \mathcal{N}} k_i > 1$, that is, there is scarcity. The set of admissible ownership structures is thus

$$\Delta_{\mathbf{k}} \equiv \left\{ \mathbf{r} \in \times_{i \in \mathcal{N}} [0, k_i] \mid \sum_{i \in \mathcal{N}} r_i = 1 \right\}.$$

We distinguish between the cases in which a firm's private information is *one-dimensional* and in which it is *multi-dimensional*. If firm i is characterized by multi-dimensional private information with dimensionality $h_i \geq 2$, then that means that it is described by a vector of maximum demands $(k_i^1, \dots, k_i^{h_i})$ and a vector of distributions $(F_i^1, \dots, F_i^{h_i})$, where to fix ideas (k_i^j, F_i^j) may be thought of as describing the downstream market j that firm i has exclusive access to serve. As usual, the differentiation between markets may be in geographic or product space. While we assume that firms do not compete downstream, foreclosure effects can still arise in the sense that shifts in asset ownership between two firms can impose externalities on firms not involved in the transaction, including reducing a nontransacting firm's expected allocation. For each $j \in \{1, \dots, h_i\}$, the constant marginal value θ_i^j is an independent draw from the distribution F_i^j with support $[0, 1]$ and positive density f_i^j. This implies that firm i, if multi-dimensional, has decreasing marginal values and a maximum demand of $k_i = \sum_{j=1}^{h_i} k_i^j$. For example, if $h_i = 2$ and $k_i^1 = k_i^2 = k$, then given a realization $\boldsymbol{\theta}_i = (\theta_i^1, \theta_i^2)$, firm i's willingness to pay for the first k units is $\max\{\theta_i^1, \theta_i^2\}$ and $\min\{\theta_i^1, \theta_i^2\}$ for the second k units. We use $H \equiv \sum_{i \in \mathcal{N}} h_i$ to denote the total number of types. In the one-dimensional case, that is, if $h_i = 1$, we simply write k_i, F_i, and f_i, that is, we drop the superscript 1. Distributions, maximum demands, and ownership are common knowledge, whereas a firm's realized types are its private information.

Given type realizations $\boldsymbol{\theta}$, the *ex post efficient* allocation (or, equivalently, ex post efficient consumption of the asset's services) is based on the ranking of the firms' types and denoted $\mathbf{Q}^e(\boldsymbol{\theta})$. For a given vector of types $\boldsymbol{\theta}$, we have

$$\mathbf{Q}^e(\boldsymbol{\theta}) \in \arg\max_{\mathbf{Q}} \sum_{i=1}^{n} \sum_{j=1}^{h_i} Q_i^j \theta_i^j \quad \text{subject to} \quad \sum_{i=1}^{n} \sum_{j=1}^{h_i} Q_i^j \leq 1 \text{ and } Q_i^j \in [0, k_i^j].$$

While $\mathbf{Q}_i^e(\boldsymbol{\theta})$ is a vector of length h_i, the total quantity allocated to firm i is $Q_i^e(\boldsymbol{\theta}) \equiv \sum_{j=1}^{h_i} Q_i^{e,j}(\boldsymbol{\theta})$, and firm i's interim expected total allocation is $q_i^e(\boldsymbol{\theta}_i) \equiv \mathbb{E}_{\boldsymbol{\theta}_{-i}}[Q_i^e(\boldsymbol{\theta})]$. The assumption that, for all $i \in \mathcal{N}$, $\sum_{j \neq i} k_j \geq 1$, implies that $q_i^e(\mathbf{0}) = 0$.

We distinguish between pure *asset ownership transfers*, which refer to transfers of ownership that do not affect the set of firms or the private information held by any firm, and *full integration*. Formally, in an asset ownership transfer, firm i sells to firm j the amount $x \in [0, \min\{r_i, k_j - r_j\}]$, so that post transaction the ownership shares are $r_i - x$ and $r_j + x$ for firms i and j and remain the same as before the transaction for all other firms. Asset ownership transfers are analyzed in Section 4. In contrast, full integration of firms i and j, which we analyze in Section 5, results in the two firms both combining their asset ownership and integrating their private information. Consequently, full integration reduces the number of firms by 1 and creates an integrated firm with a higher dimension of private information. For example, full integration of firms i and j with $h_i = h_j = 1$, results in an integrated firm with asset ownership $r_{i,j} = r_i + r_j$ and dimensionality $h_{i,j} = 2$, where the firm has maximum demand k_i associated with the type drawn from F_i and maximum demand k_j associated with the type drawn from F_j. Of course, a multi-dimensional firm can always be thought of and microfounded as having emerged from the full integration of one-dimensional firms.

Throughout, we assume that decisions to transfer asset ownership or to fully integrate, occur at the ex ante stage, that is, before the realization of any private information. This is also the stage at which we evaluate welfare. This reflects a view that private information pertains to short-term, "day-to-day" transactions, while mergers are long-term decisions and welfare and profits are naturally evaluated "on average."[14]

The timeline is as follows. First, given $\mathcal{N}$ and $\mathbf{r}$, two firms may engage in asset ownership transfers or full integration. This proposed transaction is then evaluated by antitrust authorities, including the possibility that divestitures are imposed and implemented. Second, private information is realized. Third, the day-to-day market operates and determines which firm consumes how many units of the services provided by the assets, and payments and firms' payoffs are realized.

The firms' asset ownership and maximum demands for assets determine whether they are buyers, sellers, or simply traders that have the potential to act as a buyer or a seller, depending on type realizations.[15] Firm i is a *buyer* if it has no assets of its own to sell, $r_i = 0$, or if there is no external demand for any of its assets, that is, if for all $j \in \mathcal{N}\backslash\{i\}$, $k_j = r_j$; firm i is a *seller* if it has no demand for additional assets, $k_i = r_i$ or there are no available assets to purchase, $\sum_{j \in \mathcal{N}\backslash\{i\}} r_j = 0$; and otherwise firm i is said to be a *trader*. Thus, for any trader i, we have $0 < r_i < k_i$.

[14]See Appendix B.1 for a formalization of this view in a multi-period model in which the allocation $\mathbf{Q}$ in any given period represents short-term leases and the asset ownership $\mathbf{r}$ is fixed for all periods after transactions in period 0.

[15]Lu and Robert (2001) refer to firms in this last category as "ex ante unidentified traders." Settings with traders are called *asset markets* in Loertscher and Marx (2020, 2023a) and Delacrétaz et al. (2022). Liu et al. (forth.) document the empirical relevance of traders in the context of emission permit markets, where many emitters are net buyers of emission permits in some periods and net sellers in other periods.

3.2 Market mechanism

As do Loertscher and Marx (2022) and Liu et al. (forth.), we model the market as a mechanism operated by a (fictitious) market maker or designer that, as a function of firms' types $\boldsymbol{\theta} \in [0,1]^H$, chooses a feasible allocation $\mathbf{Q}$ and determines firms' payments $\mathbf{M} = (M_i)_{i \in \mathcal{N}} \in \mathbb{R}^n_+$, where M_i is the payment from firm i to the market maker. The idea of the market being organized by a fictitious entity has a long tradition in industrial organization, and economics more generally, starting with Cournot's (1938) auctioneer, who sets the market-clearing price given firms' quantity choices, and subsequently adapted by Walras (1874) to the general equilibrium setting. Because the firms are privately informed about their types, the mechanism can make $\mathbf{Q}$ and $\mathbf{M}$ depend on the realized $\boldsymbol{\theta}$ only if it induces the firms to reveal that information to the market maker, which is why, with private information, the natural extension of a Cournot-Walras auctioneer is that of a market maker who uses a mechanism. Specifically, we stipulate that this mechanism is *direct* in that it asks every agent to report its type and *incentive compatible* (IC)—makes reporting types truthful in either a Bayes' Nash or a dominant strategy equilibrium—and *interim individually rational* (IR)—knowing its type and expected allocation and payment, every firm is weakly better off participating in the mechanism than walking away. Both IC and IR will be more formally defined below. A direct IC, IR mechanism is said to be *ex post efficient* if, for every $\boldsymbol{\theta} \in [0,1]^H$, it uses an ex post efficient allocation rule $\mathbf{Q}^e(\boldsymbol{\theta})$.

Ex post efficiency is said to be *possible* if there exists an ex post efficient, IC, IR mechanism that, in expectation, does not run a deficit, that is, $\mathbb{E}_{\boldsymbol{\theta}}[\sum_{i \in \mathcal{N}} M_i(\boldsymbol{\theta})] \geq 0$. If no such mechanism exists, then ex post efficiency is said to be *impossible*. The no-deficit constraint simply means that there is no outside source pouring money into the market to grease its wheels.[16] Our primary focus when dealing with multi-dimensional types will be on whether or not ex post efficiency is possible.[17] This question has been the focus of much of the partnership literature with one-dimensional types upon which our model builds and expands.

For the setting with one-dimensional types, which we study in the remainder of this section and in the next section, we are able to characterize the second-best social surplus when ex post efficiency is not possible, firms' incentives to transfer asset ownership ex ante,

[16] One could account for additional market frictions by replacing the no-deficit constraint by a constraint that the revenue of the market maker be greater no less than some $\kappa \geq 0$. The effect is to shrink the set of ownership structures such that ex post efficiency is possible, potentially to the empty set, and to change the right-hand side in (3) below to κ.

[17] The exceptions pertain to cases in which a firm i with dimensionality $h_i \geq 2$ is characterized by maximum demands $k_i^j \geq 1$ for all $j \in \{1, \ldots, h_i\}$, in which case all that matters to determine the firm's willingness to pay is the highest of its h_i draws, whose distribution is $\times_{j=1}^{h_i} F_i^j$, which is one-dimensional. It is precisely this property of effective one-dimensionality that is exploited in the prior literature; see, e.g., Choné et al. (forth.) and Loertscher and Marx (2019, 2022).

the effects of equalizing firms' bargaining powers, and the social surplus maximizing policy for asset ownership transfers.

3.3 Incomplete information bargaining

With one-dimensional private information, the allocation rule of the market maker's direct mechanism $\langle \mathbf{Q}, \mathbf{M} \rangle$ is a mapping $\mathbf{Q} : [0,1]^n \to \mathbb{R}^n_+$ satisfying $Q_i(\boldsymbol{\theta}) \in [0, k_i]$ and $\sum_{i \in \mathcal{N}} Q_i(\boldsymbol{\theta}) \leq 1$, and a payment rule $\mathbf{M} : [0,1]^n \to \mathbb{R}^n$, where for reports $\boldsymbol{\theta}$, $Q_i(\boldsymbol{\theta})$ specifies the quantity allocated to firm i and $M_i(\boldsymbol{\theta})$ specifies the payment from firm i to the market maker.[18] Here and in the next section, we focus on IR mechanisms that satisfy BIC and no deficit in expectation.[19] Firm i's outside option from not participating in the mechanism is $\theta_i r_i$.

For a fixed mechanism $\langle \mathbf{Q}, \mathbf{M} \rangle$, we denote firm i's interim expected allocation and payments, respectively, by

$$q_i(\theta_i) \equiv \mathbb{E}_{\boldsymbol{\theta}_{-i}}[Q_i(\theta_i, \boldsymbol{\theta}_{-i})] \quad \text{and} \quad m_i(\theta_i) \equiv \mathbb{E}_{\boldsymbol{\theta}_{-i}}[M_i(\theta_i, \boldsymbol{\theta}_{-i})].$$

The interim expected net payoff of firm i from participating in the mechanism when its type is θ and when it reports its type truthfully, with net meaning net of the outside option $r_i\theta$, is denoted by

$$u_i(\theta) \equiv \theta(q_i(\theta) - r_i) - m_i(\theta).$$

The direct mechanism $\langle \mathbf{Q}, \mathbf{M} \rangle$ is *Bayesian IC (BIC)* if for all $i \in \mathcal{N}$ and all $\theta, \theta' \in [0,1]$,

$$u_i(\theta) \geq \theta(q_i(\theta') - r_i) - m_i(\theta'), \tag{1}$$

which implies that u_i is convex. The mechanism $\langle \mathbf{Q}, \mathbf{M} \rangle$ is IR, if for all $i \in \mathcal{N}$ and all $\theta \in [0,1]$,

$$u_i(\theta) \geq 0. \tag{2}$$

Type $\hat{\theta}_i \in [0,1]$ is called a *worst-off type* of firm i if $u_i(\theta) \geq u_i(\hat{\theta}_i)$ for all $\theta \in [0,1]$.

[18] By the Revelation Principle, a focus on direct mechanisms is without loss of generality. Constraint $Q_i \in [0, k_i]$ is for convenience and can be dropped by replacing $Q_i(\boldsymbol{\theta})$ with $\min\{k_i, Q_i(\boldsymbol{\theta})\}$ in firm i's payoff.

[19] In our independent private values setting, any BIC, IR mechanism can be implemented as a dominant strategy IC, IR mechanism (see e.g. Gershkov et al., 2013). In this sense, the nature of IC is immaterial. Further, as shown in footnote 20, the focus on no deficit in expectation is also without loss of generality within the class of mechanisms that satisfy IR.

The *no-deficit constraint* is satisfied if:[20]

$$\sum_{i \in \mathcal{N}} \mathbb{E}_{\theta_i}\left[m_i(\theta_i)\right] \geq 0. \tag{3}$$

By the standard characterization (see, e.g., Myerson, 1981), BIC holds if and only if:[21]

$$q_i \text{ is nondecreasing.} \tag{4}$$

By the envelope theorem (Milgrom and Segal, 2002, Corollary 1), $u_i'(\theta) = q_i(\theta) - r_i$ wherever u_i is differentiable,[22] so that for all $\theta, \theta' \in [0,1]$,

$$u_i(\theta) = u_i(\theta') + \int_{\theta'}^{\theta} (q_i(y) - r_i) dy. \tag{5}$$

The definition of $u_i(\theta)$ and equation (5) imply that for all $\theta, \theta' \in [0,1]$,

$$m_i(\theta) = \theta(q_i(\theta) - r_i) - \int_{\theta'}^{\theta} (q_i(y) - r_i) dy - u_i(\theta'). \tag{6}$$

To model bargaining power effects, we assume that each firm has bargaining (or welfare) weight $w_i \in [0,1]$ with at least one firm having a positive bargaining weight. If multiple firms have the maximum bargaining weight, then we also specify how the market maker's revenue (if any) is divided among those firms by assuming that each firm i with $w_i = \max \mathbf{w}$ receives surplus share $\eta_i \in [0,1]$ with $\sum_{i \text{ s.t. } w_i = \max \mathbf{w}} \eta_i = 1$. The market maker's mechanism then maximizes the weighted sum of the firms' ex ante expected payoffs subject to IC, IR, and no deficit:

$$\max_{\mathbf{Q},\mathbf{M}} \mathbb{E}_{\boldsymbol{\theta}}\Big[\sum_{i \in \mathcal{N}} w_i\big(\theta_i Q_i(\boldsymbol{\theta}) - M_i(\boldsymbol{\theta})\big)\Big] \text{ subject to (1)–(3).} \tag{7}$$

The ex ante expected social surplus under incomplete information bargaining is then

$$SS(\mathbf{r}) = \sum_{i \in \mathcal{N}} \mathbb{E}_{\theta_i}[\theta_i q_i(\theta_i) - m_i(\theta_i)],$$

where q_i and m_i are the interim expected allocation and payment rules induced by the

[20]Condition (3) only requires no deficit in expectation, but this is without loss of generality because for any BIC mechanism $\langle \mathbf{Q}, \mathbf{M} \rangle$ that is such that $\sum_{i \in \mathcal{N}} \mathbb{E}_{\theta_i}\left[m_i(\theta_i)\right] = \kappa$ holds for some $\kappa \in \mathbb{R}$, there is a BIC mechanism $\langle \mathbf{Q}, \tilde{\mathbf{M}} \rangle$ with the same allocation rule and the same interim expected payments m_i whose revenue is κ ex post, i.e., that satisfies $\sum_{i \in \mathcal{N}} \tilde{M}_i(\boldsymbol{\theta}) = \kappa$ for all $\boldsymbol{\theta}$: let $\tilde{M}_i(\boldsymbol{\theta}) = m_i(\theta_i) - \sum_{j \neq i} m_j(\theta_j)/(n-1) + c_i$ with $c_i = (\mathbb{E}_{\theta_i}[m_i(\theta_i)] - \kappa)/(n-1)$. It further follows that if the mechanism $\langle \mathbf{Q}, \mathbf{M} \rangle$ satisfies IR, then so does $\langle \mathbf{Q}, \tilde{\mathbf{M}} \rangle$. For more on equivalences of this form, see Börgers and Norman (2009).

[21]To see that IC implies that q_i is nondecreasing, consider two types $\theta, \theta' \in [\underline{\theta}_i, \overline{\theta}_i]$. IC for type θ and θ' requires, respectively, $q_i(\theta)\theta - m_i(\theta) \geq q_i(\theta')\theta - m_i(\theta')$ and $q_i(\theta)\theta' - m_i(\theta) \leq q_i(\theta')\theta' - m_i(\theta')$. Subtracting the latter from the former implies that $q_i(\theta)(\theta - \theta') \geq q_i(\theta')(\theta - \theta')$, which is equivalent to q_i being nondecreasing. For more background on mechanism design, see Börgers (2015) and Krishna (2010).

[22]IC implies that u_i is a maximum of a family of affine functions, which implies that u_i is convex and so absolutely continuous and differentiable almost everywhere in the interior of its domain.

mechanism that solves (7).

That is, allocations and payments are determined by the incomplete information bargaining mechanism, which, as in Williams (1987) and Loertscher and Marx (2022), maximizes the weighted sum of firms' expected payoffs, using the firms' bargaining powers as weights, subject to IC, IR, and no deficit. This mechanism is a generalization of the K-double auction of Chatterjee and Samuelson (1983) in the sense that for the case of two firms, including one buyer and one seller, where the buyer and seller have the same type distribution and maximum demand, varying K in a K-double auction maps out the same set of expected payoffs as varying the bargaining weights in our setup.

This setup generalizes prior literature in two ways. First, we allow $k_i < 1$, with the implication that there may be multiple buyers, in contrast to a partnership dissolution setup in which all assets move to a single firm. Second, we allow unequal bargaining weights, which introduces tradeoffs between social surplus and rent extraction.[23]

To simplify the analysis with one-dimensional private information, when the firms' bargaining weights differ or when ex post efficiency is not possible, we assume that the type distributions have increasing virtual value and virtual cost functions:[24]

$$\Psi_i^S(\theta) \equiv \theta + \frac{F_i(\theta)}{f_i(\theta)} \text{ and } \Psi_i^B(\theta) \equiv \theta - \frac{1 - F_i(\theta)}{f_i(\theta)}.$$

Despite the assumption of monotone virtual value and virtual cost functions, a firm's overall virtual type function given critical type x, denoted by

$$\Psi_i(\theta, x) \equiv \begin{cases} \Psi_i^S(\theta) & \text{if } \theta \in [0, x), \\ \Psi_i^B(\theta) & \text{if } \theta \in [x, 1], \end{cases} \tag{8}$$

is nonmonotone for $x \in (0,1)$, which impacts the analysis of second-best mechanisms.

3.4 Bargaining weights effects

Using the definition in (8), we obtain the following lemma:

[23] The literature typically assumes $k_i = 1$ and equal bargaining weights, including Myerson and Satterthwaite (1983); Cramton et al. (1987); Che (2006); Figueroa and Skreta (2012); Makowski and Mezzetti (1993); Loertscher and Wasser (2019). Other research, including Gresik and Satterthwaite (1989); Lu and Robert (2001); Liu et al. (forth.), relaxes the assumption that $k_i = 1$ but continues to assume equal bargaining weights. Williams (1987) and Loertscher and Marx (2022) introduce bargaining weights but assume extremal ownership $r_i \in \{0, k_i\}$, so that each firm is either a buyer or a seller.

[24] The virtual value function Ψ_i^B captures the marginal revenue associated with firm i. To see this, consider a seller with cost c that makes a take-it-or-leave-it price offer p to firm i. The seller's problem is $\max_{p \in [0,1]} (1 - F_i(p))(p - c)$. The first-order condition is $-f_i(p)(\Psi_i^B(p) - c) = 0$, which by the standard "marginal revenue equals marginal cost" condition means that $\Psi_i^B(p)$ is the marginal revenue associated with i's demand. An analogous argument shows that Ψ_i^S captures the marginal cost associated with F_i.

Lemma 1. *Given an IC, IR mechanism $\langle \mathbf{Q}, \mathbf{M} \rangle$, the set of worst-off types for firm i is the set of $\hat{\theta}_i$ such that $q_i(\hat{\theta}_i) = r_i$ if any such $\hat{\theta}_i$ exists and otherwise is the unique $\hat{\theta}_i$ such that $q_i(\theta_i) < r_i$ for all $\theta_i < \hat{\theta}_i$ and $q_i(\theta_i) > r_i$ for all $\theta_i > \hat{\theta}_i$, and firm i's expected payment to the market maker is*

$$\mathbb{E}_{\theta_i}[m_i(\theta_i)] = \mathbb{E}_{\theta_i}\left[\Psi_i(\theta_i, \hat{\theta}_i) q_i(\theta_i)\right] - \hat{\theta}_i r_i - u_i(\hat{\theta}_i), \tag{9}$$

where $u_i(\hat{\theta}_i) \geq 0$ (and $u_i(\hat{\theta}_i) = 0$ if IR binds for firm i's worst-off type).

Proof. See Appendix A.2.

As explained in the Online Appendix, the incomplete information bargaining mechanism is found by solving a saddle-point problem, simultaneously choosing the allocation rule to maximize a Lagrangian taking as given the firms' worst-off types (inner optimization problem) and finding the worst-off types that minimize firms' interim expected payoffs (outer optimization problem).[25] This saddle-point property means that even though the worst-off types are endogenous to the allocation rule, the allocation and payment rule are still separable in the sense that one can first derive the optimal allocation rule and then adjust payments to satisfy IR.[26]

Using Lemma 1, the relevant Lagrangian for the problem of maximizing the sum of the firms' weighted expected payoffs subject to IC, IR, and no deficit is:

$$\begin{aligned} \mathcal{L} \;\equiv\; & \mathbb{E}_{\boldsymbol{\theta}}\left[\sum_{i\in\mathcal{N}} w_i \left(\theta_i Q_i(\boldsymbol{\theta}) - \Psi_i(\theta_i, \hat{\theta}_i) Q_i(\boldsymbol{\theta}) + \hat{\theta}_i r_i + u_i(\hat{\theta}_i)\right)\right] \\ & + \rho \mathbb{E}_{\boldsymbol{\theta}}\left[\sum_{i\in\mathcal{N}} \left(\Psi_i(\theta_i, \hat{\theta}_i) Q_i(\boldsymbol{\theta}) - \hat{\theta}_i r_i - u_i(\hat{\theta}_i)\right)\right] + \sum_{i\in\mathcal{N}} \mu_i u_i(\hat{\theta}_i), \end{aligned} \tag{10}$$

where ρ and $\mu_1, \ldots, \mu_n$ are the Lagrange multipliers on the no-deficit and IR constraints, respectively.[27]

[25] The solution builds on and generalizes the earlier results by Lu and Robert (2001), who assume identical distributions and maximum demands, and Loertscher and Wasser (2019), who study a partnership problem, i.e., they assume that $k_i = 1$ for all $i \in \mathcal{N}$, while allowing for heterogeneous distributions. The setup of Loertscher and Wasser (2019) is more general in that they allow interdependent values.

[26] It is difficult to see how one could solve these problems in any degree of generality without the saddle-point property. As a case in point, not being aware of it, Kittsteiner (2003) found what turns out to be the optimal dissolution mechanism for a partnership problem with two agents without being able to prove optimality (see his footnote 19).

[27] Stationarity requires that $\mathbf{Q}$ maximizes $\mathcal{L}$ among $\mathbf{Q}$ such that $q_i(\theta_i)$ is increasing for all $i \in \mathcal{N}$, and that each of $u_1(\hat{\theta}_1), \ldots, u_n(\hat{\theta}_n)$ maximizes $\mathcal{L}$, i.e., $\frac{\partial \mathcal{L}}{\partial u_i(\hat{\theta}_i)} = 0$, which we can write as $w_i - \rho + \mu_i = 0$. Primal feasibility requires that $\mathbb{E}_{\boldsymbol{\theta}}\big[\sum_{i\in\mathcal{N}} \big(\Psi_i(\theta_i, \hat{\theta}_i) Q_i(\boldsymbol{\theta}) - \hat{\theta}_i r_i - u_i(\hat{\theta}_i)\big)\big] \geq 0$, and $u_i(\hat{\theta}_i) \geq 0$. Dual feasibility requires that $\rho \geq 0$, and $\mu_i \geq 0$; and complementary slackness requires that $\rho \mathbb{E}_{\boldsymbol{\theta}}\big[\sum_{i\in\mathcal{N}} \big(\Psi_i(\theta_i, \hat{\theta}_i) Q_i(\boldsymbol{\theta}) - \hat{\theta}_i r_i - u_i(\hat{\theta}_i)\big)\big] = 0$ and $\mu_i u_i(\hat{\theta}_i) = 0$.

It is useful to isolate the allocation rule by rewriting the Lagrangian using firms' *weighted* virtual type functions, which are defined for $\alpha \in [0,1]$ by

$$\Psi_{i,\alpha}(\theta, x) \equiv \alpha\theta + (1-\alpha)\Psi_i(\theta, x).$$

In particular, (10) can be rewritten in such a way that the allocation rule appears only in the following term:

$$\rho \mathbb{E}_{\boldsymbol{\theta}}\Big[\sum_{i\in\mathcal{N}}(Q_i(\boldsymbol{\theta}) - r_i)\Psi_{i,\frac{w_i}{\rho}}(\theta_i, \hat{\theta}_i)\Big].$$

In this expression, each firm's virtual type is weighted by the ratio of its bargaining power to the Lagrange multiplier on the no-deficit constraint, where the solution value of the multiplier satisfies $\rho \geq \max \mathbf{w}$, so that all weights are in $[0,1]$.

It is apparent from this (and demonstrated formally in the Online Appendix), that, given worst-off types $\hat{\theta}_1, \ldots, \hat{\theta}_n$ and Lagrange multiplier ρ, the incomplete information bargaining allocation rule can be defined pointwise by prioritizing firms based on their weighted virtual types, as long as one is careful to maintain the monotonicity of the firms' interim expected allocation rules. To do this, we apply ironing (Myerson, 1981). Letting $\overline{\Psi}_{i,\alpha}$ denote the ironed version of $\Psi_{i,\alpha}$ (formally defined in the Online Appendix), the incomplete information bargaining allocation given types $\boldsymbol{\theta}$, worst-off types $\hat{\boldsymbol{\theta}}$, and multiplier ρ, denoted by $\mathbf{Q}^*_\rho(\boldsymbol{\theta}; \hat{\boldsymbol{\theta}})$, assigns the supply to firms up to their maximum demands in order according to their ironed weighted virtual types, $\overline{\Psi}_{i,w_i/\rho}(\theta_i, \hat{\theta}_i)$, with an appropriately chosen tie-breaking rule if required, as shown in the Online Appendix. We then have the following result:

Proposition 3. *The incomplete information bargaining allocation rule is* $\mathbf{Q}^*_{\rho^*}(\cdot; \hat{\boldsymbol{\theta}}^*_{\rho^*})$, *where given* ρ, $\hat{\boldsymbol{\theta}}^*_\rho$ *is the vector of worst-off types defined by* $\mathbb{E}_{\boldsymbol{\theta}_{-i}}[Q^*_{i,\rho}(\hat{\theta}^*_{i,\rho}, \boldsymbol{\theta}_{-i}; \hat{\boldsymbol{\theta}}^*_\rho)] = r_i$ *for all* $i \in \mathcal{N}$, *and* ρ^* *is the smallest multiplier such that the no-deficit constraint is satisfied,* $\rho^* = \min\{\rho \geq \max \mathbf{w} \mid \sum_{i\in\mathcal{N}}(\mathbb{E}_{\boldsymbol{\theta}}[\Psi_{i,0}(\theta_i, \hat{\theta}^*_{i,\rho})Q^*_{i,\rho}(\boldsymbol{\theta}; \hat{\boldsymbol{\theta}}^*_\rho)] - r_i\hat{\theta}^*_{i,\rho}) \geq 0\}$.

Proof. See the Online Appendix.

Using Proposition 3, we can analyze the effects of bargaining power on the possibility of ex post efficiency. For example, Figure 2, which assumes that $r_1 = 0.9$ and $r_2 = 0.1$, shows how changing the firms' relative bargaining power shifts the allocation rule relative to that for equal bargaining power. It also shows that, even with equal bargaining power, the allocation inefficiently favors firm 1 with its large ownership share for all but extreme type realizations. This illustrates the underlying mechanism by which both ownership structure and bargaining power have the potential to affect efficiency.

As shown in Cramton et al. (1987), the set of ex post efficiency permitting ownership structures can contain a continuum of ownership structures, possibly suggesting that there

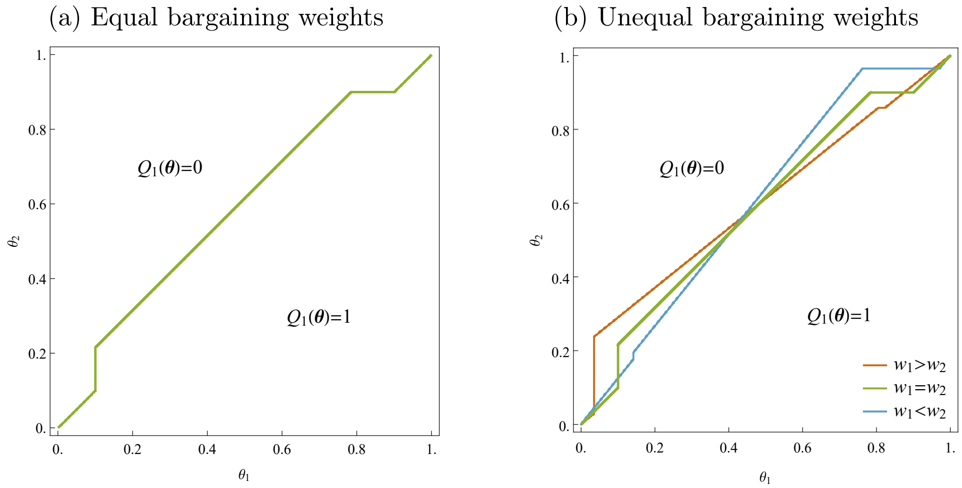

Figure 2: Incomplete information bargaining allocation rule with $n = 2$, $\mathbf{r} = (0.9, 0, 1)$, $\mathbf{k} = (1, 1)$, $\underline{\theta} = 0$, and uniformly distributed types. With $w_1 = w_2 = 1$, we have $\rho^* = 1.15$. Panel (b) also shows cases with $\mathbf{w} = (1, 0.75)$ and $\mathbf{w} = (0.75, 1)$, in which cases we have $\rho^* = 1.02$.

may also be a variety of bargaining weights that permit ex post efficiency for an appropriately chosen ownership structure. With that in mind, the following result is surprising in that it shows that ex post efficiency is not possible with unequal bargaining weights. Away from equal weights, firms with smaller weights will be discriminated against and evaluated according to their ironed weighted virtual type functions with differing weights. Hence, any imbalance of bargaining power induces allocative distortions.

Proposition 4. *Ex post efficiency requires that all firms have the same bargaining weight.*

Proof. See Appendix A.3.

We illustrate the necessity of equal bargaining weights in Figure 3. As shown there, for the case of one buyer and one seller (i.e., $r_1 = 0$), the entire frontier, which is labeled as the "Williams" frontier after Williams (1987), lies below the ex post efficient frontier. This includes the point associated with bilateral bargaining between a buyer and seller with equal bargaining weights labeled "MS," which was shown to be inefficient by Myerson and Satterthwaite (1983). The point labeled "CGK," which is for $r_1 = 0.21$ and lies on the efficient frontier, relates to Cramton et al. (1987), which shows that in a partnership problem with two agents and uniformly distributed types, ex post efficiency is possible for initial ownership shares between 0.21 and 0.79.

Proposition 4 emphasizes the importance of equal bargaining weights for ex post efficiency. Although difficult to formalize in many models, the notion that bargaining power matters for the size as well as the distribution of surplus has widespread appeal that goes

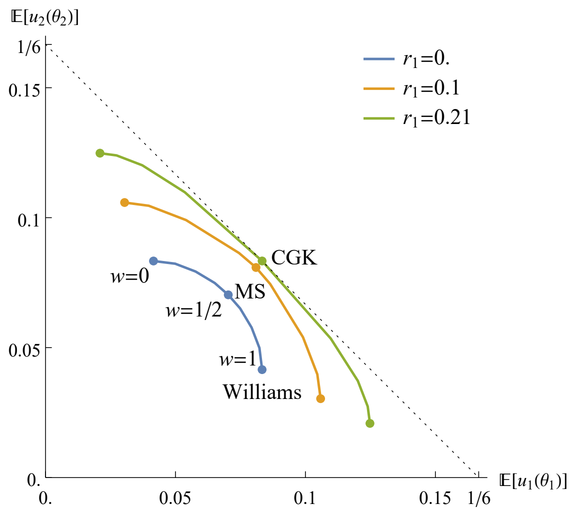

Figure 3: Expected payoffs in a market with $n = 2$ that maximizes $wu_1 + (1 - w)u_2$ for $w \in [0, 1]$. Assumes $k_1 = k_2 = 1$ and uniformly distributed types.

back to the concept of *countervailing power* introduced by Galbraith (1952) and remains prevalent.[28] For example, without providing a model of bargaining power, the Australian competition authority "has identified a range of market failures resulting from ... strong bargaining power imbalance and information asymmetry ... which ultimately cause inefficiencies" (ACCC Dairy Inquiry, 2018, p. xii). This is consistent with the incomplete information framework, which, as noted by Loertscher and Marx (2022), has the property that bargaining weights not only affect the distribution, but also the size of expected surplus. Notions of bargaining power are also connected to the ownership of productive assets. As a case in point, the U.S. *Horizontal Merger Guidelines* (p. 27) identify the ability and incentive to vertically integrate upstream as a potential source of buyer power.[29] In a labor market context, equalization of bargaining weights between firms and workers may be achieved by allowing the workers to form unions.

[28]This concept appears in antitrust practice. For example, OECD (2011, pp. 50–51) and OECD (2007, pp. 58–59) raise the possibility of a role for collective negotiation and group boycotts in counterbalancing the market power of providers of payment card services. In other examples, the U.S. DOJ and FTC recognize the potential benefits from allowing physician network joint ventures in their 1996 "Statement of Antitrust Enforcement Policy in Health Care." Krueger (2018) discusses the benefits to workers of market features that boost worker bargaining power and counterbalance monopsony power.

[29]This view also appears in the EU's evaluation of retail alliances, in which retailers cooperate in sourcing supplies and potentially other activities, with the finding that retail alliances, particularly ones that coordinate input sourcing, "generate benefits for their members by improving their bargaining position and leading to better purchasing terms" (Colen et al., 2020, p. 22). According to this report, this occurs through increased "bargaining ability," reflecting things like bargaining skill, and increased "bargaining leverage," which is "the power a firm has in the negotiation over an agreement" and relates to outside options in case no agreement is reached (Colen et al., 2020, p. 23). Alliances affect improve retailers' outside options because "pooling retail volumes increases the retailers' outside option and reduces the outside option of suppliers and thus increases bargaining leverage" (Colen et al., 2020, p. 26).

Given Proposition 4, when analyzing the effects of changes in asset ownership in what follows, we focus on the case of equal bargaining weights. This allows us to examine the best case, in a sense, for when asset ownership allows ex post efficiency. In light of the result of Myerson and Satterthwaite (1983) that in a bilateral trade setting, ex post efficiency can be achieved if the supports of the buyer's and the seller's distributions do not overlap (in which case the two firms can simply trade at a posted price between the two supports), one might think that Proposition 4 hinges on our assumption of common supports. However, as we show in Appendix B.3, in a setting with upstream and downstream firms whose distributions have different supports, with downstream firms having a "higher" support, ex post efficiency still requires that all downstream firms have the same bargaining weight and requires that weights for upstream firms with positive asset ownership are sufficiently close to those of the downstream firms, provided the gap between the supports is not too large.

4 Asset ownership transfers

In this section, we characterize the set of ex post efficiency permitting ownership structures for the model with one-dimensional types, and then examine firms' incentives for pure asset ownership transfers. In light of the impossibility of ex post efficiency away from equal bargaining weights, in what follows, we assume that firms have equal bargaining weights.[30]

4.1 Ex post efficiency permitting ownership structures

For $\mathbf{r} \in \Delta_{\mathbf{k}}$, denote by $\Pi^e(\mathbf{r})$ the maximum expected revenue to the market maker of an ex post efficient, IC, IR mechanism before any revenue is given back to the agents.[31] (Since the allocation rules is fixed, maximum expected revenue means that each firm's IR constraint is satisfied with equality at its worst-off type.) Consequently, ex post efficiency is possible if and only if $\Pi^e(\mathbf{r}) \geq 0$. We will show that $\Pi^e(\mathbf{r})$ is concave and maximized at a $\mathbf{r}^*$ such that each firm has the same worst-off type, at which point it is positive. Using the result of Lemma 1 that firm i's interim expected allocation is equal to r_i at its worst-off type, it follows that equalized worst-off types must be interior, and so at ownership $\mathbf{r}^*$, all firms are traders. Moreover, reflecting impossibility results for settings with buyers and sellers in the tradition of Vickrey (1961) and Myerson and Satterthwaite (1983), ex post efficiency is not

[30]This means that the surplus shares $\eta_1, \ldots, \eta_n$ discussed in Section 3.3 define how any revenue under binding IR for firms' worst-off types is distributed to the firms.

[31]Formally, $\Pi^e(\mathbf{r})$ is given by the expression in brackets in the second line of equation (10) with $Q_i = Q_i^e$ and $\hat{\theta}_i = \hat{\theta}_i^e$, that is, $\Pi^e(\mathbf{r}) = \mathbb{E}_{\boldsymbol{\theta}}[\sum_{i \in \mathcal{N}}(\Psi_i(\theta_i, \hat{\theta}_i^e)Q_i^e(\boldsymbol{\theta}) - \hat{\theta}_i^e r_i)]$.

possible without traders.[32]

Proposition 5. *Ex post efficiency is not possible if there are no traders; for all* $\mathbf{r}$ *in a nonempty convex set* $\mathcal{R}^e$*, ex post efficiency is possible, including when all firms are traders with ownership* $\mathbf{r}^*$*; and* $SS(\mathbf{r})$ *is concave, including being constant for* $\mathbf{r} \in \mathcal{R}^e$ *and strictly concave otherwise.*

Proof. See Appendix A.4.

To develop intuition for the concavity of $SS(\mathbf{r})$, note that a mechanism that is a convex combination of the social surplus maximizing mechanisms for two different ownership vectors is itself a feasible mechanism that satisfies IC, IR, and no deficit when the ownership vector is the same convex combination of the two different ownership vectors. But then the allocation rule can be adjusted in favor of one that generates weakly more social surplus, and strictly more outside of $\mathcal{R}^e$, which explains why $SS(\mathbf{r})$ is concave and strictly so outside of $\mathcal{R}^e$.

4.2 Social surplus maximizing merger policy

By Proposition 5, transactions that move ownership towards $\mathcal{R}^e$ increase expected social surplus. This allows us to consider the potential positive and negative effects of changes in asset ownership and the question of when and whether divestitures can play a role in remedying negative effects. For now, we assume a social surplus standard, with social surplus narrowly defined as the sum of the firms' payoffs.[33]

Because the incomplete information model with one-dimensional types satisfies all of the conditions used in Proposition 1, the social surplus maximizing policy to approve pure asset ownership transfers, block them, or approve them subject to divestitures is as described in that proposition. Further, because every symmetric, concave function is Schur concave (Marshall et al., 2011, Proposition C.2), it follows that $SS(\mathbf{r})$ is Schur concave if all firms are ex ante identical, that is, if $F_i = F$ and $k_i = k$ for all $i \in \mathcal{N}$. Consequently, all firms being ex ante identical provides a foundation for Proposition 2 and Corollary 1. In particular, with symmetric firms, greater symmetry in asset ownership increases social surplus directly and also makes divestiture-based remedies more effective in the sense used in Proposition 2.

Figure 1 illustrates the ex post efficiency permitting set $\mathcal{R}^e$ and $\mathbf{r}^*$ for $n = 3$ and different assumptions on distributions. As shown in panel (a) and observed by Cramton et al. (1987),

[32] For generalizations, see, for example, Williams (1999), Segal and Whinston (2016), or Delacrétaz et al. (2019). The proof of Delacrétaz et al. (2019) applies directly to the present setting because firms with payoff functions with constant marginal values are "decomposable" as defined there.

[33] For an overview of arguments for and against a social surplus (or welfare) standard in antitrust, see Pittman (2007) and the cites therein. Our framework can be applied to analyze transactions under a consumer surplus standard under the extension to downstream consumers in Section 6.1.

when firms are symmetric, $\mathbf{r}^*$ is symmetric, and ex post efficiency may be achievable even when $r_i = 0$ for any $i \in \mathcal{N}$. In this case, Corollary 1 implies that any shifts in the ownership structure towards symmetry are beneficial for social surplus. In contrast, when firms are not symmetric, $\mathbf{r}^*$ is not symmetric either and achieving ex post efficiency may require that a particular firm (firm 1 in the case of Figure 1(b)) has positive asset ownership.

4.3 Incentives for asset ownership transfers

We now show that bilateral transactions of assets that occur at the ex ante stage are quite generally profitable even if they are socially harmful. In particular, an acquisition of assets that does not disrupt the efficiency of the market is profitable for the acquiring firm if it improves the firm's outside option and so improves its expected payoff from participation in the mechanism. Further, when $n > 2$, transactions that disrupt the efficiency of the market can also be profitable for the transacting firms.

We call an ownership structure $\mathbf{r}$ *stable* if there are no mutually beneficial pairwise transactions. We then have the following contrasting results, the first applying to the case of two firms and the second to the case of more than two firms:

Proposition 6. *Given $n = 2$, mutually beneficial transactions of assets exist if and only if $\mathbf{r} \notin \mathcal{R}^e$. Put differently, $\mathbf{r}$ is stable if and only if it is ex post efficiency permitting.*

Proof. See Appendix A.5.

Proposition 6 implies that with only two firms, one expects mutually beneficial asset transactions to allow ex post efficiency to be achieved. Accordingly, with two firms, a laissez-faire competition policy maximizes social surplus because any profitable asset ownership transfer increases social surplus.

In contrast, with more than two firms, a pair of firms may have a mutually beneficial transaction that reduces social surplus, and so imposes negative externalities on nontrading firms:

Proposition 7. *Given $n \geq 3$ and at least two firms that are traders: (i) if $\Pi^e(\mathbf{r}) > 0$, then a weakly (strictly if there are traders i and j with $\eta_i + \eta_j < 1$) mutually beneficial pairwise asset ownership transfer exists; and (ii) if $\Pi^e(\mathbf{r}) = 0$ and $F_i = F$ for all $i \in \mathcal{N}$, then there exists a strictly mutually beneficial pairwise asset ownership transfer that results in ownership structure $\mathbf{r}'$ with $\Pi^e(\mathbf{r}') < 0$.*

Proof. See Appendix A.6.

Proposition 7 states that ex post efficiency permitting market structures are unstable if there are at least two traders that jointly extract less than the full budget surplus. Negating this and noting that (i) if there are no traders, then the market structure cannot be ex post efficiency permitting, and (ii) if there are more than two traders, then there necessarily exists a pair of traders that jointly extract less than the full budget surplus, we have the implication that ex post efficiency permitting market structures are stable only if there is exactly one trader or there are exactly two traders that jointly extract less than the full budget surplus. Further, with more than two firms, ownership structures on the boundary of the ex post efficiency permitting region, i.e., with $\Pi^e(\mathbf{r}) = 0$, are also not stable if there are two traders because they then have an incentive for transactions that harm rivals and reduce social surplus below the ex post efficient level. The reason two traders are required for this result is that the kind of transaction that can be profitable even though it moves away from ex post efficiency is one that shifts assets to the firm with the weakly higher worst-off type. If there are two traders, this is always possible. In contrast, it is not possible if one firm is a seller because the seller will have the higher worst-off type, but no demand for additional assets; and it is not possible if one firm is a buyer because the buyer will have the lower worst-off type, but no assets to trade.

To summarize, we have:

Corollary 2. *For two firms, the set of stable ownership structures coincides with the ex post efficiency permitting set, but for more than two firms, ex post efficiency is not possible for any stable ownership structure with at least two traders when all firms have positive surplus shares (for all $i \in \mathcal{N}$, $\eta_i > 0$).*

Because traders have an incentive for asset ownership transfers that reduce the efficiency of the market below the ex post efficient level and because such transactions harm rivals, concerns related to raising rivals' costs can be tied to the overall efficiency effects of shifts of assets among traders.[34] The contrast in results between the case with two firms and the one with more than two firms is stark. It has a precursor in the complete information literature and the debate on the Coase Theorem, with Aivazian and Callen (1981) arguing that with more than two agents, the emptiness of the core may render efficient bargaining impossible, and Coase (1981) countering that his argument (Coase, 1960) was based on the case with two agents. With that in mind, Corollary 2 provides an incomplete information

[34]This result contrasts with the finding of Farrell and Shapiro (1990) that in a Cournot setup, a profitable reallocation of capital that reduces welfare benefits the rivals, which increase their output in response to the contraction in total output by the transacting firms. Thus, they do not get a "raising rivals' costs" effect because the welfare reduction is borne entirely by the transacting firms. In contrast, in our setup, the reduction in the efficiency of the market affects all firms. See Podwol and Raskovich (2021) for a model of vertical mergers with inputs purchased by auction, with application to the CVS-Aetna merger investigation.

formalization of these opposing views and forces. Because bargaining externalities arise when asset transactions at the ex ante stage are profitable for the firms involved but are socially harmful, divestiture policies in our setting directly relate the theory of *raising rivals' costs* in industrial organization.[35]

Of course, if all n firms negotiated simultaneously, they would find ex post efficiency permitting arrangements. But our analysis of bilateral asset transactions then also implies that these would not be immune to bilateral deviations. In that sense, this analysis also provides an explanation for why asset transactions are typically bilateral transactions.

Proposition 5 showed that traders are necessary for ex post efficiency, and having all firms be traders with appropriate ownership is sufficient for ex post efficiency. Thus, having traders contributes to the efficient functioning of the market for asset usage. In contrast, as described in Proposition 6 and Corollary 2, traders can be the source of problems at the ex ante stage by having profitable bilateral transactions that reduce social surplus, making them of particular concern for competition authorities. Thus, a tension exists: traders are potentially problematic at the ex ante stage, but they improve market functioning at the ex post stage. Our results can be viewed as good news for a focus on structural rather than behavioral remedies by competition authorities because they suggest that structural remedies alone can be valuable and effective.

5 Full integration

We now turn to the analysis of full integration. As full integration even between two one-dimensional firms creates a multi-dimensional firm, we first derive the set of ex post efficiency permitting ownership structures. Then we analyze the effects of mergers under full integration on the market's ability to allocate ex post efficiently.

5.1 Possibility of ex post efficiency with multi-dimensional types

We now derive the set of ex post efficiency permitting ownership structures when some or all firms have multi-dimensional types and decreasing marginal values. To do so, we construct a revenue-maximizing efficient mechanism subject to IC and IR constraints, and we characterize the conditions under which that mechanism satisfies the no deficit constraint.

[35]The "raising rivals' costs" theory of harm argues that following a vertical merger, the integrated firm will charge more to external buyers for the inputs that it controls. The profitability of such a strategy usually relies on diversion of downstream customers to the integrated firm (see, e.g., Salop and Scheffman, 1983, 1987; Ordover et al., 1990). Raising rivals' costs theories have played a prominent, and sometimes controversial, role in antitrust practice (see, e.g., Coate and Kleit, 1990; Salop, 2017). It is notable that raising rivals' costs effects that arise in our setting do not rely on diversion.

For the model with multi-dimensional types, we require the mechanism to be *dominant strategy incentive compatible (DIC)*. Formally, a mechanism $\langle \mathbf{Q}, \mathbf{M} \rangle$ satisfies DIC if for all $i \in \mathcal{N}$, $\boldsymbol{\theta}_i$, $\boldsymbol{\theta}'_i$, and $\boldsymbol{\theta}_{-i}$, we have $\mathbf{Q}_i(\boldsymbol{\theta}) \cdot \boldsymbol{\theta}_i - M_i(\boldsymbol{\theta}) \geq \mathbf{Q}_i(\boldsymbol{\theta}'_i, \boldsymbol{\theta}_{-i}) \cdot \boldsymbol{\theta}_i - M_i(\boldsymbol{\theta}'_i, \boldsymbol{\theta}_{-i})$. This focus contrasts with the preceding analysis insofar as, there, the nature of incentive compatibility was not material (see footnote 19). Of course, it is still the case that any DIC mechanism is BIC, but with multi-dimensional types, it is an open question whether there is also an implication arrow that points the other way. The upshot of focusing on ex post efficient DIC mechanisms is that it implies, by Holmström (1979)'s theorem, that the set of admissible mechanisms is the set of Groves' schemes (Groves, 1973). That is, at reported type profile $\boldsymbol{\theta}$, the payment from any firm i to the market maker takes the form

$$M_i^G(\boldsymbol{\theta}) \equiv K_i(\boldsymbol{\theta}_{-i}) - \left(W(\boldsymbol{\theta}) - Q_i^e(\boldsymbol{\theta}) \cdot \boldsymbol{\theta}_i\right),$$

where $W(\boldsymbol{\theta})$ is social surplus under ex post efficiency at $\boldsymbol{\theta}$ and $K_i(\boldsymbol{\theta}_{-i})$ is a constant, that is, independent of firm i's reported type. The search for expected revenue maximizing, ex post efficient mechanisms that satisfy DIC and IR thus reduces to finding the constants $K_i(\boldsymbol{\theta}_{-i})$ for $i \in \mathcal{N}$ that satisfy IR and maximize expected revenue.

Using Lemma 1 and the fact that $q_i^e(\theta)$ is strictly increasing with $q_i^e(0) = 0$ and $q_i^e(1) = k_i$, if firm i has a one-dimensional type, then its worst-off type under ex post efficiency, $\hat{\theta}_i^e$, is unique and determined by the condition $q_i^e(\hat{\theta}_i^e) = r_i$. Consequently, for one-dimensional firms, the worst-off types and IR pin down the constant $K_i(\boldsymbol{\theta}_{-i})$ as $K_i(\boldsymbol{\theta}_{-i}) = W(\hat{\theta}_i^e, \boldsymbol{\theta}_{-i}) - r_i\hat{\theta}_i^e$, where $W(\hat{\theta}_i^e, \boldsymbol{\theta}_{-i})$ is social surplus under ex post efficiency when firm i's type is $\hat{\theta}_i$ and the types of all firms other than firm i are $\boldsymbol{\theta}_{-i}$. In contrast, for a multi-dimensional firm i, there are typically multiple type vectors $\boldsymbol{\theta}_i$ that satisfy the condition for a worst-off type that $q_i^e(\boldsymbol{\theta}_i) = r_i$. This implies that the market maker (or analyst) faces the nontrivial problem of determining which of those worst-off types maximizes revenue. As we now show, constant worst-off types are revenue maximizing:

Lemma 2. *Constant worst-off types maximize expected revenue under binding IR for firms' worst-off types, and firm i's constant worst-off type is uniquely defined by* $\hat{\boldsymbol{\theta}}_i^e = (\hat{\theta}_i^e, \dots, \hat{\theta}_i^e)$ *such that* $q_i^e(\hat{\boldsymbol{\theta}}^e{}_i) = r_i$.

Proof. See Appendix A.7.

The proof of Lemma 2 shows that firm i's interim expected payoff under the ex post efficient allocation, denoted by $u_i^e(\boldsymbol{\theta}_i)$, is convex and that, using envelope theorem results of Milgrom and Segal (2002),[36] $u_i^e(\boldsymbol{\theta}_i)$ can be written in terms of a path integral with respect

[36]Specifically, we apply Milgrom and Segal's Corollary 1 under their adaptation to the case of multi-dimensional types and relaxing an assumption through an application of the Monotone Selection Theorem of Milgrom and Shannon (1994) (see Migrom and Segal, 2002, footnote 10).

to a path connecting an arbitrary $\boldsymbol{\theta}_i'$ and $\boldsymbol{\theta}_i$, which allows its gradient to be written as a function of $(k_i^1, \ldots, k_i^{h_i})$, r_i, and $q_i^e(\boldsymbol{\theta}_i)$, where that gradient has all components equal to 0 if and only if $q_i^e(\boldsymbol{\theta}_i) = r_i$ for all $i \in \mathcal{N}$. This establishes that, analogous to the case of one-dimensional types, any worst-off type vector $\boldsymbol{\theta}_i^w$ for firm i satisfies $q_i^e(\boldsymbol{\theta}_i^w) = r_i$. We then show that firm i's expected payment is maximized when its worst-off type is a constant vector.

In light of Lemma 2, the revenue maximizing choice of the constant $K_i(\boldsymbol{\theta}_{-i})$ for a firm i whose type is multi-dimensional is

$$K_i(\boldsymbol{\theta}_{-i}) = W(\hat{\boldsymbol{\theta}}_i^e, \boldsymbol{\theta}_{-i}) - r_i \hat{\theta}_i^e,$$

where $\hat{\boldsymbol{\theta}}_i^e$ is the constant worst-off type defined in Lemma 2. Note that $r_i \hat{\theta}_i^e$ is the value of i's outside option when its type is $\hat{\boldsymbol{\theta}}_i^e$ because that type has constant marginal values. Consequently, the expected revenue maximizing, ex post efficient, DIC, IR mechanism has, for $i \in \mathcal{N}$, the payment rule

$$M_i^{VCG}(\boldsymbol{\theta}) \equiv W(\hat{\boldsymbol{\theta}}_i^e, \boldsymbol{\theta}_{-i}) - W(\boldsymbol{\theta}) + \mathbf{Q}_i^e(\boldsymbol{\theta}) \cdot \boldsymbol{\theta}_i - r_i \hat{\theta}_i^e, \tag{11}$$

where $\hat{\boldsymbol{\theta}}_i^e$ is, in slight abuse of notation, a scalar if i's type is one-dimensional and consists of the constant worst-off types given by Lemma 2 otherwise. The acronym VCG in (11) stands for Vickrey (1961), Clarke (1971), and Groves (1973), who first analyzed mechanisms of this form.

With the mechanism $\langle \mathbf{Q}^e, \mathbf{M}^{VCG} \rangle$ in hand, we can now define the set $\mathcal{R}^e$ of ex post efficiency permitting asset ownership vectors. As shown below, like in the case with one-dimensional types, $\mathcal{R}^e$ is nonempty, convex, and includes the ownership vector $\mathbf{r}^*$ that equalizes all firms' worst-off types.[37]

Proposition 8. *There exists $\mathbf{r}^* \in \Delta_{\mathbf{k}}$ and $\hat{\theta}^* \in (0,1)$ such that each firm with a one-dimensional type has worst-off type $\hat{\theta}^*$ and each firm with a multi-dimensional type has worst-off type $(\hat{\theta}^*, \ldots, \hat{\theta}^*)$; moreover, the set of ex post efficiency permitting ownership vectors, $\mathcal{R}^e$, is nonempty, convex, and contains $\mathbf{r}^*$.*

Proof. See Appendix A.8.

[37]The result that the ownership $\mathbf{r}$ that equalizes the firms' worst-off types also permits ex post efficiency is the driving force behind the possibility result of Cramton et al. (1987) for $F_i = F$ and $k_i = 1$ for all i. Che (2006) provides the first generalization to nonidentical distributions while maintaining $k_i = 1$, while Liu et al. (forth.) show that this insight extends to $k_i \neq k_j$. Segal and Whinston (2011) derive a status quo, which can be thought of as some $\mathbf{r}$ in our context, that permits ex post efficiency for a more general allocation problem, which when applied to the present problem is not necessarily $\mathbf{r}^*$. See also Schweizer (2006) for a fixed-point based possibility result. Analyzing a partnership model with nonidentical distributions and $k_i = 1$ in which the market maker maximizes a weighted sum of social surplus and revenue, Loertscher and Wasser (2019) show that, with private values, the objective function remains concave in $\mathbf{r}$ and that the optimal $\mathbf{r}$ maintains the property that firms' worst-off types are equal if possible.

An implication of Proposition 8 is that our analysis for asset ownership transfers regarding which mergers should be approved or blocked and the potential role for social surplus restoring divestitures carries over to transactions that involve the full integration of firms, with one difference and one qualification. The difference is that with full integration, the set of ex post efficiency permitting ownership structures varies with the transaction, whereas for pure asset ownership transfers, it is fixed. Consequently, divestitures that restore ex post efficiency must be calculated with regard to the post-transaction ex post efficiency permitting set. But because divestitures only involve a change in asset ownership, that set is the same before and after divestiture. The qualification is that with full integration that induces a non-degenerate multi-dimensional firm, the analysis can only be applied to mergers such that ex post efficiency is possible before or after the transaction (and any divestitures) because with multi-dimensional types the second-best mechanism for asset ownership structures that do not permit ex post efficiency is not known. In contrast, with pure asset ownership transfers and one-dimensional types, the second-best analysis applies to any asset ownership structure.

5.2 Horizontal and vertical mergers

It is useful to distinguish different categories of full integration. We refer to the full integration of two buyers or of two sellers as a *horizontal merger* and to the full integration of one buyer and one seller as a *vertical merger*. Mergers that are neither strictly horizontal nor strictly vertical are referred to as *conglomerate mergers* and are analyzed in the next subsection.

We say that a merger *makes ex post efficiency possible* if ex post efficiency is possible after the merger whereas it was not before. Likewise, we say that a merger *makes ex post efficiency impossible* if ex post efficiency was possible before the merger but not after it. A merger is *socially desirable* if it can make ex post efficiency possible and never makes it impossible. Likewise, a merger is *socially undesirable* if it can make ex post efficiency impossible and never makes it possible.

Theorem 1. *Horizontal mergers are socially undesirable, whereas vertical mergers are socially desirable.*

Proof. See Appendix A.9.

The first part of Theorem 1 resonates with the result for one-dimensional private information of Loertscher and Marx (2022) that, in a two-sided setting, horizontal mergers weakly reduce social surplus, and the result of of Loertscher and Marx (2019) that a merger in a procurement setting harms the buyer.[38] The vertical merger result in Theorem 1 highlights an important difference to the second-best analysis afforded by one-dimensional types. To illustrate, consider a setting with, before integration, one buyer and $n-1$ sellers where $n \geq 2$. Then ex post efficiency is not possible prior to a vertical merger of the buyer with one seller, and it remains impossible post merger unless $n = 2$. If types are uniformly distributed and all firms have the same maximum demands, then, as illustrated in Appendix B.5, the vertical merger increases second-best social surplus for $n = 3$ and $n = 4$ and decreases it otherwise. This contrasts with the analysis here, which shows that expected revenue under ex post efficiency increases with vertical integration, but does not contradict the second-best analysis. It merely shows that expected revenue under ex post efficiency is not a fool-proof test for the behavior of second-best social surplus. As discussed in Section 6.3, which extends Loertscher and Marx (2022, Proposition 7), with nonoverlapping supports there exist cases in which a vertical merger reduces social surplus. For example, if there is one buyer with one-unit demand and type support $[1, 2]$ and two sellers, each with one-unit supply and type support $[0, 1]$, then ex post efficiency is possible pre-merger, but following vertical integration, the vertically integrated firm acts as a buyer vis à vis the outside seller with type distributed on $[0, 1]$, and so ex post efficiency is not possible—a vertical merger in this case creates a Myerson-Satterthwaite problem.

5.3 Conglomerate mergers

We now turn to mergers that are neither purely horizontal nor purely vertical, which, as mentioned, are referred to as conglomerate mergers. For example, the merger of two traders would be a conglomerate merger, as would the merger of a trader with a buyer or a seller. Of course, a trader that has positive but very low asset ownership is "close," in a sense, to being a buyer, and a trader that has asset ownership less than but close to its maximum demand resembles a seller. This observation, together with Theorem 1, suggests that some conglomerate mergers are socially desirable, while others are socially undesirable. As we now show, this is indeed the case.

To see that some conglomerate mergers are socially undesirable, consider the case of n pre-integration firms, each with maximum demand of 1 and a one-dimensional, uniformly distributed type. Then pre-integration, expected revenue under ex post efficiency and binding

[38] In Loertscher and Marx (2019), a horizontal merger is neutral for social surplus if the buyer uses an efficient procurement mechanism because the buyer's value is commonly known, but it increases the expected payment that the buyer has to make.

IR for the firms' worst-off types is

$$\sum_{i=1}^{n} \mathbb{E}_{\boldsymbol{\theta}}\Big[W(\hat{\theta}_i^e, \boldsymbol{\theta}_{-i}) - W(\boldsymbol{\theta}) + Q_i^e(\boldsymbol{\theta})\theta_i - r_i\hat{\theta}_i^e\Big] = \big(n - (n+1)\sum_{i=1}^{n} r_i^{\frac{n}{n-1}}\big)\phi_n, \quad (12)$$

where $\phi_n > 0$. The expression in (12) is positive when it is evaluated at $n = 3$ with $r_3 = 0$ and $r_1 = 1 - r_2 \in (0.3, 0.7)$, indicating that ex post efficiency is possible in this case (see Figure 4(a)).[39] But, a conglomerate merger involving firms 1 and 2 results in a market with one buyer (firm 3) and one seller (the integrated firm with its two-dimensional type), and so ex post efficiency is not possible, providing us with an example of an undesirable conglomerate merger. In a sense, the merging firms in this example are sufficiently substitutable that the result from Theorem 1 for horizontal mergers applies.

In contrast, if $r_3 \in (0.12, 0.30)$ and either $r_1 = 0$ or $r_2 = 0$, then as indicated in Figure 4(a), ex post efficiency is not possible prior to a conglomerate merger involving firms 1 and 2, but it is possible after the merger.[40] Thus, this conglomerate merger is socially desirable—the merging firms are sufficiently complementary, with one being a seller and the other having substantial demand beyond its pre-merger asset ownership, that the result from Theorem 1 for vertical mergers applies.

We summarize with the following proposition:

Proposition 9. *Some conglomerate mergers are socially desirable while others are socially undesirable.*

Of course, from Proposition 9 arises the question under which conditions conglomerate mergers are likely to be harmful or benign. Consistent with our first example above, when ex post efficiency is possible pre-merger, a conglomerate merger reduces expected social surplus if it allows the merging firms to *corner the market*, i.e., gives the merging parties ownership of all the asset, or if the merger creates a merged entity that is the only buyer in the market. This result relates to Loertscher and Marx (2022, Proposition 7) on integration rendering ex post efficiency impossible if it results in a market with only buyers and sellers (and no traders) with distributions having identical supports.

In contrast, starting from a pre-merger market in which ex post efficiency is not possible, a conglomerate merger increases expected social surplus if it equalizes the worst-off types of

[39]Interestingly, when $n = 3$ and types are drawn from $F(x) = \sqrt{x}$, expected revenue under ex post efficiency is $\frac{1}{10}(3 - 5\sum_{i=1}^{3} r_i^2)$, which is to say that it depends on the Herfindahl–Hirschman Index (HHI). For an HHI sufficiently close to its maximum value of 1, ex post efficiency is not possible, and for an HHI sufficiently close to its minimum value of 1/3, ex post efficiency is possible.

[40]The condition for ex post efficiency not to be possible prior to a merger is that (12) is negative, and the condition for ex post efficiency to be possible post merger is that $\frac{1}{12}(5 - 6r_1^2 - 12r_1r_2 - 6r_2^2 - 8r_3^{3/2}) \geq 0$, which holds with equality when $r_1 = 0$, $r_2 = 0.70$, and $r_3 = 0.30$ (and, symmetrically, when $r_1 = 0.70$, $r_2 = 0$, and $r_3 = 0.30$).

the post-merger firms: that is, for a merger of firms 1 and 2, there exists $\hat{\theta}^* \in (0,1)$ such that in the post-merger market $q^e_{1,2}(\hat{\theta}^*, \dots, \hat{\theta}^*) = r_1 + r_2$ and for $i \in \{3, \dots, n\}$, $q^e_i(\hat{\theta}^*, \dots, \hat{\theta}^*) = r_i$. Further, by the continuity of the revenue function under ex post efficiency in the firms' worst-off types, this continues to be true as long as the merger results in firms having worst-off types that are sufficiently close to one another.

Using Theorem 1 and the continuity of the market maker's expected revenue under binding IR for worst-off types, it follows that conglomerate mergers that are "sufficiently horizontal" are socially undesirable, whereas those that are "sufficiently vertical" are socially desirable, as formalized in the following corollary:

Corollary 3. *Given firm 1 with $r_1 \in [0, k_1]$ and firm 2 with $r_2 = 0$, there exist $\underline{r}_1 \in (0, k_1)$ and $\overline{r}_1 \in (0, k_1)$ such that the full integration of firms 1 and 2 is socially undesirable if $r_1 \in [0, \underline{r}_1)$ and socially desirable if $r_1 \in (\overline{r}_1, k_1]$.*

These results suggest that non-horizontal mergers are a potential source of concern if they take the form of conglomerate mergers, and that the concern tends to be more severe the more significant are the horizontal aspects of the merger. As a concrete case in point, full integration between two airlines would typically be a conglomerate merger because on some routes the two firms' services are complements, while on others their services are competing. The more important are the latter, the heavier is the weight on the horizontal aspect of the merger.

Even without having the second-best mechanism for multi-dimensional types, some results regarding firms' incentives to engage in full integration can still be obtained. For example, if $\mathbf{r} \in \mathcal{R}^e$, then any vertical merger and any conglomerate merger that results in ex post efficiency still being possible will be profitable for the merging firms. To see this, note that the allocations are the same before and after integration and the merged firm's outside option increases relative to the sum of merging firms' pre-merger outside options because the merged firm can optimize to take advantage of whichever type is larger. The same would not be true for a horizontal merger because two buyers would still have a zero outside option after the merger and two sellers would still have the same outside option after the merger. These incentive effects reinforce concerns regarding horizontal mergers and resonate with the notion that vertical mergers tend to be benign.

5.4 Illustration of merger effects

As an illustration of the effects of full integration, consider the case of $n = 3$ with uniformly distributed types and a merger of firms 1 and 2. In a partnership setup with $k_i = 1$ for all $i \in \mathcal{N}$, the merged entity applies all the assets that it is allocated to the larger of its two type draws. As a result, as mentioned in footnote 17, the merged entity behaves as a firm with a one-dimensional type that is drawn from the distribution of the maximum of independent draws from the two pre-merger distributions.

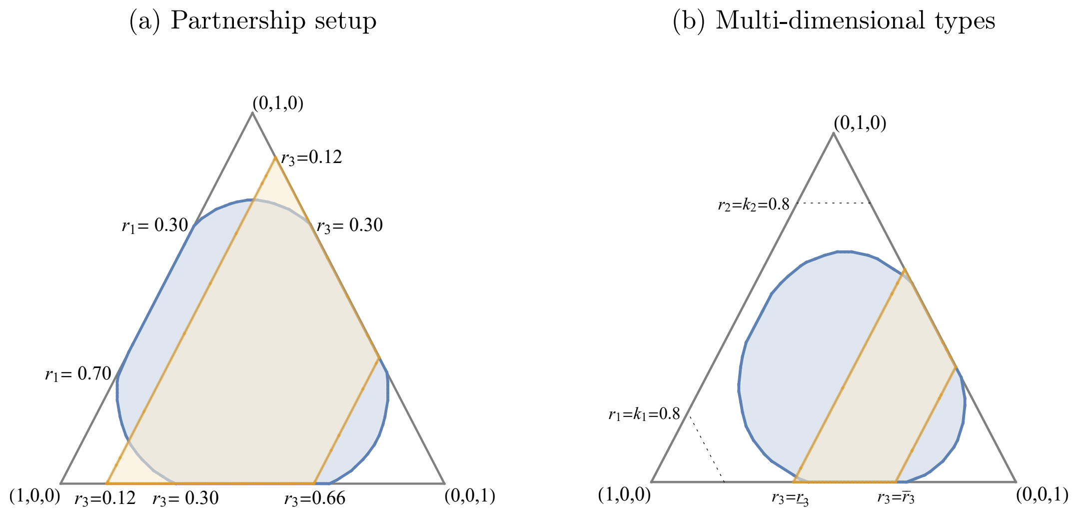

Figure 4: Effects of the full integration of firms 1 and 2 on the ex post efficiency permitting set, including the pre-merger (blue) ex post efficiency permitting set and the post-merger (orange) set of vectors (r_1, r_2, r_3) such that ex post efficiency is possible following the full integration of firms 1 and 2. Panel (a) assumes $k_1 = k_2 = k_3 = 1$. Panel (b) assumes that $k_1 = k_2 = 0.8$ and $k_3 = 1$. Both panels pre-merger types are one-dimensional and uniformly distributed.

Figure 4 illustrates that for some pre-integration asset ownership, full integration decreases expected social surplus ($\mathbf{r}$ in the blue shaded region but not in the orange shaded region), while for other pre-integration asset ownership, full integration increases expected social surplus ($\mathbf{r}$ in the orange shaded region but not in the blue shaded region). The possibility of social surplus increasing mergers is perhaps surprising—it does not arise in a one-sided or two-sided settings (Loertscher and Marx, 2019, 2022),[41] but rather relies on the presence of traders that buy for some type realizations and sell for others. For intuition, consider the example of the partnership setting shown in Figure 4(b) with $\mathbf{r} = (0, 0.8, 0.2)$. In this case, ex post efficiency is not possible because, in effect, the ownership structure is too asymmetric relative to the firms' symmetric productivities. However, the integration of firms 1 and 2 improves their distribution, allowing the new firm to, in effect, "grow into" its large ownership of 0.8. In this way, full integration can increase social surplus because it better aligns relatively large ownership with a relatively strong distribution (see Proposition

[41]For example, in a two-sided setting with one buyer and multiple sellers, where $\mathbf{r} = (0, 1/(n-1), \ldots, 1/(n-1))$ and $\mathbf{k} = (1, 1/(n-1), \ldots, 1/(n-1))$, a merger of sellers merely reduces competition among sellers, and a merger of the buyer and a seller both reduces competition among sellers and reduces the buyer's willingness to pay for outside units, both with negative consequences for social surplus.

10 below for a formalization of this). In a sense, a stronger distribution for the firm with the larger asset ownership may be just what the doctor ordered.

In the case displayed in Figure 4(b), the merger combines the two identical firms whose maximum demands are smaller than the outside firm's, giving the merger a substantial horizontal aspect. As shown in the figure, the set of ownership structures such that full integration makes ex post efficiency impossible (blue but not orange) is relatively large, while the set of ownership structures such that the reverse occurs (orange but not blue) is very small. While, consistent with Proposition 9, we see the possibility of both beneficial and harmful conglomerate mergers, the largely pessimistic view for this particular setup with its substantial horizontal aspect is supported by the result of Theorem 1 that horizontal mergers are socially undesirable.

Figure 4 also illustrates that, as was the case for transfers of asset ownership, harms associated with full integration can sometimes be remedied through divestitures.[42] This also illustrates the relevance of the three components of a competition authority's social surplus maximizing response to a merger described in Proposition 1.

5.5 Optimal ownership and revenue under ex post efficiency

Much of the discussion above has focused on whether ex post efficiency is possible. This analysis provides the foundation for the guidance that divestitures should, if possible, be designed to secure ownership structures in $\mathcal{R}^e$. But to the extent that unmodeled transactions costs or market frictions are present, a competition authority might have a preference for ownership structures that are not just an element of $\mathcal{R}^e$, but that are robust to such unmodeled costs as best possible. This can be a achieved with the ownership structure $\mathbf{r}^*$, which maximizes expected revenue under ex post efficiency and binding IR for the firms' worst-off types.[43] This brings to the forefront the question of how $\mathbf{r}^*$ varies with the size and strength of firms in the market. The possibility of differences in firms' maximum demands allows for differences in firm sizes, and differences in firms' distributions can be thought of as differences in productivity across the firms, and across types within a firm. Consider firms i and j with the same dimensionality of their types, $h_i = h_j = h$, and maximum demand

[42] For example, focusing on panel (a), when r_3 is less than 0.12 while r_1 and r_2 are similar (blue region in the middle, left part of the simplex), full integration of firms 1 and 2 reduces social surplus, but, in that case, the harm can be remedied through a divestiture of assets to firm 3. In contrast, when r_3 is greater than 0.66 while r_1 and r_2 have similar asset ownership (blue region in the lower, right corner of the simplex), the harm from full integration of firms 1 and 2 cannot be remedied through a divestiture of assets to firm 3.

[43] Following the interpretation mentioned in the introduction of asset ownership as long-term labor contracts, the social surplus benefit of symmetric ownership for symmetric firms is consistent with policies such as the Union of European Football Association's Financial Fair Play Regulations, which restrict European football clubs from consistently operating at a loss and thereby promote symmetric team rosters.

vectors $\mathbf{k}_i = (k_i^1, \ldots, k_i^h)$ and $\mathbf{k}_j = (k_j^1, \ldots, k_j^h)$. All else equal between firms i and j, $\mathbf{k}_i \geq \mathbf{k}_j$ and $\mathbf{k}_i \neq \mathbf{k}_j$ imply that $q_i^e(\boldsymbol{\theta}) > q_j^e(\boldsymbol{\theta})$ for $\boldsymbol{\theta} \in (0,1)^h$. Further, all else equal between firms i and j, if F_i^ℓ first-order stochastically dominates F_j^ℓ for all ℓ and $F_i^\ell \neq F_j^\ell$ for at least one ℓ, then $q_i^e(\boldsymbol{\theta}) > q_j^e(\boldsymbol{\theta})$ for $\boldsymbol{\theta} \in (0,1)^h$. Combining these observations, we have the following result:[44]

Proposition 10. *All else equal, firms with larger maximum demands or stronger distributions according to first-order stochastic dominance have larger asset ownership in* $\mathbf{r}^*$*: that is, given firms i and j with h-dimensional types: (i) assuming that* $F_i^\ell = F_j^\ell$ *for all* $\ell \in \{1, \ldots, h\}$*, if* $\mathbf{k}_i \geq \mathbf{k}_j$ *and* $\mathbf{k}_i \neq \mathbf{k}_j$*, then* $r_i^* > r_j^*$*; and (ii) assuming that* $\mathbf{k}_i = \mathbf{k}_j$*, if* F_i^ℓ *first-order stochastically dominates* F_j^ℓ *for all* $\ell \in \{1, \ldots, h\}$ *and* $F_i^\ell \neq F_j^\ell$ *for some* ℓ*, then* $r_i^* > r_j^*$*.*

Proposition 10 offers guidance for a divestiture strategy that, in an abundance of caution, strives to identify an ownership structure that is maximally robust to unmodeled market frictions. In this case, the target asset ownership structure should be $\mathbf{r}^*$, which gives relatively more assets to firms with relatively greater maximum demands and relatively more assets to firms with relatively greater value for their use. Further, if firms are ex ante identical, that is, have identical maximum demands and distributions, then this robustness criteria is met by having symmetric asset ownership. This provides a rationale for divestitures in markets with symmetric firms that promote symmetric asset ownership.

An open question of practical relevance remains for mergers such that ex post efficiency is possible neither before nor after the merger and such that a second-best analysis is not available because the multi-dimensionality of the integrated firm's type is nontrivial. Of course, if a post-merger divestiture exists that makes ex post efficiency possible, then a social surplus maximizing authority should approve the merger and require such a divestiture because the merger cum divestiture offers the opportunity to increase social surplus. So, the open issue pertains to the subset of mergers such that ex post efficiency is possible neither before nor after the merger, even with a divestiture, and when the second-best mechanism post merger is not known. A natural and feasible way of evaluating such a merger would be to compare expected revenue under ex post efficiency before and after the merger, which in either case is negative by the assumption that ex post efficiency is impossible. If the merger increases that revenue, then a natural rule would be to approve the merger, and to otherwise block it.[45] This rule is in the spirit of the above discussion of divestiture strategies and of the

[44] For a proof for the case of one-dimensional types, see Liu et al. (forth., Proposition 2). See also the related result of Che (2006) for one-dimensional types, $k_i = 1$ for all $i \in \mathcal{N}$, and distributions ranked by first-order stochastic dominance.

[45] Of course, the rule could be augmented by also considering all possible divestitures post-merger and to evaluate expected revenue after both the merger and the divestitures.

paper more broadly insofar as larger revenue under ex post efficiency offers more "leeway" for the market to operate "well." It also has a (partial) foundation in some one-dimensional setups, where the second-best mechanism is known. For example, with ex ante identical firms, both social surplus and expected revenue under ex post efficiency are Schur-concave. Thus, if revenue under ex post efficiency increases because the ownership structure becomes more symmetric, then social surplus increases as well. Of course, because majorization is a partial order, revenue can increase without the ownership structure becoming more symmetric, and reliability of the test would require social surplus to increase as well in such situations. While for models with interior ownership assessing whether this is the case is challenging even numerically, it is clearly the case in a bilateral trade setting à la Myerson and Satterthwaite (1983) in which the buyer's support shifts upward.[46] For an illustration of its application beyond this case, see Appendix B.4.

6 Extensions and discussion

In the following three subsections, we show how the model can be extended to accommodate downstream consumers and consumer surplus considerations, investment, and nonidentical supports of the firms' type distributions, respectively. In the last subsection, we discuss paths toward second-best mechanisms with full integration that consist of transforming multi-dimensional distributions into one-dimensional distributions that are equivalent in the sense of inducing the same ex post efficiency permitting set.

6.1 Downstream consumers and consumer surplus

We now extend the model by adding downstream consumers and provide conditions such that, taking as given that firms have market power vis à vis downstream consumers, social surplus maximization as described above is equivalent to the maximization of consumer surplus. Under these conditions, the prescriptions above are appropriate regardless of whether the focus is on social surplus or consumer surplus.

To accommodate downstream consumers and consumer surplus considerations, assume

[46] To be precise, assume firm 1 is a buyer whose type is drawn from a distribution with shifting support $[\underline{\theta}, \underline{\theta}+1]$ and firm 2 is a seller whose type is drawn from a distribution with support $[0, 1]$. Ex post revenue under ex post efficient is $\max\{\theta_2, \underline{\theta}\} - \min\{\theta_1, 1\}$, which increases in $\underline{\theta}$, and expected revenue increases because of that and because the probability that the seller is paid less than θ_1 increases as $\underline{\theta}$ increases. At the same time, social surplus under the second-best mechanism increases as $\underline{\theta}$ increases. That said, revenue under ex post efficiency is not a universally reliable indicator for the performance of the second-best mechanism; see, for example, Loertscher and Marx (2023b, Propositions 4 and 5) and the discussion after Theorem 1 above.

that asset usage is an input that improves the quality of a product that firms sell in individual downstream markets. Let $P_i(y)$ be the willingness to pay of a typical consumer in market i for the y-th unit of quality 1 and assume that the willingness to pay for the y-th unit of quality $q \in [0, k_i]$ is $qP_i(y)$. (This extends straightforwardly to the case of multi-dimensional, decreasing marginal values.) Assume further that the marginal cost of production of the downstream product is zero, the private information of each firm i pertains to the mass $\sigma_i > 0$ of identical consumers in its downstream market, and the inverse demand function $P_i(y)$ is decreasing. A maximizer of $yP_i(y)$ over y is denoted by y_i^*. For a given realization σ_i of the mass of consumers in market i, the firm's marginal willingness to pay for quality q is $\theta_i \equiv \sigma_i y_i^* P_i(y_i^*)$.

Given input quality $q \in [0, k_i]$, in equilibrium per-capita consumer surplus in market i is $q(\int_0^{y_i^*} P_i(y)dy - y_i^* P_i(y_i^*))$, per-capita profit is $qy_i^* P_i(y_i^*)$, and, dividing consumer surplus by firm profit, the pass-through rate of firm profit to consumer surplus in market i is

$$\gamma_i \equiv \frac{1}{y_i^* P_i(y_i^*)} \int_0^{y_i^*} P_i(y)dy - 1.$$

If all downstream markets have the same consumer surplus pass-through, that is, if $\gamma_i = \gamma$ for all $i \in \mathcal{N}$, then social surplus maximization, with social surplus narrowly defined as firms' profits, is the appropriate objective even for a planner that also accounts for downstream consumer surplus, including the case of a planner that only values consumer surplus.

Of course, the condition that $\gamma_i = \gamma$ for all $i \in \mathcal{N}$ will not be universally met. For example, if some firm j only serves an export market and the antitrust authority does not account for consumer surplus in other countries, then $\gamma_j = 0$ will hold, and the authority would like the market (and the market structure) to discriminate against firm j. In such settings, unequal bargaining weights are, in a sense, just what the doctor ordered because they can serve to shift the allocation towards firms whose profits pass through to consumers at a greater rate.[47]

6.2 Investment

Investment incentives feature prominently in concurrent competition policy debates and merger cases as well as in organizational economics. For example, investment incentives were at center stage in the Dow-Dupont merger in the United States or the EC's GE-Almstrom merger decision.[48] As is reasonably well known in the mechanism design literature,

[47]Le (2022) provides a model in which incomplete information bargaining weights are determined endogenously via complete information bargaining (e.g., Nash or Rubinstein) that occurs at the ex ante stage.

[48]See the U.S. DOJ's Competitive Impact Statement in the Dow-Dupont merger (https://www.justice.gov/atr/case-document/file/973951/download, pp. 2, 10, 15, 16) or the EC's statement

with incomplete information, efficient markets imply efficient investments quite generally.[49] Because the same does not appear to be the case for practitioners and scholars working in antitrust, we now provide a short, yet general, analysis of investment incentives with incomplete information.

Consider the model that allows for firms to have multi-dimensional types. Each firm i has a set of possible investments $\mathcal{I}_i$ with each investment $I_i \in \mathcal{I}_i$ for firm i being associated with a known cost $C_i(I_i)$. Investments affect the firms' type distributions without affecting their supports, so that, when accounting for investments, we now write $f_i(\theta, I_i)$ for the density of i's type θ given investment I_i if firm i's type is one-dimensional. After the investment I_i of multi-dimensional firm i, the density its j-th type θ_i^j is denoted $f_i^j(\theta_i^j, I_i)$. For a multi-dimensional firm i, let $f_i(\boldsymbol{\theta}_i, I_i) = \times_{j=1}^{h_i} f_i^j(\theta_i^j, I_i)$ denote its joint density.

Letting $f(\boldsymbol{\theta}, \mathbf{I}) \equiv \times_{i\in\mathcal{N}} f_i(\boldsymbol{\theta}_i, I_i)$, where in slight abuse of notation $\boldsymbol{\theta}_i$ is a scalar if firm i's type is one-dimensional, expected social surplus under ex post efficiency is

$$SS^{\text{invest}}(\mathbf{I}) = \int_{[0,1]^H} W(\boldsymbol{\theta}) f(\boldsymbol{\theta}, \mathbf{I}) d\boldsymbol{\theta} - \sum_{i\in\mathcal{N}} C_i(I_i),$$

where, as may be recalled, H is the number of types. We make the weak assumption that there exists a feasible vector of investments $\overline{\mathbf{I}}$ that maximizes $SS^{\text{invest}}(\mathbf{I})$.

Consider now the following investment game. In stage one, all firms simultaneously choose their investments, which are neither observable nor contractible. If a vector of investments $\mathbf{I}$ is a pure strategy Nash equilibrium outcome of the game with market structure $\mathbf{r}$, then the market operates with expected revenue computed according to $f(\boldsymbol{\theta}, \mathbf{I})$. In this setup, if the market is ex post efficient given investments $\overline{\mathbf{I}}$, then the investment game has a Nash equilibrium in which each firm $i \in \mathcal{N}$ invests $\overline{I}_i$.[50] The intuition is simple. Under the VCG mechanism, every firm is the residual claimant to the social surplus generated by its type and therefore in expectation of the social value of its investment. Thus, the individual incentives are perfectly aligned with the social planner's objective.

Moreover, there exists an asset ownership structure, denoted $\mathbf{r}^*_{\overline{\mathbf{I}}}$, that permits ex post efficiency and induces the investments $\overline{\mathbf{I}}$ as a Nash equilibrium outcome of the two-stage game. Specifically, for each i, let $q_i^e(\boldsymbol{\theta}; \mathbf{I}_{-i}) = \mathbb{E}_{\boldsymbol{\theta}_{-i}}[\mathbf{Q}_i^e(\boldsymbol{\theta}, \boldsymbol{\theta}_{-i})]$ be firm i's interim expected allocation under the ex post efficient allocation rule given investments $\mathbf{I}_{-i}$ by the other

in Annex I, paragraph 32 (https://ec.europa.eu/competition/mergers/cases/decisions/m7278_6808_3.pdf). For additional examples, see, e.g., Gilbert and Sunshine (1995) and Katz and Shelanski (2017).

[49] See, for example, Milgrom (2004) for the second-price auction and Krähmer and Strausz (2007), Loertscher and Marx (2022), and Liu et al. (forth.) for more general setups with one-dimensional types.

[50] This generalizes the insights of Krähmer and Strausz (2007), who assume that the cost functions are differentiable, and of Liu et al. (forth.), who like Krähmer and Strausz, assume one-dimensional types. The latter show further that if firm i's investment affects firm j's distribution, then the VCG mechanism does not induce efficient investments.

firms, with the expectation taken with respect to the density $f_{-i}(\boldsymbol{\theta}_{-i}, \mathbf{I}_{-i}) \equiv \times_{j \neq i} f_j(\boldsymbol{\theta}_j, I_j)$, where in abuse of notation $\mathbf{Q}_i^e$ (and $\boldsymbol{\theta}$ and $\hat{\boldsymbol{\theta}}^e$) are scalars if i's type is one-dimensional. Then there is a unique $\hat{\boldsymbol{\theta}}^e(\mathbf{I}) \in (0,1)$ such that $\sum_{i \in \mathcal{N}} q_i^e(\hat{\boldsymbol{\theta}}^e(\mathbf{I}), \mathbf{I}_{-i}) = 1$. Defining $\mathbf{r}_{\mathbf{I}}^* \equiv (q_1^e(\hat{\boldsymbol{\theta}}^e(\mathbf{I}); \mathbf{I}_{-1}), \ldots, q_n^e(\hat{\boldsymbol{\theta}}^e(\mathbf{I}); \mathbf{I}_{-n}))$, the result is an implication of Proposition 8.

Because efficient markets induce efficient investments, policies that make markets efficient are desirable with and without consideration for investment incentives. Put differently, given an efficient market, if there are concerns that investment incentives will be adversely affected by a change in asset ownership, the concern must derive from the change causing the market to no longer operate efficiently.

6.3 Integration in a setting with nonidentical supports

Up to here, we have assumed that all distributions had identical supports of $[0,1]$. We now drop this restriction. Beyond generality, the immediate purposes of this extension are that it allows us to nest settings with nonidentical supports studied in Loertscher and Marx (2022) and to generalize the analysis beyond there. Among other things, we will see that, in contrast to Theorem 1 above, vertical integration can be socially undesirable with nonidentical supports. While this basic insight was present in Loertscher and Marx (2022), the analysis there was restricted to settings with either only one buyer or only one seller before vertical integration because that paper did not tackle challenges associated with having firms whose types are multi-dimensional.[51]

To relate our results to those in that paper, we assume that firms are divided into a set $\mathcal{N}_U$ of $N_U \geq 1$ "upstream" sellers and a set $\mathcal{N}_D$ of $N_D \geq 1$ "downstream" buyers. Prior to integration, all firms are assumed to have one-dimensional types and maximum demand k, where for upstream seller i, $r_i = k$, and for downstream buyer i, $r_i = 0$. The support of the upstream sellers' type distributions is $[0,1]$, while the support of the downstream buyers' type distributions is $[\underline{\theta}, 1+\underline{\theta}]$ with $\underline{\theta} \geq 0$. This setting nests the case of identical supports studied thus far by setting $\underline{\theta} = 0$, nonidentical but overlapping supports, which corresponds to $\underline{\theta} \in (0,1)$, and nonoverlapping supports, $\underline{\theta} \geq 1$. Detailed background for the following and a discussion of the different configurations is provided in Appendix B.2.

If $\underline{\theta} < 1$ and $N_U = N_D = 1$, then ex post efficiency is not possible before vertical integration because of the impossibility result of Myerson and Satterthwaite (1983), but

[51] It is true that a merger between a buyer and seller in the setting of Loertscher and Marx (2022) creates a firm that has a two-dimensional type. However, because of the one-to-many setting before integration, the trading position of the integrated firm is predetermined—either it will trade as a buyer, which happens if there was only one buyer before integration, in which case the minimum draw from the two distributions is the relevant statistic, or it will trade as a seller, in which case it is the distribution of the maximum draw that matters.

becomes possible with vertical integration. Thus, in this case, a vertical merger is socially desirable even away from identical supports.

Now turn attention to the case of $\underline{\theta} \geq 1$. Then for any N_U and N_D, ex post efficiency is possible without vertical integration—a second-price auction to shorten the long side of the market (if any) with any reserve $p \in [1, \underline{\theta}]$ induces the ex post efficient allocation in dominant strategies without running a deficit or violating IR. Consequently, vertical integration cannot increase social surplus. Whether it is socially harmful then depends on the specifics of the environment. If $N_U = N_D$ prior to integration, then ex post efficiency remains possible after vertical integration because in the market after vertical integration, the integrated firm can without loss of generality be assumed to be self-sufficient and make in house in order to satisfy its demand, leaving a market with $N_U - 1$ upstream and $N_D - 1$ downstream firms. As just seen, ex post efficiency is possible in this case. Interestingly, the innocuous nature of vertical integration is not necessarily monotone in the number of upstream firms. That is, vertical integration can make ex post efficiency impossible when $N_U > N_D$. For $N_D = 1$, this observation was made in Loertscher and Marx (2022). Effectively, the vertically integrated firm becomes a buyer on the input market where there are $N_U - 1 \geq 1$ independent suppliers, in which case, by a generalized impossibility theorem in the spirit of Myerson and Satterthwaite (1983), ex post efficiency is not possible. As shown in Appendix B.2, the potential for socially harmful vertical integration when $\underline{\theta} \geq 1$ extends to settings with $N_D > 1$ before integration. However, any nonintegrating downstream firms always buy under ex post efficiency, and therefore always pay $\underline{\theta}$ for every unit they obtain, so for $N_D > 1$, increases in $\underline{\theta}$ translate into increases in revenue and thereby mitigate any negative effects of vertical integration.

6.4 Toward second-best mechanisms with full integration

As mentioned, for the model with multi-dimensional types, the second-best mechanism is not known, and there appears to be limited hope that it will be any time soon. A natural path forward is then to transform the multi-dimensional random variable characterizing the merged firm into a one-dimensional one that is, in a sense to be defined, equivalent. The set $\mathcal{R}^e$ that characterizes the ownership structures that permit ex post efficiency after full integration by two one-dimensional firms may be useful in that regard because a one-dimensional distribution that induces the same set $\mathcal{R}^e$, keeping all other firms the same, would be equivalent in an obvious and meaningful sense.

Consider the case with $k_i < 1$ for one or both of the merging firms prior to the merger so that after the merger, the merged firm's type is nondegenerately multidimensional (in contrast, the merged firm's type is effectively one-dimensional if the merging firms have maximum demands of 1—see footnote 17). In this case, with additional assumptions, one can still derive a one-dimensional type distribution that induces the same set $\mathcal{R}^e$. As an illustration, consider the case of $k_1 = k_2 < 1 = k_3$ and uniformly distributed types. Following a merger of firms 1 and 2, the merged firm's type is two-dimensional, with associated maximum demands of k_1 and k_2, respectively, and its asset ownership is $r_1 + r_2$. The post-merger set of ex post efficiency permitting ownership vectors has the form $\mathcal{R}^e = \{(1-r_3, r_3) \mid r_3 \in [\underline{r}_3, \overline{r}_3]\}$, where $0 < \underline{r}_3 < \overline{r}_3 < 1$. As shown in Appendix B.6, one can construct a density $\tilde{f}$ such that this same ex post efficiency permitting set $\mathcal{R}^e$ obtains when the merged entity has a one-dimensional type drawn from a distribution with density $\tilde{f}$. This density can then provide a basis for estimating effects outside $\mathcal{R}^e$ based on the (known) second-best mechanism for the case in which the merged firm has a one-dimensional type.

7 Conclusion

This paper studies an incomplete information model in which traders—firms that as a function of type realizations either make or buy or sell—are necessary for ex post efficiency. For identical supports, horizontal (vertical) mergers never make ex post efficiency possible (impossible), while some conglomerate mergers, that is, mergers that are neither horizontal nor vertical, make it possible while others foil it. The analysis provides both a rationale and guidance for divestitures that can eliminate harm from mergers. Because the analysis is based on a mechanism design approach, the effects identified derive from the primitives of the model and do not rest on contractual restrictions. Full integration, defined as a merger that combines the merging firms' assets and their private information, creates a multi-dimensional firm with decreasing marginal values. The behavior of the model with multi-dimensional types is remarkably similar to the one with one-dimensional types under ex post efficiency. For the model with one-dimensional types and identical supports, we show that equal bargaining weights are necessary for ex post efficiency and that, quite generally, firms have private incentives for bilateral asset ownership transfers ex ante even when these induce the market to operate inefficiently, suggesting a role for sustained antitrust vigilance and that bilateral transactions are the natural focus of attention. Of the many avenues for future research to explore, a particularly promising one is to allow for downstream externalities by assuming that allocating inputs to some firm affects the profits that other firms generate.

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A Appendix: Proofs

A.1 Proof of Proposition 2

Proof of Proposition 2. The social surplus results follow from the Schur-concavity of $SS(\mathbf{r})$ once we have shown that $\tilde{\mathbf{r}}'$ majorizes $\mathbf{r}'$. We prove this through the use of three lemmas, which we state and prove below. Consider a vector $\mathbf{r} \in \Delta$ with one element $r_\ell \geq \sigma > 0$. Then the amount σ can be spread among the $n-1$ elements of the vector $\mathbf{r}_{-\ell}$. Let $\bar{\mathbf{r}}(\ell, \sigma)$ be the most symmetric vector that is obtained by this procedure, that is, the most symmetric among vectors that are obtained by distributing σ among the elements in $\mathbf{r}_{-\ell}$ and replacing r_ℓ with $r_\ell - \sigma$. We show that given $\tilde{\mathbf{r}}, \mathbf{r} \in \Delta$ with $\tilde{r}_{\tilde{\ell}} = r_\ell \geq \sigma > 0$, if $\tilde{\mathbf{r}}$ majorizes $\mathbf{r}$, then $\bar{\tilde{\mathbf{r}}}(\tilde{\ell}, \sigma)$ majorizes $\bar{\mathbf{r}}(\ell, \sigma)$. We begin in Lemma A.1 by characterizing $\bar{\mathbf{r}}(\ell, \sigma)$. As a matter of notation, given vector $\mathbf{x}$, we use $x_{(i)}$ to denote the i-th highest element of $\mathbf{x}$. We define $x_{(0)} \equiv \infty$ and use $\mathbf{x}^{\downarrow}$ to denote $(x_{(1)}, \ldots, x_{(|\mathbf{x}|)})$.

Lemma A.1. *Given* $\mathbf{r} \in \Delta$ *with* $r_\ell \geq \sigma > 0$*, we have* $\bar{\mathbf{r}}(\ell, \sigma) = (\mathbf{r}'_{-\ell}, r_\ell - \sigma)$*, where* ${\mathbf{r}'_{-\ell}}^{\downarrow} = (r_{-\ell(1)}, \ldots, r_{-\ell(j)}, x, \ldots, x)$ *for* $j \in \{0, \ldots, n-1\}$ *and* $x \in [r_{-\ell(j+1)}, r_{-\ell(j)}]$ *such that* $\sum_{i=j+1}^{n-1}(x - r_{-\ell(i)}) = \sigma$*.*

Proof of Lemma A.1. Take as given $\mathbf{r} \in \Delta$ with $r_\ell \geq \sigma > 0$, and let $\mathbf{r}'$ be constructed from $\mathbf{r}$ as in the statement of the lemma. Thus, as in the statement of the lemma, there exists $j \in \{0, \ldots, n-1\}$ such that for $i \in \{0, \ldots, j\}$, $r'_{-\ell(i)} = r_{-\ell(i)}$, and for $i \in \{j+1, \ldots, n-1\}$, $r'_{-\ell(i)} = x$. Take an arbitrary vector $\hat{\mathbf{r}} \in \Delta$ with $\hat{r}_{\hat{\ell}} = r_\ell - \sigma$ for some $\hat{\ell} \in \mathcal{N}$, $\hat{\mathbf{r}}_{-\hat{\ell}} \neq \mathbf{r}'_{-\ell}$, and $\hat{\mathbf{r}}_{-\hat{\ell}} = \mathbf{r}_{-\ell} + (\varepsilon_1, \ldots, \varepsilon_{n-1})$, where $\varepsilon_i \geq 0$ and $\sum_{i=1}^{n-1} \varepsilon_i = \sigma$. We show that $\hat{\mathbf{r}}$ majorizes $\mathbf{r}'$, which then completes the proof.

We first show that $\hat{\mathbf{r}}_{-\hat{\ell}}$ majorizes $\mathbf{r}'_{-\ell}$. If $j = 0$, then $\mathbf{r}'_{-\ell} = (x, \ldots, x)$, which is majorized by any other n-dimensional vector whose elements sum to the same amount, thus including $\hat{\mathbf{r}}_{-\hat{\ell}}$. So assume that $j \geq 1$. By the construction of $\hat{\mathbf{r}}_{-\hat{\ell}}$, we have for all $i \in \{1, \ldots, j\}$, $\hat{r}_{-\hat{\ell}(i)} \geq r_{-\ell(i)} = r'_{-\ell(i)} \geq x$, which implies that for all $h \in \{1, \ldots, j\}$,

$$\sum_{i=1}^{h} \hat{r}_{-\hat{\ell}(i)} \geq \sum_{i=1}^{h} r'_{-\ell(i)}.$$

Let $\hat{\jmath}$ be the largest index such that $\hat{r}_{-\hat{\ell}(\hat{\jmath})} \geq x$. From the argument above, we know that $\hat{\jmath} \geq j$, and we know that for $h \in \{1, \ldots, \hat{\jmath}\}$,

$$\sum_{i=1}^{h} \hat{r}_{-\hat{\ell}(i)} \geq \sum_{i=1}^{h} r'_{-\ell(i)}, \tag{A.1}$$

where (A.1) holds with a strict inequality for at least one $h \in \{1, \ldots, \hat{\jmath}\}$: to see this, note that if (A.1) is not strict for $h = j$, then for all $i \in \{1, \ldots, j\}$, $\hat{r}_{-\hat{\ell}(i)} = r'_{-\ell(i)}$, and it follows from $\hat{\mathbf{r}}_{-\hat{\ell}} \neq \mathbf{r}'_{-\ell}$ and the fact that both $\hat{\mathbf{r}}_{-\hat{\ell}}$ and $\mathbf{r}'_{-\ell}$ sum to the same amount that $\hat{r}_{-\hat{\ell}(j+1)} > r'_{-\hat{\ell}(j+1)} = x$, which implies that $\hat{\jmath} \geq j+1$ and that (A.1) is strict for $h = j+1$.

Because $\hat{\mathbf{r}}_{-\hat{\ell}}$ and $\mathbf{r}'_{-\ell}$ sum to the same amount, we have $\sum_{i=1}^{n-1} \hat{r}_{-\hat{\ell}(i)} = \sum_{i=1}^{n-1} r'_{-\ell(i)}$, which we can write as

$$\sum_{i=1}^{j}(\hat{r}_{-\hat{\ell}(i)} - r'_{-\ell(i)}) + \sum_{i=j+1}^{\hat{\jmath}} (\hat{r}_{-\hat{\ell}(i)} - x) = \sum_{i=\hat{\jmath}+1}^{n-1} (x - \hat{r}_{-\hat{\ell}(i)}), \tag{A.2}$$

where the summation from $j+1$ to $\hat{\jmath}$ is defined to be zero if $\hat{\jmath} = j$. Using this, for $h \in \{\hat{\jmath}+1, \ldots, n-1\}$, we have

$$\begin{aligned}
\sum_{i=1}^{h} \hat{r}_{-\hat{\ell}(i)} - \sum_{i=1}^{h} r'_{-\ell(i)} &= \sum_{i=1}^{j}(\hat{r}_{-\hat{\ell}(i)} - r'_{-\ell(i)}) + \sum_{i=j+1}^{\hat{\jmath}} (\hat{r}_{-\hat{\ell}(i)} - x) + \sum_{i=\hat{\jmath}+1}^{h} (\hat{r}_{-\hat{\ell}(i)} - x) \\
&= \sum_{i=\hat{\jmath}+1}^{n-1} (x - \hat{r}_{-\hat{\ell}(i)}) + \sum_{i=\hat{\jmath}+1}^{h} (\hat{r}_{-\hat{\ell}(i)} - x) \\
&= (n-1-h)x - \sum_{i=h+1}^{n-1} \hat{r}_{-\hat{\ell}(i)} \\
&\geq 0,
\end{aligned}$$

where the first equality uses $r'_{-\ell(i)} = x$ for $i > j$, the second equality uses (A.2), the third equality rearranges, and the inequality uses $h \geq \hat{\jmath}+1$, which implies that for $i \in \{h+1, \ldots, n-1\}$, $\hat{r}_{-\hat{\ell}(i)} < x$. Combining this with (A.1), we conclude that $\hat{\mathbf{r}}_{-\hat{\ell}}$ majorizes $\mathbf{r}'_{-\ell}$.

We are left to show that $\mathbf{r}' = (\mathbf{r}'_{-\ell}, r_\ell - \sigma)$. Because $\hat{\mathbf{r}}$ was arbitrary, this is equivalent to showing that $\mathbf{r}'$ is majorized by $\hat{\mathbf{r}} = (\hat{\mathbf{r}}_{-\hat{\ell}}, r_\ell - \sigma)$. Let ρ be the rank of $r_\ell - \sigma$ in $\mathbf{r}'$, i.e., $r_\ell - \sigma = \mathbf{r}'_{(\rho)}$, and let $\hat{\rho}$ be the rank of $r_\ell - \sigma$ in $\hat{\mathbf{r}}$, i.e., $\hat{r}_{\hat{\ell}} = r_\ell - \sigma = \hat{\mathbf{r}}_{(\hat{\rho})}$ (breaking ties in favor of larger ρ and $\hat{\rho}$, respectively). For $h \in \{1, \ldots, \min\{\hat{\rho}-1, \rho-1\}\}$ and for $h \in \{\max\{\hat{\rho}, \rho\}, \ldots, n\}$,

$$\sum_{i=1}^{h} \hat{r}_{(i)} - \sum_{i=1}^{h} r'_{(i)} = \sum_{i=1}^{h} \hat{r}_{-\hat{\ell}(i)} - \sum_{i=1}^{h} r'_{-\ell(i)} \geq 0,$$

where the inequality uses that $\hat{\mathbf{r}}_{-\hat{\ell}}$ majorizes $\mathbf{r}'_{-\ell}$. It remains to show that $\sum_{i=1}^{h} \hat{r}_{(i)} - \sum_{i=1}^{h} r'_{(i)} \geq 0$ for $\hat{\rho} < \rho$ and $h \in \{\hat{\rho}, \ldots, \rho-1\}$ and for $\rho < \hat{\rho}$ and $h \in \{\rho-1, \ldots, \hat{\rho}\}$. If $\hat{\rho} < \rho$, then for $h \in \{\hat{\rho}, \ldots, \rho-1\}$,

$$\begin{aligned}
\sum_{i=1}^{h} \hat{r}_{(i)} - \sum_{i=1}^{h} r'_{(i)} &= \sum_{i=1}^{h-1} \hat{r}_{-\hat{\ell}(i)} + r_\ell - \sigma - \sum_{i=1}^{h} r'_{-\ell(i)} \geq \sum_{i=1}^{h-1} \hat{r}_{-\hat{\ell}(i)} + \hat{r}_{-\hat{\ell}(h)} - \sum_{i=1}^{h} r'_{-\ell(i)} \\
&= \sum_{i=1}^{h} \hat{r}_{-\hat{\ell}(i)} - \sum_{i=1}^{h} r'_{-\ell(i)} \geq 0,
\end{aligned}$$

where the first inequality uses that for $i \geq \hat{\rho}$, $\hat{r}_{-\hat{\ell}(i)} \leq \hat{r}_{(i)} \leq \hat{r}_{(\hat{\rho})} = r_\ell - \sigma$, and the second inequality uses that $\hat{\mathbf{r}}_{-\hat{\ell}}$ majorizes $\mathbf{r}'_{-\ell}$. If $\rho < \hat{\rho}$, then for $h \in \{\rho, \ldots, \hat{\rho} - 1\}$,

$$\begin{aligned}
\sum_{i=1}^{h} \hat{r}_{(i)} - \sum_{i=1}^{h} r'_{(i)} &= \sum_{i=1}^{h} \hat{r}_{-\hat{\ell}(i)} - \sum_{i=1}^{h-1} r'_{-\ell(i)} - (r_\ell - \sigma) \\
&= \sum_{i=1}^{h-1} \hat{r}_{-\hat{\ell}(i)} - \sum_{i=1}^{h-1} r'_{-\ell(i)} + \hat{r}_{-\hat{\ell}(h)} - (r_\ell - \sigma) \geq \sum_{i=1}^{h-1} \hat{r}_{-\hat{\ell}(i)} - \sum_{i=1}^{h-1} r'_{-\ell(i)} \geq 0,
\end{aligned}$$

where the first inequality uses that for $i < \hat{\rho}$, $\hat{r}_{-\hat{\ell}(i)} = \hat{r}_{(i)} \geq \hat{r}_{(\hat{\rho})} = r_\ell - \sigma$, and the second inequality uses that $\hat{\mathbf{r}}_{-\hat{\ell}}$ majorizes $\mathbf{r}'_{-\ell}$. Thus, we conclude that $\hat{\mathbf{r}}$ majorizes $\mathbf{r}'$, which completes the proof. □

Next, in Lemma A.2, we show that majorization extends to subvectors when a common element is removed:

Lemma A.2. *If $\tilde{\mathbf{r}}$ majorizes $\mathbf{r}$ and $\tilde{r}_{\tilde{\ell}} = r_\ell$, then $\tilde{\mathbf{r}}_{-\tilde{\ell}}$ majorizes $\mathbf{r}_{-\ell}$.*

Proof of Lemma A.2. Let $\tilde{j}$ and j be the rank of $\tilde{r}_{\tilde{\ell}}$ and r_ℓ, respectively, breaking ties in favor of larger $\tilde{j}$ and j. If $j \leq \tilde{j}$, then the result follows because then $\sum_{i=1}^{h} \tilde{r}_{-\hat{\ell}(i)} - \sum_{i=1}^{h} r_{-\ell(i)}$ is either the same as $\sum_{i=1}^{h} \tilde{r}_{(i)} - \sum_{i=1}^{h} r_{(i)}$, and so nonnegative (and positive for at least one h), or even greater by the amount $r_{(j)} - r_{(j+1)} \geq 0$. So, suppose that $j > \tilde{j}$. It is sufficient to check that for $h \in \{\tilde{j}+1, \ldots, j-1\}$ we have $\sum_{i=1}^{h} \tilde{r}_{-\hat{\ell}(i)} \geq \sum_{i=1}^{h} r_{-\ell(i)}$. By the definition of j, we have $r_{(i)} \geq r_\ell$ for $i < j$, and by the definition of $\tilde{j}$, we have $\tilde{r}_{(i)} \leq \tilde{r}_{\tilde{\ell}}$ for $i \geq \tilde{j} + 1$. Thus, for $h \in \{\tilde{j}, \ldots, j-1\}$, we have

$$r_{-\ell(h)} = r_{(h)} \geq r_\ell = \tilde{r}_{\tilde{\ell}} \geq \tilde{r}_{(h)} \geq \tilde{r}_{-\tilde{\ell}(h)} \tag{A.3}$$

and

$$\begin{aligned}
\sum_{i=1}^{h} \tilde{r}_{-\tilde{\ell}(i)} &= \sum_{i=1}^{h} \tilde{r}_{(i)} - \tilde{r}_{\tilde{\ell}} + \tilde{r}_{-\tilde{\ell}(h)} \geq \sum_{i=1}^{h} r_{(i)} - r_\ell + \tilde{r}_{-\tilde{\ell}(h)} \\
&= \sum_{i=1}^{h} r_{-\ell(i)} + r_\ell - r_{-\ell(h)} - r_\ell + \tilde{r}_{-\tilde{\ell}(h)} = \sum_{i=1}^{h} r_{-\ell(i)} - r_{-\ell(h)} + \tilde{r}_{-\tilde{\ell}(h)} \geq \sum_{i=1}^{h} r_{-\ell(i)},
\end{aligned}$$

where the first inequality uses that $\tilde{\mathbf{r}}$ majorizes $\mathbf{r}$ and the inequality second uses (A.3). □

Finally, in Lemma A.3, we show use Lemmas A.1 and A.2 to show that majorization extends to optimally divested ownership structures:

Lemma A.3. *Given $\tilde{\mathbf{r}}, \mathbf{r} \in \Delta_R^{n-1}$ with $\tilde{r}_{\tilde{\ell}} = r_\ell$ and $\sigma \in (0, r_\ell]$, if $\tilde{\mathbf{r}}$ majorizes $\mathbf{r}$, then $\overline{\tilde{\mathbf{r}}}(\ell, \sigma)$ majorizes $\overline{\mathbf{r}}(\ell, \sigma)$.*

Proof of Lemma A.3. Let $\tilde{j}$ and $\tilde{x}$ be the parameters of the optimal divestiture given $\tilde{\mathbf{r}}$, $\tilde{\ell}$, and σ, and let j and x be the parameters of the optimal divestiture given $\mathbf{r}$, ℓ, and σ as derived in Lemma A.1. That is, $\tilde{x} \in [\tilde{r}_{-\hat{\ell}(\tilde{j}+1)}, \tilde{r}_{-\hat{\ell}(\tilde{j})}]$ and $x \in [r_{-\ell(j+1)}, r_{-\ell(j)}]$ and

$$\sum_{i=\tilde{j}+1}^{n-1} (\tilde{x} - \tilde{r}_{-\tilde{\ell}(i)}) = \sum_{i=j+1}^{n-1} (x - r_{-\ell(i)}) = \sigma. \tag{A.4}$$

Assume that $\tilde{\mathbf{r}}$ majorizes $\mathbf{r}$. It follows from Lemma A.2 that $\tilde{\mathbf{r}}_{-\tilde{\ell}}$ majorizes $\mathbf{r}_{-\ell}$.

Let $\overline{\tilde{\mathbf{r}}}(\ell, \sigma) \equiv (\tilde{\mathbf{r}}'_{-\tilde{\ell}}, r_{\tilde{\ell}} - \sigma)$ and $\overline{\mathbf{r}}(\ell, \sigma) \equiv (\mathbf{r}'_{-\ell}, r_{\ell} - \sigma)$ and suppose that $\tilde{\mathbf{r}}'_{-\tilde{\ell}}$ does not majorize $\mathbf{r}'_{-\ell}$. Then there exists a smallest $\hat{h} \in \{1, \ldots, n-1\}$ such that $\sum_{i=1}^{\hat{h}} (\tilde{r}'_{-\tilde{\ell}(i)} - r'_{-\ell(i)}) < 0$. Given that $\tilde{\mathbf{r}}_{-\tilde{\ell}}$ majorizes $\mathbf{r}_{-\ell}$ and that $\tilde{r}'_{-\tilde{\ell}(i)}$ and $r'_{-\ell(i)}$ coincide with $\tilde{r}_{-\tilde{\ell}(i)}$ and $r_{-\ell(i)}$ for $i \le \min\{j, \tilde{j}\}$ and that both $\tilde{\mathbf{r}}'_{-\tilde{\ell}}$ and $\mathbf{r}'_{-\ell}$ are constant for $i > \max\{j, \tilde{j}\}$, it must be that $\hat{h} \in (\min\{j, \tilde{j}\}, \max\{j, \tilde{j}\}]$.

Case 1. $\tilde{j} < \hat{h} \le j$. In this case, we can rewrite (A.4) as

$$\sum_{i=\tilde{j}+1}^{j} (\tilde{x} - \tilde{r}_{-\tilde{\ell}(i)}) + \sum_{i=j+1}^{n-1} (\tilde{x} - \tilde{r}_{-\tilde{\ell}(i)}) = \sum_{i=j+1}^{n-1} (x - r_{-\ell(i)})$$

or, using $\tilde{x} \in [\tilde{r}_{-\hat{\ell}(\tilde{j}+1)}, \tilde{r}_{-\hat{\ell}(\tilde{j})}]$,

$$\sum_{i=\tilde{j}+1}^{j} \underbrace{(\tilde{x} - \tilde{r}_{-\tilde{\ell}(i)})}_{\text{positive}} + (n-1-j)(\tilde{x} - x) = \sum_{i=j+1}^{n-1} (\tilde{r}_{-\tilde{\ell}(i)} - r_{-\ell(i)}) \le 0,$$

which implies that $\tilde{x} < x$. But then

$$\begin{aligned} \sum_{i=1}^{n-1} \left(r'_{-\ell(i)} - \tilde{r}'_{-\tilde{\ell}(i)} \right) &> \sum_{i=\hat{h}+1}^{n-1} \left(r'_{-\ell(i)} - \tilde{r}'_{-\tilde{\ell}(i)} \right) = \sum_{i=\hat{h}+1}^{j} \left(r'_{-\ell(i)} - \tilde{x} \right) + \sum_{i=j+1}^{n-1} (x - \tilde{x}) \\ &\ge \sum_{i=\hat{h}+1}^{j} (x - \tilde{x}) + \sum_{i=j+1}^{n-1} (x - \tilde{x}) \ge 0, \end{aligned}$$

where the first inequality uses $\sum_{i=1}^{\hat{h}} (\tilde{r}'_{-\tilde{\ell}(i)} - r'_{-\ell(i)}) < 0$, the second inequality uses $x < r'_{-\ell(i)}$ for $i \le j$, and the third inequality uses $\tilde{x} < x$. This violates the summing up condition, which requires that $\sum_{i=1}^{n-1} r'_{-\ell(i)} = \sum_{i=1}^{n-1} \tilde{r}'_{-\ell(i)}$, giving us a contradiction.

Case 2. $j < \hat{h} \le \tilde{j}$. For this case, we know that at $\hat{h}$-th highest element, $\mathbf{r}'_{-\ell}$ is already equal to x, but $\tilde{\mathbf{r}}'_{-\tilde{\ell}}$ is still equal to $\tilde{\mathbf{r}}_{-\ell}$. By the definition of $\hat{h}$, it must be that the change from $\sum_{i=1}^{\hat{h}-1} r'_{-\ell(i)}$ to $\sum_{i=1}^{\hat{h}} r'_{-\ell(i)}$, which is equal to x, is larger than the change from $\sum_{i=1}^{\hat{h}-1} \tilde{r}'_{-\tilde{\ell}(i)}$ to $\sum_{i=1}^{\hat{h}} \tilde{r}'_{-\tilde{\ell}(i)}$. Thus, we have $x > \tilde{r}'_{-\tilde{\ell}(\hat{h})}$, which means that $x > \tilde{r}'_{-\tilde{\ell}(\hat{h})} \ge \cdots \ge \tilde{r}'_{-\tilde{\ell}(n-1)}$, which means that $\sum_{i=\hat{h}+1}^{n-1} r'_{-\ell(i)} > \sum_{i=\hat{h}+1}^{n-1} \tilde{r}'_{-\tilde{\ell}(i)}$, so we have

$$\sum_{i=1}^{n-1} r'_{-\ell(i)} = \sum_{i=1}^{\hat{h}} r'_{-\ell(i)} + \sum_{i=\hat{h}+1}^{n-1} r'_{-\ell(i)} > \sum_{i=1}^{\hat{h}} \tilde{r}'_{-\tilde{\ell}(i)} + \sum_{i=\hat{h}+1}^{n-1} \tilde{r}'_{-\tilde{\ell}(i)} = \sum_{i=1}^{n-1} \tilde{r}'_{-\ell(i)},$$

which contradicts the summing up condition, which requires that $\sum_{i=1}^{n-1} r'_{-\ell(i)} = \sum_{i=1}^{n-1} \tilde{r}'_{-\ell(i)}$.

Thus, we conclude that $\tilde{\mathbf{r}}'_{-\tilde{\ell}}$ majorizes $\mathbf{r}'_{-\ell}$, which completes the proof of Lemma A.3. $\square$

Combining Lemmas A.1–A.3 completes the proof of Proposition 2. $\blacksquare$

A.2 Proof of Lemma 1

Proof of Lemma 1. As noted in Section 3, an implication of IC is that $u_i'(\theta) = q_i(\theta) - r_i$ wherever u_i is differentiable, which by IC is almost everywhere. Given this, the monotonicity of u_i implies the following characterization of the set of worst-off types for firm i, denoted by $\Omega_i \equiv \arg\min_{\theta_i \in [0,1]} u_i(\theta_i)$ (see also Cramton et al. (1987, Lemma 2) and Loertscher and Wasser (2019)):

$$\Omega_i = \begin{cases} \{\theta_i \in [0,1] \mid q_i(\theta_i) = r_i\} & \text{if } \exists \theta_i \in [0,1] \text{ s.t. } q_i(\theta_i) = r_i, \\ \{\theta_i \in [0,1] \mid q_i(z) < r_i \ \forall z < \theta_i \text{ and } q_i(z) > r_i \ \forall z > \theta_i\} & \text{otherwise.} \end{cases}$$

In the first case in which there exists $\theta_i \in [0,1]$ such that $q_i(\theta_i) = r_i$, the set Ω_i is a (possibly degenerate) interval, and in the second case, Ω_i is a singleton.

Taking the expression for $m_i(\theta)$ in (6), with θ' replaced by $\hat{\theta}_i$, we have $\int_0^1 m_i(\theta)dF_i(\theta) = \int_0^1 (q_i(\theta)-r_i)\theta dF_i(\theta) - \int_{\hat{\theta}_i}^1 \int_{\hat{\theta}_i}^{\theta} (q_i(x)-r_i)dxdF_i(\theta) + \int_0^{\hat{\theta}_i} \int_{\theta}^{\hat{\theta}_i} (q_i(x)-r_i)dxdF_i(\theta) - u_i(\hat{\theta}_i)$. Changing the order of integration in the double integrals and substituting the virtual type functions and noting that $\mathbb{E}_{\theta_i}[\Psi_i(\theta_i, \hat{\theta}_i)] = \hat{\theta}_i$ gives the result. $\blacksquare$

A.3 Proof of Proposition 4

Proof of Proposition 4. We first show that the ranking of firms matters for efficiency. Then we show that the ranking of two firms' actual types cannot be the same as the ranking of their ironed weighted virtual types if the bargaining weights differ.

Let $X \equiv \sum_{\ell \in \mathcal{N} \setminus \{i,j\}} k_\ell$ be the maximum capacity of firms other than firms i and j. By the assumption of overall excess demand, $1 - X < k_i + k_j$. Let θ_i^r and θ_j^r denote the basis for ranking firms i and j. We consider two cases.

Case 1: $X < 1$. When all firms other than i and j have types equal to 1, then for $\theta_i^r, \theta_j^r \in (0,1)$, assets $1 - X \in (0, k_i + k_j)$ are allocated to firms i and j based on their ranking, and so the ranking matters for efficiency.

Case 2: $X \geq 1$. When all firms other than i and j have types equal to $x \in (0,1)$, then if $0 < \theta_i^r < x < \theta_j^r < 1$, assets of 1 are available to be allocated to firm j, but, because $1 - X \leq 0$, no assets are available to be allocated to firm i. And the situation is reversed if the rankings of i and j are reversed. Thus, the ranking of firms i and j around any $x \in (0,1)$

matters for efficiency. Thus, for every pair i and j of firms, there exists an open interval subset of $[0,1]$ such that the ranking of firms i and j on that interval matters for efficiency.

We now show that the ironed weighted virtual type functions for two firms cannot be the same for all types in an open interval if the firms' bargaining weights differ. Suppose that $w_i \neq w_j$ and let A be an open interval that is a subset of $[0,1]$. Suppose that for all $\theta \in A$, the ranking of firms i and j according to their types is the same as the ranking of those firms according to their weighted ironed virtual types under incomplete information bargaining:

$$\overline{\Psi}_{j,\frac{w_j}{\rho^*}}(\theta_j;\omega_j^*) > \overline{\Psi}_{i,\frac{w_i}{\rho^*}}(\theta_i;\omega_i^*) \quad \Leftrightarrow \quad \theta_j > \theta_i.$$

Then we require that for all $\theta \in A$, $\overline{\Psi}_{j,w_j/\rho^*}(\theta;\omega_j^*) = \overline{\Psi}_{i,w_i/\rho^*}(\theta;\omega_i^*)$ and that the ironed weighted virtual type functions are increasing in this region (if they are equal, but constant, as in the ironed portions, then the allocation is random and so ex post efficiency is not achieved). Given that the ironed weighted virtual type functions are increasing, they are either equal to the weighted virtual value or the weighted virtual cost, and so cannot be the same on an open interval if the weights differ. To see this, suppose that for $\theta \in A$, $\overline{\Psi}_{j,w_j/\rho^*}(\theta;\omega_j^*) = \Psi^B_{j,w_j/\rho^*}(\theta) = \overline{\Psi}_{i,w_i/\rho^*}(\theta;\omega_i^*) = \Psi^S_{i,w_i/\rho^*}(\theta)$. Then we have $\theta - (1-\frac{w_j}{\rho^*})\frac{1-F_j(\theta)}{f_j(\theta)} = \theta + (1-\frac{w_i}{\rho^*})\frac{F_i(\theta)}{f_i(\theta)}$, which we can rewrite as

$$\frac{F_i(\theta)f_j(\theta)}{(1-F_j(\theta))\,f_i(\theta) + F_i(\theta)f_j(\theta)} = \frac{\rho^* - w_j}{w_i - w_j} \equiv C$$

and, alternatively, as

$$\frac{(1-F_j(\theta))\,f_i(\theta)}{F_i(\theta)f_j(\theta) + (1-F_j(\theta))\,f_i(\theta)} = \frac{\rho^* - w_i}{w_j - w_i} \equiv C'$$

Using $\rho^* \geq \max\{w_j, w_i\}$, if $w_j < w_i$, then $C \geq 1$, and we require that for all $\theta \in A$, $\frac{1-F_j(\theta)}{f_j(\theta)} / \frac{F_i(\theta)}{f_i(\theta)} = \frac{1-C}{C}$, which is a contradiction because the left side is positive and $(1-C)/C \leq 0$; and if $w_j > w_i$, then $C' \geq 1$, and we require that $\frac{F_i(\theta)}{f_i(\theta)} / \frac{(1-F_j(\theta))}{f_j(\theta)} = \frac{1-C'}{C'}$, which is similarly a contradiction. Analogous contradictions obtain for the cases with $\overline{\Psi}_{j,w_j/\rho^*}(\theta;\omega_j^*) = \Psi^B_{j,w_j/\rho^*}(\theta) = \overline{\Psi}_{i,w_2/\rho^*}(\theta;\omega_i^*) = \Psi^B_{i,w_i/\rho^*}(\theta)$ and $\overline{\Psi}_{j,w_j/\rho^*}(\theta;\omega_j^*) = \Psi^S_{j,w_j/\rho^*}(\theta) = \overline{\Psi}_{i,w_2/\rho^*}(\theta;\omega_i^*) = \Psi^S_{i,w_i/\rho^*}(\theta)$, which completes the proof. ■

A.4 Proof of Proposition 5

Proof of Proposition 5. We begin with a lemma:

Lemma A.4. *Given equal bargaining weights,* $\Pi^e(\mathbf{r})$ *is strictly concave in* $\mathbf{r}$*, which implies that* $\mathcal{R}^e$ *is convex.*[52]

[52] $\Pi^e(\mathbf{r})$ is strictly concave in $\mathbf{r}$, but $\mathcal{R}^e$ is only convex (and not necessarily strictly convex) because $\mathcal{R}^e \equiv \{\mathbf{r} \mid \Pi^e(\mathbf{r}) \geq 0\} \cap \Delta_{\mathbf{k}}$. So, $\mathcal{R}^e$ is not strictly convex where it intersects with the boundary of $\Delta_{R,\mathbf{k}}$.

Proof of Lemma A.4. Using Lemma 1, we have

$$\Pi^e(\mathbf{r}) = \sum_{i\in\mathcal{N}} \left(\mathbb{E}_{\theta_i}\left[\Psi_i(\theta_i, \hat{\theta}_i^e(r_i))q_i^e(\theta_i)\right] - r_i\hat{\theta}_i^e(r_i)\right), \tag{A.5}$$

where $\hat{\theta}_i^e(r_i)$ is firm i's worst-off type under ex post efficiency. This can be rewritten as

$$\Pi^e(\mathbf{r}) = \sum_{i\in\mathcal{N}} \left(\int_0^{\hat{\theta}_i^e(r_i)} \Psi_i^S(\theta)q_i^e(\theta)dF_i(\theta) + \int_{\hat{\theta}_i^e(r_i)}^1 \Psi_i^B(\theta)q_i^e(\theta)dF_i(\theta) - r_i\hat{\theta}_i^e(r_i)\right).$$

Differentiating with respect to r_i, the three terms involving $\frac{d\hat{\theta}_i}{dr_i}$ cancel, and we are left with $\frac{\partial\Pi^e(\mathbf{r})}{\partial r_i} = -\hat{\theta}_i^e(r_i)$. Because $\hat{\theta}_i^e(r_i)$ increases in r_i, all second partial derivatives are negative. Further, all cross-partial derivatives are zero. This completes the proof. □

Because $-\hat{\theta}_i^e(r_i)$ is the derivative of $\Pi^e(\mathbf{r})$ with respect to r_i and because $\Pi^e(\mathbf{r})$ is strictly concave, it follows that the unique ownership structure that maximizes $\Pi^e(\mathbf{r})$ subject to the constraint that $\sum_{i\in\mathcal{N}} r_i = 1$, denoted $\mathbf{r}^*$, is such that all firms have the same worst-off types.

Further, the proof of Proposition 8, which is stated for multi-dimensional types but also encompasses one-dimensional types, shows that $\Pi^e(\mathbf{r}^*)$ is positive, implying that $\mathbf{r}^* \in \mathcal{R}^e$. We now show that $\mathbf{r}^*$ exists that induces equal worst-off types.

Lemma A.5. *There exists a unique ownership structure* $\mathbf{r}^* \in \Delta_{\mathbf{k}}$ *such that* $\hat{\theta}_i^e(r_i^*) = \hat{\theta}^e \in (0,1)$ *for all* $i \in \mathcal{N}$.

Proof of Lemma A.5. Given Lemma 1, we need only show that there exists $\hat{\theta} \in (0,1)$ such that $\mathbf{r}^* = (q_1^e(\hat{\theta}), \dots, q_n^e(\hat{\theta})) \in \Delta_{\mathbf{k}}$. Define $\mathcal{N}_{-i} \equiv \mathcal{N}\backslash\{i\}$. By the definition of ex post efficiency, for all $i \in \mathcal{N}$ and $\theta \in [0,1]$,

$$q_i^e(\theta_i) = \sum_{\mathcal{A}\subset\mathcal{N}_{-i}} \max\{0, \min\{k_i, 1 - \sum_{j\in\mathcal{A}} k_j\}\} \prod_{j\in\mathcal{N}_{-i}\backslash\mathcal{A}} F_j(\theta_i) \prod_{j\in\mathcal{A}}(1 - F_j(\theta_i)),$$

which is continuous and increasing in θ_i on $[0,1]$. Under our maintained assumption that $\sum_{j\neq i} k_j \geq 1$, $q_i^e(0) = 1$, so we have $\sum_{i\in\mathcal{N}} q_i^e(0) < 1$. The assumption of excess demand implies that $\sum_{i\in\mathcal{N}} q_i^e(1) > 1$, so we have

$$\sum_{i\in\mathcal{N}} q_i^e(0) < 1 < \sum_{i\in\mathcal{N}} q_i^e(1).$$

By the continuity and monotonicity of $q_i^e(\cdot)$ on $[0,1]$, there exists a unique $\hat{\theta} \in (0,1)$ such that $\sum_{i\in\mathcal{N}} q_i^e(\hat{\theta}) = 1$. Further, $q_i^e(\hat{\theta}) \in [0, k_i]$ for all θ. So, $(q_1^e(\hat{\theta}), \dots, q_n^e(\hat{\theta})) \in \Delta_{\mathbf{k}}$. □

It remains to show that $SS(\mathbf{r})$ is concave ans strictly concave outside of $\mathcal{R}^e$. For any $\mathbf{r} \in \Delta_{\mathbf{k}}$, let $\langle \mathbf{Q}_{\mathbf{r}}, \mathbf{M}_{\mathbf{r}}\rangle$ denote the expected social surplus maximizing mechanism, subject to IC, IR, and no deficit. Let $\mathbf{r}, \mathbf{r}' \in \Delta_{\mathbf{k}}$ and $\mu \in [0,1]$ be given. Because $\langle \mathbf{Q}_{\mathbf{r}}, \mathbf{M}_{\mathbf{r}}\rangle$ and $\langle \mathbf{Q}_{\mathbf{r}'}, \mathbf{M}_{\mathbf{r}'}\rangle$ satisfy IC, IR, and no deficit, when the ownership structure is $\mu\mathbf{r} + (1-\mu)\mathbf{r}'$, the mechanism

$\langle \hat{\mathbf{Q}}, \hat{\mathbf{M}} \rangle$ such that $\hat{q}_i(\theta_i) = \mu q_{\mathbf{r},i}(\theta_i) + (1-\mu) q_{\mathbf{r}',i}(\theta_i)$ and $\hat{m}_i(\theta_i) = \mu m_{\mathbf{r},i}(\theta_i) + (1-\mu) m_{\mathbf{r}',i}(\theta_i)$ also satisfies these constraints. Total expected social surplus from $\langle \hat{\mathbf{Q}}, \hat{\mathbf{M}} \rangle$ is $\mu SS(\mathbf{r}) + (1-\mu) SS(\mathbf{r}')$, but $\langle \mathbf{Q}_{\mu\mathbf{r}+(1-\mu)\mathbf{r}'}, \mathbf{M}_{\mu\mathbf{r}+(1-\mu)\mathbf{r}'} \rangle$ maximizes expected social surplus subject to the constraints, so

$$\mu SS(\mathbf{r}) + (1-\mu) SS(\mathbf{r}') \leq SS(\mu\mathbf{r} + (1-\mu)\mathbf{r}'),$$

which implies that $SS(\mathbf{r})$ is concave.

For $\mathbf{r} \in \mathcal{R}^e$, the market mechanism with the efficient allocation rule satisfies the no-deficit constraint. If $\mathbf{r} \notin \mathcal{R}^e$, then the no-deficit constraint cannot be satisfied with the efficient mechanism, implying that $SS(\mathbf{r}) < SS^e \equiv SS(\mathbf{r}^*)$. Thus, $SS(\mathbf{r})$ is strictly concave for $\mathbf{r} \notin \mathcal{R}^e$.

This completes the proof of Proposition 5. ■

A.5 Proof of Proposition 6

Proof of Proposition 6. If $\Pi^e(\mathbf{r}) < 0$, then ex post efficiency is not achieved under $\mathbf{r}$. The two firms can increase their joint payoff through, for example, a transaction that shifts the ownership structure to $\mathbf{r}^*$, where $\Pi^e(\mathbf{r}^*) \geq 0$ so ex post efficiency is achieved (and the ex post efficient surplus is divided between the two firms). If $\Pi^e(\mathbf{r}) \geq 0$, then ex post efficiency is achieved prior to any transaction (and the associated surplus is divided between the two firms), so no further increases in joint surplus are possible. This completes the proof. ■

A.6 Proof of Proposition 7

Proof of Proposition 7. We begin with a lemma.

Lemma A.6. *Given an IC mechanism $\langle \mathbf{Q}, \mathbf{M} \rangle$ and ownership structure $\mathbf{r}$ with $\hat{\theta}_i^{\mathbf{Q}}(r_i) \geq \hat{\theta}_j^{\mathbf{Q}}(r_j)$, where $\hat{\theta}_i^{\mathbf{Q}}(r_i)$ is the worst-off type of firm i with assets r_i under allocation rule $\mathbf{Q}$, if $\mathbf{r}'$ is derived from $\mathbf{r}$ by a T-transform that shifts assets to firm i from firm j, then*

$$\Pi^{\mathbf{Q}}(\mathbf{r}') - \Pi^{\mathbf{Q}}(\mathbf{r}) < 0. \tag{A.6}$$

Proof of Lemma A.6. Because $\Pi^{\mathbf{Q}}$ is strictly concave and $\nabla \Pi^{\mathbf{Q}}(\mathbf{r}) = -(\hat{\theta}_1, \ldots, \hat{\theta}_n)$, for any $\lambda \in (0,1)$ we have:

$$\begin{aligned}
\Pi^{\mathbf{Q}}(\mathbf{r}') - \Pi^{\mathbf{Q}}(\mathbf{r}) &= \Pi^{\mathbf{Q}}(\lambda r_i + (1-\lambda) r_j, \lambda r_j + (1-\lambda) r_i, \mathbf{r}_{-i,j}) - \Pi^{\mathbf{Q}}(r_i, r_j, \mathbf{r}_{-i,j}) \\
&< ((1-\lambda)(r_j - r_i), (1-\lambda)(r_i - r_j), \mathbf{0}_{-i,j}) \cdot (-\hat{\theta}_i, -\hat{\theta}_j, -\hat{\boldsymbol{\theta}}_{-i,j}) \\
&= (1-\lambda)(r_i - r_j)(\hat{\theta}_i - \hat{\theta}_j) \leq 0,
\end{aligned}$$

which completes the proof. □

The remainder of the proof of Proposition 7 proceeds in two parts:

Part (i). Suppose that $\Pi^e(\mathbf{r}) > 0$ and there exist two traders indexed by 1 and 2 with $\eta_1 + \eta_2 \leq 1$. By virtue of the firms being traders, $0 < r_1 < k_1$ and $0 < r_2 < k_2$. Without loss of generality, we can assume that $\hat{\theta}^e_2(r_2) \leq \hat{\theta}^e_1(r_1)$. Because $r_1 < k_1$ and $0 < r_2$, there exists $\Delta > 0$ sufficiently small that the ownership vector $\tilde{\mathbf{r}}(\Delta)$ defined by $\tilde{r}_1(\Delta) \equiv r_1 + \Delta$, $\tilde{r}_2(\Delta) \equiv r_2 - \Delta$, and $\tilde{\mathbf{r}}_{-\{1,2\}}(\Delta) \equiv \mathbf{r}_{-\{1,2\}}$ is a feasible ownership vector (i.e., $r_1 + \Delta \leq k_1$ and $0 \leq r_2 - \Delta$). Further, using the continuity of Π^e and the assumption that $\Pi^e(\mathbf{r}) > 0$, there exists $\Delta > 0$ sufficiently small that $\Pi^e(\tilde{\mathbf{r}}(\Delta)) > 0$. Taking Δ to satisfy these conditions, ex post efficiency is achieved under both $\mathbf{r}$ and $\tilde{\mathbf{r}}(\Delta)$, and by Lemma A.6,

$$\Pi^e(\tilde{\mathbf{r}}(\Delta)) < \Pi^e(\mathbf{r}). \tag{A.7}$$

Defining $\hat{m}_i(r_i) \equiv \mathbb{E}_{\theta_i}\left[\Psi_{i,0}(\theta_i, \hat{\theta}^e_i(r_i))q^e_i(\theta_i)\right] - r_i\hat{\theta}^e_i(r_i)$ and noting that $\Pi^e(\mathbf{r}) = \sum_{i\in\mathcal{N}} \hat{m}_i(r_i)$, it follows that

$$\Pi^e(\mathbf{r}) - \Pi^e(\tilde{\mathbf{r}}(\Delta)) = \hat{m}_1(r_1) + \hat{m}_2(r_2) - \hat{m}_1(r_1 + \Delta) - \hat{m}_2(r_2 - \Delta). \tag{A.8}$$

Because firm i's expected net surplus under ex post efficiency is

$$u^e_i(\mathbf{r}) \equiv \mathbb{E}_{\theta_i}\left[\theta_i(q^e_i(\theta_i) - r_i)\right] - \hat{m}_i(r_i) + \eta_i\Pi^e(\mathbf{r}),$$

the change in the joint expected gross surplus of firms 1 and 2 from a change in ownership structure from $\mathbf{r}$ to $\tilde{\mathbf{r}}(\Delta)$ is

$$\begin{aligned}
& u^e_1(\tilde{\mathbf{r}}(\Delta)) + u^e_2(\tilde{\mathbf{r}}(\Delta)) - u^e_1(\mathbf{r}) - u^e_2(\mathbf{r}) + (\tilde{r}_1(\Delta) - r_1)\mathbb{E}_{\theta_1}[\theta_1] + (\tilde{r}_2(\Delta) - r_2)\mathbb{E}_{\theta_2}[\theta_2] \\
= \; & -\hat{m}_1(r_1 + \Delta) + \eta_1\Pi^e(\tilde{\mathbf{r}}(\Delta)) - \hat{m}_2(r_2 - \Delta) + \eta_2\Pi^e(\tilde{\mathbf{r}}(\Delta)) \\
& +\hat{m}_1(r_1) - \eta_1\Pi^e(\mathbf{r}) + \hat{m}_2(r_2) - \eta_2\Pi^e(\mathbf{r}) \\
= \; & (1 - \eta_1 - \eta_2)\left(\Pi^e(\mathbf{r}) - \Pi^e(\tilde{\mathbf{r}}(\Delta))\right) \\
\geq \; & 0,
\end{aligned}$$

where the first equality uses the definition of $u^e_i(\cdot)$, the second equality uses (A.8), and the inequality uses the assumption that $\eta_1 + \eta_2 \leq 1$ and (A.7). The inequality is strict if $\eta_1 + \eta_2 < 1$. Thus, the joint expected gross payoff of firms 1 and 2 increases (weakly $\eta_1 + \eta_2 \leq 1$ and strictly if $\eta_1 + \eta_2 < 1$) as a result of shifting amount Δ of firm 2's assets to firm 1, which completes the proof of the first part of the proposition.

Part (ii). Assume, as in the statement of the proposition, that $n \in \{3, 4, \dots\}$, $\Pi^e(\mathbf{r}) = 0$, and firms 1 and 2 are traders. Without loss of generality, assume that $\hat{\theta}^e_1(r_1) \geq \hat{\theta}^e_2(r_2)$. Define ownership structure $\tilde{\mathbf{r}}(\Delta)$ by $\tilde{r}_1(\Delta) \equiv r_1 + \Delta$, $\tilde{r}_2(\Delta) \equiv r_2 - \Delta$, and $\tilde{\mathbf{r}}_{-\{1,2\}}(\Delta) \equiv \mathbf{r}_{-\{1,2\}}$, which is feasible for $\Delta \in [0, \min\{k_1 - r_1, r_2\}]$, which is a nonempty interval because firms 1 and 2 are traders. Because we are considering a shift from the firm with the weakly lower worst-off type to the firm with the weakly higher worst-off type, by Lemma A.6, for all $\Delta > 0$ in the feasible range, we have

$$\Pi^e(\tilde{\mathbf{r}}(\Delta)) < \Pi^e(\mathbf{r}) = 0. \tag{A.9}$$

The expected gross payoff of firm i with type θ_i is $\theta_i q_i(\theta_i) - m_i(\theta_i)$, and by Lemma 1,

$$\mathbb{E}_{\theta_i}[m_i(\theta_i)] = \mathbb{E}_{\theta_i}[\Psi_{i,0}(\theta_i, \hat{\theta}_i) q_i(\theta_i)] - u_i(\hat{\theta}_i) - \hat{\theta}_i r_i,$$

where binding IR for the firms' worst-off types implies that $u_i(\hat{\theta}_i) = 0$. Thus, given Δ, the expected gross payoff of firm i is

$$\tilde{u}_i(\Delta) \equiv \mathbb{E}_{\theta_i}\left[\left(\theta_i - \Psi_{i,0}(\theta_i, \hat{\theta}_i^*(\Delta))\right) q_i^*(\theta_i; \hat{\boldsymbol{\theta}}^*(\Delta), \rho^*(\Delta))\right] + \hat{\theta}_i^*(\Delta)\tilde{r}_i(\Delta),$$

where $\hat{\boldsymbol{\theta}}^*(\Delta)$ and $\rho^*(\Delta)$ are, respectively, the solution values for the worst-off types and the Lagrange multiplier on the no-deficit constraint as functions of Δ, and q_i^* is the solution value for the interim expected allocation rule for firm i.

Given the assumption that $F_i = F$ for all $i \in \mathcal{N}$, we have

$$q_i^*(\theta_i; \hat{\boldsymbol{\theta}}^*(\Delta), \rho^*(\Delta)) = \begin{cases} r_i & \text{if } \underline{z}_i(\rho^*(\Delta)) \leq \theta_i \leq \overline{z}_i(\rho^*(\Delta)), \\ q_i^e(\theta_i) & \text{otherwise,} \end{cases} \tag{A.10}$$

where $[\underline{z}_i(\rho^*(\Delta)), \overline{z}_i(\rho^*(\Delta))]$ is the ironing range for firm i and for ρ sufficiently close to 1,

$$\underline{z}_i'(\rho) < 0 \quad \text{and} \quad \overline{z}_i'(\rho) > 0. \tag{A.11}$$

Further, $\hat{\theta}_i^*(\Delta) \in [\underline{z}_i(\rho^*(\Delta)), \overline{z}_i(\rho^*(\Delta))]$ and $r_i > q_i^e(\theta_i)$ for $\theta_i < \hat{\theta}_i^*(\Delta)$ and $r_i < q_i^e(\theta_i)$ for $\theta_i > \hat{\theta}_i^*(\Delta)$. This is illustrated in Figure A.1.

(a) Virtual type functions (b) Interim expected allocation

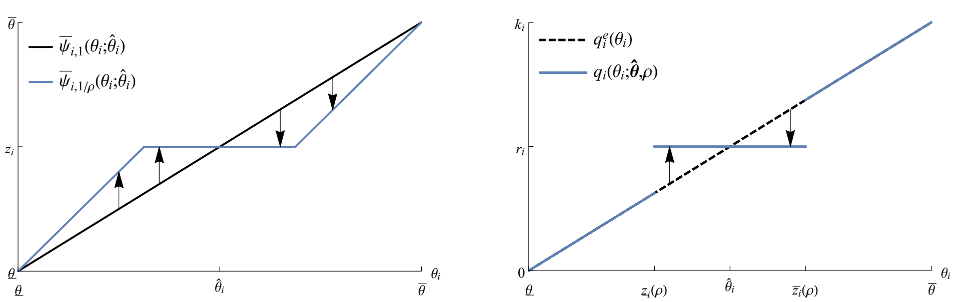

Figure A.1: Illustration of the effects of an increase in ρ above 1 on the ironed weighted virtual types and interim expected allocation. Assumes $\rho > 1$.

To establish that the envisioned transaction between firms 1 and 2 is strictly mutually beneficial, we need to show that for $\Delta > 0$ sufficiently small,

$$\sum_{i \in \{1,2\}} \tilde{u}_i(\Delta) > \sum_{i \in \{1,2\}} \tilde{u}_i(0). \tag{A.12}$$

By (A.9), the solution value for the mechanism's budget surplus, Π^*, satisfies

$$\Pi^*(\tilde{\mathbf{r}}(\Delta); \hat{\boldsymbol{\theta}}^*(\Delta), \rho^*(\Delta)) = 0. \tag{A.13}$$

Further, because $\Pi^e(\tilde{\mathbf{r}}(0)) = \Pi^e(\mathbf{r}) = 0$, it follows that $\rho^*(0) = 1$ and that $\rho^*(\Delta)$ increases as Δ increases above 0, i.e.

$$\left.\frac{\partial \rho^*(\Delta)}{\partial \Delta}\right|_{\Delta \downarrow 0} \geq 0 \text{ and } \rho^*(\Delta) > \rho^*(0) \text{ for } \Delta > 0. \tag{A.14}$$

Using the definitions of $\tilde{u}_i(\Delta)$ and Π^*, and dropping the argument Δ for $\hat{\boldsymbol{\theta}}^*(\Delta)$ and $\rho^*(\Delta)$, we can write

$$\begin{aligned} \sum_{i\in\{1,2\}} \tilde{u}_i(\Delta) &= \sum_{i\in\{1,2\}} \mathbb{E}_{\theta_i}\left[\theta_i q_i^*(\theta_i; \hat{\boldsymbol{\theta}}^*, \rho^*)\right] - \Pi^*(\tilde{\mathbf{r}}(\Delta); \hat{\boldsymbol{\theta}}^*, \rho^*) \\ &\quad + \sum_{j\in\mathcal{N}\setminus\{1,2\}} \left(\mathbb{E}_{\theta_j}\left[\Psi_{j,0}(\theta_j, \hat{\theta}_j^*) q_j^*(\theta_j; \hat{\boldsymbol{\theta}}^*, \rho^*)\right] - \hat{\theta}_j^* \tilde{r}_j(\Delta)\right) \\ &= \sum_{i\in\{1,2\}} \mathbb{E}_{\theta_i}\left[\theta_i q_i^*(\theta_i; \hat{\boldsymbol{\theta}}^*, \rho^*)\right] + \sum_{j\in\mathcal{N}\setminus\{1,2\}} \left(\mathbb{E}_{\theta_j}\left[\Psi_{j,0}(\theta_j, \hat{\theta}_j^*) q_j^*(\theta_i; \hat{\boldsymbol{\theta}}^*, \rho^*)\right] - \hat{\theta}_j^* r_j\right), \end{aligned} \tag{A.15}$$

where the second equality uses (A.13) and $\tilde{r}_j(\Delta) = r_j$ for $j \in \mathcal{N}\setminus\{1,2\}$. Thus, the joint expected payoff of firms 1 and 2 is equal to their expected utility from consumption plus the expected payments by their rivals.

By construction of the virtual type functions and $q_i^*(\hat{\theta}_i^*; \hat{\boldsymbol{\theta}}^*, \rho^*) = r_i$, we have:[53]

$$\frac{\partial}{\partial \hat{\theta}_i}\left(\mathbb{E}_{\theta_i}\left[\Psi_{i,0}(\theta_i, \hat{\theta}_i^*) q_i^*(\theta_i; \hat{\boldsymbol{\theta}}^*, \rho^*)\right] - \hat{\theta}_i^* r_i\right) = \int_{\underline{\theta}}^{\overline{\theta}} \Psi_{i,0}(\theta_i, \hat{\theta}_i^*) \frac{\partial q_i^*(\theta_i; \hat{\boldsymbol{\theta}}^*, \rho^*)}{\partial \hat{\theta}_i} dF_i(\theta_i). \tag{A.16}$$

Using (A.16) and noting that $q_i^*(\theta_i; \hat{\boldsymbol{\theta}}, \rho)$ is independent of $\hat{\boldsymbol{\theta}}$ when $\rho = 1$, it then follows that the derivative of the right side of (A.15) with respect to $\hat{\boldsymbol{\theta}}$ is zero when evaluated at $\Delta = 0$. Thus, when considering the effect of a marginal change in Δ on $\sum_{i\in\{1,2\}} \tilde{u}_i(\Delta)$ at $\Delta = 0$, we need only consider effects that come through $\rho^*(\Delta)$. But note that, by the envelope theorem, the partial derivative of

$$\mathcal{L}(\rho) = \mathbb{E}_{\boldsymbol{\theta}}\left[\sum_{i\in\mathcal{N}} \theta_i q_i^*(\theta_i; \hat{\boldsymbol{\theta}}^*, \rho)\right] - (1-\rho)\Pi^*(\tilde{\mathbf{r}}(\Delta); \hat{\boldsymbol{\theta}}^*, \rho) + \sum_{i\in\mathcal{N}} \mu_i u_i(\hat{\theta}_i^*)$$

[53] To see this, note that:

$$\begin{aligned} &\frac{\partial}{\partial \hat{\theta}_i}\left(\mathbb{E}_{\theta_i}\left[\Psi_{i,0}(\theta_i, \hat{\theta}_i) q_i(\theta_i; \hat{\boldsymbol{\theta}}, \rho)\right] - \hat{\theta}_i r_i\right) = \frac{\partial}{\partial \hat{\theta}_i}\left(\int_{\underline{\theta}}^{\hat{\theta}_i} \Psi_{i,0}^S(\theta_i) q_i(\theta_i; \hat{\boldsymbol{\theta}}, \rho) dF_i(\theta_i) + \int_{\hat{\theta}_i}^{\overline{\theta}} \Psi_{i,0}^B(\theta_i) q_i(\theta_i; \hat{\boldsymbol{\theta}}, \rho) dF_i(\theta_i)\right) - r_i \\ &= (\Psi_{i,0}^S(\hat{\theta}_i) - \Psi_{i,0}^B(\hat{\theta}_i)) q_i(\hat{\theta}_i; \hat{\boldsymbol{\theta}}, \rho) f_i(\hat{\theta}_i) + \int_{\underline{\theta}}^{\hat{\theta}_i} \Psi_{i,0}^S(\theta_i) \frac{\partial q_i(\theta_i; \hat{\boldsymbol{\theta}}, \rho)}{\partial \hat{\theta}_i} dF_i(\theta_i) + \int_{\hat{\theta}_i}^{\overline{\theta}} \Psi_{i,0}^B(\theta_i) \frac{\partial q_i(\theta_i; \hat{\boldsymbol{\theta}}, \rho)}{\partial \hat{\theta}_i} dF_i(\theta_i) \\ &= q_i(\hat{\theta}_i; \hat{\boldsymbol{\theta}}, \rho) + \int_{\underline{\theta}}^{\overline{\theta}} \Psi_{i,0}(\theta_i, \hat{\theta}_i) \frac{\partial q_i(\theta_i; \hat{\boldsymbol{\theta}}, \rho)}{\partial \hat{\theta}_i} dF_i(\theta_i) - r_i = \int_{\underline{\theta}}^{\overline{\theta}} \Psi_{i,0}(\theta_i, \hat{\theta}_i) \frac{\partial q_i(\theta_i; \hat{\boldsymbol{\theta}}, \rho)}{\partial \hat{\theta}_i} dF_i(\theta_i). \end{aligned}$$

with respect to ρ evaluated at $\rho = \rho^*$ is 0, so we have

$$\frac{\partial}{\partial \rho} \sum_{i \in \mathcal{N}} \mathbb{E}_{\theta_i} \left[\theta_i q_i^*(\theta_i; \hat{\boldsymbol{\theta}}^*, \rho) \right] \Bigg|_{\rho=\rho^*} = 0. \tag{A.17}$$

Thus, differentiating (A.15) with respect to Δ, we are left with only the effects that come through $\rho^*(\Delta)$ (as mentioned above), and we obtain, noting that $\hat{\theta}_j^* = \hat{\theta}_j^e$ at $\Delta = 0$,

$$\begin{aligned}
\sum_{i \in \{1,2\}} \tilde{u}_i'(0) &= \rho^{*\prime}(0) \frac{\partial}{\partial \rho} \left(\sum_{i \in \{1,2\}} \mathbb{E}_{\theta_i} \left[\theta_i q_i^*(\theta_i; \hat{\boldsymbol{\theta}}^*, \rho) \right] + \sum_{j \in \mathcal{N} \setminus \{1,2\}} \mathbb{E}_{\theta_j} \left[\Psi_{j,0}(\theta_j, \hat{\theta}_j) q_j^*(\theta_j; \hat{\boldsymbol{\theta}}^*, \rho) \right] \right) \Bigg|_{\rho=\rho^*(0)} \\
&= \rho^{*\prime}(0) \frac{\partial}{\partial \rho} \left(\sum_{i \in \mathcal{N}} \mathbb{E}_{\theta_i} \left[\theta_i q_i^*(\theta_i; \hat{\boldsymbol{\theta}}^*, \rho) \right] + \sum_{j \in \mathcal{N} \setminus \{1,2\}} \mathbb{E}_{\theta_j} \left[\left(\Psi_{j,0}(\theta_j, \hat{\theta}_j) - \theta_j \right) q_j^*(\theta_j; \hat{\boldsymbol{\theta}}^*, \rho) \right] \right) \Bigg|_{\rho=\rho^*(0)} \\
&= \rho^{*\prime}(0) \frac{\partial}{\partial \rho} \left(\sum_{j \in \mathcal{N} \setminus \{1,2\}} \mathbb{E}_{\theta_j} \left[\left(\Psi_{j,0}(\theta_j, \hat{\theta}_j) - \theta_j \right) q_j^*(\theta_j; \hat{\boldsymbol{\theta}}^*, \rho) \right] \right) \Bigg|_{\rho=\rho^*(0)} \\
&= \rho^{*\prime}(0) \frac{\partial}{\partial \rho} \left(\sum_{j \in \mathcal{N} \setminus \{1,2\}} \left(\int_{\underline{\theta}}^{\hat{\theta}_j^*} F_j(\theta_j) q_j^*(\theta_j; \hat{\boldsymbol{\theta}}^*, \rho) d\theta_j - \int_{\hat{\theta}_j^*}^{\overline{\theta}} (1 - F_j(\theta_j)) \, q_j^*(\theta_j; \hat{\boldsymbol{\theta}}^*, \rho) d\theta_j \right) \right) \Bigg|_{\rho=\rho^*(0)} \\
&= \rho^{*\prime}(0) \frac{\partial}{\partial \rho} \left(\sum_{j \in \mathcal{N} \setminus \{1,2\}} \left(\int_{\underline{\theta}}^{\underline{z}_j(\rho)} F_j(\theta_j) q_j^e(\theta_j) d\theta_j + \int_{\underline{z}_j(\rho)}^{\hat{\theta}_j^*} F_j(\theta_j) r_j d\theta_j \right. \right. \\
&\qquad \left. \left. - \int_{\hat{\theta}_j^*}^{\overline{z}_j(\rho)} (1 - F_j(\theta_j)) \, r_j d\theta_j - \int_{\overline{z}_j(\rho)}^{\overline{\theta}} (1 - F_j(\theta_j)) \, q_j^e(\theta_j) d\theta_j \right) \right) \Bigg|_{\rho=\rho^*(0)} \\
&= \rho^{*\prime}(0) \sum_{j \in \mathcal{N} \setminus \{1,2\}} \Bigg(\underbrace{\underline{z}_j'(\rho^*(0))}_{\text{negative}} F_j(\underline{z}_j(\rho^*(0))) \underbrace{\left(q_j^e(\underline{z}_j(\rho^*(0))) - r_j \right)}_{\text{negative}} \\
&\qquad - \underbrace{\overline{z}_j'(\rho^*(0))}_{\text{positive}} (1 - F_j(\overline{z}_j(\rho^*(0)))) \underbrace{\left(r_j - q_j^e(\overline{z}_j(\rho^*(0))) \right)}_{\text{negative}} \Bigg) \\
&> 0,
\end{aligned}$$

where the first equality uses the definition of $\tilde{u}_i$, the second equality rearranges, the third equality uses (A.17), the fourth equality uses the definition of $\Psi_{i,0}$, the fifth equality uses (A.10), the sixth equality differentiates and rearranges, and the inequality uses the observations above, including (A.11) and (A.14), which hold strictly for $\Delta > 0$ sufficiently small. Thus, $\sum_{i \in \{1,2\}} \tilde{u}_i'(0) \geq 0$ and for $\Delta > 0$ sufficiently small $\sum_{i \in \{1,2\}} \tilde{u}_i(\Delta) > \sum_{i \in \{1,2\}} \tilde{u}_i(0)$, which implies that transactions between firms 1 and 2 of $\Delta > 0$ sufficiently small are mutually beneficial. By Lemma A.6, such transactions result in $\Pi^e(\tilde{\mathbf{r}}(\Delta)) < 0$, which completes the proof. ■

A.7 Proof of Lemma 2

Proof of Lemma 2. Define $v_i : [0,1]^{h_i} \times [0,1] \to \mathbb{R}$ such that $v_i(\boldsymbol{\theta}_i, x)$ is firm i's willingness to pay when its type is $\boldsymbol{\theta}_i$ and its allocation is x, i.e.,

$$v_i(\boldsymbol{\theta}_i, x) \equiv \max_{\mathbf{Q}_i \text{ s.t. } Q_i^j \in [0,k_i^j],\ \sum_{j=1}^{h_i} Q_i^j \leq x} \mathbf{Q}_i \cdot \boldsymbol{\theta}_i.$$

Given ex post efficient, DIC mechanism $\langle \mathbf{Q}^e, \mathbf{M} \rangle$, let $u_i^e(\boldsymbol{\theta}_i)$ denote firm i's interim expected payoff net of its outside option:

$$u_i^e(\boldsymbol{\theta}_i) \equiv \mathbb{E}_{\boldsymbol{\theta}_{-i}}[v_i(\boldsymbol{\theta}_i, Q_i^e(\boldsymbol{\theta})) - v_i(\boldsymbol{\theta}_i, r_i) - M_i(\boldsymbol{\theta})].$$

Step 1. As a first step in the proof, we show that for firm i with multi-dimensional type, u_i^e is convex with gradient that exists almost everywhere.

Consider firm i with h_i-dimensional type and type space $[0,1]^{h_i}$. Because the type space, which is the finite product of unit intervals, is smoothly connected, the focus on Groves schemes is without loss of generality (Holmström, 1979), and because we are interested in revenue-maximizing, efficient, DIC mechanisms that respect IR, we consider a version of the VCG mechanism. Define the VCG transfer for firm i at type profile $\boldsymbol{\theta}$ given an arbitrary critical type $\boldsymbol{\theta}_i^c$ by (to conserve on notation here, we ignore the constant term required to ensure that IR is satisfied):

$$M_i(\boldsymbol{\theta}) \equiv \sum_{j \in \mathcal{N}\setminus\{i\}} \mathbf{Q}_j^e(\boldsymbol{\theta}_{-i}, \boldsymbol{\theta}_i^c) \cdot \boldsymbol{\theta}_j + \mathbf{Q}_i^e(\boldsymbol{\theta}_{-i}, \boldsymbol{\theta}_i^c) \cdot \boldsymbol{\theta}_i^c - \sum_{j \in \mathcal{N}\setminus\{i\}} \mathbf{Q}_j^e(\boldsymbol{\theta}_{-i}, \boldsymbol{\theta}_i) \cdot \boldsymbol{\theta}_j. \qquad \text{(A.18)}$$

By the usual arguments, this mechanism endows the agents with dominant strategies to report types truthfully. To see this, note that at type profile $\boldsymbol{\theta}$, i's payoff is

$$\mathbf{Q}_i^e(\boldsymbol{\theta}) \cdot \boldsymbol{\theta}_i - M_i(\boldsymbol{\theta}) = \sum_{j \in \mathcal{N}} \mathbf{Q}_j^e(\boldsymbol{\theta}_{-i}, \boldsymbol{\theta}_i) \cdot \boldsymbol{\theta}_j - \sum_{j \in \mathcal{N}\setminus\{i\}} \mathbf{Q}_j^e(\boldsymbol{\theta}_{-i}, \boldsymbol{\theta}_i^c) \cdot \boldsymbol{\theta}_j - \mathbf{Q}_i^e(\boldsymbol{\theta}_{-i}, \boldsymbol{\theta}_i^c) \cdot \boldsymbol{\theta}_i^c,$$

where first term on the right side is maximized social surplus at type profile $\boldsymbol{\theta}$, so by changing the report to induce a different allocation that expression cannot be increased. Because the other terms on the right side are independent of $\boldsymbol{\theta}_i$, it follows that the mechanism endows the agents with dominant strategies.

Because $\langle \mathbf{Q}^e, \mathbf{M} \rangle$ is then also Bayesian IC, it follows that for all $\boldsymbol{\theta}_i, \boldsymbol{\theta}_i' \in [0,1]^{h_i}$,

$$u_i^e(\boldsymbol{\theta}_i) \geq \mathbb{E}_{\boldsymbol{\theta}_{-i}}[\mathbf{Q}_i^e(\boldsymbol{\theta}_i', \boldsymbol{\theta}_{-i}) \cdot \boldsymbol{\theta}_i] - v_i(\boldsymbol{\theta}_i, r_i) - \mathbb{E}_{\boldsymbol{\theta}_{-i}}[M_i(\boldsymbol{\theta}_i', \boldsymbol{\theta}_{-i})].$$

This implies that u_i^e is the maximum of a family of affine functions, which implies that it is convex and so absolutely continuous and differentiable almost everywhere in the interior of its domain.

Step 2. We now show that, letting $\tilde{\theta}_i^j$ be the j-th largest element of $\boldsymbol{\theta}_i$ and $\tilde{k}_i^j$ be the maximum demand associated with the j-th largest element of $\boldsymbol{\theta}_i$, we have

$$\nabla u_i^e(\tilde{\boldsymbol{\theta}}_i) = (\tilde{k}_i^1, \ldots, \tilde{k}_i^\ell, q_i^e(\boldsymbol{\theta}_i) - \sum_{j=1}^{\ell} \tilde{k}_i^j, 0, \ldots, 0) - (\tilde{k}_i^1, \ldots, \tilde{k}_i^{\ell'}, r_i - \sum_{j=1}^{\ell'} \tilde{k}_i^j, 0, \ldots, 0), \quad \text{(A.19)}$$

where ℓ is such that $0 < q_i^e(\boldsymbol{\theta}_i) - \sum_{j=1}^{\ell} \tilde{k}_i^j \leq \tilde{k}_i^{\ell+1}$ and ℓ' is such that $0 < r_i - \sum_{j=1}^{\ell'} \tilde{k}_i^j \leq \tilde{k}_i^{\ell'+1}$.

Given type $\boldsymbol{\theta}_i$, allocation z, and transfer y, firm i's payoff net of its outside option is $\phi_i(\boldsymbol{\theta}_i, z, y) \equiv v_i(\boldsymbol{\theta}_i, z) - y - v_i(\boldsymbol{\theta}_i, r_i)$, which is differentiable and absolutely continuous in $\boldsymbol{\theta}_i$ for all $(z, y) \in [0,1] \times \mathbb{R}$. Further, v_i has strictly increasing differences in $(\boldsymbol{\theta}_i, z)$—intuitively, increases in types make the quantity more valuable and increases in the quantity make higher types more valuable.

Denote the set of outcomes that are accessible to agent i in the VCG mechanism by $X_i \equiv \{\mathbb{E}_{\boldsymbol{\theta}_{-i}}[(Q_i^e(\boldsymbol{\theta}_{-i}, \boldsymbol{\theta}_i), M_i(\boldsymbol{\theta}_{-i}, \boldsymbol{\theta}_i))] \mid \boldsymbol{\theta}_i \in [0,1]^{h_i}\}$, and let

$$X_i^*(\boldsymbol{\theta}_i) \equiv \{(z, y) \in X_i \mid \phi_i(\boldsymbol{\theta}_i, z, y) = \sup_{(z', y') \in X_i} \phi_i(\boldsymbol{\theta}_i, z', y')\}.$$

Then the Monotone Selection Theorem of Milgrom and Shannon (1994) implies that for any $(z^*(\boldsymbol{\theta}_i), y^*(\boldsymbol{\theta}_i)) \in X_i^*(\boldsymbol{\theta}_i)$, $z^*(\boldsymbol{\theta}_i)$ is nondecreasing in $\boldsymbol{\theta}_i$ and $\frac{\partial v_i(\boldsymbol{\theta}_i, z)}{\partial \theta_i^j}$ is nondecreasing in z. Therefore, for all $\boldsymbol{\theta}'_i \in [0,1]^{h_i}$,

$$\frac{\partial \phi(\boldsymbol{\theta}_i, z^*(\boldsymbol{\theta}'_i), y^*(\boldsymbol{\theta}'_i))}{\partial \theta_i^j} = \frac{\partial v_i(\boldsymbol{\theta}_i, z^*(\boldsymbol{\theta}'_i))}{\partial \theta_i^j} - \frac{\partial v_i(\boldsymbol{\theta}_i, r_i)}{\partial \theta_i^j}$$

is bounded below by $\frac{\partial v_i(\boldsymbol{\theta}_i, z^*(\mathbf{0}))}{\partial \theta_i^j} - \frac{\partial v_i(\boldsymbol{\theta}_i, r_i)}{\partial \theta_i^j}$ and bounded above by $\frac{\partial v_i(\boldsymbol{\theta}_i, z^*(\mathbf{1}))}{\partial \theta_i^j} - \frac{\partial v_i(\boldsymbol{\theta}_i, r_i)}{\partial \theta_i^j}$, which implies that $\frac{\partial \phi(\boldsymbol{\theta}_i, z, y)}{\partial \theta_i^j}$ is uniformly bounded on $(\boldsymbol{\theta}_i, z, y) \in [0,1]^{h_i} \times X^*([0,1]^{h_i})$.

Further, because $[0,1]^{h_i}$ is smoothly connected, given any $\boldsymbol{\theta}'_i, \boldsymbol{\theta}_i \in [0,1]^{h_i}$, there is a path $\mathcal{C}$ in $[0,1]^{h_i}$ described by a continuously differentiable function $\boldsymbol{\tau} : [0,1] \to [0,1]^{h_i}$ such that $\boldsymbol{\tau}(0) = \boldsymbol{\theta}'_i$ and $\boldsymbol{\tau}(1) = \boldsymbol{\theta}_i$. Because $\phi(\boldsymbol{\theta}_i, z, y)$ is differentiable in $\boldsymbol{\theta}_i \in [0,1]^{h_i}$ and $\frac{\partial \phi(\boldsymbol{\theta}_i, z, y)}{\partial \theta_i^j}$ is uniformly bounded on $(\boldsymbol{\theta}_i, z, y) \in [0,1]^{h_i} \times X^*([0,1]^{h_i})$, the function $\hat{\phi} : [0,1] \times [0,1] \times \mathbb{R} \to \mathbb{R}$ defined by $\hat{\phi}(a, z, y) \equiv \phi(\boldsymbol{\tau}(a), z, y)$ satisfies the assumptions of Milgrom and Segal's (2002) Theorem 2 and Corollary 1 (as modified in their footnote 10). It then follows that we can express $u_i^e(\boldsymbol{\theta}_i) = \phi_i(\boldsymbol{\theta}_i, z^*(\boldsymbol{\theta}_i), y^*(\boldsymbol{\theta}_i))$ in terms of the path integral with respect to $\boldsymbol{\tau}$ as follows:

$$u_i^e(\boldsymbol{\theta}_i) = u_i^e(\boldsymbol{\theta}'_i) + \int_{\mathcal{C}} \Big(\times_{j=1}^{h_i} \frac{\partial \mathbb{E}_{\boldsymbol{\theta}_{-i}}[v_i(\boldsymbol{\tau}, Q_i^e(\boldsymbol{\theta}_{-i}, \boldsymbol{\tau}))]}{\partial \theta_i^j} - \times_{j=1}^{h_i} \frac{\partial v_i(\boldsymbol{\tau}, r_i)}{\partial \theta_i^j} \Big) d\boldsymbol{\tau}.$$

This implies that

$$\frac{\partial u_i^e(\boldsymbol{\theta}_i)}{\partial \theta_i^j} = \frac{\partial \mathbb{E}_{\boldsymbol{\theta}_{-i}}[v_i(\boldsymbol{\theta}_i, Q_i^e(\boldsymbol{\theta})]}{\partial \theta_i^j} - \frac{\partial v_i(\boldsymbol{\theta}_i, r_i)}{\partial \theta_i^j},$$

which, using the implication of the definition of v_i that

$$v_i(\boldsymbol{\theta}_i, z) = \sum_{j=1}^{\ell} \tilde{\theta}_i^j \tilde{k}_i^j + \tilde{\theta}_i^{\ell+1}(z - \sum_{j=1}^{\ell} \tilde{k}_i^j),$$

where ℓ satisfies $0 < z - \sum_{j=1}^{\ell} k_i^j \leq \tilde{k}_i^{\ell+1}$, gives us the result that (A.19) holds.

Step 3. Using the result that u_i^e is convex, it follows that at any worst-off type $\boldsymbol{\theta}_i^w$ for firm i, we have $\nabla u_i^e(\boldsymbol{\theta}_i^w) = (0, \ldots, 0)$, which, using the expression for ∇u_i^e from (A.19), implies that $q_i^e(\boldsymbol{\theta}_i^w) = r_i$. Because $q_i^e(\boldsymbol{\theta}_i)$ is increasing in $\boldsymbol{\theta}_i$, the second-order condition is also satisfied.

We now show that the constant worst-off type $\hat{\boldsymbol{\theta}}_i = (\hat{\theta}_i, \ldots, \hat{\theta}_i)$ such that $q_i^e(\hat{\boldsymbol{\theta}}_i) = r_i$ is revenue maximizing. First note that because $q_i^e(\mathbf{0}) < \min\{1, \sum_{j=1}^{h_i} k_i^j\} = q_i^e(\mathbf{1})$ and because q_i^e is continuously increasing, such a type $\hat{\boldsymbol{\theta}}_i$ exists and is unique. Second, consider the payment of firm i under a VCG mechanism defined with respect to some, at this point arbitrary, critical type $\boldsymbol{\theta}_i^c$. That is, firm i's payment at type profile $\boldsymbol{\theta}$ is

$$M_i^{VCG}(\boldsymbol{\theta}_i^c, \boldsymbol{\theta}) \equiv W(\boldsymbol{\theta}_i^c, \boldsymbol{\theta}_{-i}) - W(\boldsymbol{\theta}) + \mathbf{Q}_i^e(\boldsymbol{\theta}) \cdot \boldsymbol{\theta}_i - v_i(\boldsymbol{\theta}_i^c, r_i), \tag{A.20}$$

where $v_i(\boldsymbol{\theta}_i^c, r_i)$ is the outside option for firm i's critical type. This implies that firm i's expected payment is $m_i^{VCG}(\boldsymbol{\theta}_i^c) \equiv \mathbb{E}_{\boldsymbol{\theta}}[M_i^{VCG}(\boldsymbol{\theta}_i^c, \boldsymbol{\theta})]$. By the envelope theorem and (A.19), the gradient of $m_i^{VCG}(\boldsymbol{\theta}_i^c)$ with respect to $\boldsymbol{\theta}_i^c$ is

$$(\tilde{k}_i^1, \ldots, \tilde{k}_i^\ell, q_i^e(\boldsymbol{\theta}_i) - \sum_{j=1}^{\ell} \tilde{k}_i^j, 0, \ldots, 0) - (\tilde{k}_i^1, \ldots, \tilde{k}_i^h, r_i - \sum_{j=1}^{h} \tilde{k}_i^j, 0, \ldots, 0).$$

This derivative is nonnegative for all $\boldsymbol{\theta}_i^c$ such that $q_i^e(\boldsymbol{\theta}_i^c) \geq r_i$, which is the set of types for firm i that are of interest from here on, and it is strictly positive if $q_i^e(\boldsymbol{\theta}_i^c) > r_i$. Because $q_i^e(\cdot)$ is strictly increasing, that is, $q_i^e(\boldsymbol{\theta}_i) > q_i^e(\boldsymbol{\theta}_i')$ for any $\boldsymbol{\theta}_i$ and $\boldsymbol{\theta}_i'$ satisfying $\theta_i^j \geq \theta_i^{j\prime}$ for all $j \in \{1, \ldots, h_i\}$ with at least one strict inequality, it follows that among all types $\boldsymbol{\theta}_i^c$ satisfying $q_i^e(\boldsymbol{\theta}_i^c) \geq r_i$ and $\max \boldsymbol{\theta}_i^c = \theta^1$, revenue is maximized by the vector of constant types $(\theta^1, \ldots, \theta^1)$. Consequently, the critical type that maximizes revenue while respecting firm i's IR constraint is the unique constant type $\hat{\boldsymbol{\theta}}_i$ satisfying $q_i^e(\hat{\boldsymbol{\theta}}_i) = r_i$. ■

A.8 Proof of Proposition 8

Proof of Proposition 8. Firm i's VCG payment given constant worst-off type $\hat{\boldsymbol{\theta}}_i$ is given by (11). The derivative of firm i's payment with respect to r_i is $-\hat{\theta}_i$ and the second derivative is $-\frac{d\hat{\theta}_i}{dr_i} \leq 0$, with cross-derivatives that are zero. Thus, the expected revenue under ex post efficiency is concave in $\mathbf{r}$, implying that $\mathcal{R}^e$ is convex.

Assume that all firms have the same constant worst-off type $\hat{\theta}^*$. That is, firms with a one-dimensional type have worst-off type $\hat{\theta}^*$ and firm i with a multi-dimensional type has worst-off type $\hat{\boldsymbol{\theta}}_i^* = (\hat{\theta}^*, \ldots, \hat{\theta}^*)$. A simple revealed preference argument establishes that

$$W(\boldsymbol{\theta}) - W(\hat{\boldsymbol{\theta}}_i^*, \boldsymbol{\theta}_{-i}) \leq (\boldsymbol{\theta}_i - \hat{\boldsymbol{\theta}}_i^*) \cdot \mathbf{Q}_i^e(\boldsymbol{\theta}) = \boldsymbol{\theta}_i \cdot \mathbf{Q}_i^e(\boldsymbol{\theta}) - \hat{\theta}^* Q_i^e(\boldsymbol{\theta}), \tag{A.21}$$

because as firm i's type changes from $\boldsymbol{\theta}_i$ to $\hat{\boldsymbol{\theta}}_i^*$, the planner could keep the allocation fixed at $\mathbf{Q}^e(\boldsymbol{\theta})$. Optimizing, it means that it can do weakly better, and strictly better for a positive measure of types given that we assume positive densities on $(0, 1)$. (Inequality (A.21) is the multi-unit generalization of (A.6) in Liu et al. (forth.).) Because $\sum_{i\in\mathcal{N}} Q_i^e(\boldsymbol{\theta}) = 1$, summing up both sides in (A.21) yields

$$\sum_{i\in\mathcal{N}} (W(\boldsymbol{\theta}) - W(\hat{\boldsymbol{\theta}}_i^*, \boldsymbol{\theta}_{-i})) \leq W(\boldsymbol{\theta}) - \hat{\theta}^*,$$

which is equivalent to

$$0 \leq \sum_{i\in\mathcal{N}} W(\hat{\boldsymbol{\theta}}_i^*, \boldsymbol{\theta}_{-i}) - (n-1)W(\boldsymbol{\theta}) - \hat{\theta}^*. \tag{A.22}$$

Using (11), firm i's VCG transfer given worst-off type $\hat{\boldsymbol{\theta}}_i^*$ is

$$M_i^{VCG}(\boldsymbol{\theta}) \equiv W(\hat{\boldsymbol{\theta}}_i^*, \boldsymbol{\theta}_{-i}) - W(\boldsymbol{\theta}) + \mathbf{Q}_i^e(\boldsymbol{\theta}) \cdot \boldsymbol{\theta}_i - r_i \hat{\theta}^*. \tag{A.23}$$

Summing these transfers across all firms and using $\sum_{i\in\mathcal{N}} r_i = 1$, we have

$$\sum_{i\in\mathcal{N}} M_i^{VCG}(\boldsymbol{\theta}) = \sum_{i\in\mathcal{N}} W(\hat{\boldsymbol{\theta}}_i^*, \boldsymbol{\theta}_{-i}) - (n-1)W(\boldsymbol{\theta}) - \hat{\theta}^* \geq 0,$$

where the inequality uses (A.22). Thus, the revenue of the VCG mechanism in which all agents have the same worst-off type $\hat{\theta}^*$ is never negative, and the inequality is strict for a positive measure of types because (A.21) is strict for a positive measure of types, implying that the revenue is positive in expectation. Moreover, it satisfies IR for each firm i because firm i's interim expected payoff net of its outside option for its worst-off type is

$$\mathbb{E}_{\boldsymbol{\theta}_{-i}} \left[\hat{\boldsymbol{\theta}}_i^* \cdot \mathbf{Q}_i^e(\hat{\boldsymbol{\theta}}_i^*, \boldsymbol{\theta}_{-i}) - M_i^{VCG}(\hat{\boldsymbol{\theta}}_i^*, \boldsymbol{\theta}_{-i}) \right] - r_i \hat{\theta}^* = 0,$$

where the equality uses (A.23), and, by the definition of the worst-off type, firm i's interim expected payoff net of its outside option for any other type is weakly greater.

It remains only to establish that there exists $\mathbf{r}^* \in \Delta_{\mathbf{k}}$ such that equalized worst-off types exist. Thus, we now state and prove the following lemma:

Lemma A.7. *There exists $\mathbf{r}^* \in \Delta_{\mathbf{k}}$ and $\hat{\theta}^* \in (0, 1)$ such that for all i with one-dimensional types, $r_i^* = q_i^e(\hat{\theta}^*)$, and for all i with multi-dimensional types, $r_i^* = q_i^e(\hat{\boldsymbol{\theta}}_i^*)$, where $\hat{\boldsymbol{\theta}}_i^* \equiv (\hat{\theta}^*, \ldots, \hat{\theta}^*) \in [0, 1]^{h_i}$.*

Proof of Lemma A.7. We decompose the multi-dimensional firms to create a setup with the same number of firms as the total number of types, $\sum_{i\in\mathcal{N}} h_i$. For $i \in \mathcal{N}$, we let $\tilde{\mathcal{N}}_i$ be the indices of the extended set of firms derived from firm i's types, with $|\tilde{\mathcal{N}}_i| = h_i$, $\tilde{\mathcal{N}}_i \cap \tilde{\mathcal{N}}_j = \varnothing$ for $i \neq j$, and $\tilde{\mathcal{N}} = \cup_{i\in\mathcal{N}} \tilde{\mathcal{N}}_i$. Further, define $\tilde{\mathcal{N}}_{-i} \equiv \cup_{j\in\mathcal{N}\setminus\{i\}} \tilde{\mathcal{N}}_j$ to be the set of extended firms derived from firms other than firm i. For each $i \in \tilde{\mathcal{N}}$, let $\tilde{k}_i$ denote the associated maximum demand and $\tilde{F}_i$ the associated distribution for extended firm i. For each of our actual firms $i \in \mathcal{N}$, we define the interim expected allocation under ex post efficiency when firm i has the constant type $(\theta_i, \dots, \theta_i)$ by

$$\tilde{q}_i^e(\theta_i) = \sum_{\mathcal{A}\subset\tilde{\mathcal{N}}_{-i}} \max\{0, \min\{k_i, 1 - \sum_{j\in\mathcal{A}} \tilde{k}_j\}\} \prod_{j\in\mathcal{A}} (1 - \tilde{F}_j(\theta_i)) \prod_{j\in\tilde{\mathcal{N}}_{-i}\setminus\mathcal{A}} \tilde{F}_j(\theta_i),$$

which is continuous and increasing in θ_i on $[0,1]$ and satisfies $\tilde{q}_i^e(\theta_i) \in [0, k_i]$. Then by an argument analogous to that of Lemma A.5, we have

$$\sum_{i\in\mathcal{N}} \tilde{q}_i^e(0) < 1. \tag{A.24}$$

We now define a real-valued function on $[0,1]$, denoted g, that will allow us to identify a common worst-off type:

$$g(t) \equiv \sum_{i\in\mathcal{N}} \tilde{q}_i^e(t).$$

Using (A.24), we have $g(0) = 0$, and using the assumption of excess demand, we have $g(1) > 1$. Further, $g(t)$ is continuously increasing. Thus, there exists a unique $\hat{\theta}^* \in (0,1)$ such that $g(\hat{\theta}^*) = 1$. Given $\hat{\theta}^*$, define for $i \in \mathcal{N}$, $r_i^* \equiv g_i(\hat{\theta}^*)$, which satisfies $\mathbf{r}^* \in \Delta_{\mathbf{k}}$, which completes the proof of Lemma A.7. □

Together, these results complete the proof of Proposition 8. ■

A.9 Proof of Theorem 1

Proof of Theorem 1. In what follows, Lemma A.8 proves the result for a merger of buyers, Lemma A.9 applies to a merger of sellers, and Lemma A.10 applies to a vertical merger. In these lemmas, we use the notation $v_i(x, \boldsymbol{\theta}_i)$ to denote firm i's willingness to pay for quantity x when its type is $\boldsymbol{\theta}_i$, i.e., $v_i(x, \boldsymbol{\theta}_i) \equiv \max_{\mathbf{Q}_i \text{ s.t. } Q_i^\ell \in [0,k_i^\ell],\ \sum_{\ell=1}^{h_i} Q_i^\ell \leq x} \mathbf{Q}_i \cdot \boldsymbol{\theta}_i$. Analogously, we use $v_{i,j}(\boldsymbol{\theta}_i, \boldsymbol{\theta}_j, x)$ to denote the willingness to pay of the integrated firm that combines firms i and j for quantity x, $v_{i,j}(x, \boldsymbol{\theta}_i, \boldsymbol{\theta}_j) \equiv \max_{\mathbf{Q}_i, \mathbf{Q}_j \text{ s.t. } Q_i^\ell \in [0,k_i^\ell],\ Q_j^\ell \in [0,k_j^\ell],\ \sum_{\ell=1}^{h_i} Q_i^\ell + \sum_{\ell=1}^{h_j} Q_j^\ell \leq x} \mathbf{Q}_i \cdot \boldsymbol{\theta}_i + \mathbf{Q}_j \cdot \boldsymbol{\theta}_j$.

Lemma A.8. *A merger of two buyers decreases expected revenue under ex post efficiency.*

Proof of Lemma A.8. Consider full integration between two buyers, b_1 and b_2 with $r_{b_1} = r_{b_2} = 0$. Let $\boldsymbol{\theta}_{b_1}$, $\boldsymbol{\theta}_{b_2}$, and $\boldsymbol{\theta}_o$ be the vectors of types for buyer b_1, buyer b_2, and all the other firms, respectively. Because the VCG payments of the nonintegrating firms are not affected by the integration, we can focus on the change in payments made by the integrating firms.

It is sufficient to show that the sum of the VCG payments of buyers b_1 and b_2 is greater than or equal to (and strictly greater for a positive measure set of type realizations) the VCG payment of the firm formed through the integration of buyers b_1 and b_2. Thus, it is sufficient to show that the following inequality holds (and strictly for a positive measure set of type realizations), where the left side is the sum of the VCG payments of buyers b_1 and b_2, and the right side is the VCG payment of the firm formed through the integration of buyers b_1 and b_2:

$$\begin{aligned}
& W(\mathbf{0},\boldsymbol{\theta}_{b_2},\boldsymbol{\theta}_o) - W(\boldsymbol{\theta}_{b_1},\boldsymbol{\theta}_{b_2},\boldsymbol{\theta}_o) + v_{b_1}(Q^e_{b_1}(\boldsymbol{\theta}_{b_1},\boldsymbol{\theta}_{b_2},\boldsymbol{\theta}_o),\boldsymbol{\theta}_{b_1}) \\
& + W(\boldsymbol{\theta}_{b_1},\mathbf{0},\boldsymbol{\theta}_o) - W(\boldsymbol{\theta}_{b_1},\boldsymbol{\theta}_{b_2},\boldsymbol{\theta}_o) + v_{b_2}(Q^e_{b_2}(\boldsymbol{\theta}_{b_1},\boldsymbol{\theta}_{b_2},\boldsymbol{\theta}_o),\boldsymbol{\theta}_{b_2}) \\
\geq \; & W(\mathbf{0},\mathbf{0},\boldsymbol{\theta}_o) - W(\boldsymbol{\theta}_{b_1},\boldsymbol{\theta}_{b_2},\boldsymbol{\theta}_o) + v_{b_1,b_2}(Q^e_{b_1}(\boldsymbol{\theta}_{b_1},\boldsymbol{\theta}_{b_2},\boldsymbol{\theta}_o) + Q^e_{b_2}(\boldsymbol{\theta}_{b_1},\boldsymbol{\theta}_{b_2},\boldsymbol{\theta}_o),\boldsymbol{\theta}_{b_1},\boldsymbol{\theta}_{b_2}).
\end{aligned}$$

Noting that $v_{b_1,b_2}(Q^e_{b_1} + Q^e_{b_2},\boldsymbol{\theta}_{b_1},\boldsymbol{\theta}_{b_2}) = v_{b_1}(Q^e_{b_1},\boldsymbol{\theta}_{b_1}) + v_{b_2}(Q^e_{b_2},\boldsymbol{\theta}_{b_2})$ (where we drop the argument $(\boldsymbol{\theta}_{b_1},\boldsymbol{\theta}_{b_2},\boldsymbol{\theta}_o)$ on the allocations $Q^e_{b_1}$ and $Q^e_{b_2}$), we can rewrite this as

$$W(\mathbf{0},\boldsymbol{\theta}_{b_2},\boldsymbol{\theta}_o) + W(\boldsymbol{\theta}_{b_1},\mathbf{0},\boldsymbol{\theta}_o) \geq W(\boldsymbol{\theta}_{b_1},\boldsymbol{\theta}_{b_2},\boldsymbol{\theta}_o) + W(\mathbf{0},\mathbf{0},\boldsymbol{\theta}_o). \tag{A.25}$$

Let type vector $\boldsymbol{\theta} = (\boldsymbol{\theta}_{b_1},\boldsymbol{\theta}_{b_2},\boldsymbol{\theta}_o)$ be given. Let $(\theta_{(1)},...,\theta_{(h)})$ be the ranked list of types from largest to smallest and (with some abuse of notation) let $k_{(j)}$ be the maximum demand corresponding to the j-th highest type. Letting ℓ be such that $\sum_{i=1}^{\ell-1} k_{(i)} < 1 \leq \sum_{i=1}^{\ell} k_{(i)}$, it follows that under ex post efficiency, the types $\boldsymbol{\theta}^{top} \equiv (\theta_{(1)},...,\theta_{(\ell-1)})$ are served up to their maximum demands, with the remainder of the supply, $k^r \equiv 1 - \sum_{i=1}^{\ell-1} k_{(i)}$, going to type $\theta_{(\ell)}$, i.e.,

$$W(\boldsymbol{\theta}_{b_1},\boldsymbol{\theta}_{b_2},\boldsymbol{\theta}_o) = \sum_{i=1}^{\ell-1} \theta_{(i)} k_{(i)} + \theta_{(\ell)} k^r. \tag{A.26}$$

For $i \in \{1,\ldots,\ell\}$, we let $I(\theta_{(i)})$ denote the firm associated with $\theta_{(i)}$. Further, for $j \in \{b_1, b_2, o\}$, define $k_j^{top} \equiv \sum_{i=1}^{\ell-1} k_{(i)} \cdot 1_{I(\theta_{(i)})=j}$ and $k_j^r \equiv k^r \cdot 1_{I(\theta_{(\ell)})=j}$, where, by definition

$$k_{b_1}^{top} + k_{b_2}^{top} + k_o^{top} + k_{b_1}^r + k_{b_2}^r + k_o^r = 1. \tag{A.27}$$

With some abuse of notation, for $i \in \{1,2\}$, we define $\tilde{v}_{b_i,o}\left(x, \boldsymbol{\theta}_{b_i}\backslash\boldsymbol{\theta}^{top}, \boldsymbol{\theta}_o\backslash\boldsymbol{\theta}^{top}\right)$ to be the joint willingness to pay of buyer b_i and all firms other than buyer b_1 and buyer b_2 for quantity x when buyer b_i's type is $\tilde{\boldsymbol{\theta}}_{b_i}$ given by $\tilde{\theta}^j_{b_i} = 0$ if $\theta^j_{b_i}$ is in the top $\ell - 1$ types and $\tilde{\theta}^j_{b_i} = \theta^j_{b_i}$ otherwise, and analogously for the other firms' types, and adjusting the maximum demand for type $\theta_{(\ell)}$ to be $k_{(\ell)} - k^r$.

We can now write

$$\begin{aligned}
& W(\mathbf{0}, \boldsymbol{\theta}_{b_2}, \boldsymbol{\theta}_o) + W(\boldsymbol{\theta}_{b_1}, \mathbf{0}, \boldsymbol{\theta}_o) \\
= \; & \textstyle\sum_{i=1}^{\ell-1} \theta_{(i)} k_{(i)} \left(1_{I(\theta_{(i)})=b_2} + 1_{I(\theta_{(i)})=o}\right) + \theta_{(\ell)}(k^r_{b_2} + k^r_o) + \tilde{v}_{b_2,o}\left(1 - k^{top}_{b_2} - k^{top}_o - k^r_{b_2} - k^r_o, \boldsymbol{\theta}_{b_2}\backslash\boldsymbol{\theta}^{top}, \boldsymbol{\theta}_o\backslash\boldsymbol{\theta}^{top}\right) \\
& + \textstyle\sum_{i=1}^{\ell-1} \theta_{(i)} k_{(i)} \left(1_{I(\theta_{(i)})=b_1} + 1_{I(\theta_{(i)})=o}\right) + \theta_{(\ell)}(k^r_{b_1} + k^r_o) + \tilde{v}_{b_1,o}\left(1 - k^{top}_{b_1} - k^{top}_o - k^r_{b_1} - k^r_o, \boldsymbol{\theta}_{b_1}\backslash\boldsymbol{\theta}^{top}, \boldsymbol{\theta}_o\backslash\boldsymbol{\theta}^{top}\right) \\
= \; & W(\boldsymbol{\theta}_{b_1}, \boldsymbol{\theta}_{b_2}, \boldsymbol{\theta}_o) + \textstyle\sum_{i=1}^{\ell-1} \theta_{(i)} k_{(i)} 1_{I(\theta_{(i)})=o} + \theta_{(\ell)} k^r_o \\
& + \tilde{v}_{b_2,o}\left(1 - k^{top}_{b_2} - k^{top}_o - k^r_{b_2} - k^r_o, \boldsymbol{\theta}_{b_2}\backslash\boldsymbol{\theta}^{top}, \boldsymbol{\theta}_o\backslash\boldsymbol{\theta}^{top}\right) + \tilde{v}_{b_1,o}\left(1 - k^{top}_{b_1} - k^{top}_o - k^r_{b_1} - k^r_o, \boldsymbol{\theta}_{b_1}\backslash\boldsymbol{\theta}^{top}, \boldsymbol{\theta}_o\backslash\boldsymbol{\theta}^{top}\right) \\
\geq \; & W(\boldsymbol{\theta}_{b_1}, \boldsymbol{\theta}_{b_2}, \boldsymbol{\theta}_o) + \textstyle\sum_{i=1}^{\ell-1} \theta_{(i)} k_{(i)} 1_{I(\theta_{(i)})=o} + \theta_{(\ell)} k^r_o \\
& + \tilde{v}_{b_2,o}\left(1 - k^{top}_{b_2} - k^{top}_o - k^r_{b_2} - k^r_o, \mathbf{0}, \boldsymbol{\theta}_o\backslash\boldsymbol{\theta}^{top}\right) + \tilde{v}_{b_1,o}\left(1 - k^{top}_{b_1} - k^{top}_o - k^r_{b_1} - k^r_o, \mathbf{0}, \boldsymbol{\theta}_o\backslash\boldsymbol{\theta}^{top}\right) \\
= \; & W(\boldsymbol{\theta}_{b_1}, \boldsymbol{\theta}_{b_2}, \boldsymbol{\theta}_o) + v_o(2 - k^{top}_{b_1} - k^{top}_{b_2} - k^{top}_o - k^r_{b_1} - k^r_{b_2} - k^r_o, \boldsymbol{\theta}_o) \\
= \; & W(\boldsymbol{\theta}_{b_1}, \boldsymbol{\theta}_{b_2}, \boldsymbol{\theta}_o) + v_o(1, \boldsymbol{\theta}_o) \\
= \; & W(\boldsymbol{\theta}_{b_1}, \boldsymbol{\theta}_{b_2}, \boldsymbol{\theta}_o) + W(\mathbf{0}, \mathbf{0}, \boldsymbol{\theta}_o),
\end{aligned}$$

where the second equality uses (A.26), the first inequality follows because we reduce the types in the $\tilde{v}$ expressions for b_1 and b_2 to zero, the third equality adds up all the units being allocated to outside types, and the second-to-last equality uses (A.27). This establishes (A.25), and with a strict inequality for a positive measure of type realizations, which completes the proof. □

Lemma A.9. *A merger of two sellers decreases expected revenue under ex post efficiency.*

Proof of Lemma A.9. Consider full integration between two sellers, s_1 and s_2 with $r_{s_1} = k_{s_1}$ and $r_{s_2} = k_{s_2}$. Let $\boldsymbol{\theta}_{s_1}$, $\boldsymbol{\theta}_{s_2}$, and $\boldsymbol{\theta}_o$ be the vectors of types for seller s_1, seller s_2, and all the other firms, respectively. As in the proof of Lemma A.8, the VCG payments of the nonintegrating firms are not affected by the integration, so we can focus on the change in payments made by the integrating firms. It is sufficient to show that the following inequality holds (and strictly for a positive measure set of types):

$$\begin{aligned}
& W(\mathbf{0}, \boldsymbol{\theta}_{s_2}, \boldsymbol{\theta}_o) - W(\boldsymbol{\theta}_{s_1}, \boldsymbol{\theta}_{s_2}, \boldsymbol{\theta}_o) + v_{s_1}(Q^e_{s_1}(\boldsymbol{\theta}_{s_1}, \boldsymbol{\theta}_{s_2}, \boldsymbol{\theta}_o), \boldsymbol{\theta}_{s_1}) - r_{s_1} \\
& + W(\boldsymbol{\theta}_{s_1}, \mathbf{0}, \boldsymbol{\theta}_o) - W(\boldsymbol{\theta}_{s_1}, \boldsymbol{\theta}_{s_2}, \boldsymbol{\theta}_o) + v_{s_2}(Q^e_{s_2}(\boldsymbol{\theta}_{s_1}, \boldsymbol{\theta}_{s_2}, \boldsymbol{\theta}_o), \boldsymbol{\theta}_{s_2}) - r_{s_2} \\
\geq \; & W(\mathbf{0}, \mathbf{0}, \boldsymbol{\theta}_o) - W(\boldsymbol{\theta}_{s_1}, \boldsymbol{\theta}_{s_2}, \boldsymbol{\theta}_o) + V_{s_1,s_2}(Q^e_{s_1}(\boldsymbol{\theta}_{s_1}, \boldsymbol{\theta}_{s_2}, \boldsymbol{\theta}_o) + Q^e_{s_2}(\boldsymbol{\theta}_{s_1}, \boldsymbol{\theta}_{s_2}, \boldsymbol{\theta}_o), \boldsymbol{\theta}_{s_1}, \boldsymbol{\theta}_{s_2}) - r_{s_1} - r_{s_2}.
\end{aligned}$$

Because the merging sellers' outside options drop out of this expression, the problem is identical to that in Lemma A.8, but with "s" replacing "b", and so the result of Lemma A.8 applies, thereby completing the proof. □

Lemma A.10. *A merger of a buyer and a seller increases expected revenue under ex post efficiency.*

Proof. Consider full integration between a buyer b with $r_b = 0$ and a seller s with $r_s = k_s \leq 1$. Let $\boldsymbol{\theta}_b$, $\boldsymbol{\theta}_s$, and $\boldsymbol{\theta}_o$ be the vectors of types for buyer b, seller s, and all the other firms, respectively. Let $v_{all}(x, \boldsymbol{\theta}_s, \boldsymbol{\theta}_b, \boldsymbol{\theta}_o)$ be the joint willingness to pay of all firms for quantity x units and have types $\boldsymbol{\theta}_s$, $\boldsymbol{\theta}_b$, and $\boldsymbol{\theta}_o$. Because the VCG payments of the nonintegrating firms are not affected by the integration of firms s and b, we can focus on the change in payments made by the integrating firms. We show that the following inequality holds (and strictly for a positive measure set of type realizations), where the left side is the sum of the VCG payments of buyer b and seller s, and the right side is the VCG payment of the firm formed through the integration of b and s, where that firm's worst-off type is denoted $(\omega, \ldots, \omega)$:

$$\begin{aligned}
& W(\mathbf{1}, \boldsymbol{\theta}_b, \boldsymbol{\theta}_o) - W(\boldsymbol{\theta}_s, \boldsymbol{\theta}_b, \boldsymbol{\theta}_o) + v_s(Q_s^e(\boldsymbol{\theta}_s, \boldsymbol{\theta}_b, \boldsymbol{\theta}_o), \boldsymbol{\theta}_s) - r_s \\
& + W(\boldsymbol{\theta}_s, \mathbf{0}, \boldsymbol{\theta}_o) - W(\boldsymbol{\theta}_s, \boldsymbol{\theta}_b, \boldsymbol{\theta}_o) + v_b(Q_b^e(\boldsymbol{\theta}_s, \boldsymbol{\theta}_b, \boldsymbol{\theta}_o), \boldsymbol{\theta}_b) \\
\leq \; & W(\boldsymbol{\omega}, \boldsymbol{\omega}, \boldsymbol{\theta}_o) - W(\boldsymbol{\theta}_s, \boldsymbol{\theta}_b, \boldsymbol{\theta}_o) + v_{s,b}(Q_b^e(\boldsymbol{\theta}_s, \boldsymbol{\theta}_b, \boldsymbol{\theta}_o) + Q_s^e(\boldsymbol{\theta}_s, \boldsymbol{\theta}_b, \boldsymbol{\theta}_o), \boldsymbol{\theta}_s, \boldsymbol{\theta}_b) - \omega r_s.
\end{aligned}$$

Noting that $k_s = r_s$ and that $v_{s,b}(Q_b^e + Q_s^e, \boldsymbol{\theta}_s, \boldsymbol{\theta}_b) = v_s(Q_s^e, \boldsymbol{\theta}_s) + v_b(Q_b^e, \boldsymbol{\theta}_b)$ (where we drop the argument $(\boldsymbol{\theta}_s, \boldsymbol{\theta}_b, \boldsymbol{\theta}_o)$ on the allocations Q_s^e and Q_b^e), we can rewrite this as

$$W(\mathbf{1}, \boldsymbol{\theta}_b, \boldsymbol{\theta}_o) - W(\boldsymbol{\omega}, \boldsymbol{\omega}, \boldsymbol{\theta}_o) - (1-\omega)k_s \leq W(\boldsymbol{\theta}_s, \boldsymbol{\theta}_b, \boldsymbol{\theta}_o) - W(\boldsymbol{\theta}_s, \mathbf{0}, \boldsymbol{\theta}_o). \tag{A.28}$$

By the definition of the ex post efficient allocation and since $Q_s^e(\mathbf{1}, \boldsymbol{\theta}_b, \boldsymbol{\theta}_o) = k_s$, we have

$$\begin{aligned}
W(\boldsymbol{\omega}, \boldsymbol{\omega}, \boldsymbol{\theta}_o) & \geq \omega Q_s^e(\mathbf{1}, \boldsymbol{\theta}_b, \boldsymbol{\theta}_o) + v_{b,o}(1 - Q_s^e(\mathbf{1}, \boldsymbol{\theta}_b, \boldsymbol{\theta}_o), \boldsymbol{\omega}, \boldsymbol{\theta}_o) \qquad \text{(A.29)} \\
& = \omega k_s + v_{b,o}(1 - k_s, \boldsymbol{\omega}, \boldsymbol{\theta}_o),
\end{aligned}$$

with a strict inequality for a positive measure set of types realizations. It then follows that

$$\begin{aligned}
& W(\mathbf{1}, \boldsymbol{\theta}_b, \boldsymbol{\theta}_o) - W(\boldsymbol{\omega}, \boldsymbol{\omega}, \boldsymbol{\theta}_o) - (1-\omega)k_s \\
\leq \; & k_s + v_{b,o}(1 - k_s, \boldsymbol{\theta}_b, \boldsymbol{\theta}_o) - \omega k_s - v_{b,o}(1 - k_s, \boldsymbol{\omega}, \boldsymbol{\theta}_o) - (1-\omega)k_s \\
\leq \; & k_s + v_{b,o}(1 - k_s, \boldsymbol{\theta}_b, \boldsymbol{\theta}_o) - \omega k_s - v_o(1 - k_s, \boldsymbol{\theta}_o) - (1-\omega)k_s \\
= \; & v_{b,o}(1 - k_s, \boldsymbol{\theta}_b, \boldsymbol{\theta}_o) - v_o(1 - k_s, \boldsymbol{\theta}_o) \\
\leq \; & v_{b,o}(1 - k_s, \boldsymbol{\theta}_b, \boldsymbol{\theta}_o) - v_o(1 - k_s, \boldsymbol{\theta}_o) + v_{all}(k_s, \boldsymbol{\theta}_s, \boldsymbol{\theta}_b, \boldsymbol{\theta}_o) - v_{s,o}(k_s, \boldsymbol{\theta}_s, \boldsymbol{\theta}_o) \\
= \; & W(\boldsymbol{\theta}_s, \boldsymbol{\theta}_b, \boldsymbol{\theta}_o) - W(\boldsymbol{\theta}_s, \mathbf{0}, \boldsymbol{\theta}_o),
\end{aligned}$$

where the first inequality uses (A.29). This establishes that (A.28) holds, including with a strict inequality for a positive measure set of type realizations. Thus, a vertical merger increases expected revenue under ex post efficiency and binding IR for firms' worst-off types, which establishes the result. □

Together, Lemmas A.8–A.10 complete the proof of Theorem 1. ■

B Appendix: Extensions and discussion

B.1 Extension to multiple periods

Here, we define a T-period extension of our static model. Fix an ownership structure $\mathbf{r}$. Within each period, types are realized (independently across firms and time), firms participate in the market to lease assets to or from other firms, and then firms realize payoffs associated with their total asset holdings (initial assets, minus assets leased to others, plus assets leased from others). At the end of the period, the leases expire and firms' asset holdings revert to the ownership structure $\mathbf{r}$.

If one assumes that a firm's participation decision in one period has no implications for whether it can participate in future periods, then the IR constraints in each period remain the same as in our static model. Thus, the ex post efficiency permitting set remains simply $\mathcal{R}^e$.

If, instead, a firm that chooses not to participate in period t cannot participate in any future period, then firm i's IR constraint is relaxed. Firm i participates in period t as long as $u_i(\theta_i) \geq -\sum_{\tau=t+1}^{a} \mathbb{E}_i[u(\theta_i)]$, rather than simply as long as $u_i(\theta_i) \geq 0$. Thus, the set of ownership structures that result in ex post efficiency in period $t < T$ is a superset of $\mathcal{R}^e$. However, only ownership structures in $\mathcal{R}^e$ permit ex post efficiency in every period. In this sense, the key features of the static setup extend to the multi-period model.

B.2 Extension to nonidentical supports

In this appendix, we provide additional details related ot the extension to nonidentical supports discussed in Section 6.3. The extension to nonidentical supports allows us to connect with prior results related to vertical integration.

As described in Section 6.3, for this extension to nonidentical supports, we assume that firms are divided into a set $\mathcal{N}_U$ of $N_U \geq 1$ "upstream" sellers with support $[0,1]$ and a set $\mathcal{N}_D$ of $N_D \geq 1$ "downstream" buyers with support $[\underline{\theta}, 1+\underline{\theta}]$, where prior to integration, all firms have one-dimensional types and have maximum demands of k. Thus, each seller i has $r_i = k$ and each buyer i has $r_i = 0$, which implies that $k = 1/N_U$.

In this context, integration between an upstream seller s (i.e., $s \in \mathcal{N}_U$ with $r_s = k$) and a downstream buyer b (i.e., $b \in \mathcal{N}_D$ with $r_b = 0$) creates an integrated firm i that is a trader with $r_i = k$ and $k_i = 2k$, which is naturally thought of as a vertically integrated firm. Consequently, in what follows, we use the term *vertical integration* to refer to full integration between an upstream seller and a downstream buyer.

Case I. *If $N_U = 1$, $N_D = 1$, and $\underline{\theta} < 1$ (overlapping supports), then vertical integration* *permits ex post efficiency whereas prior to full integration ex post efficiency is not possible.* In the case of one upstream seller and one downstream buyer and $\underline{\theta} < 1$, because the supports overlap, ex post efficiency is not possible by Myerson and Satterthwaite (1983). But, because full integration eliminates an agency problem within the firm, ex post efficiency is achieved following a merger of the buyer and seller. Thus, as observed in Loertscher and Marx (2022, Proposition 6), in this case vertical integration increases social surplus.

Case II. *If $N_U > 1$, $N_D = 1$, and $\underline{\theta} \geq 1$ (nonoverlapping supports), then vertical integration reduces social surplus because ex post efficiency is possible before but not after vertical integration.*
We obtain a contrasting result for the setting with one downstream buyer, multiple upstream sellers, and $\underline{\theta} \geq 1$. In the pre-integration market, ex post efficiency is possible—for example based on a fixed price of $p \in [1, \underline{\theta}]$. Integration between the downstream buyer and one upstream seller leaves us with $N_U - 1 \geq 1$ sellers and one integrated firm that has a two-dimensional type and k units of the asset. Ex post efficiency requires that k units be allocated to the integrated firm's buyer-side type, which is drawn from $[\underline{\theta}, 1 + \underline{\theta}]$, and for the remaining $1 - k$ units of supply to be allocated to the N_U largest seller types, whether that is the integrated firm's seller-side type or one of the independent sellers' types. In this setting, one can essentially remove the integrated buyer and the integrated firm's k units from the problem because that allocation occurs for all type realizations. What remains then are N_U entities all with the same support, where one (the seller side of the integrated firm) acts as a buyer and the remaining as sellers. This is a two-sided setting with a common support in which ex post efficiency is not possible (Delacrétaz et al., 2019). Thus, in this case, as in Loertscher and Marx (2022, Proposition 7), vertical integration decreases social surplus.

Case III. *If $N_U > 1$, $N_D = 1$, and $\underline{\theta} \in [0, 1)$ (overlapping supports), then the social surplus effects of vertical integration depend, in general, on the number of sellers.*
As in Loertscher and Marx (2022, Online Appendix F.2.B), with one buyer, multiple sellers, and overlapping supports, the social surplus effects of vertical integration depend, in general, on the number of sellers. We know from Williams (1999) and Makowski and Mezzetti (1993) that ex post efficiency is possible with nonidentical supports if N_U is large enough. Because vertical integration between buyer 1 and seller 1 creates an integrated firm that is a buyer (because all other firms are sellers, the integrated firm can only act as a buyer) with willingness to pay for k units from the independent sellers of $\min\{v_1, c_1\}$, which has support $[\underline{\theta}, 1]$. The results of Williams (1999) for this case imply that ex post efficiency is not possible. Hence, vertical integration is socially harmful whenever N_S and $\underline{\theta}$ are such that ex post efficiency is possible pre-integration. With identical supports, ex post efficiency

is not possible with or without vertical integration if $N_U > 1$ (see, e.g., Williams, 1999). Further, with identical supports and uniformly distributed types, as the number of sellers grows large, the change in social surplus due to vertical integration is nonmonotone in the number of outside suppliers and, in the limit, approaches zero from below (see Loertscher and Marx (2022, Figure F.1(a)) and Figure B.3 in Section B.5 below).

So far, these results have stayed within a "one-to-many" setting. The focus on one buyer and multiple sellers ensures that the post-integration firm can be viewed as effectively having a one-dimensional type because, in the absence of any other buyers or traders to sell to, the integrated firm can only act as a buyer vis á vis the other firms, so only its maximum willingness to pay for an external unit is relevant.

However, the methodology developed in this paper gives us the ability to go beyond these cases and consider many-to-many markets. Because our approach in this paper allows us to handle multi-dimensional types, we are able to extend the results of Loertscher and Marx (2022) on vertical integration beyond the one-to-many case considered there.

Case IV. *If $N_U = N_D$ and $\underline{\theta} \geq 1$ (nonoverlapping supports), then vertical integration does not affect social surplus because ex post efficiency is possible before and after vertical integration.*

With an equal number N of upstream sellers and downstream buyers and $\underline{\theta} \geq 1$, ex post efficiency is possible. For example, trade at a fixed price $p \in [1, \underline{\theta}]$ achieves ex post efficiency. Following the integration of one buyer and one seller, ex post efficiency continues to require that the buyer-side type of the integrated firm receives k units. We can think of the integrated firm's k units as allocated to its buyer-side type, leaving us with $N-1$ downstream buyers with supports $[\underline{\theta}, 1+\underline{\theta}]$ and maximum demands of k, and $k(N-1)$ units to be allocated to them from the upstream sellers with supports $[0,1]$ (the seller-side of the integrated firm acts a buyer with support $[0,1]$ and so is never allocated anything). Ex post efficiency can then be achieved in this residual market with a posted price $p \in [1, \underline{\theta}]$, with the result that vertical integration in this case is neutral for social surplus.

Case V. *If $N_U > N_D \geq 2$ and $\underline{\theta} \geq 1$ (nonoverlapping supports), then whether vertical integration increases or decreases social surplus depends on market details.*

With $N_U > N_D \geq 2$, and $\underline{\theta} \geq 1$, ex post efficiency is possible in the pre-integration market, for example based on a posted price of $p \in [1, \underline{\theta}]$. Following integration between an upstream seller and a downstream buyer, ex post efficiency requires that k units be allocated to the integrated firm's buyer type, and $k(N_D - 1)$ to the independent buyers. The remaining $k(N_U - N_D)$ units must then be allocated to the firms with the highest $N_U - N_D$ types among the $N_S - 1$ independent sellers and the integrated firm's seller type. A payment of $\underline{\theta}$ can be required from each of the independent buyers, which may or may not be sufficient to "grease the wheels" for the remaining transactions, which must occur among firms with a common support.

For example, consider the case with $N_U = 3$ and $N_D = 2$. Let c_1 denote the type of the integrated seller and c_2 and c_3 denote the types of the independent sellers. Then ex post revenue based on VCG payments with binding IR for the firms' worst-off types, where the integrated firm's worst-off type is $(\underline{\theta}, 0)$, is $k(\underline{\theta} + \max\{c_2, c_3\} - 2\max\{c_1, c_2, c_3\})$. Thus, revenue is increasing in $\underline{\theta}$ and positive for all type realizations if $\underline{\theta} \geq 2$. For $\underline{\theta} \in [0, 2)$, the sign of expected revenue depends on the distributions of the seller's types. For example, with $\underline{\theta} = 1$ and uniformly distributed types, expected revenue is negative, so ex post efficiency is not possible post-integration because the no-deficit constraint cannot be satisfied. However, if instead c_1 is drawn from the distribution $G(c) = c^{1/4}$ (with expected value of $1/5$), then expected revenue is positive and so the no-deficit constraint is satisfied, giving us the result that ex post efficiency is possible. This establishes that in this case the effects of vertical integration can go either way and depend, in general, on $\underline{\theta}$ and the sellers' type distributions.

Figure B.1 illustrates that for $N_U > 2 = N_D$ and $\underline{\theta} = 1$, the expected revenue in the post-integration market under binding IR for the firms' worst-off type varies with the number of upstream sellers N_U. Using $(\mathbf{x})_{(j)}$ to denote the j-th highest element of $\mathbf{x}$ and assuming that $N_U > N_D$, ex post revenue as a function of N_U and N_D is given by

$$\frac{1}{N_U}\left((N_D - 1)\underline{\theta} + \sum_{j=1}^{N_U - N_D} (\mathbf{c}_{-1})_{(j)} - (N_U - 1)\sum_{j=1}^{N_U - N_D} (\mathbf{c})_{(j)} + \sum_{i=2}^{N_U} \sum_{j=1}^{N_U - N_D - 1} (\mathbf{c}_{-i})_{(j)}\right). \tag{B.30}$$

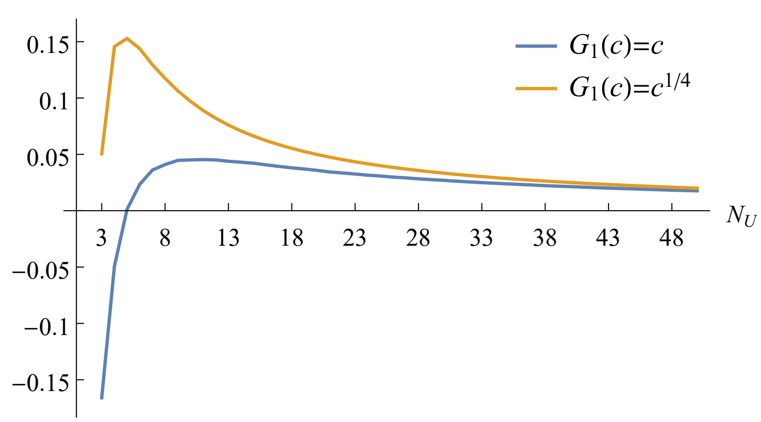

Figure B.1: Expected revenue following the integration of one upstream seller with one downstream buyer under the ex post efficient allocation and binding IR for firms' worst-off types. Assumes $N_U > 2 = N_D$, $\underline{\theta} = 1$, and common maximum demands of $k = 1/N_U$.

As N_U grows large, the sums of all but the j lowest order statistics for $j \in \{1, \ldots, N_D + 1\}$

approach $(N_U - j)\mu$, where μ is the expected cost of a seller. Thus, as N_U grows large, (B.30) approaches $\frac{\underline{\theta}-(N_D-1)\mu}{N_U}$, which then approaches zero, and from above if $N_D = 2$, as illustrated in Figure B.1. Further, increasing the number of downstream firms has the effect of increasing the number of firms that pay $\underline{\theta}$, which for $\underline{\theta}$ sufficiently large increases expected revenue and makes vertical integration less harmful.

B.3 Bargaining weights and efficiency with nonidentical supports

As we show in this appendix, the result of Proposition 4 that ex post efficiency requires identical bargaining weights continues to apply in settings with nonidentical supports under certain conditions.

Here we assume, as in Section 6.3 and Appendix B.2, that firms are divided into a set $\mathcal{N}_U$ of $N_U \geq 1$ "upstream" sellers with support $[0, 1]$ and a set $\mathcal{N}_D$ of $N_D \geq 1$ "downstream" buyers with support $[\underline{\theta}, 1 + \underline{\theta}]$, where all firms have one-dimensional types. We show that, assuming excess demand downstream, for ex post efficiency to be possible, all downstream firms must have the same bargaining weight, and we provide conditions under with the bargaining weights of upstream firms are constrained to be equal to or close to those of the downstream firms.

Proposition B.11. *Assume that there is excess demand downstream, $\sum_{i\in\mathcal{N}_D} k_i > 1$. Ex post efficiency requires that all downstream firms have the same bargaining weight, w_D; (ii) for $\underline{\theta} \in [0, 1)$, any upstream firm i has $w_i = w_D$; (iii) for $\underline{\theta} \geq 1$, any upstream firm i with $r_i > 0$ has*

$$\frac{\max\{\overline{w}_U, w_D\} - w_i}{\max\{\overline{w}_U, w_D\}} \leq (\underline{\theta} - 1)\, f_i(1).$$

Further, for $n = 2$ with firm 1 upstream and firm 2 downstream, if $r_1 > 0$ and $\underline{\theta} \geq 1$, then ex post efficiency is possible if and only if

$$\underline{\theta} \geq 1 + \frac{1}{f_1(1)} \frac{\max\{w_1, w_2\} - w_1}{\max\{w_1, w_2\}} + \frac{1}{f_2(\underline{\theta})} \frac{\max\{w_1, w_2\} - w_2}{\max\{w_1, w_2\}}. \tag{B.31}$$

Proof. The proof proceeds with a series of four lemmas. We first show in Lemma B.11 that the ranking of a downstream firm i with $k_i > r_i$, of which there is at least one, relative to other downstream firms matters for efficiency. Then we show in Lemma B.12 that for $\underline{\theta} < 1$, the ranking of an upstream firm relative to a downstream firm i with $k_i > r_i$, of which there is at least one, matters for efficiency. Then we show in Lemma B.13 that for two firms with overlapping supports, the ranking of their types cannot be the same as the ranking of their ironed weighted virtual types for all types in an open interval of the support overlap if the bargaining weights differ. Together these lemmas imply that for ex post efficiency, all downstream firms must have the same bargaining weight and that when $\underline{\theta} < 1$, all downstream and upstream firms must have the same bargaining weight. Then we address the case of $\underline{\theta} \geq 1$ in Lemma B.14. It is convenient as part of Lemma B.14 to also state an additional result for the case of $n = 2$.

Lemma B.11. *For every pair i and j of downstream firms, there exists an open interval subset of $(\underline{\theta}, 1+\underline{\theta})$ such that the ranking of firms i and j on that interval matters for efficiency.*

Proof. Let $X \equiv \sum_{\ell \in \mathcal{N}_D \setminus \{i,j\}} k_\ell$ be the demand by downstream firms other than firms i and j. By the assumption of excess demand downstream, $1 - X < k_i + k_j$. Let θ_i^r and θ_j^r denote the basis for ranking firms i and j.
Case 1: $X < 1$. When all upstream firms' types are equal to $\min\{1, \underline{\theta}\}$ and all downstream firms other than i and j have types equal to $1 + \underline{\theta}$, then for $\theta_i^r, \theta_j^r \in (\underline{\theta}, 1 + \underline{\theta})$, assets $1 - X \in (0, k_i + k_j)$ are allocated to firms i and j based on their ranking, and so the ranking matters for efficiency.
Case 2: $X \geq 1$. When all upstream firms' types are equal to $\min\{1, \underline{\theta}\}$ and downstream firms other than i and j have types equal to $x \in (\underline{\theta}, 1 + \underline{\theta})$, then if $\underline{\theta} < \theta_i^r \leq x < \theta_j^r < 1 + \underline{\theta}$, assets are available to be allocated to firm j, but, because $1 - X \leq 0$, no assets are available to be allocated to firm i. And the situation is reversed if the rankings of i and j are reversed. Thus, the ranking of firms i and j around any $x \in (\underline{\theta}, 1 + \underline{\theta})$ matters for efficiency. □

Lemma B.12. *If $\underline{\theta} < 1$, then for every upstream firm i, there exists a downstream firm j and an open interval subset of $(\underline{\theta}, 1)$ such that the ranking of firms i and j on that interval matters for efficiency.*

Proof. Let j be a downstream firm with $k_j > r_j$. Let θ_i^r and θ_j^r denote the basis for ranking of firms i and j.
Case 1. $k_i = r_i$. Then firm i is a seller. When all other upstream firms' types are equal to 1 and all downstream firms other than j have types equal to $\underline{\theta}$, then if $\underline{\theta} < \theta_i^r < \theta_j^r < 1$, then demand for firm i's assets is positive, but if $\underline{\theta} < \theta_j^r < \theta_i^r < 1$, then demand for firm i's assets is zero. Thus, the ranking matters for efficiency.
Case 2. $k_i > r_i$. Let $X \equiv \sum_{\ell \in \mathcal{N}_D \setminus \{j\}} k_\ell$ be the demand by downstream firms other than firm j. By the assumption of excess demand downstream, $1 - X < k_j$. We consider two subcases.
Case 2a: $X < 1$. When all upstream firms other than i have types equal to $\min\{1, \underline{\theta}\}$ and all downstream firms other than j have types equal to $1 + \underline{\theta}$, then for $\theta_i^r, \theta_j^r \in (\underline{\theta}, 1)$, assets $1 - X \in (0, k_j)$ are allocated to firms i and j based on their ranking, and so the ranking matters for efficiency.
Case 2b: $X \geq 1$. When all upstream firms other than i have types equal to $\min\{1, \underline{\theta}\}$ and downstream firms other than j have types equal to $x \in (\underline{\theta}, 1)$, then if $\underline{\theta} < \theta_i^r \leq x < \theta_j^r < 1$,

assets are available to be allocated to firm j, but, because $1 - X \leq 0$, no assets are available to be allocated to firm i. And the situation is reversed if the rankings of i and j are reversed. Thus, the ranking of firms i and j around any $x \in (\underline{\theta}, 1)$ matters for efficiency. $\square$

Lemma B.13. *The ironed weighted virtual type functions for two firms cannot be the same for all types in an open interval in the support of both firms' type distributions if the firms' bargaining weights differ.*

Proof. Suppose that $w_i \neq w_j$ and let A be an open interval that is a subset of the interval of overlap in the supports of the distributions of firms i and j. Suppose that for all $\theta \in A$, the ranking of firms i and j according to their types is the same as the ranking of those firms according to their weighted ironed virtual types under incomplete information bargaining:

$$\overline{\Psi}_{j,\frac{w_j}{\rho^*}}(\theta_j; \omega_j^*) > \overline{\Psi}_{i,\frac{w_i}{\rho^*}}(\theta_i; \omega_i^*) \quad \Leftrightarrow \quad \theta_j > \theta_i.$$

Then we require that for all $\theta \in A$, $\overline{\Psi}_{j,w_j/\rho^*}(\theta; \omega_j^*) = \overline{\Psi}_{i,w_i/\rho^*}(\theta; \omega_i^*)$ and that the ironed weighted virtual type functions are increasing in this region (if they are equal, but constant, as in the ironed portions, then the allocation is random and so ex post efficiency is not achieved). Given that the ironed weighted virtual type functions are increasing, they are either equal to the weighted virtual value or the weighted virtual cost, and, as we now show, these cannot be the same on an open interval if the weights differ.

To see this, suppose that for $\theta \in A$, $\overline{\Psi}_{j,w_j/\rho^*}(\theta; \omega_j^*) = \Psi^B_{j,w_j/\rho^*}(\theta) = \overline{\Psi}_{i,w_i/\rho^*}(\theta; \omega_i^*) = \Psi^S_{i,w_i/\rho^*}(\theta)$. Then we have

$$\theta - (1 - \frac{w_j}{\rho^*})\frac{1 - F_j(\theta)}{f_j(\theta)} = \theta + (1 - \frac{w_i}{\rho^*})\frac{F_i(\theta)}{f_i(\theta)},$$

which we can rewrite as

$$\frac{F_i(\theta) f_j(\theta)}{(1 - F_j(\theta))\, f_i(\theta) + F_i(\theta) f_j(\theta)} = \frac{\rho^* - w_j}{w_i - w_j} \equiv C$$

and, alternatively, as

$$\frac{(1 - F_j(\theta))\, f_i(\theta)}{F_i(\theta) f_j(\theta) + (1 - F_j(\theta))\, f_i(\theta)} = \frac{\rho^* - w_i}{w_j - w_i} \equiv C'$$

Using $\rho^* \geq \max\{w_j, w_i\}$, if $w_j < w_i$, then $C \geq 1$, and we require that for all $\theta \in A$, $\frac{1-F_j(\theta)}{f_j(\theta)} / \frac{F_i(\theta)}{f_i(\theta)} = \frac{1-C}{C}$, which is a contradiction because the left side is positive and $(1-C)/C \leq 0$; and if $w_j > w_i$, then $C' \geq 1$, and we require that $\frac{F_i(\theta)}{f_i(\theta)} / \frac{(1-F_j(\theta))}{f_j(\theta)} = \frac{1-C'}{C'}$, which is similarly a contradiction.

Now suppose that $\overline{\Psi}_{j,w_j/\rho^*}(\theta; \omega_j^*) = \Psi^B_{j,w_j/\rho^*}(\theta) = \overline{\Psi}_{i,w_2/\rho^*}(\theta; \omega_i^*) = \Psi^B_{i,w_i/\rho^*}(\theta)$. Then we require that $\theta - (1 - \frac{w_j}{\rho^*})\frac{1-F_j(\theta)}{f_j(\theta)} = \theta - (1 - \frac{w_j}{\rho^*})\frac{1-F_i(\theta)}{f_i(\theta)}$. We can write this as $\frac{1-F_i(\theta)}{f_i(\theta)} / \left(\frac{1-F_j(\theta)}{f_j(\theta)} - \frac{1-F_i(\theta)}{f_i(\theta)}\right) = \frac{\rho^* - w_j}{w_j - w_i} \equiv C''$, which implies that $\frac{1-F_j(\theta)}{f_j(\theta)} / \frac{1-F_i(\theta)}{f_i(\theta)} = \frac{1+C''}{C''}$, giving us a contradiction for $w_j < w_i$

because the left side is positive and the right side is negative. An analogous contradiction obtains if $w_j > w_i$. The remaining case, $\overline{\Psi}_{j,w_j/\rho^*}(\theta;\omega_j^*) = \Psi^S_{j,w_j/\rho^*}(\theta) = \overline{\Psi}_{i,w_2/\rho^*}(\theta;\omega_i^*) = \Psi^S_{i,w_i/\rho^*}(\theta)$, provides an analogous contradiction, which completes the proof. $\square$

Lemma B.14. *Letting* $\overline{w}_U \equiv \max_{j\in\mathcal{N}_U \text{ s.t. } r_j>0} w_j$*, given upstream firm* i *with* $r_i > 0$*, if* $\underline{\theta} \geq 1$*, then ex post efficiency requires* $\frac{\max\{\overline{w}_U, w_D\}-w_i}{\max\{\overline{w}_U, w_D\}} \leq (\underline{\theta}-1)\, f_i(1)$*. Further, if there is only one downstream firm* j*, then ex post efficiency is possible if and only if* $1+(1-\frac{w_i}{\max\{w_i,w_j\}})/f_i(1) \leq \underline{\theta} - (1-\frac{w_j}{\max\{w_i,w_j\}})/f_j(\underline{\theta})$.

Proof. Ex post efficiency (with the assumption of excess demand for downstream firms) requires that assets move from firm i to a downstream firm for all type realizations. For incomplete information bargaining to result in assets moving from firm i to a downstream firm, we require that for some downstream firm j,

$$\overline{\Psi}_{i,\frac{w_i}{\rho^*}}(\theta_i;\omega_i^*) \leq \overline{\Psi}_{j,\frac{w_j}{\rho^*}}(\theta_j;\omega_j^*). \tag{B.32}$$

Under ex post efficiency, firm i is a seller with worst-off type 1, all downstream firms have common bargaining weight w_D, and $\rho^* = \overline{w} \equiv \max\{\overline{w}_U, w_D\}$. So, (B.32) becomes $\Psi^S_{i,\frac{w_i}{\overline{w}}}(\theta_i) \leq \overline{\Psi}_{j,\frac{w_D}{\overline{w}}}(\theta_j;\omega_j^*)$, and this must hold even for $\theta_i = 1$ and even if all downstream firms have type $\underline{\theta}$, so we have

$$1 + \frac{1-\frac{w_i}{\overline{w}}}{f_i(1)} \leq \overline{\Psi}_{j,\frac{w_D}{\overline{w}}}(\underline{\theta};\omega_j^*) \in (\Psi^B_{j,\frac{w_D}{\overline{w}}}(\underline{\theta}), \Psi^S_{j,\frac{w_D}{\overline{w}}}(\underline{\theta})),$$

which, using $\Psi^S_{j,\frac{w_D}{\overline{w}}}(\underline{\theta}) = \underline{\theta}$, implies that $1+\left(1-\frac{w_i}{\overline{w}}\right)/f_i(1) \leq \underline{\theta}$. Thus, for all upstream firms i with $r_i > 0$, $\frac{\overline{w}-w_i}{\overline{w}} \leq (\underline{\theta}-1)\, f_i(1)$, which completes the proof of the first part of the lemma.

If there is only one downstream firm j, then firm j is a buyer with worst-off type $\underline{\theta}$, and so $\overline{\Psi}_{j,w_D/\overline{w}}(\underline{\theta};\omega_j^*) = \Psi^B_{j,w_D/\overline{w}}(\underline{\theta}) = \underline{\theta} - \frac{1-\frac{w_D}{\overline{w}}}{f_j(\underline{\theta})}$, which gives us

$$1 + \frac{1-\frac{w_i}{\overline{w}}}{f_i(1)} \leq \underline{\theta} - \frac{1-\frac{w_D}{\overline{w}}}{f_j(\underline{\theta})}. \tag{B.33}$$

Under condition (B.33), trade occurs for all type realizations, and so the expected revenue under ex post efficiency and binding IR for the firms' worst-off types is

$$\begin{aligned}
\Pi^e &\equiv \mathbb{E}\left[\Psi_j^B(\theta_j)q_j^e(\theta_j) + \Psi_i^S(\theta_i)q_i^e(\theta_i)\right] - r_j\underline{\theta} - r_i \\
&= \underline{\theta}\min\{k_j, R\} + R - \min\{k_j, R\} - r_j\underline{\theta} - (R - r_j) \\
&= (\underline{\theta}-1)(\min\{k_j, R\} - r_j) \\
&\geq 0,
\end{aligned}$$

where the second equality uses $\mathbb{E}\left[\Psi_j^B(\theta_j)\right] = \underline{\theta}$ and $\mathbb{E}\left[\Psi_i^S(\theta_i)\right] = 1$. Because Π^e is nonnegative, it follows that (B.33) is necessary and sufficient for ex post efficiency. Efficient trade can be achieved in an IC, IR, no-deficit way and that allocates Π^e to the firm with the higher bargaining weight using a posted price of 1 if $w_i > w_j$ and $\underline{\theta}$ if $w_j > w_i$; and if $w_i = w_j$, then Π^e can be allocated in any proportions to the firms using posted prices between 1 and $\underline{\theta}$. This completes the proof of the second part of the lemma. □

Combining Lemmas B.11–B.14 completes the proof of Proposition B.11. ■

B.4 Illustration of revenue under ex post efficiency as a guide

Here we provide an illustration in a partnership setup, which is essentially one-dimensional, of the rule of approving a merger if the merger increases revenue under ex post efficiency. Consider the case of three firms and suppose that pre-merger ownership is $\mathbf{r} = (0.85, 0.1, 0.05)$. Figure B.2, particularly panel (b) indicates that the iso-expected social surplus curve associated with $\mathbf{r} = (0.85, 0.1, 0.05)$ lies largely, if not entirely, to the right of the iso-ex post efficient revenue curve through that point. Thus, increases in ex post efficient revenue due to merger imply increases in expected social surplus.

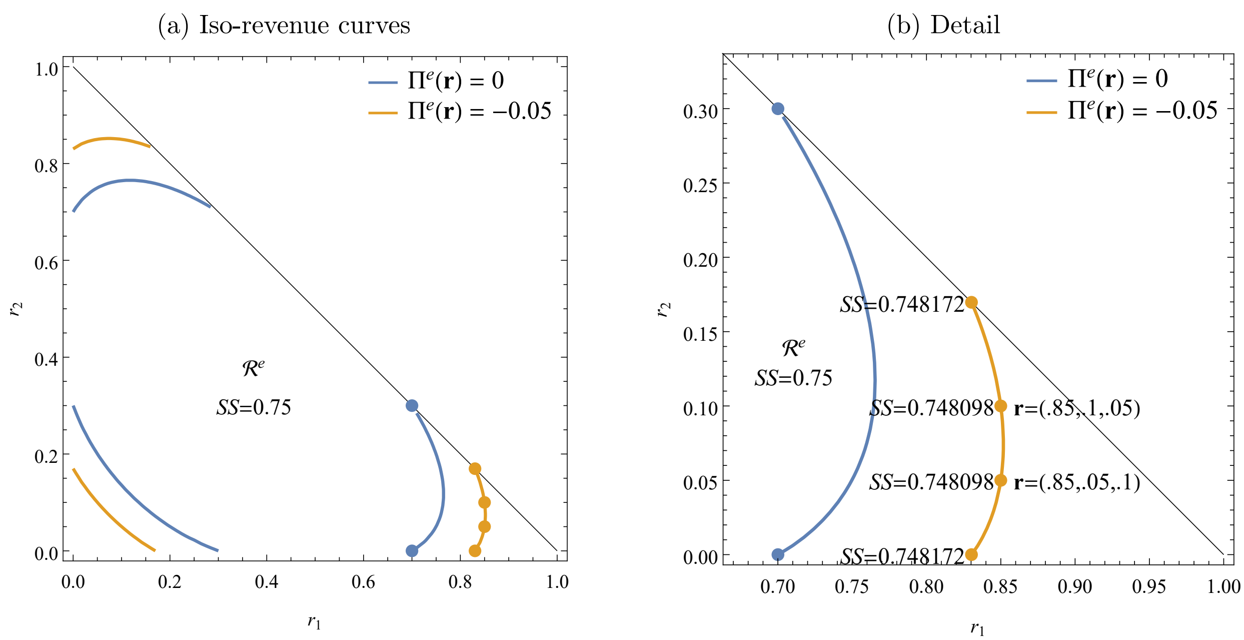

Figure B.2: Iso-ex post efficient revenue curves, i.e., constant $\Pi^e(\mathbf{r})$, and expected second-best social surplus. Assumes $n = 3$, $k_i = 1$, and uniformly distributed types. The iso-revenue curve for revenue -0.050976 includes $\mathbf{r} = (0.85, 0.05, 0.1)$ and $\mathbf{r} = (0.830275, 0.169725, 0)$, and the related vectors by symmetry. At $\mathbf{r} = (0.830275, 0.169725, 0)$, we have $\rho^* = 1.078037$ and the worst-off types are $\hat{\boldsymbol{\theta}} = (0.877443, 0.413357, 0)$. At $\mathbf{r} = (0.85, 0.05, 0.1)$, we have $\rho^* = 1.081305$ and $\hat{\boldsymbol{\theta}} = (0.886987, 0.258573, 0.318155)$.

B.5 Effects of vertical mergers on second-best social surplus

Consider a setup along the lines of Loertscher and Marx (2022) in which pre merger firm 1 is a buyer with $k_1 = 1$ and firms $2, \ldots, n$ are sellers, each with $r_i = k_i = 1$ and one-dimensional types. In this case, ex post efficiency is not possible before and after vertical integration, but the second-best mechanism is known (see, e.g., Loertscher and Marx, 2022). In the pre-merger market, there is trade between the buyer and the lowest-type seller whenever $\Psi_{1,1/\rho^b}(\theta_1, 0) > \min_{i\in\{2,\ldots,n\}} \Psi_{i,1/\rho^b}(\theta_i, 1)$, where $\rho^b > 1$ is the Lagrange multiplier on the no-deficit constraint before the merger. In the post-merger market following the merger of the buyer and firm 2, the new merged firm continues to be a buyer, but its willingness to pay for a unit from the outside sellers is only $\min\{\theta_1, \theta_2\}$, implying that the integrated firm's type distribution is $F_{1,2}(\theta) \equiv 1 - (1 - F_1(\theta))(1 - F_2(\theta))$. Thus, the merger induces a change in the buyer's distribution, and hence its virtual type function, a reduction in the number of outside sellers, and a change in the Lagrange multiplier ρ. In the post-merger market, there is trade between the vertically integrated firm and the lowest-type outside seller whenever $\Psi_{1,2,1/\rho^a}(\min\{\theta_1, \theta_2\}, 0) > \min_{i\in\{3,\ldots,n\}} \Psi_{i,1/\rho^a}(\theta_i, 1)$, where $\rho^a > 1$ is the post-merger multiplier. Although a vertical merger eliminates a double-markup (of information rents), it also makes the outside market less competitive, and possibly less efficient through the effect on ρ. As n grows large, the probability that the vertically integrated firm sources internally goes to zero, and the outside market becomes close to efficient. Because all effects become small, it is hard to prove general results analytically, but for uniformly distributed pre-merger types, social surplus effects are nonmonotone and in the limit approach zero from below as shown in Figure B.3.[54]

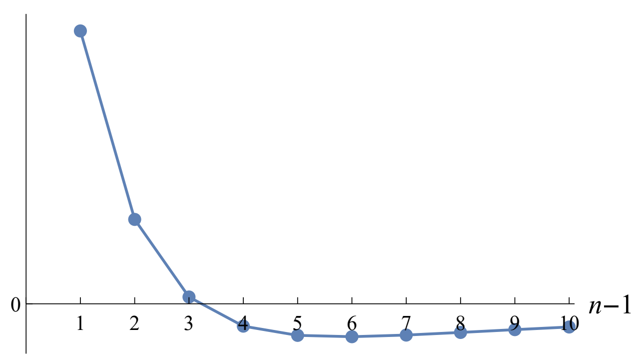

Figure B.3: Change in expected social surplus as a result of a vertical merger starting from a pre-merger market with one buyer and $n-1$ sellers. Assumes $k_1 = 1$ and for $i \in \{2, \ldots, n\}$, $r_i = k_i = 1$, and uniformly distributed pre-merger types.

[54]This corresponds to Figure F.1(a) in the Online Appendix of Loertscher and Marx (2022), but extended here to include the case of $n-2$.

B.6 Details for transforming a setting with multi-dimensional types into one with one-dimensional types

In this appendix, we provide details related to the discussion in Section 6.4 on how the two-dimensional types that arise due to the full integration of two firms might reasonably be transformed into a one-dimensional type.

Consider a setting with three pre-merger firms with one-dimensional types, ex post efficient allocation $\mathbf{Q}^e(\boldsymbol{\theta})$, and ex post welfare $W(\boldsymbol{\theta})$. We analyze the merger of firms 1 and 2 to create a post-merger firm with a two-dimensional type and asset ownership $r_{1,2} \equiv r_1 + r_2$. For this post-merger setting, the ex post efficient allocation for the merged firm is $Q^e_{1,2}(\boldsymbol{\theta}) \equiv Q^e_1(\boldsymbol{\theta}) + Q^e_2(\boldsymbol{\theta})$. The worst-off types (ω, ω) for the merged firm and $\hat{\theta}_3$ for firm 3 satisfy $q^e_{1,2}(\omega, \omega) = r_1 + r_2$ and $q^e_3(\hat{\theta}_3) = r_3$. Thus, the VCG revenue accounting for the merged entity's two-dimensional type is

$$\begin{aligned}\Pi^e(r_{1,2}, r_3) \quad = \quad & \mathbb{E}_{\boldsymbol{\theta}}[W(\omega, \omega, \theta_3) - W(\theta_1, \theta_2, \theta_3) + Q^e_1(\theta_1, \theta_2, \theta_3)\theta_1 + Q^e_2(\theta_1, \theta_2, \theta_3)\theta_2] - \omega r_{1,2} \\ & + \mathbb{E}_{\boldsymbol{\theta}}[W(\theta_1, \theta_2, \hat{\theta}_3) - W(\theta_1, \theta_2, \theta_3) + Q^e_3(\theta_1, \theta_2, \theta_3)\theta_3] - \hat{\theta}_3 r_3,\end{aligned}$$

where the expectations are taken with respect to the pre-merger distributions.

As we have shown, $\Pi^e(r_{1,2}, r_3)$ is concave and positive at a unique $\mathbf{r}^*$, which implies that we have unique cutoffs $\underline{r}_3$ and $\overline{r}_3$ such that $\Pi^e(1 - \underline{r}_3, \underline{r}_3) = 0$ and $\Pi^e(1 - \overline{r}_3, \overline{r}_3)$, where $\Pi^e(1 - r_3, r_3) > 0$ for all $r_3 \in (\underline{r}_3, \overline{r}_3)$. Thus, for the post-merger setting with two-dimensional types, $\mathcal{R}^e = \{(1 - r_3, r_3) \mid r_3 \in [\underline{r}_3, \overline{r}_3]\}$.

We can then construct a density $\tilde{f}$ such that when the merged entity has a one-dimensional type drawn from a distribution with density $\tilde{f}$, along with asset ownership $r_{1,2}$ and maximum demand of $\tilde{k} \equiv \min\{1, k_1 + k_2\}$, then the ex post efficiency permitting set is once again $\mathcal{R}^e$. In the one-dimensional setup, $\tilde{Q}^e_{1,2}(\theta_{1,2}, \theta_3) \equiv \tilde{k} \cdot 1_{\theta_{1,2} > \theta_3} + (1 - k_3) \cdot 1_{\theta_{1,2} < \theta_3}$ and $\tilde{Q}^e_3(\theta_{1,2}, \theta_3) \equiv k_3 \cdot 1_{\theta_{1,2} < \theta_3} + (1 - \tilde{k}) \cdot 1_{\theta_{1,2} > \theta_3}$, and welfare is $\tilde{W}(\theta_{1,2}, \theta_3) \equiv \tilde{Q}^e_{1,2}(\theta_{1,2}, \theta_3)\theta_{12} + \tilde{Q}^e_3(\theta_{1,2}, \theta_3)\theta_3$. The merged entity's worst-off type is $\tilde{\omega}(r_{1,2})$ satisfying $\tilde{q}^e_{1,2}(\tilde{\omega}(r_{1,2})) = r_{1,2}$, which does not depend on $\tilde{f}$. The nonmerging firm has interim expected allocation rule $\tilde{q}^e_3(\theta_3; \tilde{f}) \equiv \int_0^1 \tilde{Q}^e_3(\theta_{1,2}, \theta_3)\tilde{f}(\theta_{1,2})d\theta_{1,2}$ and worst-off type $\tilde{\theta}_3(r_3; \tilde{f})$ defined by $\tilde{q}^e_3(\tilde{\theta}_3(r_3; \tilde{f}); \tilde{f}) = r_3$, where we explicitly note the dependence on $\tilde{f}$.

We parameterize the density $\tilde{f}$ as the piecewise uniform density

$$\tilde{f}(\theta) = (\tilde{f}_1, \ldots, \tilde{f}_\ell) \cdot (1_{\theta \in [0, 1/\ell)}, 1_{\theta \in [1/\ell, 2/\ell)}, \ldots, 1_{\theta \in [(\ell-1)/\ell, 1]}),$$

where ℓ is a sufficiently large integer, $\tilde{f}_i > 0$, and $\sum_{i=1}^{\ell} \frac{1}{\ell} \tilde{f}_i = 1$.

The expected budget surplus under ex post efficiency with one-dimensional types, as a function of the firms' asset ownership as well as firm 3's worst-off type, is then

$$
\begin{aligned}
\tilde{\Pi}^e(r_{1,2}, r_3; \tilde{f}) \equiv & \int_0^1 \int_0^1 \Big(\tilde{W}(\tilde{\omega}(r_{1,2}), \theta_3) - 2\tilde{W}(\theta_{1,2}, \theta_3) + \tilde{Q}^e_{1,2}(\theta_{1,2}, \theta_3)\theta_{1,2} \\
& + \tilde{W}(\theta_{1,2}, \tilde{\theta}_3(r_3; \tilde{f})) + \tilde{Q}^e_3(\theta_{1,2}, \theta_3)\theta_3 \Big) \tilde{f}(\theta_{12}) f(\theta_3) d\theta_{12} d\theta_3 \\
& - \tilde{\omega}(r_{1,2}) r_{1,2} - \tilde{\theta}_3(r_3; \tilde{f}) r_3.
\end{aligned}
$$

One can then solve for $(\tilde{f}_1, \ldots; \tilde{f}_\ell)$, $\tilde{\theta}_3(\underline{r}_3; \tilde{f})$, and $\tilde{\theta}_3(\overline{r}_3; \tilde{f})$ such that we have, simultaneously, $\tilde{\Pi}^e(1 - \underline{r}_3, \underline{r}_3; \tilde{f}) = \tilde{\Pi}^e(1 - \overline{r}_3, \overline{r}_3; \tilde{f}) = 0$, $\tilde{q}^e_3(\tilde{\theta}_3(\underline{r}_3; \tilde{f}); \tilde{f}) = \underline{r}_3$, and $\tilde{q}^e_3(\tilde{\theta}_3(\overline{r}_3; \tilde{f}); \tilde{f}) = \overline{r}_3$. This gives us the density $\tilde{f}$ such that the ex post permitting set in the one-dimensional setting is, as in the two-dimensional setting, $\mathcal{R}^e = \{(1 - r_3, r_3) \mid r_3 \in [\underline{r}_3, \overline{r}_3]\}$, which is illustrated in Figure B.4.

(a) Ex post efficiency permitting sets

(b) Virtual types for transformed distribution

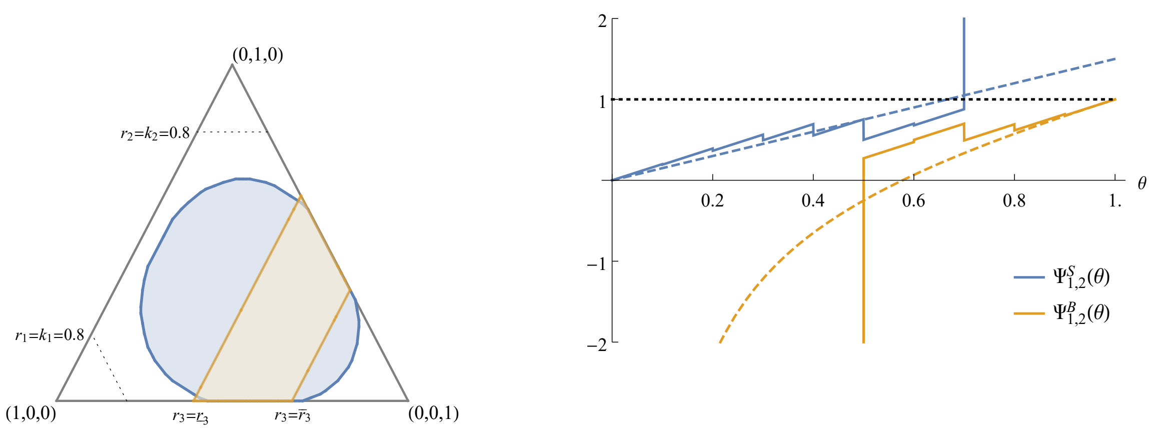

Figure B.4: Panel (a) shows the pre-merger (blue) and post-merger (orange) ex post efficiency permitting set based on two-dimensional types. Panel (b) shows the corresponding virtual types (solid lines) when the merged firm's type is transformed to be one-dimensional based on the piecewise uniform type distribution with 10 segments ($\ell = 10$). For comparison, dashed lines show the virtual type functions for the distribution for the maximum of two types. Assumes that the pre-merger types are uniformly distributed and that $k_1 = k_2 = 0.8$ and $k_3 = 1$.