Online Appendix

to accompany

"Make or buy or sell"

by

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In this Online Appendix, we do two things. In Section OA, we provide details on the existence and uniqueness of the incomplete information bargaining mechanism. In Section APP, we illustrate based on the Republic-Santek transaction how the divestiture policies that we discuss can be implemented in practice using market data that is typically available in a merger review process.

OA Derivation of the market mechanism

OA.1 Overview

Solving the problem in (7) requires overcoming two intertwined obstacles that are familiar from partnership problems. First, the firms' worst-off types are endogenous to the allocation rule (see Lemma OA.1), which means that it is not a priori known for which types the individual rationality constraints will bind. Second, because of this endogeneity, it is not a priori clear whether the allocation and payment rule are separable in the sense that the market maker can first derive the (monotone) allocation rule that maximizes its objective and then adjust the transfers to satisfy the individual rationality constraints.[1] Equation (5) is customarily referred to as the payoff equivalence theorem because it means that the interim expected payoffs and payments are pinned down by the allocation rule up to a constant. Even though this holds in our setting, the payment and allocation rule interact via the utility of the worst-off type.

Solving for the second-best mechanism involves addressing the two intertwined problems. First, the pointwise maximizer of the Lagrangian, given in equation (10), over $\mathbf{Q}$ will typically fail to be monotone, which is a problem because monotonicity is required for incentive compatibility. Second, the worst-off types are endogenous to the allocation rule.

[1] This contrasts with mechanism design problems à la Myerson (1981) or Myerson and Satterthwaite (1983), where incentive compatibility implies that for buyers' (sellers') the worst-off types are the lowest (highest) possible types for *any* allocation rule compatible with incentive compatibility. This immediately implies that the allocation and payment rules are separable in this sense.

To maintain monotonicity, the optimal allocation rule $\mathbf{Q}^*$ prioritizes on the basis of ironed weighted virtual types, where the weights are $\alpha_i = w_i/\rho^*$, with ρ^* denoting the solution value of the Lagrange multiplier on the no-deficit constraint. If more than one firm have the same ironing parameter, ties among these firms occur with positive probability. Because for every firm i the worst-off type ω_i^* is the critical type and hence inside the interval $[(\Psi_{i,\alpha_i}^S)^{-1}(z_i), (\Psi_{i,\alpha_i}^B)^{-1}(z_i)]$ over which the ironed weighted virtual type is equal to z_i, the tie-breaking has to be such that $q_i^*(\omega_i^*) = r_i$.[2] The proof follows along the same arguments as those invoked by Loertscher and Wasser (2019), but is more involved because not all of the structure carries over from the partnership model to the problem with $k_i < 1$.

OA.2 Setup

OA.2.1 Worst-off types

The set of firm i's *worst-off types* is the set of types that minimize firm i's interim expected payoff from participation in the mechanism,

$$\Omega_i(\mathbf{Q}) \equiv \arg\min_{\theta\in[0,1]} u_i(\theta), \tag{OA.1}$$

which depends on the allocation rule. While these same two problems are present in a partnership setting as well, significant additional complexities arise in our setting because the firms' maximum demands can differ. In particular, in our setup, the allocation is affected not only by how a firm is ranked by the mechanism, but also by the identities of the rivals that are ranked ahead of it.

Because q_i is nondecreasing, it follows that the first-order condition $u_i'(\theta) = q_i(\theta) - r_i = 0$ characterizes a global minimum, provided that it is satisfied for some θ. The following lemma, a version of which was first established by Cramton et al. (1987), characterizes the set of worst-off types for any allocation rule such that q_i is nondecreasing:

Lemma OA.1. *Given an incentive compatible, individually rational mechanism $\langle \mathbf{Q}, \mathbf{M}\rangle$, if there is a θ_i such that $q_i(\theta_i) = r_i$, then $\Omega_i(\mathbf{Q})$ is a (possibly degenerate) interval and $\Omega_i(\mathbf{Q}) = \{\theta_i \mid q_i(\theta_i) = r_i\}$. If $q_i(\theta_i) \neq r_i$ for all $\theta_i \in [0,1]$, then $\Omega_i(\mathbf{Q})$ is a singleton and $\Omega_i(\mathbf{Q}) = \{\theta_i \mid q_i(\theta) < r_i \ \forall \theta < \theta_i \text{ and } q_i(\theta) > r_i \ \forall\theta > \theta_i\}$.*

As observed by Cramton et al. (1987), intuitively, the worst-off type expects on average to be neither a buyer nor a seller, and therefore a firm with the worst-off type has no incentive

[2] While the ironing procedure is as in Myerson (1981), in contrast to Myerson, the ties here cannot be broken arbitrarily because of the need to respect the individual rationality constraints.

to overstate or understate its valuation and so does not need to be compensated to induce truthful reporting, which is why it is the worst-off type.[3]

OA.2.2 Ironing

The solution builds on and generalizes the earlier results by Lu and Robert (2001), who assume identical distributions and maximum demands, and Loertscher and Wasser (2019), who study a partnership problem, i.e., they assume that $k_i = 1$ for all $i \in \mathcal{N}$ while allowing for heterogeneous distributions.[4] As mentioned in the body of the paper, the construction of the second-best mechanism uses the concept of the weighted virtual type functions, which for $\alpha \in [0, 1]$ are defined as

$$\Psi_{i,\alpha}^S(\theta) \equiv \theta + (1-\alpha)\frac{F_i(\theta)}{f_i(\theta)} \quad \text{and} \quad \Psi_{i,\alpha}^B(\theta) \equiv \theta - (1-\alpha)\frac{1-F_i(\theta)}{f_i(\theta)},$$

where the assumed monotonicity of $\Psi_i^S(\theta)$ and $\Psi_i^B(\theta)$ implies monotonicity of $\Psi_{i,\alpha}^S(\theta)$ and $\Psi_{i,\alpha}^B(\theta)$. With $\Psi_i(\theta, x)$ as defined in (8), we let

$$\Psi_{i,\alpha}(\theta, x) \equiv \alpha\theta + (1-\alpha)\Psi_i(\theta, x).$$

To satisfy monotonicity, the allocation rule of the second-best mechanism will be based on the *ironed* weighted virtual type functions,

$$\overline{\psi}_{i,\alpha}(\theta, z_i) \equiv \begin{cases} \Psi_{i,\alpha}^S(\theta) & \text{if } \Psi_{i,\alpha}^S(\theta) < z_i, \\ z_i & \text{if } \Psi_{i,\alpha}^B(\theta) \leq z_i \leq \Psi_{i,\alpha}^S(\theta), \\ \Psi_{i,\alpha}^B(\theta) & \text{if } z_i < \Psi_{i,\alpha}^B(\theta), \end{cases} \tag{OA.2}$$

where for $\omega_i \in [0, 1]$, the ironing parameter $z_i \in [\Psi_{i,\alpha}^B(\omega_i), \Psi_{i,\alpha}^S(\omega_i)]$ is the unique solution to

$$\mathbb{E}_{\theta_i}\left[\Psi_{i,\alpha}(\theta_i, \omega_i)\right] = \mathbb{E}_{\theta_i}\left[\overline{\psi}_{i,\alpha}(\theta_i, z_i)\right],$$

which defines $z_{i,\alpha}^*(\omega_i)$. In the body of the paper, we write the ironed virtual type as $\overline{\Psi}_{i,\alpha}(\theta, \omega_i)$,

[3] In simpler mechanism design settings, such as a sales auction, procurement auction, or two-sided setting as in Myerson and Satterthwaite (1983) or Gresik and Satterthwaite (1989), Lemma OA.1 together with the payoff equivalence theorem and the monotonicity of the allocation due to incentive compatibility imply that, for any $\mathbf{Q}$ satisfying (4), $\Omega_i(\mathbf{Q})$ contains 0 if i is a buyer (i.e., $r_i = 0$) and 1 if i is a seller (i.e., $r_i = k_i$). Thus, for settings like these, it is known a priori for which type of a firm the individual rationality constraint will bind, irrespective of the specifics of the allocation rule. This means that when looking for optimal mechanisms, the market maker can focus on the allocation rule without worrying about repercussions on the individual rationality constraint. In contrast, if $0 < r_i < k_i$, then firm i's worst-off types will typically be interior and depend on the allocation rule. For example, under the first-best allocation rule $\mathbf{Q}^e$, which allocates 1 to the firms with the highest types, $\Omega_i(\mathbf{Q}^e)$ is a singleton $\hat{\theta}_i \in (0, 1)$ if i is vertically integrated, i.e., if $0 < r_i < k_i$. Of course, because the allocation rule is fixed under the first-best and so is not a choice variable, the worst-off types only depend $\mathbf{r}$, $\mathbf{k}$, and the distributions. Away from the first-best, this will, however, not be the case.

[4] The setup of Loertscher and Wasser (2019) is more general in that they allow interdependent values.

where

$$\overline{\Psi}_{i,\alpha}(\theta,\omega_i) \equiv \overline{\psi}_{i,\alpha}(\theta, z^*_{i,\alpha}(\omega_i)). \tag{OA.3}$$

The parameter $\alpha_i \in [0,1]$ is a firm-specific Ramsey parameter that, in general, depends on the solution value of the Lagrange multiplier ρ in (10). An immediate result from (10) is that the solution value of the Lagrange multiplier ρ must be greater than or equal to $\max \mathbf{w}$ because KKT necessary conditions require that $\mu_j = \rho - w_j$, which would be negative and so violate dual feasibility for $\rho < \max \mathbf{w}$.

Observe that for any $\alpha_i \in [0,1]$ and any $z_i \in [\Psi^B_{i,\alpha}(0), \Psi^S_{i,\alpha}(1)]$, the function $\overline{\psi}_{i,\alpha_i}(\theta, z_i)$ is monotone in θ because $\Psi^S_{i,\alpha_i}(\theta)$ and $\Psi^B_{i,\alpha_i}(\theta)$ are monotone. Consequently, any allocation rule $\mathbf{Q}_{\boldsymbol{\alpha}}$ that for each $\boldsymbol{\theta}$ serves firms in order, according to their ironed weighted virtual types, up to their maximum demands, with ties among firms with the same ironed weighted virtual type broken randomly, satisfies the monotonicity requirement (4). For brevity, we refer to such allocation rules as allocation rules that *prioritize on the basis of ironed weighted virtual types*. As we show, the market mechanism always uses such an allocation rule. Under the first-best, we have $\alpha_i = 1$ for all i and the ironing parameters $\mathbf{z}$ can be chosen arbitrarily because $\overline{\psi}_{i,1}(\theta, z) = \overline{\psi}_{i,1}(\theta, z') = \theta$. Away from the first-best, the ironing parameters are uniquely pinned down, as will be shown.

OA.2.3 Saddle point problem

Building on the insights of Loertscher and Wasser (2019), we show that optimal mechanisms have a saddle point property. To state this property, we first define the firms' *virtual net surplus* by

$$\tilde{W}_\rho(\mathbf{Q}, \hat{\boldsymbol{\theta}}) \equiv \rho \mathbb{E}_{\boldsymbol{\theta}}\left[\sum_{i\in\mathcal{N}} (Q_i(\boldsymbol{\theta}) - r_i)\Psi_{i,\frac{w_i}{\rho}}(\theta_i, \hat{\theta}_i)\right].$$

For a given value of $\rho > \max \mathbf{w}$ and a given vector of worst-off types $\boldsymbol{\omega}^*$, the optimal allocation rule, denoted $\mathbf{Q}^*$, maximizes $\tilde{W}_\rho(\mathbf{Q}, \boldsymbol{\omega}^*)$ over monotone allocation rules $\mathbf{Q}$, while $\boldsymbol{\omega}^*$ is a $\hat{\boldsymbol{\theta}}$ that minimizes $\tilde{W}_\rho(\mathbf{Q}^*, \hat{\boldsymbol{\theta}})$. This saddle point property means that even though the worst-off types are endogenous to the allocation rule, the allocation and payment rule are still separable in the sense that one can first derive the optimal allocation rule and then adjust payments to satisfy individual rationality.[5]

[5] It is difficult to see how one could solve these problems in any degree of generality without the saddle point property. As a case in point, not being aware of it, Kittsteiner (2003) found what turns out to be the optimal dissolution mechanism for a partnership problem with two agents without being able to prove optimality (see his footnote 19).

OA.3 Optimization problem

As described in the paper, the market mechanism maximizes the weighted sum of the firms' expected surpluses. That is, the market mechanism $\langle \mathbf{Q}, \mathbf{M} \rangle$ solves:[6]

$$\max_{\mathbf{Q},\mathbf{M}} \sum_{i\in\mathcal{N}} \mathbb{E}_{\boldsymbol{\theta}}\big[w_i\theta_i Q_i(\boldsymbol{\theta}) - M_i(\boldsymbol{\theta})\big] \quad \text{subject to (1)–(3),} \tag{OA.4}$$

The payoff equivalence theorem, (5), allows us to identify the set of worst-off types and to eliminate $\mathbf{M}$ from the market problem, rewriting it as a function of the interim payoff of an arbitrarily fixed critical type for each firm and the virtual surplus generated by $\mathbf{Q}$ under these critical types.

Defining

$$W_\rho(\mathbf{Q}, \mathbf{M}) \equiv \sum_{i\in\mathcal{N}} w_i \mathbb{E}_{\boldsymbol{\theta}}[\theta_i Q_i(\boldsymbol{\theta}) - M_i(\boldsymbol{\theta})] + \rho \sum_{i\in N} \mathbb{E}_{\boldsymbol{\theta}}\left[M_i(\boldsymbol{\theta})\right]$$

and

$$\tilde{W}_\rho(\mathbf{Q}, \hat{\boldsymbol{\theta}}) \equiv \rho \mathbb{E}_{\boldsymbol{\theta}}\left[\sum_{i\in\mathcal{N}} (Q_i(\boldsymbol{\theta}) - r_i)\Psi_{i,\frac{w_i}{\rho}}(\theta_i, \hat{\theta}_i)\right],$$

we can use standard techniques to obtain the following lemma:

Lemma OA.2. *Suppose the mechanism* $\langle \mathbf{Q}, \mathbf{M} \rangle$ *satisfies* (4) *and* (5). *Then for all* $\hat{\boldsymbol{\theta}} \in [0,1]^n$,

$$W_\rho(\mathbf{Q}, \mathbf{M}) = \tilde{W}_\rho(\mathbf{Q}, \hat{\boldsymbol{\theta}}) - \sum_{i\in\mathcal{N}} (\rho - w_i) u_i(\hat{\theta}_i) + \sum_{i\in\mathcal{N}} r_i w_i \mathbb{E}_{\theta_i}[\theta_i]. \tag{OA.5}$$

Moreover,

$$\Omega(\mathbf{Q}) = \arg\min_{\hat{\boldsymbol{\theta}}\in[0,1]^n} \tilde{W}_\rho(\mathbf{Q}, \hat{\boldsymbol{\theta}}). \tag{OA.6}$$

Proof. Recall from Lemma 1 and equation (9) that

$$\mathbb{E}_{\theta_i}[m_i(\theta_i)] = \mathbb{E}_{\theta_i}[\Psi_{i,0}(\theta_i;\hat{\theta}_i) q_i(\theta_i)] - u_i(\hat{\theta}_i) - \hat{\theta}_i r_i.$$

In what follows, it will be useful to note that:

$$\mathbb{E}_{\theta_i}\left[\Psi_{i,\frac{w_i}{\rho}}(\theta_i;\hat{\theta}_i)\right] = \frac{w_i}{\rho}\mathbb{E}_{\theta_i}[\theta_i] + (1 - \frac{w_i}{\rho})\hat{\theta}_i. \tag{OA.7}$$

[6] As in the paper, we impose feasibility constraint that $Q_i(\boldsymbol{\theta}) \in [0, k_i]$. Alternatively, one can replace $Q_i(\boldsymbol{\theta})$ with $\min\{k_i, Q_i(\boldsymbol{\theta})\}$ and drop this constraint.

We can then write:

$$
\begin{aligned}
& W_\rho(\mathbf{Q},\mathbf{M}) \\
= \; & \sum_{i\in\mathcal{N}} w_i \mathbb{E}_{\theta_i}[\theta_i q_i(\theta_i) - m_i(\theta_i)] + \rho \sum_{i\in\mathcal{N}} \mathbb{E}_{\theta_i}\left[m_i(\theta_i)\right] \\
= \; & \sum_{i\in\mathcal{N}} w_i \mathbb{E}_{\theta_i}[\theta_i q_i(\theta_i) - \Psi_{i,0}(\theta_i;\hat{\theta}_i) q_i(\theta_i) + u_i(\hat{\theta}_i) + \hat{\theta}_i r_i] + \rho \sum_{i\in\mathcal{N}} \Big(\mathbb{E}_{\theta_i}[\Psi_{i,0}(\theta_i;\hat{\theta}_i) q_i(\theta_i)] - u_i(\hat{\theta}_i) - \hat{\theta}_i r_i\Big) \\
= \; & \sum_{i\in\mathcal{N}} \mathbb{E}_{\theta_i}\Big[w_i \theta_i q_i(\theta_i) + (\rho - w_i)\Psi_{i,0}(\theta_i;\hat{\theta}_i) q_i(\theta_i)\Big] + \sum_{i\in\mathcal{N}} (w_i - \rho)\Big(u_i(\hat{\theta}_i) + \hat{\theta}_i r_i\Big) \\
= \; & \sum_{i\in\mathcal{N}} \rho \mathbb{E}_{\theta_i}\Big[\Psi_{i,\frac{w_i}{\rho}}(\theta_i;\hat{\theta}_i) q_i(\theta_i)\Big] + \sum_{i\in\mathcal{N}} (w_i - \rho)\Big(u_i(\hat{\theta}_i) + \hat{\theta}_i r_i\Big) \\
= \; & \sum_{i\in\mathcal{N}} \rho \mathbb{E}_{\theta_i}\Big[\Psi_{i,\frac{w_i}{\rho}}(\theta_i;\hat{\theta}_i)\left(q_i(\theta_i) - r_i\right)\Big] + \sum_{i\in\mathcal{N}} (w_i - \rho)\Big(u_i(\hat{\theta}_i) + \hat{\theta}_i r_i\Big) + \sum_{i\in\mathcal{N}} \rho \mathbb{E}_{\theta_i}\Big[\Psi_{i,\frac{w_i}{\rho}}(\theta_i;\hat{\theta}_i) r_i\Big] \\
= \; & \tilde{W}_\rho(\mathbf{Q},\hat{\boldsymbol{\theta}}) + \sum_{i\in\mathcal{N}} (w_i - \rho)\Big(u_i(\hat{\theta}_i) + \hat{\theta}_i r_i\Big) + \sum_{i\in\mathcal{N}} r_i \Big(w_i \mathbb{E}_{\theta_i}[\theta_i] + (\rho - w_i)\hat{\theta}_i\Big) \\
= \; & \tilde{W}_\rho(\mathbf{Q},\hat{\boldsymbol{\theta}}) + \sum_{i\in\mathcal{N}} (w_i - \rho)\, u_i(\hat{\theta}_i) + \sum_{i\in\mathcal{N}} r_i w_i \mathbb{E}_{\theta_i}[\theta_i],
\end{aligned}
$$

where the first equality uses the definition of W_ρ, the second uses (9), the third rearranges, the fourth uses the definition of $\Psi_{i,\frac{w_i}{\rho}}$, which implies that $\mathbb{E}_{\theta_i}\left[\Big(w_i\theta_i + (\rho - w_i)\Psi_{i,0}(\theta_i;\hat{\theta}_i)\Big)\, q_i(\theta_i)\right] = \rho\mathbb{E}_{\theta_i}\left[\Psi_{i,\frac{w_i}{\rho}}(\theta_i;\hat{\theta}_i) q_i(\theta_i)\right]$, the fifth rearranges, the sixth uses (OA.7), and the last rearranges. which establishes (OA.5). According to (OA.5), for any exogenously fixed critical types $\hat{\boldsymbol{\theta}}$, we can write $W_\rho(\mathbf{Q},\mathbf{M})$ as being equal to $\tilde{W}_\rho(\mathbf{Q},\hat{\boldsymbol{\theta}})$ minus $\sum_{i\in\mathcal{N}}(\rho - w_i)u_i(\hat{\theta}_i)$ and plus $\sum_{i\in\mathcal{N}} r_i w_i \mathbb{E}_{\theta_i}[\theta_i]$, which does not depend on $\hat{\boldsymbol{\theta}}$. Because for a given allocation rule $\mathbf{Q}$, (OA.5) is constant over all $\hat{\boldsymbol{\theta}}$, the set of critical types that minimize $u_i(\hat{\theta}_i)$ must also be the set of critical types that minimize $\tilde{W}_\rho(\mathbf{Q},\hat{\boldsymbol{\theta}})$, implying (OA.6). ■

OA.3.1 Writing the payment rule as a function of the allocation rule

From the above, it follows that we can replace constraints (1) and (2) with (4), (5), and $u_i(\omega_i) \geq 0$ for all $i \in \mathcal{N}$ and $\omega_i \in \Omega(\mathbf{Q})$. Define

$$\mathcal{Q} \equiv \{\mathbf{Q} \mid q_i \text{ is nondecreasing for each } i \in \mathcal{N}\}.$$

Consequently, (4) is equivalent to $\mathbf{Q} \in \mathcal{Q}$.

Consider an allocation rule $\mathbf{Q} \in \mathcal{Q}$ and some worst-off types $\boldsymbol{\omega} = (\omega_1, \ldots, \omega_n) \in \Omega(\mathbf{Q})$. Under (5), equation (OA.5) in Lemma OA.2 implies that we can write the Lagrangian for (OA.4) as

$$
\begin{aligned}
\mathcal{L} \;&=\; W_\rho(\mathbf{Q},\mathbf{M}) + \textstyle\sum_{i\in\mathcal{N}} \mu_i u_i(\omega_i) \\
&=\; \tilde{W}_\rho(\mathbf{Q},\boldsymbol{\omega}) + \textstyle\sum_{i\in\mathcal{N}} (w_i - \rho + \mu_i)\, u_i(\omega_i) + \textstyle\sum_{i\in\mathcal{N}} r_i w_i \mathbb{E}_{\theta_i}[\theta_i],
\end{aligned}
\tag{OA.8}
$$

where $u_i(\omega_i)$ can be treated parametrically. Because $u_i(\omega_i) = \omega_i(q_i(\omega_i) - r_i) - m_i(\omega_i)$, no matter what the pointwise maximizer implies for $q_i(\omega_i)$, one can achieve any value for $u_i(\omega_i)$ by appropriately varying the fixed payment in $m_i(\omega_i)$, which by arguments analogous to those for Lemma 1, can be written as

$$\mathbb{E}_{\theta_i}[m_i(\theta_i)] = \mathbb{E}_{\theta_i}[\Psi_{i,0}(\theta_i;\omega_i)q_i(\theta_i)] - u_i(\omega_i) - \omega_i r_i. \tag{OA.9}$$

Using (OA.9), the expected revenue, not including fixed payments, is

$$\pi(\mathbf{Q},\boldsymbol{\omega}) \equiv \sum_{i\in\mathcal{N}} \mathbb{E}_{\theta_i}[\Psi_{i,0}(\theta_i;\omega_i)q_i(\theta_i)] - \sum_{i\in\mathcal{N}} \omega_i r_i.$$

If $\pi(\mathbf{Q},\boldsymbol{\omega}) > 0$ then the objective in (OA.4) is maximized by allocating $\pi(\mathbf{Q},\boldsymbol{\omega})$ among the firms with bargaining weights equal to $\max \mathbf{w}$ (as a normalization, one can assume that $\max \mathbf{w} = 1$), which is accomplished by having interim expected payoffs to the firms' worst-off types of

$$u_i(\omega_i) = \eta_i \pi(\mathbf{Q},\boldsymbol{\omega}),$$

where $\boldsymbol{\eta} \in \Delta^{n-1}$ and $\eta_i = 0$ for any firm that does not have the maximum bargaining weight, i.e., $\eta_i = 0$ if $w_i < \max \mathbf{w}$.[7] Here we use Δ^{n-1} (with no subscript $\mathbf{k}$) to denote the standard $(n-1)$-dimensional simplex defined by:

$$\Delta^{n-1} \equiv \big\{\mathbf{x} \in [0,1]^n \mid \sum_{i\in\{1,\ldots,n\}} x_i = 1\big\}.$$

This implies $\mathbf{M}$ such that interim expected payments satisfy, for all $i \in \mathcal{N}$,

$$m_i(\theta_i) = \theta_i\,(q_i(\theta_i) - r_i) - \int_{\omega_i}^{\theta_i} (q_i(y) - r_i)dy - \eta_i \pi(\mathbf{Q},\boldsymbol{\omega}).$$

Given this, we can turn our attention to the allocation rule.

OA.3.2 Determining the allocation rule

It remains to determine the allocation rule. Because the second term on the right side of (OA.8) can be treated parametrically and the third term is independent of the allocation rule, we can restrict attention to maximizing $\tilde{W}_\rho(\mathbf{Q},\boldsymbol{\omega}) = \min_{\hat{\boldsymbol{\theta}}} \tilde{W}_\rho(\mathbf{Q},\hat{\boldsymbol{\theta}})$, where the equality follows from (OA.6) in Lemma OA.2. Consequently, an optimal allocation rule $\mathbf{Q}_\rho^{\mathbf{r}}$ has to satisfy

$$\mathbf{Q}_\rho^{\mathbf{r}} \in \arg\max_{\mathbf{Q}\in\mathcal{Q}} \min_{\hat{\boldsymbol{\theta}}\in[0,1]^n} \tilde{W}_\rho(\mathbf{Q},\hat{\boldsymbol{\theta}}). \tag{OA.10}$$

Instead of directly solving the max-min problem in (OA.10), we look for a *saddle point*

[7] For completeness, note that when $\pi(\mathbf{Q}^*,\boldsymbol{\omega}^*) > 0$, we have $\rho^* = 1$ and, by stationarity, $\mu_i^* = \rho^* - w_i$, which implies that $\mu_i^* = 0$ if $w_i = \max \mathbf{w} = 1$ and $u_i(\omega^*) = 0$ if $w_i < 1$, ensuring that the associated complementary slackness condition is satisfied. If $\pi(\mathbf{Q}^*,\boldsymbol{\omega}^*) = 0$, then $\rho^* \geq 1$, $\mu_i^* = \rho^* - w_i \geq 0$, and $u_i(\omega^*) = 0$, so again complementary slackness is satisfied.

$(\mathbf{Q}^*, \boldsymbol{\omega}^*)$ of $\tilde{W}_\rho$ that satisfies

$$\mathbf{Q}^* \in \arg\max_{\mathbf{Q}\in\mathcal{Q}} \tilde{W}_\rho(\mathbf{Q}, \boldsymbol{\omega}^*) \tag{OA.11}$$

$$\boldsymbol{\omega}^* \in \arg\min_{\hat{\boldsymbol{\theta}}\in[0,1]^n} \tilde{W}_\rho(\mathbf{Q}^*, \hat{\boldsymbol{\theta}}). \tag{OA.12}$$

For a saddle point, (OA.11) requires that the allocation rule $\mathbf{Q}^*$ maximizes the virtual objective $\tilde{W}_\rho$ under critical types $\boldsymbol{\omega}^*$, whereas (OA.12) requires that he critical types $\boldsymbol{\omega}^*$ are worst-off types under allocation rule $\mathbf{Q}^*$, i.e., $\boldsymbol{\omega}^* \in \Omega(\mathbf{Q}^*)$.

If a saddle point $(\mathbf{Q}^*, \boldsymbol{\omega}^*)$ exists, then $\mathbf{Q}_\rho^{\mathbf{r}}$ solves the problem in (OA.10) if and only if $(\mathbf{Q}_\rho^{\mathbf{r}}, \boldsymbol{\omega}^*)$ is a saddle point.[8] In what follows, we show that a saddle point $(\mathbf{Q}^*, \boldsymbol{\omega}^*)$ exists and that $\mathbf{Q}^*$ is essentially unique.

We proceed, as in Loertscher and Wasser (2019), by first determining the class of allocation rules that is consistent with (OA.11). Then we argue that an essentially unique member of this class also satisfies (OA.12).

Consider the optimization problem in (OA.11). Pointwise maximization of

$$\tilde{W}_\rho(\mathbf{Q}, \boldsymbol{\omega}^*) \equiv \rho\mathbb{E}_{\boldsymbol{\theta}}\left[\sum_{i\in\mathcal{N}}(q_i(\theta_i) - r_i)\Psi_{i,\frac{w_i}{\rho}}(\theta; \omega_i^*)\right]$$

would require allocation the supply to the firms with the highest weighted virtual types $\Psi_{i,\frac{w_i}{\rho}}(\theta; \omega_i^*)$ in order up to their maximum demands. However, $\Psi_{i,\frac{w_i}{\rho}}(\theta; \omega_i^*)$ is not monotone at ω_i^*, resulting in a violation of the monotonicity constraint $\mathbf{Q} \in \mathcal{Q}$. The solution to (OA.11) therefore involves ironing (Myerson 1981): the goods are allocated to the firms by prioritizing on the basis of the ironed weighted virtual types $\overline{\psi}_{i,\frac{w_i}{\rho}}(\theta, z_i)$ defined in (OA.2). According to (OA.3), there is a one-to-one relation between the critical type ω_i^* and the corresponding ironing parameter z_i, which we write as

$$\omega_i^* = \omega_{i,\rho}(z_i). \tag{OA.13}$$

Note that $\omega_{i,\rho}(\cdot)$ is a continuous and strictly increasing function.

OA.3.3 Tie-breaking rules

We must address the possibility that $z_i^* = z_j^*$ for some i and j, in which case ties between the ironed weighted virtual types arise with positive probability.

Before discussing tie-breaking rules, however, it is useful to note settings in which ties do not arise. Specifically, in the partnership setup with $F_i = F$ and $k_i = 1$ for all $i \in \mathcal{N}$, if the

[8] As noted by Loertscher and Wasser (2019, footnote 11): Suppose that $(\mathbf{Q}^*, \boldsymbol{\omega}^*)$ satisfies (OA.11) and (OA.12). Then $\min_{\hat{\boldsymbol{\theta}}} \tilde{W}_\rho(\mathbf{Q}^*, \hat{\boldsymbol{\theta}}) = \tilde{W}_\rho(\mathbf{Q}^*, \boldsymbol{\omega}^*) \geq \tilde{W}_\rho(\mathbf{Q}, \boldsymbol{\omega}^*) \geq \min_{\hat{\boldsymbol{\theta}}} \tilde{W}_\rho(\mathbf{Q}, \hat{\boldsymbol{\theta}})$ for all $\mathbf{Q} \in \mathcal{Q}$ and hence $\mathbf{Q}^*$ solves the problem in (OA.10). Conversely, for all $\mathbf{Q}_\rho^{\mathbf{r}}$ that satisfy (OA.10), the above has to hold with equality, implying that $(\mathbf{Q}_\rho^{\mathbf{r}}, \boldsymbol{\omega}^*)$ is a saddle point.

first-best is not possible, then $z_i^* \neq z_j^*$ for some i and j holds, which means that ties between those agents occur with probability zero. For a partnership model in which the market maker's objective function consists of a convex combination of expected social surplus and revenue, Loertscher and Wasser (2019) show that if the firms have identical distributions, then the set of optimal ownership structures is convex, includes $\mathbf{r}^*$, and increases in the set inclusion sense as the weight on revenue increases (see their Corollary 4). Moreover, optimal ownership structures equalize, if possible, the worst-off types (see their Theorem 2). Because the set of optimal ownership structures is convex and includes $\mathbf{r}^*$, the value of the market maker's objective is the same as for any optimal ownership structure for a given weight on revenue as when the ownership structure is $\mathbf{r}^*$. Consequently, the market maker's revenue is positive for any optimal ownership structure. Because identical distributions and optimal identical ironing parameters imply equal worst-off types, it follows that the ownership structure must be optimal as defined in Loertscher and Wasser (2019) for the weight on revenue implied by this mechanism. But then this revenue must be positive, contradicting that it is a second-best mechanism.

As in Loertscher and Wasser (2019), let H denote the set of all $n!$ permutations $(h_1, h_2, \dots, h_{n!})$ of $(1, 2, \dots, n)$. We call each $h \in H$ a hierarchy among the firms in $\mathcal{N}$. A *hierarchical tie-breaking rule* breaks ties in favor of the firm that is the highest in the hierarchy. Under a *split hierarchical tie-breaking rule* $\mathbf{a} \in \Delta^{n!-1}$, one hierarchy h is randomly selected from H according to the probability distribution $\mathbf{a}$ over H and then ties are broken according to h. The outcome in terms of the interim expected allocation of any tie-breaking rule can equivalently be obtained by a split hierarchical tie-breaking rule $\mathbf{a}$.

Given $\boldsymbol{\theta}$, $\mathbf{z}$, ρ, and $\mathbf{w}$, the allocation rule of interest depends on the ranking of the ironed weighted virtual types $\overline{\psi}_{i,\frac{w_i}{\rho}}(\theta_i, z_i)$. Let

$$K_{i,\rho}^{A,\mathbf{z}}(\boldsymbol{\theta}) \equiv \sum_{j \in \mathcal{N}} k_j \cdot \mathbf{1}_{\overline{\psi}_{i,\frac{w_i}{\rho}}(\theta_i, z_i) < \overline{\psi}_{j,\frac{w_j}{\rho}}(\theta_j, z_j)}$$

denote the sum of the maximum demands of firms with ironed weighted virtual types above that of firm i. Analogously, let

$$K_{i,\rho}^{T,\mathbf{z}}(\boldsymbol{\theta}) \equiv \sum_{j \in \mathcal{N}} k_j \cdot \mathbf{1}_{\overline{\psi}_{i,\frac{w_i}{\rho}}(\theta_i, z_i) = \overline{\psi}_{j,\frac{w_j}{\rho}}(\theta_j, z_j)}$$

denote the sum of the maximum demands of firms with ironed weighted virtual types tied with that of firm i (including k_i). Further, let

$$\mathcal{T}_{i,\rho}^{\mathbf{z}}(\boldsymbol{\theta}) \equiv \left\{ j \in \mathcal{N} \mid \overline{\psi}_{i,\frac{w_i}{\rho}}(\theta_i, z_i) = \overline{\psi}_{j,\frac{w_j}{\rho}}(\theta_j, z_j) \right\}$$

denote the set of firms with the same ironed weighted virtual type as firm i (including firm

i), which means that for all parameters, $\{i\} \subseteq \mathcal{T}^{\mathbf{z}}_{i,\rho}(\boldsymbol{\theta})$.

The *ironed virtual type allocation rule* $Q^{\mathbf{z},\mathbf{a}}_{i,\rho}(\boldsymbol{\theta})$ with split hierarchical tie-breaking rule $\mathbf{a}$ and expectations taken with respect to any tie-breaking randomization is then given by:

$$Q^{\mathbf{z},\mathbf{a}}_{i,\rho}(\boldsymbol{\theta}) \equiv \begin{cases} 0 & \text{if } 1 - K^{A,\mathbf{z}}_{i,\rho}(\boldsymbol{\theta}) \leq 0, \\ \sum_{h \in H} a_h \max\{0, \min\{k_i, 1 - K^{A,\mathbf{z}}_{i,\rho}(\boldsymbol{\theta}) - \sum_{j \in \mathcal{T}^{\mathbf{z}}_{i,\rho}(\boldsymbol{\theta}) \setminus \{i\}} k_j \cdot \mathbf{1}_{h(j) > h(i)}\}\} & \\ & \text{if } 0 < 1 - K^{A,\mathbf{z}}_{i,\rho}(\boldsymbol{\theta}) < K^{T,\mathbf{z}}_{i,\rho}(\boldsymbol{\theta}), \\ k_i & \text{if } K^{T,\mathbf{z}}_{i,\rho}(\boldsymbol{\theta}) \leq 1 - K^{A,\mathbf{z}}_{i,\rho}(\boldsymbol{\theta}). \end{cases} \tag{OA.14}$$

The first row in the definition of $Q^{\mathbf{z},\mathbf{a}}_{i,\rho}(\boldsymbol{\theta})$ says that if the sum of the maximum demands of firms with ironed weighted virtual types that are above that of firm i is greater than or equal to 1, then firm i gets zero units. The final row says that if the total demand of firms with ironed weighted virtual types that are greater than or equal to that of firm i is less than or equal to 1, then firm i gets is full demand. In the intermediate range, there is rationing: if no firms are tied with firm i, i.e., $\mathcal{T}^{\mathbf{z}}_{i,\rho}(\boldsymbol{\theta}) = \{i\}$ and $K^{T,\mathbf{z}}_{i,\rho}(\boldsymbol{\theta}) = k_i$, then firm i is allocated $1 - K^{A,\mathbf{z}}_{i,\rho}(\boldsymbol{\theta}) < k_i$, but if firms are tied with firm i, then there is randomization—hierarchy h is chosen with probability a_h and the available units $1 - K^{A,\mathbf{z}}_{i,\rho}(\boldsymbol{\theta})$ are allocated to firms in order according to the randomly selected hierarchy up to their maximum demands.

OA.4 Solving for the market mechanism

For a given $\boldsymbol{\omega}^*$, the allocation rule $\mathbf{Q}^* = \mathbf{Q}^{\mathbf{z},\mathbf{a}}_{\rho}$ solves the problem in (OA.11) for $\mathbf{z} = (\omega^{-1}_{1,\rho}, \ldots, \omega^{-1}_{n,\rho})$ and any tie-breaking rule $\mathbf{a} \in \Delta^{n!-1}$. Having established that all allocation rules consistent with (OA.11) are equivalent to ironed weighted virtual type allocation rules $\mathbf{Q}^{\mathbf{z},\mathbf{a}}_{\rho}$, we now turn to the second requirement for a saddle point. Condition (OA.12) requires that the critical types $\boldsymbol{\omega}^*$ are worst-off types under allocation rule $\mathbf{Q}^*$. A simultaneous solution to (OA.11) and (OA.12) hence corresponds to a vector of ironing parameters $\mathbf{z}$ and a tie-breaking rule $\mathbf{a}$ such that $\omega_{i,\rho}(z_i) \in \Omega_i(\mathbf{Q}^{\mathbf{z},\mathbf{a}}_{\rho})$ for each firm $i \in \mathcal{N}$. The interim expected share $q^{\mathbf{z},\mathbf{a}}_{i,\rho}(\theta_i)$ is constant for an interval of types θ_i that contains the critical type $\omega_{i,\rho}(z_i)$. The characterization of the set of worst-off types above hence implies that for critical types to be worst off, we must have for all $i \in \mathcal{N}$ and $\mathbf{r} \in \Delta_{\mathbf{k}}$,

$$q^{\mathbf{z},\mathbf{a}}_{i,\rho}(\omega_{i,\rho}(z_i)) = r_i.$$

We show that there is typically a unique z such that $q^{\mathbf{z},\mathbf{a}}_{i}(\omega_{i,\rho}(z_i)) = r_i$ for all $i \in \mathcal{N}$ for some $\mathbf{a}$, yielding the existence of a saddle point and a characterization of the market mechanism. To prove this result and make its statement precise, the following definitions are useful. Define the correspondence

$$\Gamma_{\mathcal{N},\rho}(\mathbf{z}) : [0,1]^n \rightrightarrows \times_{i\in\mathcal{N}}[0, \min\{1, k_i\}]$$

such that

$$\Gamma_{\mathcal{N},\rho}(\mathbf{z}) \equiv \{q^{\mathbf{z},\mathbf{a}}_{1,\rho}(\omega_{1,\rho}(z_1)), \dots, q^{\mathbf{z},\mathbf{a}}_{n,\rho}(\omega_{n,\rho}(z_n)) \mid \mathbf{a} \in \Delta^{n!-1}\}.$$

The correspondence $\Gamma_{\mathcal{N},\rho}(\mathbf{z})$ yields the set of all vectors of expected allocations for critical types $\omega_{1,\rho}(z_1), \dots, \omega_{n,\rho}(z_n)$ that can be obtained with ironing parameters $\mathbf{z} \in [0,1]^n$ and some tie-breaking rule $\mathbf{a}$. If $z_i = z_j$ for two firms i and j, then there is a positive probability of a tie and the expected shares depend on tie breaking. The correspondence $\Gamma_{\mathcal{N},\rho}(\mathbf{z})$ is singleton-valued if and only if $z_i \neq z_j$ for all i and $j \neq i$.

Define $\Delta^{n-1}_{\mathbf{k}}$ to be the same as $\Delta_{\mathbf{k}}$ as defined in the body of the paper, but here we add the superscript $n-1$ to be explicit that we mean an $(n-1)$-dimensional object,

$$\Delta^{n-1}_{\mathbf{k}} \equiv \Big\{\mathbf{x} \in \times_{i\in\{1,\dots,n\}}[0, k_i] \mid \sum_{i=1}^{n} x_i = 1\Big\}.$$

Note that $\Delta^{n-1}_{\mathbf{k}}$ differs from a standard simplex in that the included vectors are constrained not only to add to 1 but are also constrained to be feasible allocation vectors given the firms' maximum demands, i.e., each x_i is less than or equal to k_i and greater than or equal to $\max\{0, 1 - \sum_{j\in\mathcal{N}\setminus\{i\}} k_j\} = 0$.

OA.5 Statement of the theorem

We are now ready to state the analog of Theorem 1 in Loertscher and Wasser (2019). The proof of Theorem OA.1 makes use of the geometric structure of $\Gamma_{\mathcal{N},\rho}$, but in our setup with the maximum demands not equal across firms and not equal to 1, this structure differs in significant ways from that of Loertscher and Wasser (2019), so a new proof is required.[9]

Theorem OA.1. *For each $\rho \geq \max \mathbf{w} = 1$ and $\mathbf{r} \in \Delta^{n-1}_{\mathbf{k}}$, there exists a unique $\mathbf{z} \in [0,1]^n$ such that $\mathbf{r} \in \Gamma_{\mathcal{N},\rho}(\mathbf{z})$, denoted by $\mathbf{z}^* = \Gamma^{-1}_{\mathcal{N},\rho}(\mathbf{r})$. All market mechanisms $\langle \mathbf{Q}^{\mathbf{r}}_{\rho}, \mathbf{M}^{\mathbf{r}}_{\rho}\rangle$ that solve $\max_{\mathbf{Q},\mathbf{M}} W_\rho(\mathbf{Q},\mathbf{M})$ subject to incentive compatibility and individual rationality consist of an allocation rule $\mathbf{Q}^{\mathbf{r}}_{\rho}$ that allocates the supply to the firms with the greatest ironed weighted virtual types $\overline{\psi}_{i,\frac{w_i}{\rho}}(\theta_i, z^*_i)$ up to their maximum demands, where ties are broken such that for*

[9]Loertscher and Wasser (2019) are able to prove results for a partnership problem, i.e., $k_1 = \dots = k_n = 1$, for general n by making use of the recursive structure of $\Gamma_{\mathcal{N},\rho}$ in that setting to argue by induction. That same recursive structure does not exist when firms' maximum demands differ or are less than 1 (see footnote 10 below), although sufficient structure continues to exist to facilitate the proof. Lu and Robert (2001) analyze the problem with $k_i = k < 1$ for all $i \in \mathcal{N}$. But because they also assume $F_i = F$ and $w_i = 1$ for all i, their results are not applicable here either.

all $i \in \mathcal{N}$, $q^{\mathbf{r}}_{i,\rho}(\omega_{i,\rho}(z_i^*)) = r_i$, *and a payment rule* $\mathbf{M}^{\mathbf{r}}_{\rho}$ *such that interim expected payments satisfy for all* $i \in \mathcal{N}$,

$$m^{\mathbf{r}}_{i,\rho}(\theta_i) = \theta_i(q^{\mathbf{r}}_{i,\rho}(\theta_i) - r_i) - \int_{\omega_{i,\rho}(z_i^*)}^{\theta_i} (q^{\mathbf{r}}_{i,\rho}(y) - r_i)dy - \eta_i \pi(\mathbf{Q}^{\mathbf{r}}_{\rho}, \omega_{1,\rho}(z_1^*), \ldots, \omega_{n,\rho}(z_n^*)).$$

A split hierarchical tie-breaking rule $\mathbf{a}^*$ *and unique Lagrange multiplier* $\rho^* \geq \max \mathbf{w}$ *exist such that* $\mathbf{Q}^{\mathbf{z}^*,\mathbf{a}^*}_{\rho^*}$, *defined in (OA.14), with a corresponding payment rule* $\mathbf{M}^{\mathbf{z}^*,\mathbf{a}^*}_{\rho^*}$, *solves* $\max_{\mathbf{Q},\mathbf{M}} W_\rho(\mathbf{Q},\mathbf{M})$ *subject to incentive compatibility, individual rationality, and no deficit.*

Proof. In Section OA.6, we prove that there exists a unique $\mathbf{z} \in [0,1]^n$ such that $\mathbf{r} \in \Gamma_{\mathcal{N},\rho}(\mathbf{z})$. Given this, it follows that for this unique $\mathbf{z}$ and some tie-breaking rule $\mathbf{a}$, the ironed virtual type allocation rule $\mathbf{Q}^{\mathbf{z},\mathbf{a}}_{\rho}$ and the critical types $(\omega_{1,\rho}(z_1), \ldots, \omega_{n,\rho}(z_n))$ constitute a saddle point satisfying (OA.11) and (OA.12), making $\mathbf{Q}^{\mathbf{z},\mathbf{a}}_{\rho}$ an optimal allocation rule consistent with (OA.10).

By restricting the definition of $\Gamma_{\mathcal{N},\rho}$ and the statement of Theorem OA.1 to $z_i \in [0,1]$, we have confined attention to critical types $\omega_i^* \in [\omega_{i,\rho}(0), \omega_{i,\rho}(1)] \subset [0,1]$. This restriction is without loss when we are looking for incomplete information bargaining allocation rules. As becomes apparent below, for $\mathbf{z} = \Gamma^{-1}_{\mathcal{N},\rho}(\mathbf{r})$, we have $z_i = 0$ if and only if $r_i = 0$. Hence for all $\mathbf{r}$, $z_j > 0$ for at least one j. Accordingly, $q^{\mathbf{z},\mathbf{a}}_{i,\rho}(\omega_{i,\rho}(z_i)) = 0$ for all $z_i \leq 0$. If there is a saddle point involving critical type $\omega_i^* = \omega_{i,\rho}(0)$, then there is also a saddle point for each $\omega_i^* \in [0, \omega_{i,\rho}(0))$. However, all these saddle points are equivalent in terms of the implied allocation rule $\mathbf{Q}^*$ and i's worst-off types $\Omega_i(\mathbf{Q}^*) = \{\theta_i \mid q_i^*(\theta_i) = 0\} = [0, (\Psi^B_{i,\frac{w_i}{\rho}})^{-1}(0)]$. A similar line of argument can be invoked for $z_i \geq 1$, which only occurs if $r_i = k_i$.

From the preceding paragraph, we conclude that whereas there can be multiple saddle points satisfying (OA.11) and (OA.12), the corresponding allocation rule $\mathbf{Q}^*$ is unique up to the tie-breaking rule and can be defined as allocating to the greatest ironed weighted virtual types, up to their maximum demands, for ironing parameters $\mathbf{z} = \Gamma^{-1}_{\mathcal{N},\rho}(\mathbf{r})$. Whereas the exact specification of the tie-breaking rule may differ, all market allocation rules result in the same interim expected allocations, which in turn pin down interim expected payments, as explained above.

This lets us conclude that for any $\rho \geq \max \mathbf{w}$ there exists a split hierarchical tie-breaking rule $\mathbf{a}^*$ such that $\mathbf{Q}^{\mathbf{z}^*,\mathbf{a}^*}_{\rho}$ and $\mathbf{M}^{\mathbf{z}^*,\mathbf{a}^*}_{\rho}$ solve $\max_{\mathbf{Q},\mathbf{M}} W_\rho(\mathbf{Q},\mathbf{M})$ subject to incentive compatibility and individual rationality.

To complete the proof, note that in the limit at ρ goes to infinity, the allocation rule approaches that for the mechanism that maximizes the market maker's revenue. But, of course, that mechanism has positive expected revenue because the market maker can, for example, propose the mechanism that matches pairs of firms where one firm is willing to buy a unit at the fixed price of 2/3 and the other firm is willing to sell a unit at the fixed price of 1/3, with randomization used to select which firms trade if the number of willing buyers and willing sellers at those prices differ. The mechanism is individually rational because a firm could report a type of 1/2 and guarantee that it does not trade. Thus, there exists finite ρ^* such that the expected revenue is positive. The continuity and monotonicity of our problem then guarantee that $\rho^* = \min\{\rho \geq \max \mathbf{w} \mid \sum_{i \in \mathcal{N}} \mathbb{E}[M_{i,\rho}^{\mathbf{z}^*,\mathbf{a}^*}(\boldsymbol{\theta})] \geq 0\}$ is well defined and unique. This completes the proof of the second part of Theorem OA.1. It remains to prove the first sentence of the theorem, which we do in Section OA.6.

OA.6 Proof of the first sentence of Theorem OA.1

Here we show that given $\rho \geq \max \mathbf{w}$, for each $\mathbf{r} \in \Delta_{\mathbf{k}}^{n-1}$, there exists a unique $\mathbf{z} \in [0,1]^n$ such that $\mathbf{r} \in \Gamma_{\mathcal{N},\rho}(\mathbf{z})$.

The first part of the proof establishes two properties of $\Gamma_{\mathcal{N},\rho}$ that are sufficient for the result. Then we show that these two properties hold. We rely on an induction argument, first showing that the properties hold for any $\mathcal{N}$ with $|\mathcal{N}| = n = 2$, and then showing that this implies that the properties hold for any $\mathcal{N}$ with $|\mathcal{N}| \geq 2$.

For any $A \subseteq [0,1]^n$, let

$$\Gamma_{\mathcal{N},\rho}(A) \equiv \{\mathbf{y} \in [0,1]^n \mid \mathbf{y} \in \Gamma_{\mathcal{N},\rho}(\mathbf{z}) \text{ for some } \mathbf{z} \in A\}$$

denote the image of A under $\Gamma_{\mathcal{N},\rho}$. To prove that for each $\mathbf{r} \in \Delta_{\mathbf{k}}^{n-1}$, there is a unique $\mathbf{z} \in [0,1]^n$ such that $\mathbf{r} \in \Gamma_{\mathcal{N},\rho}(\mathbf{z})$, we show that $\Gamma_{\mathcal{N},\rho}$ has the following two properties:

Property 1. For every $\mathbf{y} \in \Gamma_{\mathcal{N},\rho}([0,1]^n)$, there is a unique $\mathbf{z}$ such that $\mathbf{y} \in \Gamma_{\mathcal{N},\rho}(\mathbf{z})$.

Property 2. We have $\Delta_{\mathbf{k}}^{n-1} \in \Gamma_{\mathcal{N},\rho}([0,1]^n)$.

Property 1 implies the uniqueness part. It says that every point in the image of $\Gamma_{\mathcal{N},\rho}$ corresponds to exactly one $\mathbf{z}$. Put differently, the inverse correspondence $\Gamma_{\mathcal{N},\rho}^{-1}(\mathbf{y}) \equiv \{\mathbf{z} \in [0,1]^n \mid \mathbf{y} \in \Gamma_{\mathcal{N},\rho}(\mathbf{z})\}$ is singleton-valued for all $\mathbf{y} \in \Gamma_{\mathcal{N},\rho}([0,1]^n)$. Property 2 implies the existence part. It says that the image of $\Gamma_{\mathcal{N},\rho}$ contains $\Delta_{\mathbf{k}}^{n-1}$.

As a **roadmap** for what follows, we build up the proof that Properties 1 and 2 hold from a series of lemmas, many of which have counterparts in Loertscher and Wasser (2019), as noted below:

- Preliminaries (Section OA.6.1)

- **Lemma OA.3** provides an expression for the interim expected allocations of the firms' critical types $(\omega_{1,\rho}(z_1), \ldots, \omega_{n,\rho}(z_n))$ under tie-breaking hierarchy h, which we denote by $\mathbf{p}_\rho(\mathbf{z}, h) \in [0,1]^n$. As we show, these points can be used to define $\Gamma_{\mathcal{N},\rho}([0,1]^n)$. Indeed, $\Gamma_{\mathcal{N},\rho}(\mathbf{z})$ =Conv($\{\mathbf{p}_\rho(\mathbf{z}, h) \mid h \in H\}$). (The analogous result in Loertscher and Wasser (2019) is the displayed equation on p. 1097 showing $p_i(\mathbf{z}, h)$ for their setup. The expression for $\mathbf{p}_\rho(\mathbf{z}, h)$ is more complex in our setup because more than one firm can have a positive allocation and a firm's allocation depends on the maximum demands of the firms that trade ahead of it.)
- **Lemma OA.4** (analog to Lemma 2 in Loertscher and Wasser (2019)) observes that the characterization of $\Gamma_{\mathcal{N},\rho}(\mathbf{z})$ as Conv($\{\mathbf{p}_\rho(\mathbf{z}, h) \mid h \in H\}$) implies that $\Gamma_{\mathcal{N},\rho}(z)$ is nonempty and convex, and the lemma shows that the correspondence $\Gamma_{\mathcal{N},\rho}$ is upper hemicontinuous.
- **Lemma OA.5** uses the partition of the domain of $\Gamma_{\mathcal{N},\rho}$ into $\xi_{\mathcal{N}}(z)$ for $z \in [0,1]$ and uses the definition $O_{\mathcal{N}}(z) \equiv \Gamma_{\mathcal{N}}(\xi_{\mathcal{N}}(z))$ to note that the image of $\Gamma_{\mathcal{N},\rho}$ can be written as $\Gamma_{\mathcal{N},\rho}([0,1]^n) = \cup_{z\in[0,1]} O_{\mathcal{N},\rho}(z)$. (The analogous result in Loertscher and Wasser (2019) is the middle of p. 1098.)

- Properties 1 and 2 hold for $n = 2$ (Section OA.6.2)
  - **Lemma OA.6** states that Property 1 holds for $n = 2$. (The analogous result in Loertscher and Wasser (2019) is in their Section A.2.)
  - **Lemma OA.7** states that Property 2 holds for $n = 2$. (The analogous result in Loertscher and Wasser (2019) is in their Section A.2.)

- Characterizing $O_{\mathcal{N},\rho}$ and $\Gamma_{\mathcal{N},\rho}$ for $n > 2$ (Section OA.6.3)
  - **Lemma OA.8** establishes that $\Gamma_{\mathcal{N},\rho}$ is an $(n-1)$-dimensional polytope. (The analogous result in Loertscher and Wasser (2019) is stated in the middle of p. 1100.)
  - **Lemma OA.9** makes use of the definition of polytopes $o^{\mathcal{K}}_{\mathcal{N},\rho}(z)$ for each $\mathcal{K} \subset \mathcal{N}$, which gives us $O_{\mathcal{N},\rho}(z) = \cup_{\mathcal{K}\subset\mathcal{N}} o^{\mathcal{K}}_{\mathcal{N},\rho}(z)$. The lemma shows that $o^{\mathcal{K}}_{\mathcal{N},\rho}(z)$ is defined by convex combinations of a particular set of points of the form $\mathbf{p}_\rho(\mathbf{z}, h)$. (The analogous result in Loertscher and Wasser (2019) is stated on their p. 1102, however, in stating the result, they rely on the particular recursive structure of their setup, which does not apply in our setup—see footnote 10 below.)
  - **Lemma OA.10** (analog to Lemma 3 in Loertscher and Wasser (2019)) then shows that all the points that define $O_{\mathcal{N},\rho}(z)$ are nondecreasing in z, and increasing in at least one coordinate. Thus, $O_{\mathcal{N},\rho}(z)$ increases with z.

- **Lemma OA.11** (analog to Lemma 4 in Loertscher and Wasser (2019)) uses the monotonicity result of Lemma OA.10 to show that if Property 1 holds for $\Gamma_{\mathcal{K},\rho}$ for all $\mathcal{K} \subset \mathcal{N}$, then it holds for $\Gamma_{\mathcal{N},\rho}$.
- **Lemma OA.12** combines Lemmas OA.6 and OA.11 to conclude that Property 1 holds. (This corresponds to the "Final step" on p. 1106 of Loertscher and Wasser (2019) as it relates to Property 1.)
- **Lemma OA.13** (analog to Lemma 5 in Loertscher and Wasser (2019)) shows that the image of $\Gamma_{\mathcal{N},\rho}([z,1]^n)$ is an n-dimensional convex polytope for all $z \leq 1$ and has boundary given by $O_{\mathcal{N},\rho}(z) \cup S_{\mathcal{N},\rho}(z)$.
- **Lemma OA.14** shows that the image of $\Gamma_{\mathcal{N},\rho}([0,1]^n)$ is defined by vertices $\mathbf{p}_\rho(\mathbf{z}, h_\ell)$ with $\ell \in \{1,\dots,n!\}$ and $z_i \in \{0,1\}$ for all $i \in \mathcal{N}$. (This property is also true in the setup of Loertscher and Wasser (2019) and follows from the discussion on their p. 1105 and the monotonicity properties of their setup, but they do not call it out specifically.)
- **Lemma OA.15** shows that $\Gamma_{\mathcal{N},\rho}(0,\dots,0)$, which is one component of $O_{\mathcal{N},\rho}(0)$, lies below $\Delta^{n-1}_{\mathbf{k}}$. (The analogous point is made by Loertscher and Wasser (2019) on their p. 1105. It is straightforward in their setup because, there, the analog to $\Gamma_{\mathcal{N},\rho}(0,\dots,0)$, which they denote by $\Gamma_n(\underline{z},\dots,\underline{z})$, is, as they note, a simplex with only n vertices rather than a polytope with $n!$ vertices. In contrast, in our setup, we continue to have the polytope.)

• Condition for $O_{\mathcal{N},\rho}(z)$ to "pass through" all of $\Delta^{n-1}_{\mathbf{k}}$ and so to have $\Delta^{n-1}_{\mathbf{k}} \in \Gamma([0,1]^n)$ (Section OA.6.4)

- **Lemma OA.16** shows that if the n facets of $\Delta^{n-1}_{\mathbf{k}}$ that are subsets of the n facets of Δ^{n-1} are contained in $O_{\mathcal{N},\rho}(0)$, then as z increases from 0 to 1, $O_{\mathcal{N},\rho}(z)$ must "pass through" all of $\Delta^{n-1}_{\mathbf{k}}$, i.e., for all $\mathbf{y} \in \Delta^{n-1}_{\mathbf{k}}$, there exists $z \in [0,1]$ such that $\mathbf{y} \in O_{\mathcal{N},\rho}(z)$, and so $\Delta^{n-1}_{\mathbf{k}} \in \Gamma([0,1]^n)$. Thus, we have the desired result if we can show that the n facets of $\Delta^{n-1}_{\mathbf{k}}$ that are subsets of the n facets of Δ^{n-1} are contained in $O_{\mathcal{N},\rho}(0)$. Specifically, we show that the intersection of the boundary of Δ^{n-1} and $\Delta^{n-1}_{\mathbf{k}}$ that has i-th component equal to $\max\{0, 1-\sum_{j\in\mathcal{N}\setminus\{i\}} k_j\}$ lies in $o^{\mathcal{N}\setminus\{i\}}_{\mathcal{N},\rho}(0)$. (An informal argument along the same lines is made in Loertscher and Wasser (2019) at the bottom of their p. 1105 in the discussion of their Figure 10.)

• $O_{\mathcal{N},\rho}(0)$ contains $\Gamma_{\mathcal{N}\setminus\{i\},\rho}([0,1]^{n-1})$ (Section OA.6.5)

– **Lemma OA.17** shows that $O_{\mathcal{N},\rho}(0)$ contains $\Gamma_{\mathcal{N}\setminus\{i\},\rho}([0,1]^{n-1})$. (In Loertscher and Wasser (2019), this result is stated in the displayed equation on p. 1105, just before Lemma 6. It is more straightforward in their setup because, essentially, $o_{\mathcal{N},\rho}^{\mathcal{N}\setminus\{i\}}$ is equal to $\Gamma_{\mathcal{N}\setminus\{i\},\rho}([0,1]^{n-1})$, whereas in our setup $\Gamma_{\mathcal{N}\setminus\{i\},\rho}([0,1]^{n-1})$ is a subset of $o_{\mathcal{N},\rho}^{\mathcal{N}\setminus\{i\}}$.)

– **Lemma OA.18** (analog to Lemma 6 in Loertscher and Wasser (2019)) shows that if Property 2 holds or $\Gamma_{\mathcal{N}\setminus\{i\},\rho}$, then it holds for $\Gamma_{\mathcal{N},\rho}$, because by Lemma OA.17, for all $i \in \mathcal{N}$, $\Delta^{n-2}_{\mathbf{k}^{\mathcal{N}\setminus\{i\}}}$ is contained in the image of $\Gamma_{\mathcal{N}\setminus\{i\},\rho}$, which is then contained in $O_{\mathcal{N},\rho}(0)$, which implies that $\Delta^{n-1}_{\mathbf{k}}$ is contained in $O_{\mathcal{N},\rho}(0)$.

– **Lemma OA.19** (analog to Lemma 7 in Loertscher and Wasser (2019)) makes the induction argument that, given that Property 2 holds for $n = 2$, Lemma OA.18 can be applied iteratively to show that it holds for $\Gamma_{\mathcal{N},\rho}$. (This also relates to the "Final step" on p. 1106 of Loertscher and Wasser (2019) as it relates to Property 2.)

OA.6.1 Preliminaries

Define the distributions of the weighted virtual types as follows: for $J \in \{S, B\}$,

$$G^J_{i,\rho}(y) \equiv \begin{cases} 0 & \text{if } y < \Psi^J_{i,\frac{w_i}{\rho}}(0), \\ F_i((\Psi^J_{i,\frac{w_i}{\rho}})^{-1}(y)) & \text{if } y \in [\Psi^J_{i,\frac{w_i}{\rho}}(0), \Psi^J_{i,\frac{w_i}{\rho}}(1)], \\ 1 & \text{if } y > \Psi^J_{i,\frac{w_i}{\rho}}(1). \end{cases} \tag{OA.15}$$

The distributions $G^S_{i,\rho}$ and $G^B_{i,\rho}$ are increasing, $G^S_{i,\rho}(z_i) < G^B_{i,\rho}(z_i)$ for all $z_i \in [0,1]$, $G^S_{i,\rho}(0) = 0$, and $G^B_{i,\rho}(1) = 1$. Observe that for every $i \in \mathcal{N}$, $\rho \geq \max \mathbf{w}$, and $y \in \mathbb{R}$, we have $G^S_{i,\rho}(y) \leq F_i(y) \leq G^B_{i,\rho}(y)$.

Suppose that $z_i > z_j$. Then firm i's critical type $\omega_{i,\rho}(z_i)$ expects that its ironed virtual type $\overline{\psi}_{i,\frac{w_i}{\rho}}(\omega_{i,\rho}(z_i), z_i) = z_i$ is greater than the ironed virtual type $\overline{\psi}_{j,\frac{w_i}{\rho}}(\theta_j, z_j)$ of firm j with probability $G^B_{j,\rho}(z_i)$. And if $z_i < z_j$, then firm i's critical type $\omega_{i,\rho}(z_i)$ expects that its ironed virtual type $\overline{\psi}_{i,\frac{w_i}{\rho}}(\omega_{i,\rho}(z_i), z_i) = z_i$ is greater than the ironed virtual type $\overline{\psi}_{j,\frac{w_i}{\rho}}(\theta_j, z_j)$ of firm j with probability $G^S_{j,\rho}(z_i)$.

As a point of reference, we start by writing out the expression for a firm's interim expected allocation under the efficient allocation rule. To do this, denote by $P(\mathcal{X})$ the set of all subsets of $\mathcal{X}$, including the empty set, and define the function $\varphi_i : P(\mathcal{N}\setminus\{i\}) \to [0, k_i]$ that maps subsets $\mathcal{A}$ of firms in $\mathcal{N}$ other than firm i to the allocation of firm i when the supply of 1 is allocated to the firms in $\mathcal{A}$ up to their maximum capacities before any remaining supply is allocated to firm i:

$$\varphi_i(\mathcal{A}) \equiv \min\{k_i, \max\{0, 1 - \sum_{j\in\mathcal{A}} k_j\}\}.$$

Accordingly, firm i's interim expected allocation under the efficient allocation rule when its type is θ_i is

$$q_i^e(\theta_i) = \sum_{\mathcal{A}\in P(\mathcal{N}\backslash\{i\})} \Big(\prod_{j\in\mathcal{A}}(1 - F_j(\theta_i))\Big)\Big(\prod_{j\in\mathcal{N}\backslash(\mathcal{A}\cup\{i\})} F_j(\theta_i)\Big)\varphi_i(\mathcal{A}),$$

where we follow the convention that for all $x \neq 0$, $\prod_{j\in\varnothing} x = 1$ and $\prod_{j\in\varnothing} 0 = 0$. As reflected in this expression, under the efficient allocation rule and given θ_i, the probability that firm i's allocation is $\varphi(\mathcal{A})$ is the probability that the firms in $\mathcal{A}$ have types greater than θ_i and that the other firms have types less than θ_i.

Now depart from ex post efficiency and consider firm i and a vector of ironing parameters $\mathbf{z}$. Let the set of firms other than i that have an ironing parameter less than z_i be denoted, respectively, by

$$\mathcal{L}_i(\mathbf{z}) \equiv \{j \mid j \neq i \text{ and } z_j < z_i\}.$$

Similarly, let the sets of firms with ironing parameter equal to and greater than z_i be denoted by

$$\mathcal{E}_i(\mathbf{z}) \equiv \{j \mid j \neq i \text{ and } z_j = z_i\}$$

and

$$\mathcal{G}_i(\mathbf{z}) \equiv \{j \mid j \neq i \text{ and } z_j > z_i\}.$$

If $\mathcal{E}_i(\mathbf{z}) \neq \varnothing$ for some i, then ties in terms of ironed, weighted virtual types have positive probability. Suppose that ties are broken hierarchically according to h. For each firm i, let

$$\underline{\mathcal{E}}_i(\mathbf{z}, h) \equiv \{j \in \mathcal{E}_i(\mathbf{z}) \mid h(j) < h(i)\}$$

and

$$\overline{\mathcal{E}}_i(\mathbf{z}, h) \equiv \{j \in \mathcal{E}_i(\mathbf{z}) \mid h(j) > h(i)\}.$$

In what follows, in an attempt to balance clarity and the compactness of the mathematical expressions, given $\mathcal{X}$ and $\mathcal{Y}$ that are disjoint subsets of $\mathcal{N}$, we use $\prod_{\substack{j\in\mathcal{X}\\ j\in\mathcal{Y}}} x_j$ to mean $\prod_{j\in\mathcal{X}} x_j \prod_{j\in\mathcal{Y}}$. In the analog to the displayed equation on p. 1097 of Loertscher and Wasser (2019), we have:[10]

[10]For comparison, in a partnership setup, $k_1 = \cdots = k_n = 1$, so $\varphi_i(\mathcal{A}) = 0$ whenever $\mathcal{A}$ is nonempty and $\varphi_i(\varnothing) = 1$. Thus, the expression in Lemma OA.3 becomes

$$p_{i,\rho}(\mathbf{z}, h) = \prod_{\substack{\ell\in\mathcal{L}_i(\mathbf{z})\\ \ell\in\underline{\mathcal{E}}_i(\mathbf{z},h)}} G^B_{\ell,\rho}(z_i) \prod_{\substack{\ell\in\mathcal{G}_i(\mathbf{z})\\ \ell\in\overline{\mathcal{E}}_i(\mathbf{z},h)}} G^S_{\ell,\rho}(z_i).$$

Lemma OA.3. *Under hierarchy h, the interim expected allocation of critical type $\omega_{i,\rho}(z_i)$ of firm i, $q_{i,\rho}(\omega_{i,\rho}(z_i))$, is given by*

$$p_{i,\rho}(\mathbf{z},h) \equiv \sum_{\mathcal{A}\in P(\mathcal{N}\setminus\{i\})} \left(\prod_{\substack{j\in\mathcal{A}\cap\mathcal{L}_i(\mathbf{z}) \\ j\in\mathcal{A}\cap\underline{\mathcal{E}}_i(\mathbf{z},h)}} (1-G^B_{j,\rho}(z_i)) \prod_{\substack{j\in(\mathcal{N}\setminus\mathcal{A})\cap\mathcal{L}_i(\mathbf{z}) \\ j\in(\mathcal{N}\setminus\mathcal{A})\cap\underline{\mathcal{E}}_i(\mathbf{z},h)}} G^B_{j,\rho}(z_i) \prod_{\substack{j\in\mathcal{A}\cap\mathcal{G}_i(\mathbf{z}) \\ j\in\mathcal{A}\cap\overline{\mathcal{E}}_i(\mathbf{z},h)}} (1-G^S_{j,\rho}(z_i)) \prod_{\substack{j\in(\mathcal{N}\setminus\mathcal{A})\cap\mathcal{G}_i(\mathbf{z}) \\ j\in(\mathcal{N}\setminus\mathcal{A})\cap\overline{\mathcal{E}}_i(\mathbf{z},h)}} G^S_{j,\rho}(z_i)\varphi_i(\mathcal{A}) \right).$$

Proof. View $\mathcal{A}$ as the set of firms that trade ahead of firm i. We sum over all such sets $\mathcal{A}$. The probability of a given set $\mathcal{A}$ is the probability that: (i) each firm $j\in\mathcal{A}$ either has an ironed weighted virtual type that is greater than z_i or has an ironed weighted virtual type that is equal to z_i and has $h(j)>h(i)$; and (ii) each firm $j\in\mathcal{N}\setminus(\mathcal{A}\cup\{i\})$ either has an ironed weighted virtual type that is less than z_i or has an ironed weighted virtual type that is equal to z_i and has $h(j)<h(i)$. We can divide firms in $\mathcal{A}$ in to four groups according to whether the firms are also in $\mathcal{L}_i(\mathbf{z})$, $\mathcal{G}_i(\mathbf{z})$, $\underline{\mathcal{E}}_i(\mathbf{z})$, or $\overline{\mathcal{E}}_i(\mathbf{z})$. For any firm $j\in\mathcal{A}\cap\mathcal{L}_i(\mathbf{z})$, the probability that firm j's ironed weighted virtual type is greater than z_i is $1-G^B_{j,\rho}(z_i)$, and the probability that it is equal to z_i is zero. For any firm $j\in\mathcal{A}\cap\mathcal{G}_i(\mathbf{z})$, the probability that firm j's ironed weighted virtual type is greater than z_i is $1-G^S_{j,\rho}(z_i)$, and the probability that it is equal to z_i is zero. For any firm $j\in\mathcal{A}\cap\underline{\mathcal{E}}_i(\mathbf{z})$, the probability that firm j's ironed weighted virtual type is greater than z_i is $1-G^B_{j,\rho}(z_i)$, and the probability that it is equal to z_i and has $h(j)>h(i)$ is zero. Finally, for any firm $j\in\mathcal{A}\cap\overline{\mathcal{E}}_i(\mathbf{z})$, the probability that firm j's ironed weighted virtual type is greater than z_i is $1-G^B_{j,\rho}(z_i)$, and the probability that it is equal to z_i and has $h(j)>h(i)$ is $G^B_{j,\rho}(z_i)-G^S_{j,\rho}(z_i)$, implying that the probability that firm i is served before j is $1-G^B_{j,\rho}(z_i)+G^B_{j,\rho}(z_i)-G^S_{j,\rho}(z_i)=1-G^S_{j,\rho}(z_i)$.

Turning to the firms in $\mathcal{N}\setminus(\mathcal{A}\cup\{i\})$, we can also divide those firms according to whether they are also in $\mathcal{L}_i(\mathbf{z})$, $\mathcal{G}_i(\mathbf{z})$, $\underline{\mathcal{E}}_i(\mathbf{z})$, or $\overline{\mathcal{E}}_i(\mathbf{z})$. For any firm $j\in\mathcal{N}\setminus(\mathcal{A}\cup\{i\})\cap\mathcal{L}_i(\mathbf{z})=(\mathcal{N}\setminus\mathcal{A})\cap\mathcal{L}_i(\mathbf{z})$ the probability that firm j's ironed weighted virtual type is less than z_i is $G^B_{j,\rho}(z_i)$, and the probability that it is equal to z_i is zero. For any firm $j\in\mathcal{N}\setminus(\mathcal{A}\cup\{i\})\cap\mathcal{G}_i(\mathbf{z})=(\mathcal{N}\setminus\mathcal{A})\cap\mathcal{G}_i(\mathbf{z})$, the probability that firm j's ironed weighted virtual type is less than z_i is $G^S_{j,\rho}(z_i)$, and the probability that it is equal to z_i is zero. For any firm $j\in\mathcal{N}\setminus(\mathcal{A}\cup\{i\})\cap\underline{\mathcal{E}}_i(\mathbf{z})=(\mathcal{N}\setminus\mathcal{A})\cap\underline{\mathcal{E}}_i(\mathbf{z})$, the probability that firm j's ironed weighted virtual type is less than z_i is $G^S_{j,\rho}(z_i)$, and the probability that it is equal to z_i and has $h(j)<h(i)$ is $G^B_{j,\rho}(z_i)-G^S_{j,\rho}(z_i)$, yielding $G^B_{j,\rho}(z_i)$ as the probability that i is served ahead of j. Finally, for any firm $j\in\mathcal{N}\setminus(\mathcal{A}\cup\{i\})\cap\overline{\mathcal{E}}_i(\mathbf{z})=(\mathcal{N}\setminus\mathcal{A})\cap\overline{\mathcal{E}}_i(\mathbf{z})$, the probability that

This has a useful recursive structure: if firms $i\in\mathcal{K}$ have $z_i>z$ and firms $j\in\mathcal{N}\setminus\mathcal{K}$ have $z_j=z$, then all firms in $\mathcal{N}\setminus\mathcal{K}$ are in $\mathcal{L}_i(\mathbf{z})$, so we have

$$p_{i,\rho}(\mathbf{z},h) = \prod_{\substack{\ell\in\mathcal{K}\cap\mathcal{L}_i(\mathbf{z}) \\ \ell\in\mathcal{K}\cap\underline{\mathcal{E}}_i(\mathbf{z},h) \\ \ell\in(\mathcal{N}\setminus\mathcal{K})\cap\mathcal{L}_i(\mathbf{z})}} G^B_{\ell,\rho}(z_i) \prod_{\substack{\ell\in\mathcal{K}\cap\mathcal{G}_i(\mathbf{z}) \\ \ell\in\mathcal{K}\cap\overline{\mathcal{E}}_i(\mathbf{z},h)}} G^S_{\ell,\rho}(z_i) = p^{\mathcal{K}}_{i,\rho}(\mathbf{z},h) \prod_{\ell\in(\mathcal{N}\setminus\mathcal{K})\cap\ \mathcal{L}_i(\mathbf{z})} G^B_{\ell,\rho}(z_i).$$

firm j's ironed weighted virtual type is less than z_i is $G^S_{j,\rho}(z_i)$, and the probability that it is equal to z_i and has $h(j) < h(i)$ is zero.

Given the set of firms $\mathcal{A}$ that trade ahead of firm i, firm i's allocation is $\varphi_i(\mathcal{A}) = \min\{k_i, \max\{0, 1 - \sum_{j\in\mathcal{A}} k_j\}\}$, which completes the proof. ■

Using $p_{i,\rho}(\mathbf{z}, h)$ as defined in Lemma OA.3, define $\mathbf{p}_\rho(\mathbf{z}, h) \equiv (p_{1,\rho}(\mathbf{z}, h), \ldots, p_{n,\rho}(\mathbf{z}, h))$. The outcome $(q^{\mathbf{z},\mathbf{a}}_{1,\rho}(\omega_{1,\rho}(z_1)), \ldots, q^{\mathbf{z},\mathbf{a}}_{n,\rho}(\omega_{n,\rho}(z_n)))$ of every split hierarchical tie-breaking rule $\mathbf{a}$ is equal to a convex combination of $\mathbf{p}_\rho(\mathbf{z}, h)$ for different hierarchies $h \in H$. Consequently, the set of all possible expected allocation vectors given $\mathbf{z}$ is equal to the convex hull of the expected allocations under fixed hierarchies:

$$\Gamma_{\mathcal{N},\rho}(\mathbf{z}) = \text{Conv}(\{\mathbf{p}_\rho(\mathbf{z}, h) \mid h \in H\}).$$

Depending on $\mathbf{z}$, we may have $\mathbf{p}_\rho(\mathbf{z}, h_1) = \mathbf{p}_\rho(\mathbf{z}, h_2)$ for some $h_1 \neq h_2$. In particular, if all n elements of $\mathbf{z}$ are distinct, i.e., $\mathcal{E}_i(z) = \varnothing$ for all $i \in \mathcal{N}$, then tie-breaking has no bite and all $\mathbf{p}_\rho(\mathbf{z}, h)$ coincide. In this case, $\Gamma_{\mathcal{N},\rho}(\mathbf{z})$ is a singleton. By contrast, if $\mathbf{z}$ is such that $z_i = z$ for all $i \in \mathcal{N}$, i.e., $\mathcal{L}_i(\mathbf{z}) = \mathcal{G}_i(\mathbf{z}) = \varnothing$, then all $n!$ points $\mathbf{p}_\rho(\mathbf{z}, h)$ are distinct extreme points of the convex hull $\Gamma_{\mathcal{N},\rho}(\mathbf{z})$. In general, if $\mathbf{z}$ is such that its elements take $\ell \le n$ distinct values $z^1, \ldots, z^\ell$, then $\Gamma_{\mathcal{N},\rho}(\mathbf{z})$ is equal to the convex hull of $\prod_{j=1}^{\ell} t_j!$ distinct extreme points, where t_j denotes the number of firms i with $z_i = z^j$.

Lemma OA.4. *The correspondence $\Gamma_{\mathcal{N},\rho}$ has the following properties:*

(i) for all $\mathbf{z} \in [0, 1]^n$, $\Gamma_{\mathcal{N},\rho}(\mathbf{z})$ is nonempty and convex;

(ii) the correspondence $\Gamma_{\mathcal{N},\rho}$ is upper hemicontinuous.

Proof. The proof follows from the same argument as in the proof of Lemma 2 in Loertscher and Wasser (2019). Part (i) immediately follows from the discussion above. For part (ii), we have to show that for any two sequences $\mathbf{z}^s \to \mathbf{z}$ and $\mathbf{y}^s \to \mathbf{y}$ such that $\mathbf{y}^s \in \Gamma_{\mathcal{N},\rho}(\mathbf{z}^s)$, we have $\mathbf{y} \in \Gamma_{\mathcal{N},\rho}(\mathbf{z})$. If $\mathbf{z}$ is such that all its components are distinct, then $\Gamma_{\mathcal{N},\rho}(\mathbf{z})$ is a singleton that is continuous at $\mathbf{z}$. Moreover, if the sequence $\mathbf{z}^s \to \mathbf{z}$ is such that the sets of firms for which ironing parameters coincide stay the same over the whole sequence, then $\Gamma_{\mathcal{N},\rho}(\mathbf{z}^s)$ and $\Gamma_{\mathcal{N},\rho}(\mathbf{z})$ are all equal to the convex hull of the same number of extreme points. Because these extreme points are continuous in $\mathbf{z}^s$, $\mathbf{y}^s \in \Gamma_{\mathcal{N},\rho}(\mathbf{z}^s)$ and $\mathbf{y}^s \to \mathbf{y}$ imply that $\mathbf{y} \in \Gamma_{\mathcal{N},\rho}(\mathbf{z})$ in this case. Finally, suppose that there are some i and j for which $z^s_i > z^s_j$ but $z_i = z_j$. Then, if $\mathbf{y}^s \to \mathbf{y}$ such that $\mathbf{y}^s \in \Gamma_{\mathcal{N},\rho}(\mathbf{z}^s)$, there exists a hierarchical tie-breaking rule for $\mathbf{z}$ where $h(i) > h(j)$ for all i and j with $z^s_i > z^s_j$ and $z_i = z_j$ that induces $\mathbf{y}$. Hence $\mathbf{y} \in \Gamma_{\mathcal{N},\rho}(\mathbf{z})$. ■

So as to study properties of the image of $\Gamma_{\mathcal{N},\rho}$, it proves useful to consider the following partition of the domain $[0,1]^n$. Define

$$\xi_{\mathcal{N}}(z) \equiv \{\mathbf{z} \in [z,1]^n \mid z_i = z \text{ for at least one } i \in \mathcal{N}\}.$$

Note that $\xi_{\mathcal{N}}(z) \cap \xi_{\mathcal{N}}(z') = \varnothing$ for all $z \neq z'$. Moreover, $\cup_{z\in[0,1]}\xi_{\mathcal{N}}(z) = [0,1]^n$. Thus, $\xi_{\mathcal{N}}$ represents a partition of the domain of $\Gamma_{\mathcal{N},\rho}$. In addition, define

$$O_{\mathcal{N},\rho}(z) \equiv \Gamma_{\mathcal{N},\rho}(\xi_{\mathcal{N}}(z)).$$

Hence, we have the following characterization of the image of $\Gamma_{\mathcal{N},\rho}$:

Lemma OA.5. *The image of* $\Gamma_{\mathcal{N},\rho}$ *can be written as*

$$\Gamma_{\mathcal{N},\rho}([0,1]^n) = \cup_{z\in[0,1]} O_{\mathcal{N},\rho}(z).$$

Below, we determine properties of $O_{\mathcal{N},\rho}(z)$ and their implications for $\Gamma_{\mathcal{N},\rho}([0,1]^n)$.

OA.6.2 Properties 1 and 2 hold for $n = 2$

Suppose that $n = 2$. There are only two possible hierarchies between two firms, i.e., $H = \{h_1, h_2\}$. Let h_1 be the hierarchy where firm 1 wins ties and h_2 be the hierarchy where firm 2 wins ties. The expected allocation under h_1 when types are $(\omega_{1,\rho}(z_1), \omega_{2,\rho}(z_2))$ is

$$\hat{\boldsymbol{\zeta}}_{1,\rho}(z_1, z_2) \equiv \Big(G^B_{2,\rho}(z_1)k_1 + (1 - G^B_{2,\rho}(z_1))(1 - k_2),\ G^S_{1,\rho}(z_2)k_2 + (1 - G^S_{1,\rho}(z_2))(1 - k_1)\Big)$$

and the expected allocation under h_2 when types are $(\omega_{1,\rho}(z_1), \omega_{2,\rho}(z_2))$ is

$$\hat{\boldsymbol{\zeta}}_{2,\rho}(z_1, z_2) \equiv \Big(G^S_{2,\rho}(z_1)k_1 + (1 - G^S_{2,\rho}(z_1))(1 - k_2),\ G^B_{1,\alpha}(z_2)k_2 + (1 - G^B_{1,\alpha}(z_2))(1 - k_1)\Big).$$

(Here we use our maintained assumptions that for all i, $k_i \leq 1$ and that $1 < \sum_{i\in\mathcal{N}} k_i$.) Further, define for $i \in \{1, 2\}$,

$$\boldsymbol{\zeta}_{i,\rho}(z) \equiv \hat{\boldsymbol{\zeta}}_{i,\rho}(z, z).$$

Hence, $\mathbf{p}_\rho(z, z, h_i) = \boldsymbol{\zeta}_{i,\rho}(z)$ for $i \in \{1, 2\}$. The description of $\Gamma_{\mathcal{N},\rho}$ above, including Lemma OA.4 implies that

$$\Gamma_{\{1,2\},\rho}(z_1, z_2) = \begin{cases} \hat{\boldsymbol{\zeta}}_{1,\rho}(z_1, z_2) & \text{if } z_1 > z_2, \\ \text{Conv}(\{\boldsymbol{\zeta}_{1,\alpha}(z), \boldsymbol{\zeta}_{2,\alpha}(z)\}) & \text{if } z_1 = z_2 = z, \\ \hat{\boldsymbol{\zeta}}_{2,\rho}(z_1, z_2) & \text{if } z_1 < z_2. \end{cases}$$

Suppose that $z_1 = z_2 = z$. Geometrically, $\Gamma_{\{1,2\},\rho}(z, z)$ is equal to all the points on the line segment from $\boldsymbol{\zeta}_{1,\rho}(z)$ to $\boldsymbol{\zeta}_{2,\rho}(z)$, i.e., all points in $\{a\boldsymbol{\zeta}_{1,\rho}(z) + (1 - a)\boldsymbol{\zeta}_{2,\rho}(z) \mid a \in [0, 1]\}$, where a is the probability that firm i wins ties (i.e., according to hierarchy h_i).

Now consider $O_{\{1,2\},\rho}(z) = \Gamma_{\{1,2\},\rho}(\xi_2(z))$ for some $z \in (0, 1)$. In Figure OA.5, $O_{\{1,2\},\rho}(z)$ is represented by a polygonal chain. Geometrically, $O_{\{1,2\},\rho}(z)$ consists of the line segment

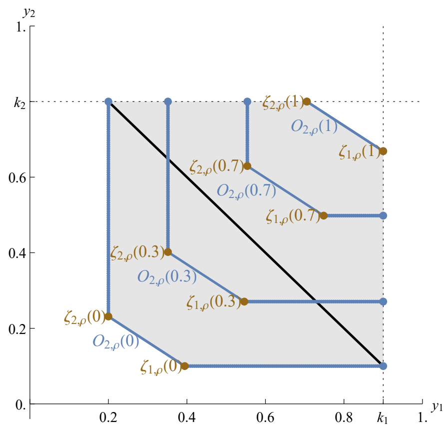

Figure OA.5: Illustration of the image of $\Gamma_{\{1,2\},\rho}$, which is $\Gamma_{\{1,2\},\rho}([0,1]^2) = \cup_{z\in[0,1]} O_{\{1,2\},\rho}(z)$ for $\rho = 1.3$. The figure shows $O_{\{1,2\},\rho}(z)$ for $z \in \{0, 0.3, 0.7, 1\}$, for a market with $\mathbf{k} = (0.9, 0.8)$ and $\mathbf{w} = (1, 0.8)$. The set $\Gamma_{\{1,2\},\rho}([0,1]^2)$ is the shaded polygon. The black diagonal line segment is $\Delta^1_{\mathbf{k}}$. Assumes that types are uniformly distributed.

from $\boldsymbol{\zeta}_{2,\rho}(z)$ to $\boldsymbol{\zeta}_{1,\rho}(z)$ that represents $\Gamma_{\{1,2\},\rho}(z,z)$ with two line segments attached to its endpoints: a vertical line segment from $\boldsymbol{\zeta}_{2,\rho}(z)$ to $(G^S_{2,\rho}(z)k_1 + (1 - G^S_{2,\rho}(z))(1-k_2),\ k_2)$ that represents $\Gamma_{\{1,2\},\rho}(z, z_2)$ for all $z_2 \in (z, 1]$ and a horizontal line segment from $\boldsymbol{\zeta}_{1,\rho}(z)$ to $(k_1,\ G^S_{1,\rho}(z)k_2 + (1 - G^S_{1,\rho}(z))(1-k_1))$ that represents $\Gamma_{\{1,2\},\rho}(z_1, z)$ for all $z_1 \in (z, 1]$.

Observe that both coordinates of the vertices $\boldsymbol{\zeta}_{1,\rho}(z)$ and $\boldsymbol{\zeta}_{2,\rho}(z)$ are continuous and increasing in z. Hence, for $z' > z$, $O_{\{1,2\},\rho}(z') \cap O_{\{1,2\},\rho}(z) = \varnothing$ and $O_{\{1,2\},\rho}(z')$ is further away from the origin than $O_{\{1,2\},\rho}(z)$. Put differently, $O_{\{1,2\},\rho}$ has the following monotonicity property: if $z' > z$, then for all $\mathbf{y}' \in O_{\{1,2\},\rho}(z')$ and $\mathbf{y} \in O_{\{1,2\},\rho}(z)$, we have $y'_i = y_i$ for at least one $i \in \{1, 2\}$. Hence, for every $\mathbf{y} \in \Gamma_{\{1,2\},\rho}([0,1]^2)$, there is a unique z such that $\mathbf{y} \in O_{\{1,2\},\rho}(z)$. Moreover, for each $\mathbf{y} \in O_{\{1,2\},\rho}(z)$, there is a unique point $(z_1, z_2) \in \xi_{\{1,2\}}(z)$ such that $\mathbf{y} \in \Gamma_{\{1,2\},\rho}(z_1, z_2)$. Consequently, for every $\mathbf{y} \in \Gamma_{\{1,2\},\rho}([0,1]^2)$, there is a unique $\mathbf{z} \in [0,1]^2$ such that $\mathbf{y} \in \Gamma_{\{1,2\},\rho}(\mathbf{z})$, that is:

Lemma OA.6. *Property* 1 *holds for* $\Gamma_{\{1,2\},\rho}$.

Turning to Property 2, consider $O_{\{1,2\},\rho}(0)$ and note that

$$\boldsymbol{\zeta}_{1,\rho}(0) = \big(G^B_{2,\rho}(0)k_1 + (1 - G^B_{2,\rho}(0))(1-k_2),\ 1-k_1\big)$$

and

$$\boldsymbol{\zeta}_{2,\rho}(0) = \big(1-k_2,\ G^B_{1,\rho}(0)k_2 + (1 - G^B_{1,\rho}(0))(1-k_1)\big).$$

Hence, the points $\mathbf{y} \in \Gamma_{\{1,2\},\rho}(0,0)$ all lie below $\Delta^1_{\mathbf{k}}$, which in Figure OA.5 is represented by the black line segment from $(1-k_2, k_2)$ to $(k_1, 1-k_1)$. Moreover, the vertical and horizontal parts of $O_{\{1,2\},\rho}(0)$ intersect with $\Delta^1_{\mathbf{k}}$ exactly at its boundary because the endpoint of the vertical segment of $O_{\{1,2\},\rho}(0)$ is

$$\big(G^S_{2,\rho}(0)k_1 + (1 - G^S_{2,\rho}(0))(1 - k_2),\ k_2\big) = (1 - k_2,\ k_2),$$

and the endpoint of the horizontal segment of $O_{\{1,2\},\rho}(0)$ is

$$\big(k_1,\ G^S_{1,\rho}(z)k_2 + (1 - G^S_{1,\rho}(z))(1 - k_1)\big) = (k_1,\ 1 - k_1).$$

Let us increase z. For z small enough, the line segment $\Gamma_{\{1,2\},\rho}(z,z)$ still lies below $\Delta^1_{\mathbf{k}}$ such that the vertical and horizontal parts of $O_{\{1,2\},\rho}(z)$ intersect with $\Delta^1_{\mathbf{k}}$ because the endpoints $(G^S_{2,\rho}(z)k_1 + (1 - G^S_{2,\rho}(z))(1 - k_2),\ k_2)$ and $(k_1,\ G^S_{1,\rho}(z)k_2 + (1 - G^S_{1,\rho}(z))(1 - k_1))$ of $O_{\{1,2\},\rho}(z)$ are above and to the right of $\Delta^1_{\mathbf{k}}$ for all $z > 0$. As z increases, the two intersection points move inward on $\Delta^1_{\mathbf{k}}$. As z becomes large enough, one of the two vertices $\boldsymbol{\zeta}_{1,\rho}$ and $\boldsymbol{\zeta}_{2,\rho}$ crosses $\Delta^1_{\mathbf{k}}$ such that one intersection point lies in $\Gamma_{\{1,2\},\rho}(z,z)$, and the two intersection points approach each other until they coincide when the second vertex also crosses $\Delta^1_{\mathbf{k}}$.[11] Finally, for z sufficiently close to 1, both $\boldsymbol{\zeta}_{1,\rho}(z)$ and $\boldsymbol{\zeta}_{2,\rho}(z)$, and therefore the entire polygonal chain $O_{\{1,2\},\rho}(z)$, lie above $\Delta^1_{\mathbf{k}}$. To see this, note that

$$\boldsymbol{\zeta}_{1,\rho}(1) = \big(k_1,\ G^S_{1,\rho}(1)k_2 + (1 - G^S_{1,\rho}(1))(1 - k_1)\big)$$

and

$$\boldsymbol{\zeta}_{2,\rho}(1) = \big(G^S_{2,\rho}(1)k_1 + (1 - G^S_{2,\rho}(1))(1 - k_2),\ k_2\big),$$

whose components sum to more than 1.[12]

We have just shown that for every $\mathbf{y} \in \Delta^1_{\mathbf{k}}$, there is a z such that $\mathbf{y} \in O_{2,\rho}(z)$. Thus, $\Delta^1_{\mathbf{k}} \subset \Gamma_{\{1,2\},\rho}([0,1]^2) = \cup_{z\in[0,1]} O_{\{1,2\},\rho}(z)$, that is

Lemma OA.7. *Property 2 holds for* $\Gamma_{\{1,2\},\rho}$.

In Figure OA.5, $\Gamma_{\{1,2\},\rho}([0,1]^2)$ is the shaded area between $O_{\{1,2\},\rho}(0)$ and $O_{\{1,2\},\rho}(1)$, representing a hexagon.

At this point, we have completed the proof of Theorem OA.1 for the case of $n = 2$. In what follows, we extend the argument to general n.

OA.6.3 Characterizing $O_{\mathcal{N},\rho}$ and $\Gamma_{\mathcal{N},\rho}$ for $n > 2$

We now extend the approach above to $n > 2$. Characterizing $O_{\mathcal{N},\rho}$ and $\Gamma_{\mathcal{N},\rho}$ turns out to be significantly more complex in this case. To handle this complexity, we first uncover the underlying structure of $O_{\mathcal{N},\rho}$. Exploiting this structure, we show that Property 1 and

[11] In the special case in which the slope of $\Gamma_{2,\rho}(z,z)$ is -1 at the value of z where it crosses $\Delta^1_{\mathbf{k}}$, then both vertices cross simultaneously and all of $\Gamma_{2,\rho}(z,z)$ intersects $\Delta^1_{\mathbf{k}}$ for that value of z.

[12] To see this, note that the some of the components of $\boldsymbol{\zeta}_{1,\rho}(1)$ is $1 - G^S_{1,\rho}(1)(1 - k_1 - k_2) > 1$, where the inequality uses the assumption of excess demand, $1 < k_1 + k_2$, and similarly for $\boldsymbol{\zeta}_{2,\rho}(1)$.

Property 2 hold for $\mathcal{N}$ if they hold for all $\mathcal{K} \subset \mathcal{N}$. Using the two-firm results as the base case, the two properties then hold by induction for all $\mathcal{N}$.

Suppose that $z_1 = z_2 = \cdots = z_n = z$ and consider $\Gamma_{\mathcal{N},\rho}(z, \ldots, z) =$Conv$(\{\mathbf{p}_\rho(z, \ldots, z, h) \mid h \in H\})$. For each of the $n!$ different hierarchies $h \in H$,

$$p_{i,\rho}(z,\ldots,z,h) = \sum_{\mathcal{A}\in P(\mathcal{N}\setminus\{i\})} \prod_{j\in\mathcal{A}\cap\underline{\mathcal{E}}_i(h)} (1-G^B_{j,\rho}(z)) \prod_{j\in(\mathcal{N}\setminus\mathcal{A})\cap\underline{\mathcal{E}}_i(h)} G^B_{j,\rho}(z) \prod_{j\in\mathcal{A}\cap\overline{\mathcal{E}}_i(h)} (1-G^S_{j,\rho}(z)) \prod_{j\in(\mathcal{N}\setminus\mathcal{A})\cap\overline{\mathcal{E}}_i(h)} G^S_{j,\rho}(z)\varphi_i(\mathcal{A}).$$

where we have simplified the notation by writing $\underline{\mathcal{E}}_i(h)$ instead of $\underline{\mathcal{E}}_i(z, \ldots, z, h)$ and $\overline{\mathcal{E}}_i(h)$ instead of $\overline{\mathcal{E}}_i(z, \ldots, z, h)$. If each $z > 0$, then each $h \in H$ yields a distinct $\mathbf{p}_\rho(z, \ldots, z, h)$. We show that all points $\mathbf{p}_\rho(z, \ldots, z, h)$ lie in the same $(n-1)$-dimensional hyperplane. For example, for $n = 2$, we have

$$\begin{aligned}
&\sum_{i\in\{1,2\}} \left(G^B_{i,\rho}(z) - G^S_{i,\rho}(z)\right) p_{i,\rho}(z,z,h) \\
=\ &\left(\prod_{i\in\mathcal{N}} G^B_{i,\rho}(z) - \prod_{i\in\mathcal{N}} G^S_{i,\rho}(z)\right)(k_1+k_2-1) + \sum_{i\in\mathcal{N}} \left(G^B_{i,\rho}(z) - G^S_{i,\rho}(z)\right)\varphi_i(\mathcal{N}\setminus\{i\}),
\end{aligned}$$

which is independent of h.

Lemma OA.8. *The points in the set* $\{\mathbf{p}_\rho(z, \ldots, z, h) \mid h \in H\}$ *all lie in the same* $(n-1)$-*dimensional hyperplane.*

Proof. Consider tie-breaking hierarchy h_1 with $h_1(1) > h_1(2) > \cdots > h_1(n)$ and tie-breaking hierarchy $\hat{h}_1$ that is the same as h_1 except that it switches the order of the ℓ-th and $(\ell+1)$-st firms for some $\ell \in \{1, \ldots, n-1\}$. We show that

$$\sum_{i\in\mathcal{N}} \left(G^B_{i,\rho}(z) - G^S_{i,\rho}(z)\right) p_{i,\rho}(z,\ldots,z,h_1) = \sum_{i\in\mathcal{N}} \left(G^B_{i,\rho}(z) - G^S_{i,\rho}(z)\right) p_{i,\rho}(z,\ldots,z,\hat{h}_1). \quad \text{(OA.16)}$$

Analogous arguments then imply that for any two tie-breaking hierarchies h and h' that differ only in that the order for two adjacent firms in h is switched in h', we have

$$\sum_{i\in\mathcal{N}} \left(G^B_{i,\rho}(z) - G^S_{i,\rho}(z)\right) p_{i,\rho}(z,\ldots,z,h) = \sum_{i\in\mathcal{N}} \left(G^B_{i,\rho}(z) - G^S_{i,\rho}(z)\right) p_{i,\rho}(z,\ldots,z,h'),$$

which then implies that $\sum_{i\in\mathcal{N}} \left(G^B_{i,\rho}(z) - G^S_{i,\rho}(z)\right) p_{i,\rho}(z, \ldots, z, h)$ is constant for all $h \in H$, implying that all points $\mathbf{p}_\rho(z, \ldots, z, h)$ lie in the same $(n-1)$-dimensional hyperplane, which is the desired result.

In what follows, we show that (OA.16) holds. Because h_1 and $\hat{h}_1$ only differ in the switching of the positions of firms ℓ and $\ell + 1$ in the hierarchy, for all other firms $i \in \mathcal{N}\setminus\{\ell, \ell + 1\}$, we have $p_{i,\rho}(z, \ldots, z, h_1) = p_{i,\rho}(z, \ldots, z, \hat{h}_1)$, and so in order to show that

(OA.16) holds, it is sufficient to show that

$$\sum_{i\in\{\ell,\ell+1\}} \left(G^B_{i,\rho}(z) - G^S_{i,\rho}(z)\right) p_{i,\rho}(z,\ldots,z,h_1) - \sum_{i\in\{\ell,\ell+1\}} \left(G^B_{i,\rho}(z) - G^S_{i,\rho}(z)\right) p_{i,\rho}(z,\ldots,z,\hat{h}_1) = 0. \quad \text{(OA.17)}$$

Letting

$$\tilde{\mu}(\mathcal{A}) \equiv \prod_{j\in\mathcal{A}\cap\{\ell+2,\ldots,n\}} (1 - G^B_{j,\rho}(z)) \prod_{j\in(\mathcal{N}\setminus\mathcal{A})\cap\{\ell+2,\ldots,n\}} G^B_{j,\rho}(z) \prod_{j\in\mathcal{A}\cap\{1,\ldots,\ell-1\}} (1 - G^S_{j,\rho}(z)) \prod_{j\in(\mathcal{N}\setminus\mathcal{A})\cap\{1,\ldots,\ell-1\}} G^S_{j,\rho}(z),$$

and using the definition of $p_{i,\rho}(z,\ldots,z,h_1)$, the left side of (OA.17) can be written as

$$\begin{aligned}&\left(G^B_{\ell,\rho}(z) - G^S_{\ell,\rho}(z)\right)\left(G^B_{\ell+1,\rho}(z) - G^S_{\ell+1,\rho}(z)\right) \qquad \text{(OA.18)}\\ &\cdot \sum_{\mathcal{A}\in P(\mathcal{N}\setminus\{\ell,\ell+1\})} \tilde{\mu}(\mathcal{A})\left[\varphi_\ell(\mathcal{A}) + \varphi_{\ell+1}(\mathcal{A}\cup\{\ell\}) - \varphi_\ell(\mathcal{A}\cup\{\ell+1\}) - \varphi_{\ell+1}(\mathcal{A})\right].\end{aligned}$$

To understand expression (OA.18), note that:

(i) $\left(G^B_{\ell,\rho}(z) - G^S_{\ell,\rho}(z)\right)\left(G^B_{\ell+1,\rho}(z) - G^S_{\ell+1,\rho}(z)\right)$ is the probability that the types of firm ℓ and firm $\ell+1$ are both in the ironing region (otherwise, the difference between h_1 and $\hat{h}_1$ has no effect);

(ii) the sum is taken over the possible sets $\mathcal{A}$ of firms other than ℓ and $\ell+1$ that trade ahead of those two firms;

(iii) $\tilde{\mu}(\mathcal{A})$ is the probability that the firms in $\mathcal{A}$ trade ahead of firms ℓ and $\ell+1$, given that the types of firm ℓ and firm $\ell+1$ are both in the ironing region;

(iv) $\varphi_\ell(\mathcal{A}) + \varphi_{\ell+1}(\mathcal{A}\cup\{\ell\})$ is the sum of the quantities allocated to firms ℓ and $\ell+1$ given $\mathcal{A}$ and hierarchy h_1; and

(v) $\varphi_\ell(\mathcal{A}\cup\{\ell+1\}) + \varphi_{\ell+1}(\mathcal{A})$ is the sum of the quantities allocated to firms ℓ and $\ell+1$ given $\mathcal{A}$ and hierarchy $\hat{h}_1$.

For all $\mathcal{A} \in P(\mathcal{N}\setminus\{\ell,\ell+1\})$, and regardless of whether firm ℓ or firm $\ell+1$ comes first in the hierarchy, firms ℓ and $\ell+1$ are allocated the available supply after the firms in $\mathcal{A}$ have been served, $\max\left\{0, 1 - \sum_{j\in\mathcal{A}} k_j\right\}$, up to their maximum demands, giving us the result that

$$\varphi_\ell(\mathcal{A}) + \varphi_{\ell+1}(\mathcal{A}\cup\{\ell\}) = \min\left\{k_\ell + k_{\ell+1}, \max\left\{0, 1 - \sum_{j\in\mathcal{A}} k_j\right\}\right\} = \varphi_\ell(\mathcal{A}\cup\{\ell+1\}) + \varphi_{\ell+1}(\mathcal{A}),$$

which implies that (OA.18) is zero and so (OA.17) holds, which completes the proof. ■

Given Lemma OA.8, $\Gamma_{\mathcal{N},\rho}(z,\ldots,z)$ is an $(n-1)$-dimensional convex polytope (in the hyperplane discussed above) with vertices $\{\mathbf{p}_\rho(z,\ldots,z,h) \mid h \in H\}$. Each vertex is connected to $n-1$ other vertices through an edge.

Now consider a nonempty set of firms $\mathcal{K} \subset \mathcal{N}$. Define the set of hierarchies $H_\mathcal{K} \subset H$ such that for all $h \in H_\mathcal{K}$, we have $h(i) > h(j)$ for all $i \in \mathcal{K}$ and $j \in \mathcal{N}\setminus\mathcal{K}$. If ties are broken based only on hierarchies in $H_{\mathcal{K}}$, then firms in $\mathcal{K}$ always win ties against firms in $\mathcal{N}\backslash\mathcal{K}$. The $(n-2)$-dimensional polytope $\mathrm{Conv}(\{\mathbf{p}_\rho(z,\ldots,z,h) \mid h \in H_{\mathcal{K}}\})$ is a facet (i.e., an $(n-2)$-face) of the $(n-1)$-dimensional polytope $\Gamma_{\mathcal{N},\rho}(z,\ldots,z)$. The boundary of $\Gamma_{\mathcal{N},\rho}(z,\ldots,z)$ consists of 2^n-2 such facets, one for each possible nonempty $\mathcal{K}\subset\mathcal{N}$.

For each $\mathcal{K}\subset\mathcal{N}$, we define the polytope

$$o^{\mathcal{K}}_{\mathcal{N},\rho}(z) \equiv \{\mathbf{y}\in[0,1]^n \mid \mathbf{y}\in\Gamma_{\mathcal{N},\rho}(\mathbf{z}) \text{ for some } \mathbf{z}\in[z,1]^n \text{ with } z_i = z \text{ for all } i\in\mathcal{N}\backslash\mathcal{K}\}.$$

With this definition, $O_{\mathcal{N},\rho}(z) = \cup_{\mathcal{K}\subset\mathcal{N}} o^{\mathcal{K}}_{\mathcal{N},\rho}(z)$. Note that $o^{\varnothing}_{\mathcal{N},\rho}(z) = \Gamma_{\mathcal{N},\rho}(z,\ldots,z)$. Consequently, $O_{\mathcal{N},\rho}(z)$ is a polytopal complex that consists of 2^n-1 polytopes of dimension $(n-1)$: $\Gamma_{\mathcal{N},\rho}(z,\ldots,z)$ with a polytope $o^{\mathcal{K}}_{\mathcal{N},\rho}(z)$ with nonempty $\mathcal{K}\subset\mathcal{N}$ attached to each of its 2^n-2 facets. It then follows that $O_{\mathcal{N},\rho}(z)$ is defined by a set of points as follows:

Lemma OA.9. *For each $\mathcal{K}\subset\mathcal{N}$ (including the empty set), $o^{\mathcal{K}}_{\mathcal{N},\rho}(z)$ is defined by convex combinations of points in the set*

$$\mathcal{P}^{\mathcal{K}}_\rho \equiv \{\mathbf{p}_\rho(\mathbf{z},h) \mid z_i\in[z,1] \textit{ for all } i\in\mathcal{K},\ z_j = z \textit{ for all } i\in\mathcal{N}\backslash\mathcal{K} \textit{ and } h\in H_{\mathcal{K}}\}.$$

As described in Lemma OA.9, $O_{\mathcal{N},\rho}(z)$ is defined by convex combinations of points in $\mathcal{P}^{\mathcal{K}}_\rho$ for $\mathcal{K}\subset\mathcal{N}$. As we now show, analogous to the case of $n=2$, each coordinate of each of these points is continuous and nondecreasing in z, with at least one coordinate of each point increasing in z:

Lemma OA.10. *Each coordinate of each of the points in $\mathcal{P}^{\mathcal{K}}_\rho$ for $\mathcal{K}\subset\mathcal{N}$, i.e., the points that define $O_{\mathcal{N},\rho}(z)$, is continuous and nondecreasing in z, and has at least one coordinate that is increasing in z whenever it is less than its maximum value of k_i for coordinate $i\in\mathcal{N}$.*

Proof. Recall the definition of $p_{i,\rho}(\mathbf{z},h)$ from Lemma OA.3. It will be useful to define for $\mathbf{z}\in[0,1]^n$ and $h\in H$,

$$p_{i,\rho}(\mathbf{z},h) = \sum_{\mathcal{A}\in P(\mathcal{N}\backslash\{i\})} \mu_{i,\rho}(\mathbf{z},\mathcal{A},h)\varphi_i(\mathcal{A}),$$

where

$$\mu_{i,\rho}(\mathbf{z},\mathcal{A},h) \equiv \prod_{\substack{j\in\mathcal{A}\cap\mathcal{L}_i(\mathbf{z})\\ j\in\mathcal{A}\cap\underline{\mathcal{E}}_i(\mathbf{z},h)}} (1-G^B_{j,\rho}(z_i)) \prod_{\substack{j\in(\mathcal{N}\backslash\mathcal{A})\cap\mathcal{L}_i(\mathbf{z})\\ j\in(\mathcal{N}\backslash\mathcal{A})\cap\underline{\mathcal{E}}_i(\mathbf{z},h)}} G^B_{j,\rho}(z_i) \prod_{\substack{j\in\mathcal{A}\cap\mathcal{G}_i(\mathbf{z})\\ j\in\mathcal{A}\cap\overline{\mathcal{E}}_i(\mathbf{z},h)}} (1-G^S_{j,\rho}(z_i)) \prod_{\substack{j\in(\mathcal{N}\backslash\mathcal{A})\cap\mathcal{G}_i(\mathbf{z})\\ j\in(\mathcal{N}\backslash\mathcal{A})\cap\overline{\mathcal{E}}_i(\mathbf{z},h)}} G^S_{j,\rho}(z_i).$$

Note that

$$\mu_{i,\rho}(\mathbf{z},\mathcal{A},h)\in[0,1] \text{ and } \sum_{\mathcal{A}\in P(\mathcal{N}\backslash\{i\})} \mu_{i,\rho}(\mathbf{z},\mathcal{A},h) = 1$$

and that

$$\varphi_i(\varnothing) = k_i \geq \max_{\mathcal{A}\in P(\mathcal{N}\backslash\{i\})} \varphi_i(\mathcal{A}).$$

In what follows, we will use the result that if $p_{i,\rho}(\mathbf{z}, h) < k_i$ and $\mu_i(\mathbf{z}', \varnothing, h) > \mu_i(\mathbf{z}, \varnothing, h)$, then $p_{i,\rho}(\mathbf{z}, h) < p_{i,\rho}(\mathbf{z}', h)$ because in this case, the change from $\mathbf{z}$ to $\mathbf{z}'$ shifts probability weight towards $\varphi_i(\varnothing) = k_i$, which is the maximum value of $\varphi_i(\mathcal{A})$. For the result that for some i, $p_{i,\rho}(\mathbf{z}, h)$ is increasing in z, we focus on cases in which $p_{i,\rho}(\mathbf{z}, h) < k_i$, i.e., cases in which $p_{i,\rho}(\mathbf{z}, h)$ has not yet achieved its maximum value of k_i.

As shown in Lemma OA.9, $O_{\mathcal{N},\rho}(z)$ is defined by points for each $\mathcal{K} \subset \mathcal{N}$ that are of the form

$$\mathbf{p}_\rho(\mathbf{z}, h) \text{ with } z_i \in [z, 1] \text{ for all } i \in \mathcal{K},\ z_j = z \text{ for all } i \in \mathcal{N} \backslash \mathcal{K}, \text{ and } h \in H_{\mathcal{K}}.$$

Recalling the definition of $H_{\mathcal{K}}$, this means that firms with an ironing parameter greater than z are ahead in the tie-breaking hierarchy of firms with an ironing parameter of z. For example, if h_1 is the tie-breaking hierarchy with $h_1(1) > h_1(2) > \cdots > h_1(n)$, then the points are $\mathbf{p}_\rho(\hat{\mathbf{z}}^m, h_1)$, where $\hat{\mathbf{z}}^m = (\hat{z}_1, \ldots, \hat{z}_m, z, \ldots, z)$ has $m \in \{0, \ldots, n-1\}$ components greater than z. Focusing on this tie-breaking hierarchy, we have from Lemma OA.3,

$$p_{i,\rho}(\hat{\mathbf{z}}^m, h_1) = \sum_{\mathcal{A} \in P(\mathcal{N} \backslash \{i\})} \left(\prod_{\substack{j \in \mathcal{A} \cap \mathcal{L}_i(\hat{\mathbf{z}}^m) \\ j \in \mathcal{A} \cap \underline{\mathcal{E}}_i(\hat{\mathbf{z}}^m, h_1)}} (1 - G^B_{j,\rho}(\hat{z}^m_i)) \prod_{\substack{j \in (\mathcal{N} \backslash \mathcal{A}) \cap \mathcal{L}_i(\hat{\mathbf{z}}^m) \\ j \in (\mathcal{N} \backslash \mathcal{A}) \cap \underline{\mathcal{E}}_i(\hat{\mathbf{z}}^m, h_1)}} G^B_{j,\rho}(\hat{z}^m_i) \prod_{\substack{j \in \mathcal{A} \cap \mathcal{G}_i(\hat{\mathbf{z}}^m) \\ j \in \mathcal{A} \cap \overline{\mathcal{E}}_i(\hat{\mathbf{z}}^m, h_1)}} (1 - G^S_{j,\rho}(\hat{z}^m_i)) \prod_{\substack{j \in (\mathcal{N} \backslash \mathcal{A}) \cap \mathcal{G}_i(\hat{\mathbf{z}}^m) \\ j \in (\mathcal{N} \backslash \mathcal{A}) \cap \overline{\mathcal{E}}_i(\hat{\mathbf{z}}^m, h_1)}} G^S_{j,\rho}(\hat{z}^m_i) \varphi_i(\mathcal{A}) \right).$$

Consider the i-th component of $\mathbf{p}_\rho(\hat{\mathbf{z}}^m, h_1)$ for $i \in \{m+1, \ldots, n\}$. Then we have

$$\begin{aligned} p_{i,\rho}(\hat{\mathbf{z}}^m, h_1) \quad = \quad & \sum_{\mathcal{A} \in P(\mathcal{N} \backslash \{i\})} \left(\prod_{j \in \mathcal{A} \cap \{i+1, \ldots, n\}} (1 - G^B_{j,\rho}(z)) \prod_{j \in (\mathcal{N} \backslash \mathcal{A}) \cap \{i+1, \ldots, n\}} G^B_{j,\rho}(z) \prod_{\substack{j \in \mathcal{A} \cap \{1, \ldots, m\} \\ j \in \mathcal{A} \cap \{m+1, \ldots, i-1\}}} (1 - G^S_{j,\rho}(z)) \right. \\ & \left. \cdot \prod_{\substack{j \in (\mathcal{N} \backslash \mathcal{A}) \cap \{1, \ldots, m\} \\ j \in (\mathcal{N} \backslash \mathcal{A}) \cap \{m+1, \ldots, i-1\}}} G^S_{j,\rho}(z) \varphi_i(\mathcal{A}) \right) \end{aligned}$$

and

$$\mu_{i,\rho}(\hat{\mathbf{z}}^m, \varnothing, h_1) = \prod_{j \in \{i+1, \ldots, n\}} G^B_{j,\rho}(z) \prod_{\substack{j \in \{1, \ldots, m\} \\ j \in \{m+1, \ldots, i-1\}}} G^S_{j,\rho}(z),$$

which is increasing in z because $m \in \{0, \ldots, n-1\}$ and $i \in \{1, \ldots, n\}$, so for at least one of the products in the expression above, the set over which the product is taken must be nonempty. As argued above, because $\mu_{i,\rho}(\hat{\mathbf{z}}^m, \varnothing, h_1)$ is increasing in z, it follows that $p_{i,\rho}(\hat{\mathbf{z}}^m, h_1)$ is increasing in z.

In contrast, for the i-th component of $\mathbf{p}_\rho(\hat{\mathbf{z}}^m, h_1)$ with $i \in \{1, \ldots, m\}$, we have

$$\mathcal{L}_i(\hat{\mathbf{z}}^m) = \{j \in \{1, \ldots, m\} \backslash \{i\} \mid \hat{z}^m_j < \hat{z}^m_i\} \cup \{m+1, \ldots, n\},$$

$$\mathcal{G}_i(\hat{\mathbf{z}}^m) = \{j \in \{1, \ldots, m\} \backslash \{i\} \mid \hat{z}^m_j > \hat{z}^m_i\},$$

$$\underline{\mathcal{E}}_i(\hat{\mathbf{z}}^m) = \{j \in \{1, \dots, m\} \backslash \{i\} \mid \hat{z}_j^m = \hat{z}_i^m \text{ and } h_1(j) < h_1(i)\},$$

and

$$\overline{\mathcal{E}}_i(\hat{\mathbf{z}}^m) = \{j \in \{1, \dots, m\} \backslash \{i\} \mid \hat{z}_j^m = \hat{z}_i^m \text{ and } h_1(j) > h_1(i)\}.$$

Because none of these sets varies with z, it follows that $p_{i,\rho}(\hat{\mathbf{z}}^m, h_1)$ does not vary with z.

Analogous results hold for all other tie-breaking hierarchies. Thus, we conclude that each component of each point that defines $O_{\mathcal{N},\rho}(z)$ is nondecreasing in z and at least one component of each point is increasing in z. ■

Using Lemma OA.10, we obtain the induction step with which we can establish that Property 1 holds.

For the three-firm example, Lemma OA.10 implies that $O_{\{1,2,3\},\rho}(z)$ moves towards the observer as we increase z.

Monotonicity of $O_{\mathcal{N},\rho}$ implies that for each $\mathbf{y} \in \Gamma_{\mathcal{N},\rho}([0,1]^n) = \bigcup_{z\in[0,1]} O_{\mathcal{N},\rho}(z)$ there is a unique z such that $\mathbf{y} \in O_{\mathcal{N},\rho}(z)$.

Lemma OA.11. *If Property* 1 *holds for all* $\Gamma_{\mathcal{K},\rho}$ *with* $\mathcal{K} \subset \mathcal{N}$*, then Property* 1 *holds for* $\Gamma_{\mathcal{N},\rho}$.

Proof. Lemma OA.10 implies that for every $\mathbf{y} \in \Gamma_{\mathcal{N},\rho}([0,1]^n)$ there is a unique z such that $\mathbf{y} \in O_{\mathcal{N},\rho}(z)$. We show that for every $\mathbf{y} \in O_{\mathcal{N},\rho}(z)$, there is a unique $\mathcal{K} \subset \mathcal{N}$ such that $\mathbf{y} \in \Gamma_{\mathcal{N},\rho}(\xi_{\mathcal{N}}^{\mathcal{K}}(z))$. Consider $\mathcal{K}, \mathcal{T} \subset \mathcal{N}$ such that $\mathcal{K} \neq \mathcal{T}$. Without loss of generality, suppose that $\mathcal{K} \cap (\mathcal{N} \backslash \mathcal{T}) \neq \varnothing$ (i.e., $\mathcal{K}$ is not a subset of $\mathcal{T}$). Then, for all $\mathbf{y} \in \Gamma_{\mathcal{N},\rho}(\xi_{\mathcal{N}}^{\mathcal{K}}(z))$ and $\tilde{\mathbf{y}} \in \Gamma_{\mathcal{N},\rho}(\xi_{\mathcal{N}}^{\mathcal{T}}(z))$, $y_i > \tilde{y}_i$ for at least one $i \in \mathcal{K} \cap (\mathcal{N} \backslash \mathcal{T})$. To see this, consider the corresponding $\mathbf{z} \in \xi_{\mathcal{N}}^{\mathcal{K}}(z)$ and $\tilde{\mathbf{z}} \in \xi_{\mathcal{N}}^{\mathcal{T}}(z)$. For $i \in \mathcal{K} \cap (\mathcal{N} \backslash \mathcal{T})$ and $j \in \mathcal{N} \backslash \mathcal{K}$, we have $z_i > z_j = z$ but $\tilde{z}_i = z \leq \tilde{z}_j$. Hence, in the first case, the critical type of firm i has a strictly higher expected allocation against firms in $\mathcal{N} \backslash \mathcal{K}$ than in the second case. The same is true for $j \in \mathcal{K} \cap \mathcal{T}$, since $z_i > z$ whereas $\tilde{z}_i = z < \tilde{z}_j$. Finally, the expected allocation of firm i's critical type against other firms in $\mathcal{K} \cap (\mathcal{N} \backslash \mathcal{T})$ cannot be lower for all $i \in \mathcal{K} \cap (\mathcal{N} \backslash \mathcal{T})$ when considering $\mathbf{z} \in \xi_{\mathcal{N}}^{\mathcal{K}}(z)$ than when considering $\tilde{\mathbf{z}} \in \xi_{\mathcal{N}}^{\mathcal{T}}(z)$. Consequently, $y_i > \tilde{y}_i$ for at least one i.

So far we have shown that for every $\mathbf{y} \in \Gamma_{\mathcal{N},\rho}([0,1]^n)$, there are unique z and $\mathcal{K}$ such that $\mathbf{y} \in \Gamma_{\mathcal{N},\rho}(\xi_{\mathcal{N}}^{\mathcal{K}}(z))$. This already partially pins down $\mathbf{z}$: for all $i \in \mathcal{N} \backslash \mathcal{K}$, we have $z_i = z$. Moreover, $\mathbf{y} \in \Gamma_{\mathcal{N},\rho}(\xi_{\mathcal{N}}^{\mathcal{K}}(z))$ implies that $\mathbf{y} \in o_{\mathcal{N},\rho}^{\mathcal{K}}(z)$ and therefore,

$$\mathbf{y}_{\mathcal{K}} \in \{\mathbf{y} \in [0,1]^n \mid \mathbf{y} \in \Gamma_{\mathcal{N},\rho}(\mathbf{z}) \text{ for some } \mathbf{z} \in [z,1]^n \text{ with } z_i = z \text{ for all } i \in \mathcal{N} \backslash \mathcal{K}\}.$$

If Property 1 holds for $\mathcal{K} \subset \mathcal{N}$, then there is a unique $\mathbf{z}_{\mathcal{K}} \in [z,1]^{|\mathcal{K}|}$ such that $\mathbf{y}_{\mathcal{K}} \in \{\mathbf{y} \in [0,1]^n \mid \mathbf{y} \in \Gamma_{\mathcal{N},\rho}(\mathbf{z}) \text{ with } z_i = z_{\mathcal{K}} \text{ for } i \in K \text{ and } z_i = z \text{ for all } i \in \mathcal{N} \backslash \mathcal{K}\}$. This pins down z_i

also for $i \in \mathcal{K}$. ■

Lemma OA.11 together with Lemma OA.6 completes our demonstration of Property 1:

Lemma OA.12. *Property* 1 *holds.*

In a similar manner as we constructed $O_{\mathcal{N},\rho}$, define

$$S_{\mathcal{N},\rho}(z) \equiv \Gamma_{\mathcal{N},\rho}(\{\mathbf{z} \in [z,1]^n \mid z_i = 1 \text{ for at least one } i \in \mathcal{N}\}).$$

The set $S_{\mathcal{N},\rho}(z)$ represents the image under $\Gamma_{\mathcal{N},\rho}$ of the set of all $\mathbf{z}$ where $z_i \geq z$ for all i and $z_i = 1$ for at least one i. Observe that $S_{\mathcal{N},\rho}(z)$ contains all the boundary points of $O_{\mathcal{N},\rho}(\tilde{z})$ for each $\tilde{z} \in [z,1]$. Moreover, $S_{\mathcal{N},\rho}(1) = O_{\mathcal{N},\rho}(1)$. In addition, define for $\mathcal{K} \subset \mathcal{N}$,

$$s^{\mathcal{K}}_{\mathcal{N},\rho}(z) \equiv \{\mathbf{y} \in [0,1]^n \mid \mathbf{y} \in \Gamma_{\mathcal{N},\rho}(\mathbf{z}) \text{ for } \mathbf{z} \text{ with } z_j \in [z,1] \text{ for all } j \in \mathcal{K} \text{ and } z_i = 1 \text{ for all } i \in \mathcal{N}\backslash\mathcal{K}\}.$$

Lemma OA.13. *The image of $\Gamma_{\mathcal{N},\rho}([z,1]^n)$ is an n-dimensional convex polytope for all $z \leq 1$. The boundary of this polytope is $O_{\mathcal{N},\rho}(z) \cup S_{\mathcal{N},\rho}(z)$.*

Proof. This is the counterpart to Lemma 5 in Loertscher and Wasser (2019). We know that $\Gamma_{\{1,2\},\rho}([z,1]^2)$ is a hexagon, which is a polytope. We now show that if $\Gamma_{\mathcal{K},\rho}([z,1]^t)$ is a convex polytope for all $\mathcal{K} \subset \mathcal{N}$, then $\Gamma_{\mathcal{N},\rho}([z,1]^n)$ is a convex polytope. Consequently, the first statement in the lemma follows by induction.

Suppose that $\Gamma_{\mathcal{K},\rho}([z,1]^{|\mathcal{K}|})$ is a convex polytope for all $\mathcal{K} \subset \mathcal{N}$ and recall that $\Gamma_{\mathcal{N},\rho}([z,1]^n) = \bigcup_{\tilde{z}\in[z,1]} O_{\mathcal{N},\rho}(\tilde{z})$. As derived above, $O_{\mathcal{N},\rho}(z)$ is a polytopal complex. By Lemma OA.10, all coordinates of the extreme points of $\Gamma_{\mathcal{N},\rho}(z,\ldots,z)$ are continuous and nondecreasing in z, and strictly increasing for at least one coordinate, which implies that $O_{\mathcal{N},\rho}(z)$ continuously moves further away from the origin as z increases. Hence, $O_{\mathcal{N},\rho}(z)$ is part of the boundary of $\Gamma_{\mathcal{N},\rho}([z,1]^n)$. Recalling that $\Gamma_{\mathcal{N},\rho}([z,1]^n) = \bigcup_{z'\in[z,1]} O_{\mathcal{N},\rho}(z')$, in addition to $O_{\mathcal{N}\rho}(z)$ being part of the boundary of $\Gamma_{\mathcal{N},\rho}([z,1]^n)$, all boundary points of $O_{\mathcal{N},\rho}(\tilde{z})$ for each $\tilde{z} \in (z,1)$ are also part of the boundary of $\Gamma_{\mathcal{N},\rho}([z,1]^n)$, whereas all interior points of $O_{\mathcal{N},\rho}(\tilde{z})$ are in the interior of $\Gamma_{\mathcal{N},\rho}([z,1]^n)$. Lastly, note that $O_{\mathcal{N},\rho}(1)$ consists of only one convex polytope (namely $\Gamma_{\mathcal{N},\rho}(1,\ldots,1)$) and that all its points are part of the boundary of $\Gamma_{\mathcal{N},\rho}([z,1]^n)$.

The set $S_{\mathcal{N},\rho}(z)$ represents all points on the boundary of $\Gamma_{\mathcal{N},\rho}([z,1]^n)$, i.e., boundary points that are not in $O_{\mathcal{N},\rho}(z)$. Consequently, $O_{\mathcal{N},\rho}(z) \cup S_{\mathcal{N},\rho}(z)$ represents the entire boundary of $\Gamma_{\mathcal{N},\rho}([z,1]^n)$. Like $O_{\mathcal{N},\rho}(z)$, $S_{\mathcal{N},\rho}(z)$ is also a polytopal complex that consists of $2^n - 1$ convex polytopes of dimension $n-1$. The boundary of $\Gamma_{\mathcal{N},\rho}([z,1]^n)$ therefore consists of $2^{n+1} - 2$ convex polytopes ($o^{\mathcal{K}}_{\mathcal{N},\rho}(z)$ and $s^{\mathcal{K}}_{\mathcal{N},\rho}(z)$ for all $\mathcal{K} \subset \mathcal{N}$), making $\Gamma_{\mathcal{N},\rho}([z,1]^n)$ an n-dimensional polytope with $2^{n+1} - 2$ facets.

Recall that for all $z < 1$, $O_{\mathcal{N},\rho}(z)$ consists of $\Gamma_{\mathcal{N},\rho}(z,\ldots,z)$ with a $o^{\mathcal{K}}_{\mathcal{N},\rho}(z)$ attached to each facet. The points in each $o^{\mathcal{K}}_{\mathcal{N},\rho}(z)$ are further away from the origin than the points on the corresponding facet of $\Gamma_{\mathcal{N},\rho}(z,\ldots,z)$. Because of the monotonicity and continuity properties of $O_{\mathcal{N},\rho}(z)$, for all $\mathbf{y} \in \text{Conv}(O_{\mathcal{N},\rho}(z))$ such that $\mathbf{y} \notin O_{\mathcal{N},\rho}(z)$, there is a $\tilde{z} \in (z,1]$ such that $\mathbf{y} \in O_{\mathcal{N},\rho}(\tilde{z})$. Hence, the polytope $\Gamma_{\mathcal{N},\rho}([z,1]^n) = \bigcup_{\tilde{z}\in[z,1]} O_{\mathcal{N},\rho}(\tilde{z})$ is convex. ■

It follows from Lemma OA.13 and the monotonicity and continuity properties of $O_{\mathcal{N},\rho}(z)$ that the image of $\Gamma_{\mathcal{N}}([0,1]^n)$ is defined by a set of vertices as follows:

Lemma OA.14. *The image of $\Gamma_{\mathcal{N},\rho}([0,1]^n)$ is defined by vertices $\mathbf{p}_\rho(\mathbf{z},h_\ell)$ with $\ell \in \{1,\ldots,n!\}$ and $z_i \in \{0,1\}$ for all $i \in \mathcal{N}$.*

In an analog to the point made in Loertscher and Wasser (2019, p. 1105) that $\Gamma_{\mathcal{N},\rho}(0,\ldots,0)$ lies below the simplex, we have:

Lemma OA.15. *Polytope $\Gamma_{\mathcal{N},\rho}(0,\ldots,0)$ does not intersect $\Delta^{n-1}_{\mathbf{k}}$, and $\Gamma_{\mathcal{N},\rho}(0,\ldots,0)$ lies closer to the origin than $\Delta^{n-1}_{\mathbf{k}}$.*

Proof. Recall that $\Gamma_{\mathcal{N},\rho}(0,\ldots,0)$ is a polytope with $n!$ vertices, $\mathbf{p}_\rho(0,\ldots,0,h_\ell)$ for $\ell \in \{1,\ldots,n!\}$. For each $h \in H$, $\mathbf{p}_\rho(0,\ldots,0,h) = (p_{1,\rho}(0,\ldots,0,h),\ldots,p_{n,\rho}(0,\ldots,0,h))$, with

$$p_{i,\rho}(0,\ldots,0,h) = \sum_{\mathcal{A}\in P(\mathcal{N}\setminus\{i\})}\left(\prod_{j\in\mathcal{A}\cap\underline{\mathcal{E}}_i(h)}(1-G^B_{j,\rho}(0))\prod_{j\in(\mathcal{N}\setminus\mathcal{A})\cap\underline{\mathcal{E}}_i(h)}G^B_{j,\rho}(0)\prod_{j\in(\mathcal{N}\setminus\mathcal{A})\cap\overline{\mathcal{E}}_i(h)}G^S_{j,\rho}(0)\varphi_i(\mathcal{A})\right).$$

We show that $\sum_{i\in\mathcal{N}} p_{i,\rho}(0,\ldots,0,h) < 1$, which implies that each of the $n!$ vertices that defines $\Gamma_{\mathcal{N},\rho}(0,\ldots,0)$ lies closer to the origin than $\Delta^{n-1}_{\mathbf{k}}$, giving us the desired result.

Consider hierarchy h_1, where firms are ordered $1 \succ 2 \succ \cdots \succ n$. The argument for other hierarchies is the same because there is nothing firm-specific that the argument exploits and hence, up to relabeling, the argument applies for any hierarchy. For firm i that is i-th highest in the hierarchy, we can write $p_{i,\rho}(0,\ldots,0,h_1)$ as:

$$p_{i,\rho}(0,\ldots,0,h_1) = \sum_{\mathcal{A}\in\mathcal{A}_i}\mu_{i,\rho}(0,\mathcal{A},h_1)\varphi_i(\mathcal{A}),$$

where

$$\mathcal{A}_i \equiv \{\mathcal{X} \in P(\mathcal{N}\setminus\{i\}) \mid \{1,\ldots,i-1\} \subset \mathcal{X}\}$$

and

$$\mu_{i,\rho}(0,\mathcal{A},h_1) \equiv \prod_{j\in\mathcal{A}\setminus\{1,\ldots,i\}}(1-G^B_{j,\rho}(0))\prod_{j\in(\mathcal{N}\setminus\mathcal{A})\setminus\{1,\ldots,i\}}G^B_{j,\rho}(0).$$

Because $\sum_{\mathcal{A}\in\mathcal{A}_i}\mu_{i,\rho}(0,\mathcal{A},h_1) = 1$, $p_{i,\rho}(0,\ldots,0,h_1)$ is a weighted average of allocations $\{\varphi_i(\mathcal{A})\}_{\mathcal{A}\in\mathcal{A}_i}$. Further, given the definition of $\mathcal{A}_i$, we have

$$\max_{\mathcal{A}\in\mathcal{A}_i}\varphi_i(\mathcal{A}) = \min\{k_i, \max\{0, 1-\sum_{j\in\{1,\ldots,i-1\}}k_j\}\}.$$

Using this and the result that for $i \in \{1, \ldots, n-1\}$,

$$\mu_{i,\rho}(0, \{1, \ldots, i-1\}, h_1) = \prod_{j \in \{i+1,\ldots,n\}} G^B_{j,\rho}(0) < 1,$$

we have for $i \in \{1, \ldots, n-1\}$,

$$p_{i,\rho}(0, \ldots, 0, h_1) < \min\{k_i, \max\{0, 1 - \sum_{j \in \{1,\ldots,i-1\}} k_j\}\}.$$

For the n-th firm, we have

$$p_{n,\rho}(0, \ldots, 0, h_1) = \min\{k_n, \max\{0, 1 - \sum_{j \in \{1,\ldots,n-1\}} k_j\}\} = \max\{0, 1 - \sum_{j \in \{1,\ldots,n-1\}} k_j\}.$$

Summing $p_{i,\rho}(0, \ldots, 0, h_1)$ for $i \in \mathcal{N}$, we have

$$\sum_{i \in \mathcal{N}} p_{i,\rho}(0, \ldots, 0, h_1) < \sum_{i \in \mathcal{N}} \min\{k_i, \max\{0, 1 - \sum_{j \in \{1,\ldots,i-1\}} k_j\}\} = 1,$$

which completes the proof. ■

OA.6.4 Condition for $O_{\mathcal{N},\rho}(z)$ to "pass through" all of $\Delta^{n-1}_{\mathbf{k}}$ and so to have $\Delta^{n-1}_{\mathbf{k}} \in \Gamma([0,1]^n)$

Recall that $\Delta^{n-1} \equiv \{\mathbf{y} \in [0,1]^n \mid \sum_{i \in \mathcal{N}} y_i = 1\}$ is the polytope in n-dimensional space defined by n vertices $(\mathbf{v}_1, \ldots, \mathbf{v}_n)$ with $\mathbf{v}_i = (v_{i,1}, \ldots, v_{i,n})$, where $v_{i,i} = 1$ and $v_{i,j} = 0$ for all $j \in \mathcal{N} \backslash \{i\}$. The modified simplex $\Delta^{n-1}_{\mathbf{k}}$ is derived from Δ^{n-1} by truncation:

$$\Delta^{n-1}_{\mathbf{k}} = \left\{\mathbf{y} \in \Delta^{n-1} \mid \text{for all } i \in \mathcal{N},\ y_i \leq k_i\right\}.$$

Note that $\Delta^{n-1}_{\mathbf{k}}$ has n facets that are subsets of the n facets of Δ^{n-1}, and also potentially additional facets created by the truncations.

Lemma OA.16. *If the n facets of $\Delta^{n-1}_{\mathbf{k}}$ that are subsets of the n facets of Δ^{n-1} are contained in $O_{\mathcal{N},\rho}(0)$, then as z increases from 0 to 1, $O_{\mathcal{N},\rho}(z)$ must "pass through" all of $\Delta^{n-1}_{\mathbf{k}}$, i.e., for all $\mathbf{y} \in \Delta^{n-1}_{\mathbf{k}}$, there exists $z \in [0,1]$ such that $\mathbf{y} \in O_{\mathcal{N},\rho}(z)$, and so $\Delta^{n-1}_{\mathbf{k}} \in \Gamma([0,1]^n)$.*

Proof. Suppose not. Then there exists a point $\hat{\mathbf{y}}$ on one of the facets of $\Delta^{n-1}_{\mathbf{k}}$ that is not a subset of the n facets of Δ^{n-1} and that is not an element of $O_{\mathcal{N},\rho}(z)$ for any $z \in [0,1]$. Because $\hat{\mathbf{y}}$ lies on one of the facets created by truncation, it must have $\hat{y}_i = k_i$ for some i and there must be points $\hat{\mathbf{v}}$ and $\hat{\mathbf{v}}'$ that are on the edges of $\Delta^{n-1}_{\mathbf{k}}$ that do intersect Δ^{n-1} and that are contained in $O_{\mathcal{N},\rho}(z)$ such that $\hat{v}_i = \hat{v}'_i = k_i$ and whose convex combination gives $\hat{\mathbf{y}}$, i.e., there exists $a \in (0,1)$ such that $a\hat{\mathbf{v}} + (1-a)\hat{\mathbf{v}}' = \hat{\mathbf{y}}$. Because, by our supposition, $\hat{\mathbf{y}}$ is not an element of $O_{\mathcal{N},\rho}(z)$ for any $z \in [0,1]$, it must be that for all $z \in [0,1]$ and $\mathbf{y} \in O_{\mathcal{N},\rho}(z)$, $\hat{\mathbf{y}} \geq \mathbf{y}$, with a strict inequality for at least one component. But from Lemma OA.10, we know that each component of vertices $\hat{\mathbf{v}}$ and $\hat{\mathbf{v}}'$ is nondecreasing in z, which is a contradiction. ■

We now show that the intersection of the boundary of Δ^{n-1} and $\Delta^{n-1}_{\mathbf{k}}$ lies in $O_{\mathcal{N},\rho}(0)$, so that the predicate of Lemma OA.16 holds. Specifically, we show that the intersection of the boundary of Δ^{n-1} and $\Delta^{n-1}_{\mathbf{k}}$ that has i-th component equal to $\max\{0, -\sum_{j\in\mathcal{N}\setminus\{i\}} k_j\}$ lies in $o^{\mathcal{N}\setminus\{i\}}_{\mathcal{N},\rho}(0)$.

OA.6.5 $O_{\mathcal{N},\rho}(0)$ contains $\Gamma_{\mathcal{N}\setminus\{i\},\rho}([0,1]^{n-1})$, thereby allowing induction

We can now use Lemma OA.14 to connect the image of $\Gamma_{\mathcal{N}\setminus\{i\},\rho}([0,1]^{n-1})$ to $\Gamma_{\mathcal{N},\rho}$.[13] For $\mathcal{K}\subset\mathcal{N}$, define $\mathbf{k}^{\mathcal{K}}\equiv(k_i)_{i\in\mathcal{K}}$. We show that the image of $\Gamma_{\mathcal{N}\setminus\{i\},\rho}([0,1]^{n-1})$ is a subset of $o^{\mathcal{N}\setminus\{i\}}_{\mathcal{N},\rho}(0)$. This will allow us to use an induction argument because having $\Delta^{n-2}_{\mathbf{k}^{\mathcal{N}\setminus\{i\}}}\in\Gamma_{\mathcal{N}\setminus\{i\},\rho}([0,1]^{n-1})$ for all $i\in\mathcal{N}$ implies that $\Delta^{n-2}_{\mathbf{k}^{\mathcal{N}\setminus\{i\}}}\in o^{\mathcal{N}\setminus\{i\}}_{\mathcal{N},\rho}(0)$ for all $i\in\mathcal{N}$, which implies that the facets of $\Delta^{n-1}_{\mathbf{k}}$ that are subsets of the facets of Δ^{n-1} are contained in $O_{\mathcal{N},\rho}(0)$, giving us the desired result.

The definition of $o^{\mathcal{K}}_{\mathcal{N},\rho}(z)$ implies that

$$o^{\mathcal{N}\setminus\{i\}}_{\mathcal{N},\rho}(0)=\{\mathbf{y}\in[0,1]^n\mid\mathbf{y}\in\Gamma_{\mathcal{N},\rho}(\mathbf{z})\text{ for some }\mathbf{z}\in[0,1]^n\text{ with }z_i=0\}.$$

In what follows, we show that $o^{\mathcal{N}\setminus\{i\}}_{\mathcal{N},\rho}(0)$ essentially contains the image of $\Gamma_{\mathcal{N}\setminus\{i\},\rho}([0,1]^{n-1})$.

In particular, the largest of the vertices that define $o^{\mathcal{N}\setminus\{i\}}_{\mathcal{N},\rho}(0)$ are the same as the ones that define $\Gamma_{\mathcal{N}\setminus\{i\},\rho}([0,1]^{n-1})$, but other vertices in $o^{\mathcal{N}\setminus\{i\}}_{\mathcal{N},\rho}(0)$ are less (closer to the origin) than the corresponding vertices in $\Gamma_{\mathcal{N}\setminus\{i\},\rho}([0,1]^{n-1})$, with the implication that $o^{\mathcal{N}\setminus\{i\}}_{\mathcal{N},\rho}(0)$ contains $\Gamma_{\mathcal{N}\setminus\{i\},\rho}([0,1]^{n-1})$:

Lemma OA.17. *For all $i\in\mathcal{N}$, the polytope*

$$\hat{\Gamma}^i_\rho\equiv\Big\{\mathbf{y}\in[0,1]^n\mid((\mathbf{y}_{-i})_1,\ldots,(\mathbf{y}_{-i})_{n-1})\in\Gamma_{\mathcal{N}\setminus\{i\},\rho}([0,1]^{n-1})\text{ and }y_i=\max\{0,1-\sum_{j\in\mathcal{N}\setminus\{i\}}k_j\}\Big\}$$

is a subset of $o^{\mathcal{N}\setminus\{i\}}_{\mathcal{N},\rho}(0)$.

Proof. Polytope $o^{\mathcal{N}\setminus\{i\}}_{\mathcal{N},\rho}(0)$ is defined by vertices $\mathbf{p}_\rho(\mathbf{z},h_\ell)$ with h_ℓ such that firm i is last in the tie-breaking hierarchy, $z_i=0$, $z_j\in\{0,1\}$ for all $j\in\mathcal{N}\setminus\{i\}$, and if $z_j=1$ and $z_{j'}=0$, then $h_\ell(j)>h_\ell(j')$, i.e., firms with an ironing parameter of 1 are ahead in the tie-breaking hierarchy of firms with an ironing parameter of 0. For example, if h_1 is the tie-breaking hierarchy with $h_1(1)>h_1(2)>\cdots>h_1(n)$, then one of the vertices that defines $o^{\mathcal{N}\setminus\{n\}}_{\mathcal{N},\rho}(0)$

[13] If $\sum_{j\in\mathcal{N}\setminus\{i\}}k_j\le 1$ so that the full demand of the firms in $\mathcal{N}\setminus\{i\}$ can be met with supply 1, then define $\Gamma_{\mathcal{N}\setminus\{i\},\rho}(\mathbf{z})=\mathbf{k}$ for all $\mathbf{z}\in[0,1]^{n-1}$.

is $\mathbf{p}_\rho(\hat{\mathbf{z}}^m, h_1)$, where $\hat{\mathbf{z}}^m$=$(1, \ldots, 1, 0, \ldots, 0)$ has $m \in \{0, \ldots, n-1\}$ components equal to 1. For the n-th component of $\mathbf{p}_\rho(\hat{\mathbf{z}}^m, h_1)$, we have

$$p_{n,\rho}(\hat{\mathbf{z}}^m, h_1) = \max\{0, 1 - \sum_{j \in \mathcal{N}\setminus\{n\}} k_j\},$$

i.e., a firm that has an ironing parameter of zero and that is last in the tie-breaking hierarchy only gets the residual. Further, for all $i \in \{1, \ldots, n-1\}$ with $\hat{z}_i^m = 1$, we have

$$p_{i,\rho}(\hat{\mathbf{z}}^m, h_1) = p_{i,\rho}^{\mathcal{N}\setminus\{n\}}(\hat{\mathbf{z}}_{-n}^m, h_1^{\mathcal{N}\setminus\{n\}}),$$

where $\hat{\mathbf{z}}_{-n}^m = (\hat{z}_1^m, \ldots, \hat{z}_{n-1}^m)$, i.e., the i-th component with $\hat{z}_i^m = 1$ is not affected by the presence or absence of an n-th firm (at the bottom of the tie-breaking hierarchy) with $\hat{z}_n^m = 0$. Finally, for all $i \in \{1, \ldots, n-1\}$ with $\hat{z}_i^m = 0$, we have

$$p_{i,\rho}(\hat{\mathbf{z}}^m, h_1) \quad \leq \quad p_{i,\rho}^{\mathcal{N}\setminus\{n\}}(\hat{\mathbf{z}}_{-n}^m, h_1^{\mathcal{N}\setminus\{n\}})$$

because in the interim expected allocation on the right side, the i-th firm is tied in terms of its ironing parameter with firms $m+1, \ldots, n-1$, whereas on the left side, firm i is also tied with firm n. Analogous arguments apply for other tie-breaking hierarchies.

Defining for all $i \in \mathcal{N}$,

$$\mathcal{Z}^i \equiv \{(\mathbf{z}, h) \in \{0,1\}^n \times H \mid h(i) = n,\ z_i = 0, \text{ and } z_j = 1 \text{ and } z_{j'} = 0 \text{ implies } h(j) > h(j')\},$$

the vertices of $o_{\mathcal{N},\rho}^{\mathcal{N}\setminus\{\ell\}}(0)$ can be written as $\{\mathbf{p}_\rho^{\mathcal{N}\setminus\{\ell\}}(\mathbf{z}, h)\}_{(\mathbf{z},h)\in\mathcal{Z}^\ell}$. All of the vertices of $o_{\mathcal{N},\rho}^{\mathcal{N}\setminus\{\ell\}}(0)$ have the same ℓ-th coordinate equal to $\max\{0, 1 - \sum_{j\in\mathcal{N}\setminus\{\ell\}} k_j\}$. For each $(\mathbf{z}, h) \in \mathcal{Z}^\ell$, we can define a another vertex $\mathbf{v}_\rho^{\mathcal{N}\setminus\{\ell\}}(\mathbf{z}, h)$ with $v_{i,\rho}^{\mathcal{N}\setminus\{\ell\}}(\mathbf{z}, h) = p_{i,\rho}^{\mathcal{N}\setminus\{\ell\}}(\mathbf{z}_{-\ell}, h^{\mathcal{N}\setminus\{\ell\}})$ for $i \in \mathcal{N}\setminus\{\ell\}$ and $v_{\ell,\rho}^{\mathcal{N}\setminus\{\ell\}}(\mathbf{z}, h) = \max\{0, 1 - \sum_{j\in\mathcal{N}\setminus\{\ell\}} k_j\}$. The vertices $\{\mathbf{v}_\rho^{\mathcal{N}\setminus\{\ell\}}(\mathbf{z}, h)\}_{(\mathbf{z},h)\in\mathcal{Z}^\ell}$ define the polytope $\hat{\Gamma}_\rho^\ell$. Because all of the vertices that define $\hat{\Gamma}_\rho^\ell$ have the same ℓ-th coordinate as the vertices that define $o_{\mathcal{N},\rho}^{\mathcal{N}\setminus\{\ell\}}(0)$, namely $\max\{0, 1 - \sum_{j\in\mathcal{N}\setminus\{\ell\}} k_j\}$, it follows that $o_{\mathcal{N},\rho}^{\mathcal{N}\setminus\{\ell\}}(0)$ and $\hat{\Gamma}_\rho^\ell$ lie in the same $(n-1)$-dimensional hyperplane. Further, using the analysis above, for any $(\mathbf{z}, h) \in \mathcal{Z}^\ell$ with $z_i = 0$, we have $p_{i,\rho}^{\mathcal{N}\setminus\{\ell\}}(\mathbf{z}, h) \leq v_{i,\rho}^{\mathcal{N}\setminus\{\ell\}}(\mathbf{z}, h)$; and for any $(\mathbf{z}, h) \in \mathcal{Z}^\ell$ with $z_i = 1$, we have $p_{i,\rho}^{\mathcal{N}\setminus\{\ell\}}(\mathbf{z}, h) = v_i^{\mathcal{N}\setminus\{\ell\}}(\mathbf{z}, h)$. Thus, $\hat{\Gamma}_\rho^\ell$ and $o_{\mathcal{N},\rho}^{\mathcal{N}\setminus\{\ell\}}(0)$ share the same "upper" boundary points defined by $\mathbf{z}$ with $z_\ell = 0$ and $\mathbf{z}_{-\ell} = (1, \ldots, 1)$, and otherwise each vertex defining $\hat{\Gamma}_\rho^\ell$ is weakly greater than the corresponding vertex defining in $o_{\mathcal{N},\rho}^{\mathcal{N}\setminus\ell}(0)$. Using the monotonicity of the points $\mathbf{p}_\rho^{\mathcal{N}\setminus\{\ell\}}(\mathbf{z}, h)$ (see Lemma OA.10), it follows that $\hat{\Gamma}_\rho^\ell$ is a subset of $o_{\mathcal{N},\rho}^{\mathcal{N}\setminus\{\ell\}}(0)$. ■

Combining the results above, we have:

Lemma OA.18. *If for all $i \in \mathcal{N}$, Property 2 holds for $\Gamma_{\mathcal{N}\setminus\{i\},\rho}$, i.e., $\Delta_{\mathbf{k}_{-i}}^{n-2} \subset \Gamma_{\mathcal{N}\setminus\{i\},\rho}([0,1]^{n-1})$, then $O_{\mathcal{N},\rho}(0)$ contains the boundaries of $\Delta_{\mathbf{k}}^{n-1}$ that intersect Δ^{n-1}, and so Property 2 holds* *for* $\Gamma_{\mathcal{N},\rho}$.

Proof. Supposing that for all $i \in \mathcal{N}$, $\Delta^{n-2}_{\mathbf{k}^{\mathcal{N}\setminus\{i\}}} \subset \Gamma_{\mathcal{N}\setminus\{i\},\rho}([0,1]^{n-1})$, then Lemma OA.17 implies for all $i \in \mathcal{N}$, we have $\Delta^{n-2}_{\mathbf{k}^{\mathcal{N}\setminus\{i\}}} \subset o^{\mathcal{N}\setminus\{i\}}_{\mathcal{N},\rho}(0) \subset \Gamma_{\mathcal{N}}([0,1]^n)$, which implies that $\Delta^{n-1}_{\mathbf{k}} \subset \Gamma_{\mathcal{N},\rho}([0,1]^n)$ and completes the proof. ■

Using the result that Property 2 holds for $n = 2$ and applying Lemma OA.18 iteratively, we have:

Lemma OA.19. *Property* 2 *holds for* $\Gamma_{\mathcal{N},\rho}$.

APP Application to the Republic-Santek transaction

A natural question is how the divestiture policies that we discuss can be implemented in practice. In a merger review context, the parties would need to provide details of their own holdings to the competition authority and assist the competition authority in understanding the nature of upstream and downstream competitive constraints. In addition, public filings and industry analyst reports provide relevant information for assessing market structure, maximum capacities, market shares, and parties' margins. This means that $\mathbf{r}$ can plausibly be treated as observable.

In this appendix, we show how the framework of this paper can be applied to market data that is typically available in a merger review process. As we now illustrate, historical market shares on the input market and maximum allocations can be used to estimate firms' expected allocations and maximum demands. Under parametric assumptions about firms' distributions, one can estimate these parameters to match the firms' historical market shares and to determine, for any given $\mathbf{r}$, how efficient the market operates. With these estimates in hand, one can then estimate sets like $\mathcal{R}^e$ and $\mathcal{R}(\mathbf{r})$, determine whether a proposed transaction is harmful, and what (if any) divestitures are capable of offsetting that harm or would even permit the first-best if the first-best was not possible prior to the transaction.[14]

In our illustration, there are three firms. The initial ownership structure is $\mathbf{r}^b = (1/2, 1/2, 0)$ and the vector of maximum demands is $\mathbf{k} = (1, 1/2, 0)$. The transaction consist of firm 2 selling its assets to firm 1, resulting in $\mathbf{r}^a = (1, 0, 0)$. Panel (a) in Figure APP.6 displays $\mathbf{r}^b$, $\mathbf{r}^a$, and the estimated $\mathbf{r}^*$ and $\mathcal{R}^e$. As the figure shows, the first-best is not possible with or without the transaction. Panel (b) adds the estimated set $\mathcal{R}(\mathbf{r}^b)$ (in yellow) and divestitures that restore social surplus equal to $SS(\mathbf{r}^b)$ as well as divestitures that permit the first-best.

Our application is inspired by the 2021 transaction involving waste management companies Republic and Santek.[15] However, the specific data that we use are hypothetical.

Republic and Santek both held upstream waste disposal assets and also operated downstream waste collection businesses. In addition to using its own upstream assets, Republic also purchased upstream assets from Santek (and other firms). For Santek, in addition to consuming its own upstream assets, it also sold upstream assets to other firms, including to Republic and a third firm, Regional, that held no upstream assets of its own. Motivated by

[14] For inspiration, we use the 2021 transaction involving waste management companies Republic and Santek, but we use fictitious data. (For background, see Loudermilk et al. (2023) and the U.S. DOJ's website on "U.S. and Plaintiff States v. Republic Services, Inc. and Santek Waste Services, LLC" (https://www.justice.gov/atr/case/us-and-state-alabama-v-republic-services-inc-and-santek-waste-services-llc).)

[15] For background, see Loudermilk et al. (2023) and the U.S. DOJ's website on "U.S. and Plaintiff States v. Republic Services, Inc. and Santek Waste Services, LLC" (https://www.justice.gov/atr/case/us-and-state-alabama-v-republic-services-inc-and-santek-waste-services-llc).

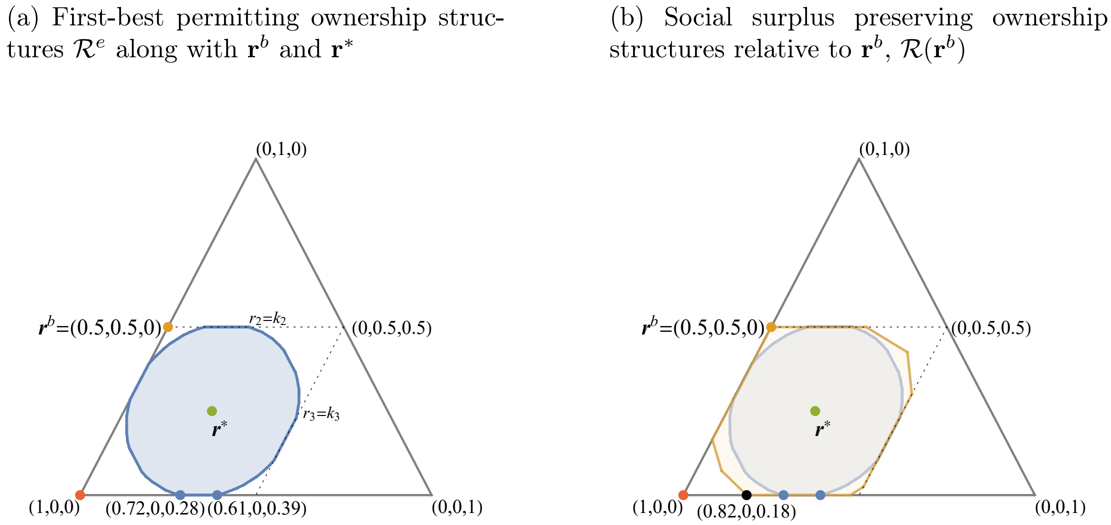

Figure APP.6: First-best permitting region (blue) and social surplus preserving region $\mathcal{R}(\mathbf{r}^b)$ (yellow) for upstream market shares $\mathbf{r} \in \Delta_{\mathbf{k}}$ and $\mathbf{k} = (1, 1/2, 1/2)$ and distributions calibrated to the waste management market, which has $\mathbf{r}^b = (1/2, 1/2, 0)$. Specifically, $F_i(\theta) = 1 - (1-\theta)^{s_i}$, with $s_1 = 1$, $s_2 = 1.2$, and $s_3 = 0.8$.

these facts, we model Republic as a trader, Santek, as a seller, and Regional as a buyer.

In general, one would expect the upstream ownership structure, $\mathbf{r}^b$, to be observable. The maximum demands $\mathbf{k}$ can be estimated using data on historical allocations. Specifically, assuming one has observations of input good market quantities q_i^a over $t \in \{1, \dots, T\}$ periods that result from T independent instances of the market we studied, the maximum demands can be estimated by $\hat{k}_i = \max_t \{q_i^a\}$. This is a good approximation if T is large enough because, regardless of how efficient the market operates, every once in a while the type realizations will be such that firm i is allocated its maximum demand k_i. Likewise, firm i's expected or average quantity $\hat{q}_i$ can be estimated by $\hat{q}_i = \frac{\sum_t q_i^a}{T}$, and firm i's estimated market share generated in the input market will be $\hat{\varsigma}_i = \hat{q}_i / R$.

For our illustration, we assume the following industry data:

Firm i	r_i^b	$\hat{k}_i$	Market share $\hat{\varsigma}_i$	Type
1. Republic	1/2	1	50%	trader
2. Santek	1/2	1/2	24%	seller
3. Regional	0	1/2	26%	buyer

To complete the specification of the market, we then need to estimate the firms' type distributions. To do so, we assume a class of parameterized distributions and calibrate the firms' parameters based on the information in the table above. Specifically, we model each firm i as having a type distribution of the form $F_i(\theta) \equiv 1-(1-\theta)^{s_i}$, where $s_i > 0$. See Waehrer and Perry (2003) for an axiomatic foundation for this basic structure.[16] The firms' market shares together with an additional identifying assumption, such as the margin for one of the firms, then determine the distributional parameters. For computational convenience, instead of using one firm's margin, we use as the identifying assumption that Republic has uniformly distributed types, i.e., $\hat{s}_{\text{Republic}} = 1$, which implies that the ironing parameter for Republic's ironed virtual type function has an analytic form. Proceeding in this way, we estimate the firms' distributional parameters as follows. Given $\mathbf{r}$, $\hat{\mathbf{k}}$, and arbitrary distributional parameters $\mathbf{s}$, one can solve numerically for ρ^* and Republic's worst-off type, $\hat{\theta}^*_{\text{Republic}}$ (because Santek is a seller, $\hat{\theta}^*_{\text{Santek}} = 1$, and because Regional is a buyer, $\hat{\theta}^*_{\text{Regional}} = 0$). Thereby one obtains the firms' interim expected allocation rules and the associated market shares. One can iterate over distributional parameters to calibrate to the market shares $\hat{\varsigma}$. The distributional parameters shown in the table below imply that $\rho^* = 1.09$ and $\hat{\theta}^*_{\text{Republic}} = 0.50$, which then imply the market shares $\hat{\varsigma}_i$ shown in the table above.[17] The table below displays the distributional parameters $\hat{s}_i$ obtained through this procedure:

Firm i	$\hat{s}_i$
1. Republic	1
2. Santek	1.2
3. Regional	0.8

With the distributional parameters in hand, we can calculate the ex post efficiency permitting region for ownership structures, $\mathcal{R}^e$, as well as the set $\mathcal{R}(\mathbf{r}^b)$, which contains all the ownership structures that generate a social surplus of at least $SS(\mathbf{r}^b)$. Both of these regions are illustrated in Figure APP.6. As the figure shows, ex post efficiency is not possible under the industry's pre-transaction upstream ownership structure $\mathbf{r}^b = (1/2, 1/2, 0)$, and the upstream ownership structure that maximizes Π^e is $\mathbf{r}^* = (1/2, 1/4, 1/4)$.[18] The

[16] Waehrer and Perry (2003) show that their three properties of no externalities, homogeneity, and constant returns are satisfied if and only if there exists a distribution function G with support $[0, 1]$ such that for all i and $c \in [0, 1)$, $F_i(c) = 1 - (1 - G(c))^{s_i}$ for $s_i > 0$.

[17] If there are traders that do not have uniformly distributed types, then one must also solve numerically for their ironing parameters. This applies, for example, for the problem of estimating $\mathcal{R}(\bar{\mathbf{r}})$, displayed in panel (b) of Figure APP.6.

[18] In the efficient allocation, Santek and Regional are allocated their full maximum demand if and only if their type is greater than the type of Republic. Thus, given that Republic's type is uniformly distributed, we have $q^e_{\text{Santek}}(\theta) = q^e_{\text{Regional}}(\theta) = \theta$. For Republic, we have $q^e_{\text{Republic}}(\theta) = 2 - (1-\theta)^{0.8} - (1-\theta)^{1.2}$. The upstream ownership structure that equalizes the firms' worst-off types to be $\hat{\theta}$ is then $\mathbf{r}^*$ satisfying $r^*_{\text{Santek}} = r^*_{\text{Regional}} = \hat{\theta}$, $r^*_{\text{Republic}} = 2 - (1-\hat{\theta})^{0.8} - (1-\hat{\theta})^{1.2}$, and $r^*_{\text{Republic}} + r^*_{\text{Santek}} + r^*_{\text{Regional}} = 2$.

post-transaction markets structure is indicated as a red dot in Figure APP.6 and given by $(1,0,0)$. The minimal divestiture that offsets the harm from that transaction, which requires Republic to divest 36% to Regional, is indicated by the black dot in Figure APP.6(b). If it divests between 57% and 78%, ex post efficiency is achieved after the transaction with divestiture whereas it was not possible before the transaction.

Ultimately, the DOJ allowed Republic's acquisition of Santek subject to the divestiture of a number of Santek's assets to approved buyers.[19]

Solving this, we get $\hat{\theta} = 0.502$, and so $r^*_{\text{Santek}} = r^*_{\text{Regional}} = 0.5024$ and $r^*_{\text{Republic}} = 0.9951$, which rounds to $\mathbf{r}^* = (1, 0.5, 0.5)$. Even though Santek and Republic's distributional parameters differ, their r^*_i's are the same.

[19]U.S. DOJ, "U.S. and Plaintiff States v. Republic Services, Inc. and Santek Waste Services, LLC" (https://www.justice.gov/atr/case/us-and-state-alabama-v-republic-services-inc-and-santek-waste-services-llc).

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