Optimal Allocation of Exclusivity Contracts

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Problem Definition: We study optimal allocation and pricing procedures for the revenue-maximizing seller facing a network of competing buyers. Buyers have private information about the value of an item being sold (such as franchise contract, good, service). Furthermore, buyers place a premium on obtaining the item exclusively, i.e., if no competitors obtain the item at the same time. *Academic/Practical Relevance:* Exclusivity is valuable and impacts allocation, pricing and purchase decisions in a variety of settings, ranging from retailing luxury goods to allocation of franchises to structuring distribution and advertising contracts. *Methodology:* We model the monopolist sellers revenue-maximization problem as a mechanism design problem and establish a connection with the maximum independent set problem on the underlying network of competing buyers. We combine methodology from optimal mechanism design and analysis of combinatorial optimization algorithms. *Results:* Our approach not only allows us to establish that the exclusivity implies suboptimaility of pricing and standard auctions, but also yields design of a novel hybrid auction-pricing procedure which we prove is optimal for a class of exclusivity valuations. This hybrid auction-pricing procedure is easy to implement and revenue-dominates posted prices. *Managerial Implications:* These findings suggest that one needs to focus on the possibility of exclusive allocation first, which is a useful guidance in structuring contracts and/or negotiation process when exclusive arrangements are a possibility. More generally, we provide a structured approach to allocating and pricing items when exclusivity is valuable by deriving methodological insights on when the seller should consider allocating exclusively, who the seller should allocate exclusively to, and how exclusivity contracts should be priced.

Key words: exclusivity, pricing, auctions and mechanism design, contracts, networks

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1. Introduction

Exclusivity rights are considered valuable in a variety of settings. For example, a contract securing rights to sell a product or offer a service is more valuable if no competitor secures the same contract. Consider a case of allocating a retail franchise, such as a car dealership, a chain restaurant, etc. If a buyer secures a franchise contract, the value of the business could be enhanced if competitors do not obtain a franchise: a portion of the business that a competitor could have served is likely to be captured by the sole franchise owner. Exclusivity contracts are commonplace for distribution agreements: a distributer often gets an exclusive contract to distribute a brand name product in a given geographic region. Exclusivity brings an additional value to contracts or purchases in many other settings, such as, advertising. Some examples of the exclusivity in advertising include having an ad shown exclusively on a webpage (i.e., without any competitors' ads showing), or exclusivity in the form of sponsorship of events, celebrity or athlete product endorsements, building naming rights, or product placement and merchandizing agreements in mass media entertainment such as movies, TV shows, or video games. Exclusivity could be perceived as valuable from a consumer's perspective as well. Luxury brands often try to create and exploit the perception of exclusivity that consumers might attach to purchasing their product. A consumer might value the product more highly if nobody else has the product, (e.g., designer clothing, a limited edition sports car, or obtaining a special feature or power-up in a game).

The notion of obtaining an exclusive contract inherently assumes that competitors are excluded from obtaining the same contract. Therefore, it is important to determine perceived competitors for each buyer, and, more generally, the structure of the competitor relationship. As is common with franchising and with exclusive sales, service and distribution agreements, the scope of exclusivity contracts is defined by the structure of the competitor relationship and could be restricted to a geographic area or a market segment. Geographical limitations are a natural limitation in a retail franchise example: unserved demand from a competitor who did not get the contract might only be captured by nearby locations. Hence, the notion of exclusivity in a franchise contract is typically limited to a geographic or demographic area (e.g., one Honda dealership in the city, no other McDonalds restaurants within 1km radius, etc.). In advertising, the scope of exclusivity might be limited to rivals who are perceived as direct competitors: for example, an online shoe retailer might perceive its ad allocation as exclusive, even if there are, say, financial institution ads shown on the same webpage, as long as no other shoe retailer ads are shown. In fact, multiple non-competing brands sponsorship or product placement is common (e.g., official credit card, official car, official drink, official ... of a sporting event). Finally, a social network could also define the scope of exclusivity. For example, designer clothing might not be considered as valuable if another person wears exactly the same item at an event; a limited edition sports car might not be perceived as valuable if an identical car is parked in the same parking lot; and even obtaining a special feature or power in a game might not be worth any bragging rights if another friend already achieved the same. Thus, a social network could be the underlying topology for understanding the notion of exclusivity in product placement, and firms would have to take that into account when making decisions on targeting (groups of) consumers with exclusive or non-exclusive offers.

In this paper, we study how to allocate and price exclusivity contracts, or any goods or services which could have added value if allocated exclusively. We consider the model in which a monopolistic seller aims to allocate items (contracts) to unit-demand buyers (one contract per buyer).

Each buyer i has a privately held value v_i for the item and another privately held value w_i for the item being allocated exclusively to them. (The difference between these two values can be thought of as the exclusivity premium and, similarly, the difference between prices for non-exclusive and exclusive allocations determined by the optimal mechanisms can be thought of as the price of exclusivity.) For example, a car dealership has a private valuation for the franchise contract that would secure the right to sell a car brand, and another (higher) private valuation if such a franchise contract is exclusive making them the only dealer in the region selling that particular brand.

We start our analysis in Section 2 by considering pricing strategies for the revenue-maximizing seller facing buyers who value exclusivity. If limited to posted prices, the seller should inflate the price that would have been posted had there been no additional value for exclusivity. High-valuation buyers are willing to buy at an inflated price since it increases the probability that no other buyers will get the item and thus unlock the exclusivity value for it. Thus, the inflated price signals the exclusivity potential to buyers who obtain the item. Many luxury good products and products with the "coolness" factor such as innovative electronic devices often appear to be overpriced. This suggests that the overpricing is due to optimal price setting aimed at capturing added value for exclusivity that buyers might associate with owning such products.

By quoting a single price to buyers who put a premium on exclusivity, the seller leaves it to buyers to evaluate risks associated with buying an item, without knowing whether they are paying for a non-exclusive or an exclusive allocation. Instead of quoting a single posted price, a seller could consider offering a two-price menu: a possibly different posted price for a non-exclusive allocation and another, presumably higher, posted price in the case that only one buyer obtains the item and thus gets it exclusively. Such a menu might not be easy to implement in everyday purchase settings, but it could potentially alleviate the uncertainty buyers face with respect to the exclusivity of their possible purchase and could, consequently, have a potential for higher seller's revenues relative to single posted price schemes. However, we show that the seller does not gain anything by quoting such a two-price menu. This makes the seller's price-setting job easy as there are no benefits in complicated posted price schemes and the optimal posted price strategy involves a single (inflated) price.

Optimal pricing strategies are more nuanced and complex in richer settings, as discussed in Section 2.3. We demonstrate this in a context of sequential pricing, in which the seller approaches buyers one at a time, allowing buyers to incorporate observed actions of those preceding them. In particular, a buyer may value an item higher (from the *ex ante* perspective) if none of the predecessors bought an item. We discuss how the revenue-maximizing seller should set prices sequentially in such informational setting. We first restrict the seller to precommit to a fixed pair of prices (for non-exclusive and exclusive purchase), identical for all buyers. Unlike the simultaneous one-shot setting, the optimal price pair cannot be reduced to a single price. We then examine the performance of sequential pricing in which the seller either precommits to price changes, or dynamically changes prices in response to realized buyer purchase decisions. It turns out that the seller can do better by committing to the sequential prices, while both pricing strategies generate higher expected revenues than one-time posted prices.

We close our analysis of pricing in Section 2, by showing that in the settings with exclusivity valuations the revenue-maximizing seller needs to consider and implement more complex mechanisms. (This contrasts the fact that the posted prices are the optimal mechanism for the revenue-maximizing seller with sufficient supply if buyers do not put any additional value on exclusivity.) Specifically, the presence of privately held exclusivity valuations introduces well-known difficulties of the multi-dimensional mechanism design and makes the problem of determining revenue-maximizing (optimal) mechanism analytically intractable.

In view of suboptimality of pricing mechanisms and the analytical hurdle of multi-dimensional mechanism design, in Section 3 we propose and analyze a simple-to-implement hybrid auction-pricing procedure for selling items in the presence of exclusivity. The main idea is to separate exclusive from non-exclusive allocations. The seller starts by running a standard ascending auction with reserve for exclusive allocations only, with the caveat that there will be no exclusive allocation should the auction price raise to a (dynamically adjusted) upper threshold value. When a buyer drops from the exclusivity auction, the upper threshold value is adjusted, based on the number of buyers left in the auction and the history of auction prices at which previous buyers dropped. The seller then continues running the ascending auction until either (i) only one buyer is left in the exclusivity auction and gets the item exclusively, or (ii) the current upper threshold value is reached in which case the auction is canceled, and the item is offered non-exclusively to all buyers at a price posted at that time. (Note that (ii) indicates that the exclusivity is overdemanded at an upper threshold value.)

In Section 3.1 we show that the hybrid auction-pricing procedure revenue dominates simultaneous posted prices and is the optimal mechanism for a practically relevant class of valuations. In particular, we establish optimality for *linear exclusivity* valuations in which a buyer who obtains the item exclusively can capture some of the value of its perceived competitors since, by virtue of exclusivity, none of them obtained the item. More formally, the exclusivity premium for such valuations is a linear combination of the non-exclusive allocation values of all other buyers. For example, a car dealer i who obtains the exclusive franchise contract covering some region, derives its exclusivity premium, $w_i - v_i$, from the values v_j, that its competitors, j, would not be able to realize (since they didn't get the franchise contract). However, while competitors are shut out of the market, buyer i, who got the contract exclusively, might only realize a fraction α_{ij} of v_j (because it might not serve j's customers as efficiently as j could have, had they obtained the contract).

In Section 3.2, we illustrate an application of the hybrid auction-pricing procedure to a supply chain contract setting, i.e., we show how a monopolistic revenue-maximizing supplier should allocate franchise contracts to competing retailers that have private information on the market share they could capture if they obtain the contract exclusively or non-exclusively.

Finally, in Section **??** we provide technical framework and describe the mechanism design approach which yields the hybrid auction-pricing procedure in the setting described above and which can be applied to other settings to design their own optimal mechanism implementations. In order to achieve this, we consider a model that allows for a limited scope of exclusivity. In this model, local exclusivity is determined by an underlying (publicly known) network topology. The network nodes correspond to buyers, while arcs define the set of perceived competitors for each buyer. A buyer considers a contract to be allocated exclusively to it if none of its neighbors (as defined by the network) obtains the item. We then localize the concept of linear exclusivity to the neighborhood of a node by introducing *local linear exclusivity* (LLE) in which a buyer who obtains the item exclusively can capture some of the value of its neighbors. The model with LLE valuations is defined precisely in Section 4.1. We then go on to establish the connection between the independent set problem on this network and the optimal mechanism design problem for LLE valuations. Exploiting this connection provides structural insights into which settings allow for design of computationally tractable implementations of optimal mechanism.

For example, it is straightforward to implement the optimal mechanism when all valuations are publicly known, with a natural condition that LLE valuations are bounded in the sense that competitors can never jointly benefit more than 100% of the value v_j from buyer j who didn't obtain the item. (We call such a setting *bounded local linear exclusivity*, BLLE, and provide a formal definition in Section 4.1.) In such situations, the seller cannot benefit from any exclusive allocations since the premium buyers put on exclusivity does not compensate the loss of the values of competitors that didn't obtain the item. Thus, the seller should simply sell non-exclusivity contracts to all buyers i and charge them v_i. This observation suggests that in many settings exclusivity should not even be considered. For example, a monopolist manufacturer of a product, facing potential retailers with similar capabilities (e.g., the value of business at a given location is the same for anyone who sells the product in that particular location) and with knowledge of retailers' pricing and costs of carrying the product, should not even consider exclusivity arrangements (such as making the product available exclusively at certain retailers) and should make the product widely available, i.e., at all retailers interested in carrying it.

In contrast, implementing the optimal mechanism within the LLE model could be a non-trivial task for the seller. Diversely held private information also induces inefficiencies and could result in exclusive allocations, even when such exclusive allocations are impossible with publicly known information. Thus, it might be optimal for the seller who does not know the buyers' values to allocate the item exclusively, even in the BLLE setting. Furthermore, private information also turns an allocation problem that is trivial to solve in the public information setting into a computationally hard problem in the private information setting.

We also show a somewhat surprising non-monotonicity of revenues with respect to buyers' valuations: if any of v_i or w_i increases or stays the same (the total value in the system increases), the seller's revenues could decrease. The reason for this is that buyers' information rents depend not only on the values but also on the underlying network structure and their locations in the network.

Even when buyers valuations are known to the seller, implementing the optimal mechanism might be an insurmountable challenge. We show in Section 4.2 that finding the optimal allocation is computationally hard, at least as hard as finding a maximum independent set in a network. This result indicates that in general, it will be impossible (unless P=NP) to define a procedure that would implement the optimal mechanism and, thus, one should not hope for developing any simplistic revenue-optimal pricing schemes or even optimal auctions (which would be guaranteed to end in a reasonable amount of time). Given the complexity results, the proposed hybrid auction-pricing procedure (or any other procedure) cannot be the optimal mechanism for all possible settings and all network structures.

While implementing optimal mechanism is hard in general, our results on the hybrid auction-pricing procedure not only demonstrate that in some settings it is possible to design sales procedures that overcome suboptimality of pricing (established in Section 2.4) and prohibitive complexity of optimal mechanisms in general (that was presented in Section 4.2), but also provide guidance on how to design other procedures that could be optimal for application-specific networks and industry structures.

We provide brief concluding remarks in Section 5. All proofs are relegated to the Appendix.

1.1. Related Literature

There is a large operations management literature that focuses on managing incentive conflicts in contracting, e.g., numerous supply chain models are reviewed in Cachon (2003). Specifically, pricing strategies are of practical importance due to the simplicity of the underlying contract form (i.e, posted prices) and have been widely studied (e.g., see Cachon and Feldman 2011, Nasiry and Popescu 2011). We show that there is a different set of issues that need to be addressed when exclusive allocations are a possibility. The structure of optimal mechanisms goes beyond pricing and we resort to the use of Myerson's mechanism design techniques (Myerson 1981). Mechanism design approach is standardly used in theoretical analyses involving privately held information in many contexts including operations management (e.g., see Gallien 2006, Chen 2007, Duenyas et al. 2013, Lobel and Xiao 2013), although exclusivity has not been the focus of this operations management literature. From the abstract modeling perspective, exclusivity is a form of an allocation-dependent negative externality. Thus, our work is somewhat related to the literature that studies retailers' stocking decisions, pricing decisions, and the supply chain performance by considering externalities among retailers in the same supply chain echelon (e.g., see Bernstein and Federgruen 2004, Netessine and Zhang 2005, Adida and DeMiguel 2011).

The underlying network topology is crucial in modeling and understanding the local exclusivity, i.e., exclusivity with a limited scope. Several recent papers study allocation and pricing procedures on networks with positive and negative externalities. In Candogan et al. (2012), a monopolistic seller's pricing strategies for a divisible good are examined in a public information setting with a local positive network effect, i.e., a buyer's utility is increasing with the usage level of its peers. The work of Bhattacharya et al. (2011) provides allocation and pricing procedures on the network structure in a public information setting as well, and, in addition, focuses on algorithmic issues. In another paper focused on algorithmic issues, Haghpanah et al. (2011), positive externalities are modeled so that a buyer's value is the product of a fixed private type and a known submodular function of the allocation of its peers, and the focus is on understanding algorithmic issues. In contrast, in this paper, a buyer's privately held value increases when none of its neighbors gets the item. This important distinction even impacts basic computational complexity findings. With exclusivity, the discrepancy of complexities of the revenue maximizing optimal solution does not stem out of the possible negative virtual valuations, but from the underlying network structure.

The allocation and pricing in the presence of different types of externalities, mostly motivated by problems arising in internet ad auctions, has been of interest to interdisciplinary research combining optimization, microeconomic theory, and algorithmic techniques and methodology. Since leveraging information on externality information may improve the efficiency or enhance the seller's revenues, mechanisms that use externality valuation information have been explored in different formats, (e.g., Ghosh and Mahdian 2008, Chen and Kempe 2009, Ghosh and Sayedi 2010, Constantin et al. 2010, Conitzer and Sandholm 2012).

There is also a large amount of economics literature that addresses interdependent valuations (e.g., see survey Maskin 2003). Unlike models of network externalities (e.g., Katz and Shapiro 1985, Parker and Alstyne 2005), in which buyers' valuations are often assumed to depend on the (expected) size of their associated network, valuations in our model depend on the allocation in the neighborhood. Moreover, a variety of models for externalities have been studied in detail in, e.g., Jehiel et al. (1996), Jehiel and Moldovanu (2001), Aseff and Chade (2008), Figueroa and Skreta (2011), and Brocas (2012). It is important to note that the notion of exclusivity, which we consider in this paper, is fundamentally different from that of externalities: exclusivity valuation, unlike most models with externalities, imposes no externality on buyers who do not get the contract. Still, the techniques used to analyze mechanisms with interdependent valuations have a similar flavor to those one could use for dealing with exclusivity. In particular, externalities in Jehiel et al. (1996) are modeled as private information of the rivals which is similar to the information structure in our LLE setting. In our analysis of direct optimal mechanisms, we use classical Myerson's methodology (Myerson 1981), and in Section 2.4, we briefly discuss why it is difficult to handle information structures that cannot be embedded into Myerson's framework.

2. Pricing Mechanisms

2.1. Model

A monopolist seller has unlimited supply of identical items (e.g., contracts) that can be allocated among $N = \{1, 2, \ldots, n\}$ unit-demand buyers. (Thus, we may assume there are $K = n$ items.) Buyer i's valuation for the item is v_i if obtaining the item non-exclusively, and w_i if obtaining the item exclusively (i.e., if none of the competitors obtains the item). Thus, buyer i's type is represented by a vector $\mathbf{v}_i = (w_i, v_i)$. We assume

$$w_i \geq v_i \geq 0,$$

where, without the loss of generality, we normalize buyer i's value for not getting an item to zero. Note that the difference between the exclusive and non-exclusive valuation $w_i - v_i$ can be thought of as the value of the exclusivity to buyer i.

We consider the setting in which $\mathbf{v}_i$ is independent private information, while the number of buyers is publicly known. The seller's valuation vector is assumed to be $(0, 0)$. Buyer i's private information $\mathbf{v}_i = (w_i, v_i)$ is a realization of a continuous two-dimensional random variable (W_i, V_i)

with joint cumulative distribution function F_i and with support $\Omega = [\underline{w}, \overline{w}] \times [\underline{v}, \overline{v}]$. The corresponding density function is denoted by f_i. Let $F_i^{\lambda w+(1-\lambda)v}$ denote the distribution of $\lambda W_i + (1-\lambda)V_i$ for $\lambda \in [0,1]$. (Thus, marginal distributions are F_i^w and F_i^v.) To make analysis tractable, we make a standard regularity assumption, i.e., $1 - F_i^{\lambda w+(1-\lambda)v}$ is log-concave for $\lambda \in [0,1]$.

The most prevalent way of facilitating trade is through pricing. Here, we discuss how the seller should exploit the fact that buyers have higher valuations for an item should they obtain it exclusively. We first examine the simplest pricing schemes, posted prices. We also compare their performance when buyers have exclusivity valuations with other pricing schemes that can be defined in many ways, e.g., sequential pricing schemes. We then discuss the optimal mechanisms, which go beyond pricing, for the revenue-maximizing seller to sell items when buyers have exclusivity valuations.

2.2. Preliminary: Posted Prices

We start by analyzing the simplest pricing schemes, (simultaneous) posted prices. A natural benchmark for assessing the effect of exclusivity valuations is the standard case in which there are no exclusivity premiums.

Example 1. Consider two buyers with independent private values v_i that are uniformly distributed on $[0,1]$. If there are no exclusivity valuations, i.e., if $w_i = v_i$, then the seller's expected revenue is maximized by posting a price $P = 0.5$. (The probability that a buyer buys the item at price P is the probability that $v_i > P$, and, thus, the seller's expected revenue per buyer is $P(1-P)$.) Therefore, the seller's expected revenue from two buyers is $2P(1-P) = 0.5$.

If buyers have exclusivity valuations $w_i = v_i + \varepsilon_i$, where ε_i is independent of v_i and also uniformly distributed on $[0,1]$, the seller can set the price to $P = 0.75 > 0.5$ and have the expected revenue of 0.75. Thus, the revenue-maximizing seller should exploit exclusivity valuations of buyers by inflating the posted price. We will show that $P = 0.75$ is an optimal posted price in this case. □

In order to determine the optimal posted price P, we first analyze a more complicated posted price mechanism. We study the seller posting a two-price menu $\{P_i^{10}, P_i^{11}\}_{i=1}^n$ for each buyer. Each buyer i can only accept or reject the entire menu. If buyer i is the only buyer who accepts the price menu, it will get the item exclusively and pay P_i^{10}. If there is more than one buyer accepting the price menu offered to them, every such buyer i gets the item non-exclusively and pays P_i^{11}. (The first digit in the superscript is the indicator of whether buyer i gets the item or not and the second digit is the indicator of whether any other buyer gets the item or not. Thus, 10 in the superscript indicates the exclusive allocation to buyer i, and 11 indicates a non-exclusive allocation to buyer

i.) Buyers are assumed to simultaneously accept or reject the two-price menu offered to them and the price they will pay will be determined only after all buyers' responses are received by the seller.

Note that a (single) posted price mechanism is equivalent to a special case of a two-price menu where two prices are identical: $(P_i^{10}, P_i^{11}) = (P_i, P_i)$. Somewhat surprisingly, single posted prices are sufficient to ensure optimality of expected revenues for the seller.

LEMMA 1. *A single posted price mechanism is revenue-optimal among two-price menus.*

Let P^* be the revenue-maximizing single posted price and let R^* be the expected seller revenue in this case. Let P^0 be the revenue-maximizing posted price for buyers with no exclusivity value, i.e., $w_i = v_i$. Also, let R^0 be the expected seller revenue when $w_i = v_i$ and when the single price is P^0.

PROPOSITION 1. *Consider ex-ante identical buyers. Then,* $P^* \geq P^0$ *and* $R^* \geq R^0$*.*

Proposition 1 establishes that, in settings with buyers that value exclusivity, $w_i - v_i > 0$, the seller can increase revenues by exploiting these buyers' exclusivity values. Interestingly, the posted price P^* in such settings should be inflated relative to the optimal posted price P^0 when the value of exclusivity is ignored. When facing $P^* > P^0$, buyers have to trade-off the possibility of obtaining the item exclusively because competitors might be priced out of the market with an inflated price, with the possibility of themselves being priced out of the market due to an inflated price. The seller is facing the same trade-off: inflating the price will bring higher revenues from high-value buyers but will also leave low-value buyers priced out of the market. The proof of Lemma 1 demonstrates how to compute price P^* and, in particular, establishes that $P^* = 0.75$ in Example 1. Even if the distribution of buyers' values are not known, Proposition 1 provides an easily implementable managerial guidance: a seller facing buyers that value exclusivity should inflate the posted price to capture some of these exclusivity values.

However, when buyers have exclusivity valuations, the posted price mechanism with an inflated price P^* is not optimal among all allocation and pricing procedures. This posted price mechanism can be dominated even by other pricing schemes. Intuitively, the reason for the suboptimality of the posted price mechanism is that buyers have multi-dimensional private valuations, but they can only provide a one-dimensional response, i.e., simultaneously deciding to accept the posted price or not. Next, we will show how the seller can obtain a larger revenue by sequentially exploiting buyers' exclusivity valuations.

2.3. Sequential Pricing

There are many ways to define sequential pricing schemes. Motivated by Lemma 1, we start with sequential pricing with a single price quoted for each buyer at any time point. The price may depend on decisions of previous buyers.

We still consider two buyers and the information structure that is similar as in Example 1. Unlike posted price mechanisms, the seller approaches buyers one at a time. Without loss of the generality, let the seller approach buyer 1 first and buyer 2 second. The seller quotes a price P_1 for buyer 1, and buyer 1 either accepts or rejects the offer. No matter what decision buyer 1 made, the seller continues to approach buyer 2. However, the price quoted to buyer 2 may depend on buyer 1's decision. In particular, the seller quotes P_2^0 when buyer 1 rejects the offer, while quotes P_2^1 when buyer 1 accepts the offer.

Similar to posted price mechanisms, buyers put exclusivity premium on the allocation when other buyers do not get the item. Hence, buyer 1 bears the risk whether to get the item exclusively, depending on buyer 2's later decision. However, due to the sequential selling procedure, buyer 2 has an information advantage on whether she could get the item exclusively. In particular, buyer 2 gets the item exclusively when buyer 1 rejected the offer, while gets the item non-exclusively when buyer 1 accepted the offer.

There are many different ways for the seller to set the three prices P_1, P_2^0, and P_2^1. To illustrate how the seller can benefit from the information advantage in the sequential selling procedure, we first examine that the seller is required to set a uniform price for both buyers under any circumstance in Example 2.

Example 2. Consider two buyers with $w_i = v_i + \varepsilon_i$, where v_i and ε_i are independently uniformly distributed on $[0,1]$. The seller is required to set a uniform price $P_1 = P_2^0 = P_2^1 = P^u$. Then, the seller will set $P^u = 0.83 > P^*$, and the seller's revenue R^u is 0.79, which is also larger than R^*. $\square$

Since buyers have more information on how likely they will get items exclusively, the seller can further exploit such exclusivity valuations and inflate the posted prices more. It is straightforward to conjecture that the seller can obtain a higher revenue by allowing different prices for different buyers under different circumstances. One such selling procedure is that the seller preannounces all three prices and commits to these prices. This case is analyzed in the following Example 3.

Example 3. Consider two buyers with $w_i = v_i + \varepsilon_i$, where v_i and ε_i are independently uniformly distributed on $[0,1]$. The seller sets and commits to P_1, P_2^0, and P_2^1. It turns out that the seller sets $P_1 = 0.92$, $P_2^0 = 0.86$, and $P_2^1 = 0.83$. The seller's revenue R^c is 0.80, which is larger than R^u and R^*. $\square$

All prices are inflated more than simultaneous prices. Note that the price quoted for buyer 2 when she is known for sure to get the item non-exclusively, which is supposed to be 0.5, is also inflated in order to guarantee that the exclusivity valuation of buyer 1 can be well exploited.

Commitment can be achieved in the short run, however, it is possible that the seller may not be able to commit to these prices in the long run. This indicates that the seller may reset the prices when approaching buyer 2. The next example analyzes the situation when the seller does not have commitment on the prices.

Example 4. Consider two buyers with $w_i = v_i + \varepsilon_i$, where v_i and ε_i are independently uniformly distributed on $[0,1]$. The seller sets P_1, P_2^0, and P_2^1 at the beginning but cannot commit. It turns out that the seller sets $P_1 = 0.81$, $P_2^0 = 0.83$, and $P_2^1 = 0.5$. The seller's revenue R^{nc} is 0.77, which is larger than R^* and smaller than R^c and R^u. $\square$

Commitment is beneficial for the seller, while the seller who uses sequential pricing and is lack of commitment can also do better than posted prices.

The intuition underlying Example 2, 3, and 4 is as follows. Buyers who are approached later have clearer pictures on whether they will get items exclusively or not. If none of the previous buyers gets the item, a buyer will value an item under such situation higher than the one under posted prices. Thus, the seller may set a higher price to exploit such higher valuations and obtain a higher revenue. Also note that, unlike posted prices, the seller can do better with state-contingent prices (as in Example 3) than with a uniform price for each buyer (Example 2).

Furthermore, there are other ways to define sequential pricing schemes. One possibility is that the seller quotes each buyer a price menu (P_i^{10}, P_i^{11}). If buyer 1 accepts P_1^{10}, the seller will not approach buyer 2 and buyer 1 will get the item exclusively. If buyer 1 accepts P_1^{11}, the seller can approach buyer 2 and will quote buyer 2 price P_2^{11}. (Since buyer 2 will only get the item non-exclusively by accepting the price menu, there is no need to quote P_2^{10}.) Upon accepting P_2^{11}, both buyers get items. However, if buyer 2 rejects P_2^{11}, buyer 1 may still get the item exclusively under the non-exclusive price. If buyer 1 rejects (P_i^{10}, P_i^{11}), the seller will approach buyer 2 with price P_2^{10}. (Similar to the previous case, there is no need for the seller to quote buyer 2 P_2^{11}.)

Unlike posted prices, sequential pricing schemes with price menus cannot always reduce to the schemes with a single price. We illustrate this with Example 5.

Example 5. Consider two buyers with $w_i = v_i + \varepsilon_i$, where v_i and ε_i are independently uniformly distributed on $[0,1]$. The seller quotes buyer i a price menu (P_i^{10}, P_i^{11}). It turns out that the seller sets $P_1^{10} = 1.33$, $P_1^{11} = 0.88$, $P_2^{10} = 0.82$, and $P_2^{11} = 0.77$. The seller's revenue R^{pm} is 0.80.

If the seller is restricted to quote the same price the first buyer, i.e., $(P_1^{10} = P_1^{11})$, the seller sets $P_1^{10} = P_1^{11} = 1.03$, $P_2^{10} = 0.83$, and $P_2^{11} = 1.0$. The seller's revenue is lower and it is 0.77. $\square$

2.4. Suboptimality of Pricing

Even though exclusivity valuations may be exploited by pricing, neither posted prices nor sequential pricing schemes are optimal among all possible selling procedures when buyers have exclusivity valuations. Thus, we next consider the structure of optimal allocation and pricing mechanisms that exploit both the exclusivity and non-exclusivity valuations. By the Revelation Principle (Myerson 1981), we consider direct mechanisms that allocate items based on buyers' reports. Reports from all buyers are $\widehat{\mathbf{v}} = (\widehat{\mathbf{v}}_i, \widehat{\mathbf{v}}_{-i}) \in \Omega^n$. A direct mechanism specifies the allocation ($p_i : \Omega^n \to \{0,1\}$ is buyer i's probability to get an item) and payments ($m_i : \Omega^n \to \mathbf{R}$ is the payment from buyer i to the seller) for each $\widehat{\mathbf{v}} \in \Omega^n$. If buyer i does not participate, it does not get any item. (Although we focus on the deterministic mechanisms here, results in this section can be shown to hold for the non-deterministic mechanisms by redefining allocation probabilities as in Section 4.2.)

Buyer i's ex post utility when reporting its type as $\widehat{\mathbf{v}}_i$, while its true type is $\mathbf{v}_i$, and when other buyers report $\mathbf{v}_{-i}$, is

$$\begin{aligned} U_i\left(\widehat{\mathbf{v}}_i, \mathbf{v}_i, \mathbf{v}_{-i}\right) = {} & w_i p_i\left(\widehat{\mathbf{v}}_i, \mathbf{v}_{-i}\right) \prod_{j \neq i} \left(1 - p_j\left(\widehat{\mathbf{v}}_i, \mathbf{v}_{-i}\right)\right) \\ & + v_i p_i\left(\widehat{\mathbf{v}}_i, \mathbf{v}_{-i}\right) \left(1 - \prod_{j \neq i} \left(1 - p_j\left(\widehat{\mathbf{v}}_i, \mathbf{v}_{-i}\right)\right)\right) - m_i\left(\widehat{\mathbf{v}}_i, \mathbf{v}_{-i}\right). \end{aligned} \tag{1}$$

Let p denote $\{p_i\}_{i=1}^n$ and m denote $\{m_i\}_{i=1}^n$ and, thus, the LP relaxation of the seller's Revenue Maximization Problem (General-RMP) is

$$\max_{p,m} \sum_{i=1}^{n} \int m_i\left(\mathbf{v}_i, \mathbf{v}_{-i}\right) dF(\mathbf{v})$$

subject to

$$\begin{aligned} \text{(EPIC)} \quad & U_i\left(\mathbf{v}_i, \mathbf{v}_i, \mathbf{v}_{-i}\right) \geq U_i\left(\widehat{\mathbf{v}}_i, \mathbf{v}_i, \mathbf{v}_{-i}\right) \text{ for all } i \text{ and all } \mathbf{v}_i, \widehat{\mathbf{v}}_i, \mathbf{v}_{-i}, \\ \text{(EPIR)} \quad & U_i\left(\mathbf{v}_i, \mathbf{v}_i, \mathbf{v}_{-i}\right) \geq 0 \text{ for all } i \text{ and all } \mathbf{v}_i, \mathbf{v}_{-i}, \\ \text{(Feasibility)} \quad & 0 \leq p_i\left(\mathbf{v}\right) \leq 1 \text{ for all } i, \end{aligned}$$

where (EPIC) is the ex post incentive compatibility constraint to ensure truth-telling and (EPIR) is the ex post individual rationality constraint to ensure participation. (Throughout the rest of this paper, we will simplify the notation by denoting $U_i\left(\mathbf{v}_i, \mathbf{v}_i, \mathbf{v}_{-i}\right)$ by $U_i\left(\mathbf{v}_i, \mathbf{v}_{-i}\right)$.)

Example 6. Suppose no buyer puts a premium on exclusivity, i.e., $w_i = v_i$ for all i. Here, given unit-demand buyers and sufficient supply to meet demand, the Problem (General-RMP) decomposes to n Myerson's optimal auctions with one item and one bidder each. The optimal auction is a second price auction with reserve (Myerson 1981), which, in the case of a single buyer, reduces to determining whether the reserve is met. Thus, the posted price mechanism is optimal: there exists a price a (auction reserve price) such that every buyer i willing to pay a gets the item. $\square$

This example demonstrates why posted prices are the most widespread way of facilitating trade when the seller has no capacity issues and can meet all demand. Note that, when the supply is smaller than the demand, a second price auction with reserve is still optimal, while this optimal mechanism can also be achieve by a standard ascending auction.

However, if exclusivity has value, not only is the use of posted prices suboptimal, but finding an optimal mechanism becomes a hard problem to tackle. The rest of this paper is devoted towards this issue.

Example 7. We consider the same valuation setting as in Example 1, i.e., $w_i = v_i + \varepsilon_i$, where v_i and ε_i are independently and uniformly distributed on $[0,1]$. Problem (General-RMP) is a linear programming problem when the support Ω is finite. We then find the seller's optimal expected revenue by discretizing the type space. As shown in the Appendix, the seller's optimal expected revenue stabilizes around 0.9. Recall the expected revenue from posted prices is 0.75. There is still a significant gap between the numerical result of the optimal mechanism and the expected revenue of posted price schemes. $\square$

Problem (General-RMP), as well as the corresponding social surplus maximization problem, is a multi-dimensional mechanism design problem and is extremely difficult to solve analytically. The core difficulty lies in defining information rents in the case of privately held multi-dimensional valuations (i.e., exclusive and non-exclusive valuations in this paper). Incentive compatibility constraints in the multi-dimensional mechanism design problem can be characterized as monotonicity and integrability conditions, and, as pointed out in Jehiel et al. (1999), the integrability condition is the primary source of difficulties in generalizing the standard Myerson approach. (We demonstrate this in the proof of Proposition **??**.) Furthermore, it is well known that solutions to multi-dimensional mechanism design problems are sensitive to various details of the environment, e.g., the seller's belief about the buyers' types. Hence, there is little hope for finding closed-form solutions. This unappealing feature of multi-dimensional mechanism design problems has been demonstrated by Rochet and Choné (1998), Armstrong (1996), and Manelli and Vincent (2007).

In general, exclusivity makes the design of revenue-maximizing sales procedures challenging for the seller. Posted prices are not optimal (Example 7) and implementable procedures for determining optimal allocation and pricing decisions are beyond reach (multi-dimensional mechanism design problem). Given the suboptimality of pricing and the analytical hurdle for finding the optimal mechanisms, we will next propose a hybrid auction-pricing procedure, which is easy to implement, revenue dominates posted prices, and is also optimal under some specific settings.

3. Hybrid Auction-Pricing Procedure

With the coexistence of exclusive and non-exclusive allocations, a natural selling procedure would associate selling exclusivity and non-exclusivity together. Recall that when selling exclusivity, ascending auction implements the optimal mechanism for exclusivity as the demand for exclusivity exceeds the supply. Meanwhile, when selling non-exclusivity, posted prices implement the optimal mechanism for non-exclusivity as the demand equals to the supply. There might be different ways to combine selling exclusivity and non-exclusivity. However, the only feasible combination is to sell exclusivity first, since exclusivity is deteriorated after a non-exclusive allocation.

In this section, we propose a simple hybrid auction-pricing procedure. The main idea behind the hybrid auction-pricing procedure is to separate exclusive and non-exclusive allocation decisions. The procedure first attempts to allocate exclusively by an ascending auction with reserve. The auction ends if the market clears before the ascending auction reaches a ceiling threshold price. If this ceiling price is achieved, the auction ends without any exclusive allocations, and items are offered to buyers at a fixed price. (Note that posted price mechanisms can be implemented by choosing not to run the exclusivity auction by setting a ceiling price at zero and by choosing the posted price at which items will be offered.) This procedure generalizes and outperforms posted prices, and is in fact an optimal mechanism for one class of information and network structures.

To illustrate this hybrid auction-pricing procedure, let's still consider two *ex ante* identical buyers. There are three parameters in the procedure, r the reserve price for the ascending auction, R the upper threshold reserve for the ascending auction, and P the price for non-exclusivity. Note that by setting $r = R = P = P^*$, the hybrid auction-pricing procedure reduces to posted prices in Section 2.2.

In Example 8, we illustrate the intuition underlying the hybrid auction-pricing procedure by examining the performance of directly associating the optimal mechanisms for exclusivity and non-exclusivity.

Example 8. Consider two buyers with $w_i = v_i + \varepsilon_i$, where v_i and ε_i are independently uniformly distributed on $[0,1]$. Let the seller set $r = 0.81$, which is the reserve in the optimal mechanism for exclusivity, and $P = 0.5$, which is the posted price in the optimal mechanism for non-exclusivity. Assume buyers truthfully report under the hybrid auction-pricing procedure. Then the seller sets $R = 1.94$ and the revenue is 0.85. □

Note that the revenue obtained here is an upper bound for the seller's revenue under asymmetric information by directly appending the optimal mechanism for non-exclusivity to the optimal mechanism for exclusivity. This revenue is larger than any of the revenue we can get from pricing mechanisms.

Next, we formally describe the hybrid auction-pricing procedure among n buyers. The procedure consists of two phases. (Note that all prices in the procedure are for each buyer and could be different across buyers when they are not *ex ante* identical. We suppress the buyer index for the purpose of presentation.)

Hybrid Auction-Pricing Procedure

Phase I (Auction).
Step 1. Run ascending auction with reserve r and upper threshold value $R\left(k, P^{10}_{-k}\right)$, where k is the number of buyers left in the ascending auction and P^{10}_{-k} is the price history at which previous $n-k$ buyers dropped.

(i) If a buyer drops at a price $P^{10}_{(n-k)+1}$ before $R\left(k, P^{10}_{-k}\right)$, go to Step 2.
(ii) If no buyer drops before $R\left(k, P^{10}_{-k}\right)$, go to Phase II.

Step 2. If $k > 1$, go to step 3. Otherwise, go to Step 4.

Step 3. Incorporate $P^{10}_{(n-k)+1}$ into P^{10}_{-k}, and update the price history to $P^{10}_{-(k-1)}$. Set $r = P^{10}_{(n-k)+1}$ and calculate a new upper threshold $R\left(k-1, P^{10}_{-(k-1)}\right)$. Go back to Step 1.

Step 4. Auction ends and allocate the item exclusively to the buyer who just dropper at price $P^{10}_{(n-k)+1}$.

Phase II (Pricing).
The seller cancels the exclusivity ascending auction with the remaining k buyers and sells items non-exclusively to each of these k buyers at a predetermined price $P^{11}\left(k, P^{10}_{-k}\right)$. Note that $P^{11}\left(k, P^{10}_{-k}\right)$ is different from $R\left(k, P^{10}_{-k}\right)$.

3.1. Optimality of Hybrid Auction-Pricing Procedure

In this section, we analyze the optimality of the hybrid auction-pricing procedure with two buyers under some specific settings. One such setting is that buyer i's exclusivity valuation w_i is derived from her privately held valuation v_i for non-exclusive allocation, and the valuations v_j of buyer i's competitor. In particular, we assume that buyer i's exclusivity premium is a proportion of the non-exclusivity valuation of buyer i's competitor j:

$$w_i = v_i + \alpha_{ij} v_j \tag{2}$$

with publicly known non-negative parameter $\alpha_{ij} \leq 1$. If (2) holds, we say that valuations satisfy *linear exclusivity* (LE).

With LE valuations, we first illustrate how the seller should set r, R, and P in the hybrid auction-pricing procedure.

EXAMPLE 9. Consider two buyers with LE valuations with $\alpha_{12} = \alpha_{21} = \frac{1}{2}$, i.e., $w_i = v_i + \frac{1}{2} v_{-i}$, where v_i is uniformly distributed on $[0,1]$. Then, it is optimal for the seller to set $r = 0.6$, $R = 1$, and $P = 0.67$. The seller's revenue is 0.58. $\square$

Next we will show that the hybrid auction-pricing procedure implements the seller revenue-maximizing mechanism under LE valuations. LE valuations enable us to find the analytical solution to the mechanism design problem with exclusivity valuations. We first describe in detail the allocation and pricing rules of the optimal mechanism in this specific setting. This helps to illustrate how one can try to design other procedures for allocating and pricing exclusivity that could turn out to be optimal in some other settings of interest. Then we use this detailed description to design the hybrid auction-pricing procedure as the only candidate for implementing the optimal mechanism, assuming that all buyers respond truthfully to announced prices. We conclude the argument by showing that truthful responding is the perfect Bayesian equilibrium.

We first present the direct optimal mechanism. Let $v_1 = x$ and $v_2 = y$, and denote buyers' exclusivity premia by $\gamma_1(y) = \alpha_{12} y$ and $\gamma_2(x) = \alpha_{21} x$. Let $x_0(y)$ and $y_0(x)$ be functions implicitly defined by

$$\psi_1(x_0(y)) = -\gamma_1(y) \text{ and } \psi_2(y_0(x)) = -\gamma_2(x), \tag{3}$$

respectively, and let $x_1(y)$ and $y_1(x)$ be functions implicitly defined by

$$\psi_1(x_1(y)) - \psi_2(y) = \gamma_2(x_1(y)) - \gamma_1(y) \text{ and } \psi_1(x) - \psi_2(y_1(x)) = \gamma_2(x) - \gamma_1(y_1(x)), \tag{4}$$

respectively. Let x_2 and y_2 denote the roots of

$$\psi_1(x_2) = \gamma_2(x_2) \text{ and } \psi_2(y_2) = \gamma_1(y_2), \tag{5}$$

respectively, and let x_* and y_* denote the roots of

$$x_0(y_*) = x_*,\ y_0(x_*) = y_*, \text{ and } x_1(y_*) = x_*\ (y_1(x_*) = y_*). \tag{6}$$

The regularity assumption ensures that these functions are well defined. Let $\alpha_{12}, \alpha_{21} \leq 1$ and, thus, $\psi_1(\cdot) - \gamma_2(\cdot)$ and $\psi_2(\cdot) - \gamma_1(\cdot)$ are increasing functions.

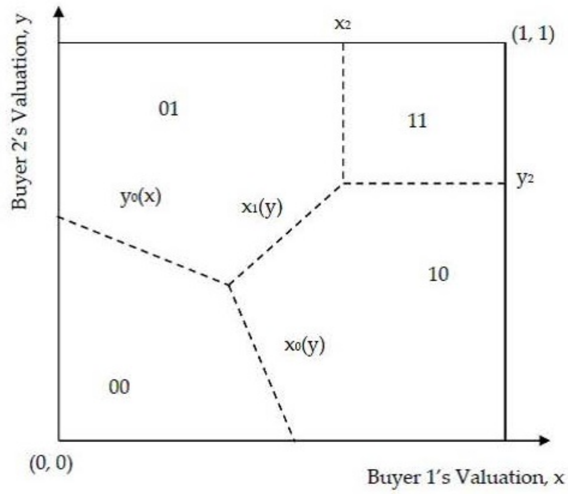

Figure 1 **Optimal Mechanism with v Uniformly Distributed on $[0,1]$.**

PROPOSITION 2. *The optimal allocation is*

$$\begin{cases} (p_1,p_2)=(0,0)\,, \text{ if } x<x_0\,(y) \text{ and } y<y_0\,(x)\,, \\ (p_1,p_2)=(1,1)\,, \text{ if } x>x_2 \text{ and } y>y_2, \\ (p_1,p_2)=(1,0)\,, \text{ if } x>\max\{x_0\,(y)\,,x_1\,(y)\} \text{ and } y<y_2, \\ (p_1,p_2)=(0,1)\,, \text{ if } y>\max\{y_0\,(x)\,,y_1\,(x)\} \text{ and } x<x_2. \end{cases}$$

The optimal payments are

$$(m_1,m_2)=\begin{cases} (0,0), & \text{if } (p_1,p_2)=(0,0)\,, \\ (x_2,y_2) & \text{if } (p_1,p_2)=(1,1)\,, \\ (\max\{x_1\,(y)\,,x_0\,(y)\}+\alpha_{12}y,0), & \text{if } (p_1,p_2)=(1,0)\,, \\ (0,\max\{y_1\,(x)\,,y_0\,(x)\}+\alpha_{21}x), & \text{if } (p_1,p_2)=(0,1)\,. \end{cases}$$

(In Appendix, we generalize the optimal mechanism to n buyers.)

Figure 1 illustrates the optimal mechanism for $F_1=F_2=\mathbf{U}\,[0,1]$. In this case, we have

$$x_0\,(y)=(1-\alpha_{12}y)\,/2,\; y_0\,(x)=(1-\alpha_{21}x)\,/2,$$

$$x_1\,(y)=y\,(2-\alpha_{12})\,/\,(2-\alpha_{21})\,,\; y_1\,(x)=x\,(2-\alpha_{21})\,/\,(2-\alpha_{12})\,,$$

$$x_2=1/\,(2-\alpha_{21})\,,\; y_2=1/\,(2-\alpha_{12})\,,$$

and

$$(x_*,y_*)=((2-\alpha_{12})\,/\,(4-\alpha_{12}\alpha_{21})\,,(2-\alpha_{21})\,/\,(4-\alpha_{12}\alpha_{21}))\,.$$

The region labeled 00 corresponds to no-allocation, $(p_1,p_2)=(0,0)$, region 10 corresponds to the exclusive allocation to player 1, $(p_1,p_2)=(1,0)$, region 01 corresponds to the exclusive allocation to player 2, $(p_1,p_2)=(0,1)$, and region 11 corresponds to the non-exclusive allocation, $(p_1,p_2)=(1,1)$.

Note that if $\alpha_{12} = \alpha_{21} = 0$, then $x_0 = y_0 = x_2 = y_2 = 1/2$, and all four regions are rectangular. If, instead, $\alpha_{12} = 1$ or $\alpha_{21} = 1$, the non-exclusive allocation region 11 disappears.

We next present an ascending price auction that implements the seller revenue-maximizing mechanism. In order to simplify the exposition, we assume $F_1 = F_2 = \mathbf{U}[0,1]$. Furthermore, without loss of generality, we focus on the non-degenerate case with $\alpha_{12} < 1$ and $\alpha_{21} < 1$. (Otherwise, a non-exclusive allocation is not possible and the problem reduces to selling the exclusive allocation to the highest bidder, i.e., the classical single item optimal auction of Myerson, 1981)

Figure 1 provides guidance for designing an ascending auction. The auction has to start with the reservation prices so that buyer 1 with $v_1 = x \leq x_*$ and buyer 2 with $v_2 = y \leq y_*$ do not even participate in the auction. This shows that auction prices are not anonymous when buyers are not symmetric (which is the case for $\alpha_{12} \neq \alpha_{21}$). Throughout the auction, each buyer faces an increasing price for the exclusive allocation (regions 10 and 01) and clinches the exclusive allocation if the rival drops from the auction because its price for exclusive allocation becomes too high. If $v_1 = x \geq x_2$ and $v_2 = y \geq y_2$, the non-exclusive allocation (region 11) is optimal: this is achieved by simply stopping the auction when auction prices imply $v_1 = x \geq x_2$ and $v_2 = y \geq y_2$. If only one of the buyers accepts the offer at the beginning of the auction, the auction goes into a second stage, in which a new take-it-or-leave-it offer will be made.

We next describe the hybrid auction-pricing procedure in detail.

Each of the two bidders is facing their own increasing price for exclusive allocation. Prices increase as time $t \in [0, T]$ increases. At time t, buyer 1 is quoted price

$$P_1^{10}(t) = \left(\frac{\alpha_{12}(2 - \alpha_{21})}{2 - \alpha_{12}} + 1 \right) \left(x_* + \frac{x_2 - x_*}{T} t \right),$$

for the exclusive allocation, while buyer 2 is quoted price

$$P_2^{10}(t) = \left(\frac{\alpha_{21}(2 - \alpha_{12})}{2 - \alpha_{21}} + 1 \right) \left(y_* + \frac{y_2 - y_*}{T} t \right).$$

for the exclusive allocation.

If neither buyer accepts the price quoted to them at time $t = 0$ (reserve price), the auction ends immediately with no allocation and no payments.

If only one buyer (say buyer 1) accepts the offer at time $t = 0$ (reserve price), the auction goes into a second stage. A take-it-or-leave-it offer is then presented to buyer 1: getting the item exclusively with price

$$\widetilde{P}_1^{10}(y) = \frac{1}{2} + \frac{1}{2}\alpha_{12} y,$$

where buyer 2 reveals its type $v_2 = y$ (to both the seller and buyer 1). If buyer 2 accepts the offer at time $t = 0$, the corresponding price is

$$\widetilde{P}_2^{10}(x) = \frac{1}{2} + \frac{1}{2}\alpha_{21}x,$$

where buyer 1 reveals its type $v_1 = x$ (to both the seller and buyer 2).

Otherwise, the auction continues until buyer i drops from the auction at time $0 < t^* < T$, which ends the auction with the rival obtaining the item exclusively at the price $P_{-i}^{10}(t^*)$. If both buyers stay in the auction until time T, the auction ends with the non-exclusive allocation that charges x_2 to buyer 1 and y_2 to buyer 2.

Note that if buyers are *ex ante* symmetric (i.e., if $\alpha_{12} = \alpha_{21}$), then $P_1^{10}(t) = P_2^{10}(t)$.

Proposition 3. *With LE valuations, there exist parameters r, R, and P such that*
i) If buyers bid/respond truthfully, the outcome of the hybrid auction-pricing procedure matches that of the seller's optimal mechanism.
ii) Bidding truthfully in the hybrid auction-pricing procedure is a perfect Bayesian equilibrium.

Establishing Proposition 3 is possible because LE valuations fit one of the restricted information structures for which the multi-dimensional mechanism design problem can be solved, e.g., $w_i = v_i + \Xi_i(v_{-i})$, where Ξ_i is a publicly known function. In fact, it is known that the standard Myersonian approach is applicable provided that there exists a one-dimensional representation of the multi-dimensional privately held information, e.g., $w_i = \Theta_i(v_i)$, where Θ_i is a publicly known function. (In fact, Figueroa and Skreta (2011) provides a rather general framework for identifying information structures for which Myersonian approach to solving the mechanism design problem is applicable. However, that work does not provide insights on implementation nor their computational tractability and practicality, which is exactly our focus here.) We illustrate this with singling out two additional valuation structures for which an analogous approach yields procedures that implement the optimal mechanism: additive exclusivity, $w_i = v_i + \theta_i^0$ (publicly known additive premium that buyer i is ready to pay for obtaining the item exclusively), and multiplicative exclusivity, $w_i = \theta_i^1 v_i$ (publicly known multiplier buyer i is ready to pay for obtaining the item exclusively), where θ_i^0 and θ_i^1 are publicly known constants. (The latter is the information structure in the model of Aseff and Chade (2008).)

By following the approach laid out here, it is straightforward to establish that the hybrid auction-pricing procedure (with minor modifications of the threshold update calculation in the ascending auction phase) is the optimal mechanism implementation for additive exclusivity valuations. However, the hybrid auction-pricing procedure cannot be optimal for all information structures that allow one-dimensional representation, as indicated by the optimal mechanism under multiplicative exclusivity (which can be established by applying our approach to establish a different implementation or by inspecting the optimal mechanism described by Aseff and Chade (2008)).

3.2. An Application to Supply Chain Contracts

Consider a two-period game between a monopolistic supplier and n retailers. In period 1, the monopolistic supplier is selling identical buyback contracts (ω, b) to n retailers in a market with stochastic demand D, where ω is the per unit price charged to the retailer and b is the per unit payment given to the retailer for any remaining goods. (Revenue-sharing contracts are also included in this framework, since they are equivalent to buyback contracts.) Note that the supplier designs the allocation and pricing procedures to sell the contracts, while (ω, b) is pre-announced and fixed all through the two-period game. Let G denote the cumulative distribution function and μ denote the mean value of D.

If retailer i gets the contract non-exclusively, i.e., at least one of its neighbors gets the contract as well, the demand that retailer i can seize in period 2 is $D_i = a_i^{11}D$; if retailer i gets the contract exclusively, i.e., none of its neighbors gets the contract, the demand that retailer i can seize in period 2 is $D_i = a_i^{10}D$; otherwise, the retailer gets zero. Furthermore, the market share vector (a_i^{10}, a_i^{11}) is retailer i's private information, and satisfies $1 > a_i^{10} \geq a_i^{11} \geq 0$ and

$$\sum_{i=1}^{n} a_i^{11} \leq 1.$$

In addition, let I^P denote the set of retailers getting the contracts and a_i denote retailer i's generic market share ($a_i = a_i^{10}$, a_i^{11}, or 0).

In period 2, retailer $i \in I^P$ decides the order quantity q_i from the supplier. Let c_s be the supplier's per unit production cost and c_r be the retailer's per unit marginal cost. If the retailer does not satisfy the demand, there incurs a per unit goodwill penalty g_r on the retailer and g_s on the supplier. Also let the supplier's salvage value be v and the exogenous revenue rate of the product be r. Furthermore, the expected sales for the ordered quantity q_i is defined as

$$ES_i(q_i) \triangleq E_D[\min(q_i, a_i D)] = q_i - a_i \int_0^{q_i/a_i} G(D)\, dD,$$

the expected leftover inventory is defined as

$$EI_i(q_i) \triangleq E_D[\max(q_i - a_i D, 0)] = q_i - ES_i(q_i),$$

and the expected lost-sales is defined as

$$EL_i(q_i) \triangleq E_D[\max(a_i D - q_i, 0)] = a_i \mu - ES_i(q_i).$$

Then, retailer i's expected profit is

$$\pi_i = \begin{cases} rES_i(q_i) + vEI_i(q_i) - g_r EL_i(q_i) - c_r q_i - T_i, & \text{if } i \in I^P; \\ 0, & \text{otherwise.} \end{cases}$$

where T_i is the payment from retailer i to the supplier. Meanwhile, the supplier's profit is

$$\pi_s = \sum_{i \in I^P} \left(T_i - g_s EL_i(q_i) - c_s q_i \right).$$

Next, we study each retailer's optimal order quantity and the corresponding expected profits of retailers and the supplier under the buyback contract (ω, b). Given the buyback contract (ω, b), the payment is $T_i = \omega q_i - bEI_i(q_i)$, and, thus, the profit of retailer $i \in I^P$ can be rewritten as

$$\pi_i = (r + g_r - c_r - \omega) q_i - (r - v + g_r - b) a_i \int_0^{q_i/a_i} G(D)\, dD - a_i g_r \mu.$$

First order conditions indicate that the optimal order quantity is

$$q_i^*(\omega, b) = a_i G^{-1}\left(\frac{r + g_r - c_r - \omega}{r - v + g_r - b} \right).$$

Therefore, retailer i's optimal profit is

$$\pi_i^*(\omega, b) = a_i \Pi_r - a_i g_r \mu,$$

where

$$\Pi_r(\omega, b) = (r + g_r - c_r - \omega) G^{-1}\left(\frac{r + g_r - c_r - \omega}{r - v + g_r - b} \right) - (r - v + g_r - b) \int_0^{G^{-1}\left(\frac{r+g_r-c_r-\omega}{r-v+g_r-b}\right)} G(D)\, dD.$$

Note that $\Pi_r(\omega, b)$ can be regarded as the aggregate profit (after compensating the retailer's goodwill penalty) of retailers. Furthermore, with $q_i^*(\omega, b)$, the supplier's profit is

$$\pi_s(\omega, b) = \sum_{i \in I^P} \left(a_i \Pi_s(\omega, b) - a_i g_s \mu \right),$$

where

$$\Pi_s(\omega, b) = (\omega + g_s - c_s) G^{-1}\left(\frac{r + g_r - c_r - \omega}{r - v + g_r - b} \right) - (b + g_s) \int_0^{G^{-1}\left(\frac{r+g_r-c_r-\omega}{r-v+g_r-b}\right)} G(D)\, dD.$$

Note that $\Pi_s(\omega, b)$ can be regarded as the aggregate profit (after compensating the supplier's goodwill penalty) of the supplier.

In order to focus on the allocation and pricing of local exclusivity in period 1, we consider the buyback contract (ω^*, b^*) that coordinates the supply chain and gives the supplier zero profit in period 2. In fact, how to split the period 2 profit in the coordinated supply chain depends on the bargaining powers between the supplier and retailers, and, thus, all splits are possible in real business scenarios. Theoretically speaking, in the First Best, i.e., (a_i^{10}, a_i^{11}) is publicly known, all buyback contracts that coordinate the supply chain, including (ω^*, b^*), can also give the supplier the maximal expected profit in the two-period game. However, in the Second Best, i.e., (a_i^{10}, a_i^{11}) is private information, the performance of the supplier in the two-period game depends on how to split the profit in the coordinated supply chain. On one extreme, the buyback contract that coordinates the supply chain and gives retailers zero profits can give the supplier the maximal expected profit in the two-period game and can also achieve the First Best. On the other extreme, the First Best can not be achieved, and the buyback contract (ω^*, b^*) provides a lower bound on the performance of the supplier in the two-period game among all possible buyback contracts that coordinate the supply chain. The reason is that (ω^*, b^*) maximizes the information rent that the supplier has to give to each retailer in order to induce truthful reporting in period 1. Therefore, we focus on this later extreme to study the allocation and pricing of local exclusivity in period 1. Note that the supplier will get non-zero profit in period 1 by selling the buyback contract to retailers, even though the supplier gets zero in period 2 under (ω^*, b^*).

We then characterize (ω^*, b^*), which coordinates the supply chain and gives the supplier zero profit in period 2. Let Π_c denote the aggregate profit of the coordinated supply chain (after compensating all goodwill penalty). Following the standard results in Cachon (2003), this buyback contract (ω^*, b^*) must satisfy

$$\begin{aligned} r + g_r - c_r - \omega^* &= \lambda \left(r + g_s + g_r - c_s - c_r \right), \\ r - v + g_r - b^* &= \lambda \left(r - v + g_s + g_r \right), \\ \text{and } \left(1 - \lambda\right) \Pi_c &= g_s \mu. \end{aligned}$$

Note that the simplest form of supply chain contracts, the wholesale-price contract with wholesale price set as c_s, is an example of (ω^*, b^*), when $g_r = g_s = v = c_r = 0$. Hence, with (ω^*, b^*), we have $\Pi_r (\omega^*, b^*) = \lambda \Pi_c$, and the profit of retailer $i \in I^P$ under (ω^*, b^*) is

$$\pi_i^* \left(\omega^*, b^*; a_i \right) = a_i \left(\Pi_c - g_r \mu \right).$$

Therefore, instead of focusing on privately held (a_i^{10}, a_i^{11}), we can directly consider privately held profits $\pi_i^* (\omega^*, b^*; a_i^{10})$ and $\pi_i^* (\omega^*, b^*; a_i^{11})$ for each retailer and, moreover, the structure of (a_i^{10}, a_i^{11}) can be carried over to $(\pi_i^* (\omega^*, b^*; a_i^{10}), \pi_i^* (\omega^*, b^*; a_i^{11}))$.

4. Local Exclusivity on a Network: Complexity of Implementing Optimal Mechanism

In general, when buyers have exclusivity valuations, there may exist another level of complexity due to competing relationships in additional to the analytical hurdle from multi-dimensional mechanisms design. In particular, even when we can analytically solve the mechanism design problem, it is not clear whether there exists a reasonable procedure to implement the optimal mechanism. In this section, we show this difficulty by presenting optimal mechanisms with a network generalization of LE valuations. We first formally describe this generalization and then illustrate the range of difficulties the seller might face with implementing the optimal mechanism.

4.1. Local Exclusivity on a Network

As in the opening examples, the scope of exclusivity might be limited to an area in a geographic or demographic network, to a market segment in a competition network, or to a group of people in a social network. We now formally define local exclusivity on a network.

Relationships among buyers are defined by a network (N, E) where E is the 0-1 adjacency matrix: $e_{ij} = 1$ if and only if buyer i considers buyer j, $j \neq i$, to be related to it (e.g., i considers j as a competitor or i and j are geographical neighbors or directly connected in a social network). Let $S(i) \subseteq N \setminus \{i\}$ denote the set of buyer i's neighbors, i.e., the set of all other buyers that i considers to be related to it: $S(i) = \{j \in N : e_{ij} = 1\}$.

Buyer i has exclusivity valuation w_i for the item if none of its *neighbors* $j \in S(i)$ gets an item, and has non-exclusivity valuation v_i if there is a neighbor $j \in S(i)$ who also obtains the item. We still consider the setting in which $\mathbf{v}_i$ is private information, while network (N, E) is publicly known.

Direct mechanisms are defined as in Section 2.4, and, thus, buyer i's ex post utility when reporting its type as $\widehat{\mathbf{v}}_i$, while its true type is $\mathbf{v}_i$, and when other buyers report $\mathbf{v}_{-i}$, is

$$\begin{aligned} U_i\left(\widehat{\mathbf{v}}_i, \mathbf{v}_i, \mathbf{v}_{-i}\right) = {} & w_i p_i\left(\widehat{\mathbf{v}}_i, \mathbf{v}_{-i}\right) \prod_{j \in S(i)} \left(1 - p_j\left(\widehat{\mathbf{v}}_i, \mathbf{v}_{-i}\right)\right) \\ & + v_i p_i\left(\widehat{\mathbf{v}}_i, \mathbf{v}_{-i}\right) \left(1 - \prod_{j \in S(i)} \left(1 - p_j\left(\widehat{\mathbf{v}}_i, \mathbf{v}_{-i}\right)\right)\right) - m_i\left(\widehat{\mathbf{v}}_i, \mathbf{v}_{-i}\right). \end{aligned} \tag{7}$$

Therefore, the LP relaxation of the seller's Revenue Maximization Problem (General-RMP) is also similar to the one in Section 2.4 except for substituting $U_i\left(\widehat{\mathbf{v}}_i, \mathbf{v}_i, \mathbf{v}_{-i}\right)$ with the formulation in (7).

Without imposing any structure on w_i, problem (General-RMP), as well as the corresponding social surplus maximization problem, is still a multi-dimensional mechanism design problem. Furthermore, a numerical approach to solve this problem also has limited potential, given that even simplistic instances exhibit computational complexity obstacles: for example, even if $\mathbf{v}_i = (1, 0)$ for all i (i.e., buyers only value exclusivity and this valuation is the same for all buyers and is publicly known, so there is no private information at all in this setting), the Problem (General-RMP) reduces to finding the maximum independent set on (N, E).

To gain theoretical insights on the impact of exclusivity when allocating items on the network, we consider a simplified private information structure that exploits the exclusivity value on the underlying network (N, E). In our model, the exclusivity valuation w_i is derived from the privately held valuation v_i for non-exclusive allocation, and the valuations v_j, $j \in S(i)$, of buyer i's neighbors. In particular, we assume that buyer i's exclusivity premium is a linear combination of non-exclusivity valuations of buyer i's neighbors $j \in S(i)$:

$$w_i = v_i + \sum_{j \in S(i)} \alpha_{ij} v_j \tag{8}$$

with publicly known non-negative matrix $A = [\alpha_{ij}]$. If (8) holds, we say that valuations satisfy *local linear exclusivity* (LLE). Note that the publicly known network structure defines LLE (through neighborhoods $S(i)$ and weights α_{ij}) and is of fundamental importance in our analysis.

If buyer i gets the item exclusively, none of its neighbors j gets the item and thus buyer i can realize some of their unrealized values. For example, in an example of buyers of advertising space (or potential buyers of a franchise), buyer j who does not get the ad space (franchise contract), will lose potential customers and buyer i might attract some fraction α_{ij} of that lost value for j, as j's potential customers will be presented by i's ad only (will be able to go to i's franchise only). LLE valuations allow for one-dimensional representation of privately held information, even though buyer types are two-dimensional and, in fact, depend on diversely held private information in buyer i's neighborhood (i.e., $\mathbf{v}_i = (w_i, v_i)$ is a function of v_i and v_j, $j \in S(i)$).

In what follows, it will be useful to distinguish a special case of LLE, where for every j,

$$\sum_{\{i: j \in S(i)\}} \alpha_{ij} \leq 1. \tag{9}$$

In other words, if buyer j does not get the item, the most other buyers i for whom j is in their neighborhood, $j \in S(i)$, can collectively benefit from buyer j's unrealized value is bounded by v_j, i.e., they cannot realize more than 100% of the value j would have realized if allocated the item. Valuations that satisfy both (8) and (9) are said to satisfy *bounded local linear exclusivity* (BLLE).

4.2. Optimal Mechanisms for LLE Valuations

We study optimal mechanisms with local exclusivity on a network in this section. We first consider a complete information setting, i.e., the setting in which there is no privately held information and v_i are known to the seller. The ex post utility (7) can be rewritten as

$$U_i\left(v_i, v_{-i}\right) = v_i p_i\left(v_i, v_{-i}\right) + \left(\sum_{j \in S(i)} \alpha_{ij} v_j\right) p_i\left(v_i, v_{-i}\right) \prod_{j \in S(i)} \left(1 - p_j\left(v_i, v_{-i}\right)\right) - m_i\left(v_i, v_{-i}\right) \tag{10}$$

Without private information, there can be no misreporting, so the (EPIC) trivially holds, and the monopolistic seller can capture the entire social surplus by setting m_i to make the (EPIR) binding. Hence, the revenue maximization problem in the perfect information setting (FB-RMP) (also known as First Best (FB) solution as it gives an upper bound for what the seller can achieve in the optimal mechanism) is equivalent to the social surplus maximization problem, i.e.,

$$\max_{\{p_i\}_{i=1}^n} \sum_{i=1}^n \left(v_i + \left(\sum_{j \in S(i)} \alpha_{ij} v_j\right) \prod_{j \in S(i)} \left(1 - p_j\left(v\right)\right)\right) p_i\left(v\right)$$

subject to

$$\text{(Feasibility) } 0 \leq p_i\left(v\right) \leq 1 \text{ for all } i.$$

Note that the problem (FB-RMP) is a network optimization problem, and thus, the optimal solution fundamentally depends on the network structure.

Proposition 4. *Suppose that buyer valuations are BLLE and publicly known. The seller maximizes revenues by allocating an item to every buyer.*

Proposition 4 is straightforward but it provides an important benchmark for further analysis. It establishes that there are no exclusive allocations when the exclusivity premium is bounded by (9). Thus, solving the problem in the complete information setting is trivial and the seller should concentrate on providing sufficient supply and not exploit the (limited) potential of local exclusivity allocations. For example, if all buyers have similar capabilities (e.g., have similar business models), then excluding any buyer will result in the loss of that buyers unrealized value which is at least as large as the additional value its neighbors in the network could have jointly realized due to his exclusion. (BLLE valuations have diseconomies of scale structure.)

In contrast, exploiting the exclusivity on the network may be necessary if exclusive valuations are not bounded (e.g., satisfying LLE). For example, if excluding a less capable buyer would allow its more capable neighbors in the network to jointly realize higher value from the buyer's exclusion than the value the buyer would realize were he to have gotten the item.

Proposition 5. *Suppose that buyer valuations are LLE and known to a revenue-maximizing seller. Allocating exclusively to some buyers could be optimal. Furthermore, finding a deterministic optimal solution to the (FB-RMP) problem is at least as hard as finding the maximum independent set in (N, E).*

If the exclusivity premium is large compared to valuations of the players, the optimal solution will tend to allocate exclusively. Hence, as shown in the proof, one can construct large enough α_{ij} such that solving (FB-RMP) finds the maximum independent set in (N, E).

We now turn to the private information setting. We will show that, in contrast to Proposition 4, exclusive allocations are possible and that the mechanism design problem becomes computationally hard even for BLLE valuations.

Following the methodology in Jehiel et al. (1996), we can rewrite the (EPIC) as follows. By the Envelope Theorem,

$$\frac{dU_i\left(v_i, v_{-i}\right)}{dv_i} = \frac{\partial U_i\left(\widehat{v}_i, v_i, v_{-i}\right)}{\partial v_i}|_{(\widehat{v}_i)=(v_i)} = p_i\left(v_i, v_{-i}\right). \tag{11}$$

Obviously, $U_i\left(v_i, v_{-i}\right)$ is increasing in v_i. Moreover, since $U_i\left(v_i, v_{-i}\right)$ is a convex function, it is equivalent to require $dp_i\left(v_i, v_{-i}\right)/dv_i \geq 0$, which means $p_i\left(v_i, v_{-i}\right)$ is increasing in v_i. Hence, we can rewrite the interim utility function as

$$U_i\left(v_i, v_{-i}\right) = U_i\left(\underline{v}_i, v_{-i}\right) + \int_{\underline{v}_i}^{v_i} p_i\left(t, v_{-i}\right) dt \tag{12}$$

We choose $\underline{v}_i$ as the bottom type and make the bottom type binds

$$U_i\left(\underline{v}_i, v_{-i}\right) = 0. \tag{13}$$

By (11) and $0 \leq p_i\left(v_i, v_{-i}\right) \leq 1$, we know that $U_i\left(v_i, v_{-i}\right) \geq 0$ for any v_i.

By (10), (12) and (13), we rewrite the ex post payment as

$$m_i\left(v_i, v_{-i}\right) = v_i p_i\left(v_i, v_{-i}\right) + \left(\sum_{j\in S(i)} \alpha_{ij} v_j\right) p_i\left(v_i, v_{-i}\right) \prod_{j\in S(i)} \left(1 - p_j\left(v_i, v_{-i}\right)\right) - \int_{\underline{v}_i}^{v_i} p_i\left(t, v_{-i}\right) dt.$$

Thus, the seller's expected revenue can now be expressed as

$$\sum_{i=1}^{N} \int m_i\left(v_i, v_{-i}\right) dF^v(v) = \int \sum_{i=1}^{n} \left(\psi_i + \gamma_i \prod_{j\in S(i)} \left(1 - p_j\left(v\right)\right)\right) p_i\left(v\right) dF^v(v), \tag{14}$$

where γ_i is the exclusivity premium, i.e., $\gamma_i \triangleq \sum_{j\in S(i)} \alpha_{ij} v_j$. Note that γ_i depends on the network structure, i.e., $S\left(i\right)$, fraction α_{ij} for $j \in S\left(i\right)$, and valuations of buyer i's neighbors (and not virtual valuations).

For any set of realizations $\{v_i\}_{i\in N}$, the seller's revenue maximization problem (SB-RMP) (also known as Second Best (SB) solution as the seller has to pay information rents due to information asymmetry) can be stated as a point-wise maximization problem, i.e.,

$$\max_{\{p_i\}_{i=1}^{n}} \sum_{i=1}^{n} \left(\psi_i + \gamma_i \prod_{j\in S(i)} \left(1 - p_j\left(v\right)\right)\right) p_i\left(v\right)$$

subject to

$$\text{(Feasibility)} \quad 0 \le p_i(v) \le 1 \text{ for all } i,$$
$$\text{(Monotonicity)} \quad p_i(v_i, v_{-i}) \text{ is increasing in } v_i.$$

Note that the last constraint is part of the (EPIC).

Also note that the existence of a negative virtual valuation $\psi_i < 0$ implies that there must be buyers who will not get an item. Looking at the objective function of the (SB-RMP) problem, buyer i with $\psi_i < 0$ will not be allocated an item non-exclusively, and if buyer i gets an exclusive allocation, then it must be that γ_i is large enough and consequently $S(i) \neq \emptyset$, which means that none of buyers j, $j \in S(i)$, will get the item.

The following proposition contrasts Proposition 4 and shows the computational hardness of the allocation problem in the private information environment.

Proposition 6. *Suppose that buyer valuations v_i are privately held. Suppose that valuations are BLLE. Then allocating exclusively to some buyers could be optimal. Furthermore, finding a deterministic optimal solution to the (SB-RMP) problem is at least as hard as finding the maximum independent set in (N, E), even if virtual valuations $\psi_i \ge 0$ for all i.*

It is important to note that the hardness is not driven just by possibly negative virtual valuations ψ_i, and could be due to the publicly known network structure. (Such a discrepancy is observed in many problems and typically stems out of the fact that virtual valuations computed using standard Myerson technique turn non-negative values into possibly negative ones, and the underlying optimization problem that allows for negative inputs has a different complexity than the problem restricted to non-negative inputs.)

Table 1 summarizes the results of Proposition 4, Proposition 5, and Proposition 6.

	BLLE	**LLE**
Public Information	Non-Exclusive only (straightforward) (Proposition 4)	Exclusive or Non-Exclusive (could be hard, depends on network) (Proposition 5)
Private Information	Exclusive or Non-Exclusive (could be hard, depends on network) (Proposition 6)	Exclusive or Non-Exclusive (could be hard, depends on network) (Propositions 5,6)

Table 1 Optimal Allocation and Complexity

These results show how the complexity of making optimal allocation and pricing decisions varies even within LLE framework. Thus, the same conclusions extend to fully general information structures for which solving Problem (General-RMP) analytically is beyond reach.

The main reason behind studying complexity (and relating the hard instances to the maximum-independent set problem) is to demonstrate that there is little hope for creating a reasonable procedure for making optimal allocation and pricing decisions. (If such procedure were to exist and were guaranteed to end in reasonable time (e.g., so that the number of queries and information updates needed grows polynomially with the growth of the number of buyers), this would establish $P = NP$, and refute a central conjecture and decades-old open problem in theoretical computer science.) Still, it is possible to have subclasses of network structures and LLE valuations for which implementing the optimal mechanism is possible with a simple procedure. The structure of results in this section indicates that both the underlying network structure and private information could be determining factors.

We conclude this section by illustrating a non-monotonicity property of optimal mechanisms with exclusivity. This property could be considered as yet another indication that there is little hope for finding simple and intuitive procedures implementing optimal outcomes in general settings. When buyers have private information, they can get informational rents from the revenue-maximizing seller. These rents obviously depend on the value of privately held information. On the other hand, it would be reasonable to expect that the seller's optimal revenues increase as the total value in the system increases (i.e., w_i, v_i increase or stay the same). However, this intuition does not hold because buyers' information rents also depend on the network structure. We show that the non-monotonicity of revenues is possible, even for a network with two buyers.

Example 10. Consider a clique of size two, i.e., $S(1) = 2$ and $S(2) = 1$. Let $F_1 = F_2 = \mathbf{U}[0, 1]$. Let $v_1 = 0.01 + 2/3$, $v_2 = 2/3$, and $\alpha_{12} = \alpha_{21} = 0.4$. Consequently, $w_1 = 0.01 + 2.8/3$ and $w_2 = 0.004 + 2.8/3$. It turns out that the optimal mechanism is to allocate items (non-exclusively) to both buyers and the seller's revenue is 1.25. However, if both α_{12} and α_{21} are increased to 0.7, the optimal mechanism is to allocate an item exclusively to buyer 1 and the seller's revenue goes down to 1.133. The similar effect can occur when only α_{12} is increased to 0.9. Computational details are provided at the end of the Appendix. □

5. Concluding Remarks

In this paper, we discuss how to allocate and price contracts, goods or services, when potential buyers have private valuations and put a premium on an exclusive allocation to them. The exclusivity is a key feature of our model since a buyer's value depends on the overall allocation, i.e., whether their competitors or peers are excluded from the allocation. The notion of exclusivity could have a limited scope and is naturally defined by proximity on a network in which buyers are represented by nodes and perceived relationships among buyers are represented by arcs. This underlying network structure is of critical importance when developing revenue-maximizing allocation and pricing procedures. In some cases, the network topology with natural limits on the value of exclusivity could provide insights that yield straightforward allocation and pricing procedures. In other cases, the network structure could be an unsurmountable obstacle to finding optimal solutions for the monopolistic seller.

We find that the revenue-maximizing seller facing buyers that put a premium on exclusivity can do better than ignoring this additional value buyers put on exclusive allocations. Even the seller who is restricted to the simplest pricing schemes (posted prices) can increase revenues by inflating the price it would have charged if there were no additional value to buyers in the exclusive allocation. However, when buyers have exclusivity valuations, posted price mechanisms can be dominated by other pricing schemes that can be defined in many ways. Examples include sequential pricing schemes. The reason is that buyers who make decisions later have more information to evaluate their likelihood to get items exclusively. We then show that optimal mechanisms go beyond pricing, but determining optimal mechanism is a daunting task in a fully generic case with two-dimensional privately held buyer valuations. The fact that buyers have two-dimensional private information, a value for a non-exclusive allocation and a value for an exclusive allocation, introduces well-known fundamental difficulties of two-dimensional optimal mechanism design problems.

Given this analytical hurdle, we propose a hybrid auction-pricing procedure, which is easily implementable and revenue dominates posted prices. Furthermore, we show that the hybrid auction-pricing procedure is the optimal mechanism under the setting of linear exclusivity (LE), in which a buyer can capture some of her competitors' non-exclusivity valuations when getting the item exclusively. The proof of the optimality of the hybrid auction-pricing procedure requires deriving the optimal mechanism, while the information structure of LE valuations permits one-dimensional representation of the private information. This information structure allows us to bypass the difficulty of multi-dimensional mechanism design and gain insights on optimal mechanisms for a large class of settings that could be of practical relevance. Our approach is not limited to LE valuations. We can derive results analogous to those presented in the paper, as long as the buyers' valuations (w_i, v_i) can be parameterized by a well-behaved function $\Gamma_i(v_1, ..., v_n)$ of one-dimensional information v_j that is privately held by each buyer j. Specifically, we can extend our results from Section 3.1 and Section 4.2 with either additive exclusivity valuations or multiplicative exclusivity valuations. While the approach is analogous, the structure of results may change. In fact, the hybrid auction-pricing procedure is also the optimal mechanism with additive exclusivity. However, an ascending auction that implements the optimal mechanism for multiplicative exclusivity valuations has to simultaneously quote exclusivity and non-exclusivity prices, and possibly quote them non-anonymously, which is different from the hybrid auction-pricing procedure presented here. This is not surprising given that there is little hope of solving a generic two-dimensional optimal mechanism design problem.

In general, even when we can analytically solve the mechanisms design problem, it is not clear whether there exists a reasonable procedure to implement the optimal mechanism. We show this difficulty by presenting optimal mechanisms with LLE valuations, in which underlying network topology defines the structure of exclusivity valuations (so that a value of exclusivity is a linear combination of non-exclusivity valuations of the neighbors in the network). If the non-exclusivity valuations in the network neighborhood limit the exclusivity valuation of a buyer, the revenue-maximizing seller who knows buyers' valuations should allocate only non-exclusive contracts to all buyers. However, if buyers' valuations are privately held, the seller would have to pay information rents and allocating exclusive contracts could be optimal. Thus, the problem of finding the maximum independent set is encoded in the problem of finding an optimal mechanism. For network structures for which the maximum independent set problem is computationally tractable (the simplest one being the complete graph which is the underlying network topology in our hybrid auction-pricing procedure), this provides no obstacle by itself (e.g., bipartite graphs might arise naturally in some applications and our approach could yield appropriate procedures). However, for generic networks max independent set problem is a canonical computationally intractable problem and one should not hope for developing any simplistic revenue-optimal pricing schemes or even optimal auctions (which would be guaranteed to end in a reasonable amount of time).

The notion that buyers put a premium on exclusive allocation is closely related to negative externalities, as the value to the buyer who gets the item decreases if any of the buyer's competitors also get the item. However, exclusivity has an all-or-nothing feature in the sense that there is no incremental benefit as the number of local competitors decreases. One way to model negative externalities, and possibly exclusivity, would be to assume buyers demand more than one item and are interested in buying items designed for their competitors. In such a setting, an exclusive allocation corresponds to the buyer obtaining $k+1$ items: an item designed for the buyer and all of the unique items designed for each of its k neighbors. The difficulty with such an approach is that buyers have non-monotonic marginal values for items they demand: the first item has value v_i, the $k+1^{\text{st}}$ has marginal value $w_i - v_i$, while all other items have marginal value zero. While there is a work on efficient mechanisms when buyers have decreasing marginal values (e.g., Ausubel 2004), we are not aware of any results or general methodologies that could handle non-monotonic marginal values (and these are fundamental features of representing exclusivity) in the revenue-maximization setting. However, our approach with LLE valuations could be extended in a way that relaxes the notion of exclusivity. We could define a more general model for valuations so that every buyer that gets a contract (exclusively or not) has a value for it that is equal to $v_i + \sum_j \alpha_{ij} v_j$, where summation is over all $j \in S(i)$ that did not get the contract. In other words, buyer i gets the value from the contract v_i and a fraction of the value of the contract for each of its neighbors that were shut out of the market (capturing a fraction α_{ij} of the unrealized neighbor's value v_j). Deriving an optimal mechanism in this case is analogous to our approach with LLE valuations.

In conclusion, we provide a structured approach to allocating and pricing items when exclusivity is valuable. Specifically, we show that for the class of linear exclusivity valuations one needs to focus on the possibility of exclusive allocation first, which is a useful guidance in structuring contracts and/or negotiation process, even if the optimal hybrid auction-pricing procedure we present is not utilized. More generally, we provide theoretical and methodological insights on when the seller should consider allocating exclusively, who the seller should allocate exclusively to, and how exclusivity contracts should be priced. Our model is generic and simple, but the setup and findings provide possible directions for further research by extending the model parameters, generalizing the information structure, and adjusting the approach to fit applications where exclusivity plays a major role.

Appendix. Proofs

Proof of Lemma 1. We first study the Bayesian Nash equilibrium among buyers for any given price menu. Let $A_i \in [0,1]$ denote buyer i's equilibrium probability to accept the price menu. Buyer i accepts the price menu if

$$\left(w_i - P_i^{10}\right) \prod_{j \neq i} (1 - A_j) + \left(v_i - P_i^{11}\right) \left(1 - \prod_{j \neq i} (1 - A_j)\right) \geq 0. \tag{15}$$

We define a function $H_i\left(\eta^1, \eta^2; \{A_j\}_{j \neq i}\right) \triangleq \eta^1 \prod_{j \neq i} (1 - A_j) + \eta^2 \left(1 - \prod_{j \neq i} (1 - A_j)\right)$, and condition (15) is equivalent to $H_i\left(w_i, v_i; \{A_j\}_{j \neq i}\right) \geq H_i\left(P_i^{10}, P_i^{11}; \{A_j\}_{j \neq i}\right)$.

Let the distribution of $H_i\left(W_i, V_i; \{A_j\}_{j \neq i}\right)$ be $F_i^H\left(\cdot; \{A_j\}_{j \neq i}\right)$. Then the equilibrium condition is

$$A_i = 1 - F_i^H\left(H_i\left(P_i^{10}, P_i^{11}; \{A_j\}_{j \neq i}\right); \{A_j\}_{j \neq i}\right). \tag{16}$$

Since there are unlimited supply of items, $\sum_{i=1}^n A_i \leq n$. Let $\mathbf{A} \triangleq \{A_i\}_{i=1}^n$ and it is clear that $\mathbf{A}$ is non-empty, compact and convex. Note that, for any given $\{P_i^{10}, P_i^{11}\}_{i=1}^n$, $H_i\left(P_i^{10}, P_i^{11}; \{A_j\}_{j \neq i}\right)$ is continuous in

$\{A_j\}_{j\neq i}$ and, thus, $F_i^H\left(H_i\left(P_i^{10}, P_i^{11}; \{A_j\}_{j\neq i}\right); \{A_j\}_{j\neq i}\right)$ is continuous in $\{A_j\}_{j\neq i}$. Therefore, by Kakutani fixed-point theorem, the equilibrium exists. Moreover, we can rewrite condition (16) as

$$H_i\left(P_i^{10}, P_i^{11}; \{A_j\}_{j\neq i}\right) = \left(F_i^H\right)^{-1}\left(1 - A_i; \{A_j\}_{j\neq i}\right). \tag{17}$$

Next, we study the optimal price menu. The seller's expected revenue is

$$\sum_{i=1}^{n}\left(P_i^{10} A_i \prod_{j\neq i}(1-A_j) + P_i^{11} A_i\left(1-\prod_{j\neq i}(1-A_j)\right)\right) = \sum_{i=1}^{n} A_i H_i\left(P_i^{10}, P_i^{11}; \{A_j\}_{j\neq i}\right) = \sum_{i=1}^{n} A_i \left(F_i^H\right)^{-1}(1-A_i; \{A_j\}_{j\neq i}).$$

Note that the last equality comes from condition (17). Let $\mathbf{A}^*$ be the optimal solution for this problem. Then any price menu $\{P_i^{10*}, P_i^{11*}\}$ is optimal if $H_i\left(P_i^{10*}, P_i^{11*}; \{A_j^*\}_{j\neq i}\right) = \left(F_i^H\right)^{-1}\left(1 - A_i^*; \{A_j^*\}_{j\neq i}\right)$.

If a single price is quoted for each buyer, i.e., $P_i^{10} = P_i^{11} = P_i$, then we construct $P_i^* = H_i\left(P_i^*, P_i^*; \{A_j^*\}_{j\neq i}\right) = \left(F_i^H\right)^{-1}\left(1 - A_i^*; \{A_j^*\}_{j\neq i}\right)$ and P_i^* is optimal. Note that P_i^* is the average price of the price menu $\{P_i^{10*}, P_i^{11*}\}$, satisfying

$$P_i^* = P_i^{10*}\prod_{j\neq i}\left(1 - A_j^*\right) + P_i^{11*}\left(1 - \prod_{j\neq i}\left(1 - A_j^*\right)\right). \tag{18}$$

□

Remarks: Some parts of the technical proof of Lemma 1 have an intuitive explanation as follows. Consider a two-price mechanism $(P_j^{10}; P_j^{11})$ for $j = 1, \ldots, n$. Pick a buyer i with $P_i^{10} \neq P_i^{11}$ and denote the probability that no other buyer among $-i$ wins a contract as a_i. Hence, if buyer i accepts the two-price menu $(P_i^{10}; P_i^{11})$, her expected payment does not depend on her type and equals $a_i P_i^{10} + (1 - a_i)P_i^{11}$. Let us now consider a mechanism that proposes the same two-price menu $(P_j^{10}; P_j^{11})$ to all other buyers $j \neq i$ and a single price menu $P_i = a_i P_i^{10} + (1 - a_i)P_i^{11}$ to buyer i. Given the same decisions of $-i$ buyers, buyer i faces the same expected payment $a_i P_i^{10} + (1 - a_i)P_i^{11}$. Since buyer i faces the same expected payment, the acceptance decisions of each buyer i's type does not change. Neither does change the expected payment of buyer i to the seller. Hence, the two-price menu can be replaced with one-price menu for one buyer without affecting the decisions of all buyers and the expected revenue of the seller. With this procedure, we can proceed sequentially.

However, this intuition does not address the existence of the equilibrium. □

Proof of Proposition 1. Consider ex ante identical buyers and then the seller quotes the same price for each buyer. Let A denote a buyer's equilibrium probability to accept the price menu.

If the seller ignores the exclusivity, it believes that buyers accept the posted price based on their non-exclusivity valuations. Hence, the seller posts a price $\widetilde{P}$ for each buyer to maximize its expected revenue $n\widetilde{P}\left(1 - F^v\left(\widetilde{P}\right)\right)$, where $1 - F^v\left(\widetilde{P}\right)$ is the probability that a buyer will accepts the price menu. The log transformation of the objective is $\log n + \log\widetilde{P} + \log\left(1 - F^v\left(\widetilde{P}\right)\right)$. Since $1 - F^v(\cdot)$ is log-concave, the first order condition is valid and the optimal P^0 is determined by

$$P^0 - \frac{1 - F^v\left(P^0\right)}{f^v\left(P^0\right)} = 0.$$

If the seller considers the exclusivity, by Lemma 1, it is sufficient to consider single posted price. The seller's expected revenue under price P is $nP\left(1 - F^H(P; A)\right)$, where

$$A = 1 - F^H(P; A). \tag{19}$$

The log transformation of the objective is $\log n + \log P + \log\left(1 - F^H(P; A)\right)$. Since $1 - F^H(P; A(P))$ is log-concave in P, the first order condition is valid and the optimal P^* is determined by

$$\frac{1}{P^*} + \frac{-f^H\left(P^*; A\left(P^*\right)\right) - \frac{\partial F^H\left(P^*; A\left(P^*\right)\right)}{\partial A}\frac{dA\left(P^*\right)}{dP}}{1 - F^H\left(P^*; A\left(P^*\right)\right)} = 0.$$

By condition (19),

$$\frac{dA}{dP} = -f^H(P;A) - \frac{\partial F^H(P;A)}{\partial A}\frac{dA}{dP},$$

and, thus,

$$\frac{dA}{dP} = \frac{-f^H(P;A)}{1+\frac{\partial F^H(P;A)}{\partial A}}.$$

Therefore, the first order condition can be rewritten as

$$P^* - \frac{1-F^H(P^*;A(P^*))}{f^H(P^*;A)}\left(1+\frac{\partial F^H(P^*;A(P^*))}{\partial A}\right) = 0$$

We next prove that $\left(1-F^H\left(\widehat{P};A\right)\right)/f^H\left(\widehat{P};A\right) \geq \left(1-F^v\left(\widehat{P}\right)\right)/f^v\left(\widehat{P}\right)$ for any A and any $\widehat{P}$. We first derive the distribution of $H(W_i,V_i;A)$ for given A. Since $\mathbf{v}_i$ is private information and $w_i \geq v_i \geq 0$, there always exist a constant $\beta' \geq 0$ and a random variable $\varepsilon_i' \geq 0$ independent of V_i such that $W_i = V_i + \beta' V_i + \varepsilon_i'$. Hence, $H(W_i,V_i;A)$ can be rewritten as $H(W_i,V_i;A) = V_i(1+\beta)+\varepsilon_i$, where $\beta = \beta'(1-A)^{n-1}$ and $\varepsilon_i = \varepsilon_i'(1-A)^{n-1}$.

Let the distribution of ε_i be F^ε and the density be f^ε. Since buyers are ex ante identical, the distribution of ε_i is identical across buyers. Then

$$F^H\left(\widehat{P};A\right) = \Pr\left(V_i(1+\beta)+\varepsilon_i \leq \widehat{P}\right) = \int_0^{\widehat{P}} F^v\left(\frac{\widehat{P}-\varepsilon_i}{1+\beta}\right) f^\varepsilon(\varepsilon_i)\,d\varepsilon_i,$$

and

$$f^H\left(\widehat{P};A\right) = \frac{1}{1+\beta}\int_0^{\widehat{P}} f^v\left(\frac{\widehat{P}-\varepsilon_i}{1+\beta}\right) f^\varepsilon(\varepsilon_i)\,d\varepsilon_i.$$

Let $LHS \triangleq \left(1-F^H\left(\widehat{P};A\right)\right)/f^H\left(\widehat{P};A\right)$, and then

$$\int_0^{\widehat{P}}\left(F^v\left(\frac{\widehat{P}-\varepsilon_i}{1+\beta}\right) + \frac{LHS}{1+\beta} f^v\left(\frac{\widehat{P}-\varepsilon_i}{1+\beta}\right)\right) f^\varepsilon(\varepsilon_i)\,d\varepsilon_i = 1.$$

By the first mean value theorem of integration, there exists $\xi \in \left[0,\widehat{P}\right]$ such that

$$\left(F^v\left(\frac{\widehat{P}-\xi}{1+\beta}\right) + \frac{LHS}{1+\beta} f^v\left(\frac{\widehat{P}-\xi}{1+\beta}\right)\right) F^\varepsilon\left(\widehat{P}\right) = 1.$$

Therefore,

$$LHS = \frac{\frac{1}{F^\varepsilon\left(\widehat{P}\right)} - F^v\left(\frac{\widehat{P}-\xi}{1+\beta}\right)}{f^v\left(\frac{\widehat{P}-\xi}{1+\beta}\right)}(1+\beta).$$

Since $1-F^v(\cdot)$ is log concave, $\left(\widehat{P}-\xi\right)/(1+\beta) \leq \widehat{P}$, $1/F^\varepsilon\left(\widehat{P}\right) \geq 1$, and $1+\beta \geq 1$, we must have $LHS \geq \left(1-F^v\left(\widehat{P}\right)\right)/f^v\left(\widehat{P}\right)$.

We also prove that $\partial F^H\left(\widehat{P};A\right)/\partial A \geq 0$. Let the distribution of ε_i' be $F^{\varepsilon'}$ and the density be $f^{\varepsilon'}$. Note that $F^{\varepsilon'}$ and $f^{\varepsilon'}$ are independent of A. Since $\varepsilon_i = \varepsilon_i'(1-A)^{n-1}$

$$f^\varepsilon(\varepsilon_i) = \frac{1}{(1-A)^{n-1}} f^{\varepsilon'}\left(\frac{\varepsilon_i}{(1-A)^{N-1}}\right).$$

Hence, $F^H\left(\widehat{P};A\right)$ can be rewritten as

$$F^H\left(\widehat{P};A\right) = \frac{1}{(1-A)^{n-1}}\int_0^{\widehat{P}} F^v\left(\frac{\widehat{P}-\varepsilon_i}{1+\beta'(1-A)^{n-1}}\right) f^{\varepsilon'}\left(\frac{\varepsilon_i}{(1-A)^{n-1}}\right) d\varepsilon_i.$$

Using the first mean value theorem of integration, there exists $\zeta \in \left[0, \widehat{P}\right]$ such that

$$F^H\left(\widehat{P};A\right) = \frac{1}{(1-A)^{n-1}} F^v\left(\frac{\widehat{P}-\zeta}{1+\beta'(1-A)^{n-1}}\right) F^{\varepsilon'}\left(\frac{\widehat{P}}{(1-A)^{n-1}}\right).$$

It is clear that $\partial F^H\left(\widehat{P};A\right)/\partial A \geq 0$.

Finally, we prove $P^* \geq P^0$. Since

$$\frac{1-F^H\left(P^0;A\left(P^0\right)\right)}{f^H\left(P^0;A\right)} \geq \frac{1-F^v\left(P^0\right)}{f^v\left(P^0\right)} \text{ and } \frac{F^H\left(P^0;A\left(P^0\right)\right)}{\partial A} \geq 0,$$

the first order condition indicates that

$$P^0 - \frac{1-F^H\left(P^0;A\left(P^0\right)\right)}{f^H\left(P^0;A\right)}\left(1+\frac{\partial F^H\left(P^0;A\left(P^0\right)\right)}{\partial A}\right) \leq P^0 - \frac{1-F^v\left(P^0\right)}{f^v\left(P^0\right)} = 0.$$

Note that $1-F^H(P;A(P))$ is log-concave in P, we must have $P^* \geq P^0$.

Furthermore, the optimal expected profit with exclusivity consideration is

$$R^* = nP^*\left(1-F^H\left(P^*;A\left(P^*\right)\right)\right) \geq nP^0\left(1-F^H\left(P^0;A\left(P^0\right)\right)\right) \geq nP^0\left(1-F^v\left(P^0\right)\right) = R^0,$$

where the first inequality comes from the fact P^* is optimal in the posted price mechanism with exclusivity consideration, while the second inequality comes from the fact that $F^H\left(\widehat{P};A\right) \leq F^v\left(\widehat{P}\right)$ for any A and $\widehat{P}$. In addition, when the seller ignores the exclusivity valuations, $nP^0\left(1-F^v\left(P^0\right)\right)$ is the expected revenue from the seller's perspective, and $nP^0\left(1-F^H\left(P^0;A\left(P^0\right)\right)\right)$ is the expected revenue from the researcher's perspective. □

Remarks: Some parts of the technical proof of Proposition 1 have an intuitive explanation as follows. Consider an optimal pricing mechanism when there is no exclusivity and denote its price as P^0. Also consider the environment when buyers have exclusivity: $w_i > v_i$ with positive probability. Let us assume that buyers use the "old" strategies (the same as when there is no exclusivity; these strategies might not be optimal), and, thus, have the same probability of contract acceptance. Since, buyer i has a premium of exclusivity (this happens if other buyers do not get the items with positive probability) more types v_i of the agent are ready to accept price P^0. Let us increase the price for buyer i until $P_i = P' \leq P^0$ such that the probability of that buyer accepts the contract remains as for the case when there is no exclusivity. For price P' buyer i is playing a best response to other buyers' (not yet optimal) strategies. We repeat this procedure with each buyer. Since buyers are identical, this procedure results in the same price P' for each buyer. Note that in the new mechanism each buyer plays a best response and accepts the contract with the same probability as for P^0 and the environment with no exclusivity. The new mechanism might not be optimal, but it has a higher price and a higher revenue. Hence, the optimal mechanism will have a higher revenue.

However, this intuitive reasoning cannot provide the exact optimal price and expected revenue. □

Proof of Example 7. Problem (General-RMP) is a linear programming problem when the support Ω is finite. We discretize the type space Ω as follows,

$$\Omega(M) = \{(i/(M-1)+j/(M-1), i/(M-1)) : i,j = 0,1,\ldots,M-1\},$$

where M is the number of possible outcomes of v_i or ε_i. Moreover, we consider discrete uniform distributions on both v_i and ε_i. The following Table 2 shows the numerical solutions of Problem (General-RMP).

The number of variables is $4|\Omega|^2 = 4M^2$ and the number of (EPIC) constraints is $2|\Omega|^2(|\Omega|-1) = 2(M^6 - M^4)$. Hence, it rather quickly becomes computationally unmanageable. □

Grid Index M	$\|\Omega\|$	Seller's Expected Revenue
2	4	1.188
3	9	1.056
4	16	0.995
5	25	0.961
6	36	0.938
7	49	0.924
8	64	0.915
9	81	0.907
10	100	0.901

Table 2 Numerical Solutions of the Seller's Expected Revenues

Proof of Proposition 2. Following the technique in Jehiel et al. (1996), we can work out the optimal mechanism. A non-exclusive allocation occurs iff $\psi_1(x)+\psi_2(y) > \psi_1(x)+\gamma_1$, $\psi_1(x)+\psi_2(y) > \psi_2(y)+\gamma_2$, and $\psi_1(x)+\psi_2(y) \geq 0$. An exclusive allocation to buyer 1 occurs iff $\psi_1(x)+\gamma_1 \geq \psi_1(x)+\psi_2(y)$, $\psi_1(x)+\gamma_1 \geq \psi_2(y)+\gamma_2$, and $\psi_1(x)+\gamma_1 \geq 0$. Similarly, an exclusive allocation to buyer 2 occurs iff $\psi_2(y)+\gamma_2 \geq \psi_1(x)+\psi_2(y)$, $\psi_2(y)+\gamma_2 \geq \psi_1(x)+\gamma_1$, and $\psi_2(y)+\gamma_2 \geq 0$. Otherwise, no allocation occurs.

Since $\gamma_1 \geq 0$ and $\gamma_2 \geq 0$, the above condition can be simplified as follows. A non-exclusive allocation occurs iff $\psi_2(y) > \gamma_1 = \alpha_{12}y$ and $\psi_1(x) > \gamma_2 = \alpha_{21}x$. An exclusive allocation to buyer 1 occurs iff $\gamma_1 \geq \psi_2(y)$, $\psi_1(x)-\psi_2(y) \geq \gamma_2-\gamma_1$, and $\psi_1(x) \geq -\gamma_1$. An exclusive allocation to buyer 2 occurs iff $\gamma_2 \geq \psi_1(x)$, $\gamma_2-\gamma_1 \geq \psi_1(x)-\psi_2(y)$, and $\psi_2(y) \geq -\gamma_2$.

Moreover, when $\psi_2(y)=\gamma_1$ and $\psi_1(x)=\gamma_2$, $\psi_1(x)-\psi_2(y)+\gamma_1-\gamma_2=0$ holds. When $\psi_1(x)=-\gamma_1$ and $\psi_2(y)=-\gamma_2$, $\psi_1(x)-\psi_2(y)+\gamma_1-\gamma_2=0$ also holds. Therefore, the valuation space is divided into four regions.

The proof is completed by noting that $\psi_1(\cdot)$, $\psi_2(\cdot)$, $\gamma_1(\cdot)$, $\gamma_2(\cdot)$, $\psi_1(\cdot)-\gamma_2(\cdot)$, and $\psi_2(\cdot)-\gamma_1(\cdot)$ are increasing functions. $\square$

Remarks: The optimal mechanism provided in Proposition 2 can be generalized to n buyers. Without loss of generality, we can order buyers according to their virtual valuations ψ_i, i.e., $\psi_1 \geq \psi_2 \geq \cdots \geq \psi_n$. Let K^* be the cutoff for ψ_i, i.e. for $i \leq K^*$, $\psi_i \geq 0$ and for $i > K^*$, $\psi_i < 0$. If $K^*=0$, let $\sum_{i=1}^{K^*}\psi_i = 0$. Let $j^* \triangleq \arg\max_{j\in N}(\gamma_j+\psi_j)$. We use the subscripted ψ_i to denote an arbitrary buyer i's virtual valuation (not ordered). We also order other buyers $-i$ according to ψ_{-i} excluding ψ_i, i.e., $\psi_{(1)} \geq \psi_{(2)} \geq \cdots \geq \psi_{(N-1)}$. K^*_{-i} is similarly defined.

Let

$$Z_i^{11} \triangleq \left\{ z_i : \psi_i(z_i) \geq 0 \text{ and } \sum_{h=1}^{K^*_{-i}} \psi_{(h)} + \psi_i(z_i) > \gamma_j + \psi_j \text{ for all } j \right\}$$

and

$$Z_i^{10} \triangleq \left\{ z_i : \gamma_i + \psi_i(z_i) \geq \gamma_j + \psi_j \text{ for all } j \text{ and } \gamma_i + \psi_i(z_i) 1_{\{\psi_i(z_i)<0\}} \geq \sum_{h=1}^{K^*_{-i}} \psi_{(h)} \right\}.$$

Then we can define thresholds $\underline{y}^{11}(v_{-i})$ for the non-exclusive allocation, and $\underline{y}^{10}(v_{-i})$ for the exclusive allocation:

$$\underline{y}^{11}(v_{-i}) \triangleq \inf\left\{z_i \in Z_i^{11}\right\} \text{ and } \underline{y}^{10}(v_{-i}) \triangleq \inf\left\{z_i \in Z_i^{10}\right\}.$$

Let Q_i^{10} denote buyer i's probability to get the item exclusively, let Q_i^{1k} denote buyer i's probability to get the item with k items allocated to its neighbors, let Q_i^{00} denote buyer i's probability not to get any item with no item allocated to its neighbors, and let Q_i^{0k} denote buyer i's probability not to get any item with k items allocated to its neighbors.

Proposition 7. *The optimal mechanism when buyers in perfect competition have BLLE valuations allocates as follows:*
$Q_i^{1K^*-1}(v)=1$, *for* $i=1,2,...,K^*$, *if* $\sum_{h=1}^{K^*}\psi_h > \max\{\gamma_{j^*}+\psi_{j^*},0\}$ *or* $\sum_{h=1}^{K^*}\psi_h = 0 > \gamma_{j^*}+\psi_{j^*}$;
$Q_{j^*}^{10}(v)=1$, *if* $\gamma_{j^*}+\psi_{j^*} \geq \max\{\sum_{h=1}^{K^*}\psi_h,0\}$;
$Q_i^{00}(v)=1$, *for* $i=1,2,...,n$, *if* $\max\{\sum_{h=1}^{K^*}\psi_h,\gamma_{j^*}+\psi_{j^*}\}<0$. *The payments are given by*

$$m_i(v)=v_i p_i(v)+\gamma_i Q_i^{10}(v)-\int_{\underline{v}_i}^{v_i} p_i(t,v_{-i})\,dt. \tag{20}$$

In particular, if $Q_i^{1K^*-1}(v)=1$, *the payment is* $m_i(v)=\underline{y}^{11}(v_{-i})$; *if* $Q_i^{10}(v)=1$, *the payment is* $m_i(v)=\gamma_i+\underline{y}^{10}(v_{-i})$; *otherwise, the payment is* $m_i(v)=0$.

Proof of Proposition 7. We prove the proposition in the following steps.

Step 1: The proposed mechanism satisfies the (EPIR).

By the construction of the payment rule, $m_i(\underline{v}_i,v_{-i})=\underline{v}_i p_i(\underline{v}_i,v_{-i})+\gamma_i Q_i^{10}(\underline{v}_i,v_{-i})$. Therefore,

$$U_i(\underline{v}_i,v_{-i})=\underline{v}_i p_i(\underline{v}_i,v_{-i})+\gamma_i Q_i^{10}(\underline{v}_i,v_{-i})-m_i(\underline{v}_i,v_{-i})=0.$$

Step 2: The proposed mechanism satisfies the (EPIC).

To prove this claim, we need only check the $p_i(v)\triangleq\sum_{k=1}^{n-1}Q_i^{1k}(v)+Q_i^{10}(v)$ is non-decreasing in v_i.

If $\max\{\sum_{h=1}^{K^*}\psi_h,\gamma_{j^*}+\psi_{j^*}\}<0$, $Q_i^{00}=1$ for $i=1,2,...,n$. We have $p_i(v)=0$. This case means $\gamma_j+\psi_j<0$. It is a trivial case to verify the monotonicity. In the following proof, we only consider the case $\max\{\sum_{h=1}^{K^*}\psi_h,\gamma_{j^*}+\psi_{j^*}\}\geq 0$.

Case 1: $j\in\{1,...,K^*\}$.
If $\sum_{h=1}^{K^*}\psi_h>\gamma_{j^*}+\psi_{j^*}$, then $Q_j^{1K^*-1}=1$. Hence, we have $p_j(v)=1$.
If $\gamma_{j^*}+\psi_{j^*}\geq\sum_{h=1}^{K^*}\psi_h$ and $j=j^*$ ($\gamma_j+\psi_j\geq\gamma_i+\psi_i$, for $i=1,...,n$), then $Q_j^{10}=1$. Hence, we have $p_j(v)=1$.
If $\gamma_{j^*}+\psi_{j^*}\geq\sum_{h=1}^{K^*}\psi_h$ and $j\neq j^*$ ($\gamma_{j^*}+\psi_{j^*}>\gamma_j+\psi_j$), then $Q_{j^*}^{10}=1$. Hence, we have $p_j(v)=0$.
From $p_j(v)=0$ to $p_j(v)=1$, we have two possibilities.
1) If $j=j^*$ when $p_j(v)=1$, then ψ_j changes from $\psi_j\leq\gamma_{j^*}+\psi_{j^*}-(\psi_1+\cdots+\psi_{j-1}+\psi_{j+1}+\cdots+\psi_{K^*})$ and $\psi_j<\gamma_{j^*}+\psi_{j^*}-\gamma_j$ to $\psi_j\geq\gamma_i+\psi_i-\gamma_j$ for any $i=1,...,n$.
2) If $j\neq j^*$ when $p_j(v)=1$, then ψ_j changes from $\psi_j\leq\gamma_{j^*}+\psi_{j^*}-(\psi_1+\cdots+\psi_{j-1}+\psi_{j+1}+\cdots+\psi_{K^*})$ and $\psi_j<\gamma_{j^*}+\psi_{j^*}-\gamma_j$ to $\psi_j>\gamma_{j^*}+\psi_{j^*}-(\psi_1+\cdots+\psi_{j-1}+\psi_{j+1}+\cdots+\psi_{K^*})$ and $\psi_j<\gamma_{j^*}+\psi_{j^*}-\gamma_j$.
In both cases, ψ_j is increasing. As ψ_j itself is increasing in v_j, we see that $p_j(v)$ is non-decreasing in v_i.

Case 2: $j\in\{K^*+1,...,n\}$.
If $\sum_{h=1}^{K^*}\psi_h>\gamma_{j^*}+\psi_{j^*}$, then $Q_i^{1K^*-1}=1$ for $i=1,2,...,K^*$. Hence, we have $p_j(v)=0$.
If $\gamma_{j^*}+\psi_{j^*}\geq\sum_{h=1}^{K^*}\psi_h$ and $j\neq j^*$ ($\gamma_{j^*}+\psi_{j^*}>\gamma_j+\psi_j$), then $Q_{j^*}^{10}=1$. Hence, we have $p_j(v)=0$.
If $\gamma_{j^*}+\psi_{j^*}\geq\sum_{h=1}^{K^*}\psi_h$ and $j=j^*$ ($\gamma_j+\psi_j\geq\gamma_i+\psi_i$, for $i=1,...,n$) , then $Q_j^{10}=1$. Hence, we have $p_j(v)=1$.
From $p_j(v)=1$ to $p_j(v)=0$, we also have two possibilities.
1) If $j=j^*$ when $p_j(v)=0$, then ψ_j changes from $\psi_j\geq\psi_1+\psi_2+\cdots+\psi_{K^*}-\gamma_j$ and $\psi_j\geq\gamma_i+\psi_i-\gamma_j$, for any $i=1,...,n$ to $\psi_j<\psi_1+\psi_2+\cdots+\psi_{K^*}-\gamma_j$ and $\psi_j\geq\gamma_i+\psi_i-\gamma_j$, for any $i=1,...,n$.
2) If $j\neq j^*$ when $p_j(v)=0$, then ψ_j changes from $\psi_j\geq\psi_1+\psi_2+\cdots+\psi_{K^*}-\gamma_j$ and $\psi_j\geq\gamma_i+\psi_i-\gamma_j$, for any $i=1,...,n$ to $\psi_j<\gamma_{j^*}+\psi_{j^*}-\gamma_j$.
In both cases, ψ_j is decreasing. As ψ_j itself is increasing in v_j, we see that $p_j(v)$ is non-decreasing in v_i.

Step 3: The proposed mechanism is the equilibrium, i.e. there is no strict incentive for the seller to deviate.

If $\max\{\sum_{h=1}^{K^*}\psi_h, \gamma_{j^*}+\psi_{j^*}\}<0$, $Q_i^{00}=1$ for $i=1,2,...,n$. Obviously, there is no incentive to deviate, or else, the seller would obtain a negative profit. Then we consider the case $\max\{\sum_{h=1}^{K^*}\psi_h, \gamma_{j^*}+\psi_{j^*}\}\geq 0$ as follows.

Case 1: If $\sum_{h=1}^{K^*}\psi_h > \gamma_{j^*}+\psi_{j^*}$, then $Q_i^{1K^*-1}=1$ for $i=1,2,...,K^*$.

Consider a deviation as $Q_{i'}^{1K^*-1}=1-\epsilon$ for $i'\in\{1,2,...,K^*\}$, where ϵ is a small positive number.

Subcase 1: $Q_{i'}^{0K^*}=\epsilon$, $Q_i^{1K^*-1}=1$ for $i\neq i'$ and $i\in\{1,2,...,K^*\}$, and $Q_{i''}^{1K^*-1}=\epsilon$ for $i''\in\{K^*+1,...,n\}$. By the definition of K^*, if $n\geq K^*$, there is no incentive because $\psi_{i''}<0$.

Subcase 2: $Q_{i'}^{0k}=\epsilon$ for $k<K^*$, $Q_i^{1k-1}=\varepsilon$ for $i\in\{1,2,...,k\}$ if $i'>k$ or $i\in\{1,2,...,k+1\}\setminus\{i'\}$ if $i'\leq k$ (this is the best deviation for allocating only k items), and $Q_{i''}^{0k}=\epsilon$ for $i''\in\{k+1$ (or $k+2),...,n\}$. Obviously, there is no strict incentive to this since $\psi_{i''}\geq 0$ for $i''\leq K^*$.

Subcase 3: $Q_{i'}^{01}=\epsilon$, $Q_j^{10}=\epsilon$ for $j\neq i'$, and $Q_i^{1K^*-1}=1-\epsilon$ for $i\neq i'$ and $i\in\{1,2,...,K^*\}$. Then the expected revenue is $(1-\epsilon)(\sum_{h=1}^{K^*}\psi_h)+\epsilon(\gamma_j+\psi_j)$. By the condition $\sum_{h=1}^{K^*}\psi_h>\gamma_{j^*}+\psi_{j^*}$ and the definition of j^*, $(1-\epsilon)(\sum_{h=1}^{K^*}\psi_h)+\epsilon(\gamma_j+\psi_j)<\sum_{h=1}^{K^*}\psi_h$. Therefore, there is no incentive to deviate.

Subcase 4: $Q_{i'}^{10}=\epsilon$ and $Q_i^{1K^*-1}=1-\epsilon$. By the similar argument in subcase 2, we know there is no incentive to deviate.

Subcase 5: $Q_{i'}^{1k}=\epsilon$ for $k<K^*$, and $Q_i^{1k}=\epsilon$ for $i\in\{1,2,...,k\}$ if $i'>k$ or $i\in\{1,2,...,k+1\}\setminus\{i'\}$ if $i'\leq k$. Obviously, there is no strict incentive as in Subcase 2.

Subcase 6: $Q_{i'}^{00}=\epsilon$. Obviously, there is no strict incentive because $v_{i'}^{00}=0$.

Case 2: If $\gamma_{j^*}+\psi_{j^*}\geq\sum_{h=1}^{K^*}\psi_h$, then $Q_{j^*}^{10}=1$.

Consider a deviation as $Q_{j^*}^{10}=1-\epsilon$, where ϵ is a small positive number.

Subcase 1: if $j^*\in\{1,2,...,k\}$ for $k\leq K^*$, $Q_i^{1k-1}=\epsilon$ for $i\in\{1,2,...,k\}$. Then the expected revenue is $(1-\epsilon)(\gamma_{j^*}+\psi_{j^*})+\epsilon(\sum_{h=1}^{K^*}\psi_h)$. By the condition $\gamma_{j^*}+\psi_{j^*}\geq\sum_{h=1}^{K^*}\psi_h$, there is no strict incentive to deviate.

Subcase 2: if $j^*\notin\{1,2,...,k\}$ for $k\leq K^*$, $Q_{j^*}^{0k}=\epsilon$, and $Q_i^{1k-1}=\epsilon$ for $i\in\{1,2,...,k\}$. By the similar argument in subcase 1, there is no strict incentive to deviate.

Subcase 3: $Q_{j^*}^{01}=\epsilon$, and $Q_{j'}^{10}=\epsilon$. The expected revenue is $(1-\epsilon)(\gamma_{j^*}+\psi_{j^*})+\epsilon(\gamma_{j'}+\psi_{j'})$. By the definition of j^*, there is no strict incentive to deviate.

Subcase 4: $Q_{j^*}^{00}=\epsilon$. Obviously, there is no strict incentive.

Therefore, there is no strict incentive for the seller to deviate from the proposed allocation. The payment rule is just a direct result from condition (20). $\square$

The proof of Proposition 7 requires that the proposed mechanism satisfy both (EPIR) and (EPIC) conditions, and there should be no incentive for the seller to deviate.

If ties need to be broken, we let the seller prefer exclusive allocation if $\sum_{h=1}^{K^*}\psi_h=\gamma_{j^*}+\psi_{j^*}$. Moreover, Z_i^{11} and Z_i^{10} reflect the requirement of buyer i's valuation given other buyers' valuations for the non-exclusive allocation and exclusive allocation, respectively. Since $\sum_{\{i:j\in S(i)\}}\alpha_{ij}\leq 1$ and the virtual valuation $\psi_j(\cdot)$ is an increasing function, $\psi_j(v_j)-\alpha_{ij}v_j$ for some $j\in S(i)$ is also increasing in v_j. By the definition of Z_i^{11} and Z_i^{10}, we know that for any $v_i\geq\underline{y}^{11}(v_{-i})$ and $Q_i^{1K^*-1}(v_i,v_{-i})=1$, $v_i\in Z_i^{11}$; while for any $v_i\geq\underline{y}^{10}(v_i)$ and $Q_i^{10}(v_i,v_{-i})=1$, $v_i\in Z_i^{10}$. As a result, we have the specified formulation of the payments.

Note that for the exclusive allocation, the buyer has to pay not only the minimum requirement of the allocation, but also the exclusivity premium γ_i, which depends on the network structure and neighbors' valuations. However, the exclusivity premium γ_i is not affected by buyer i's valuation. Therefore, when the non-exclusive allocation prevails, the increase of buyer i''s valuation can not change the allocation into the exclusive allocation. Its own valuation will determine whether it will obtain the item or not rather than exclusivity or non-exclusivity. Therefore, buyer i's payments are irrelevant to its valuation.

Proof of Proposition 3. In order to simplify the proof, we rewrite the price system as follows.

$$P_1^{10}(0)=\frac{2+\alpha_{12}-\alpha_{12}\alpha_{21}}{4-\alpha_{12}\alpha_{21}} \text{ and } P_2^{10}(0)=\frac{2+\alpha_{21}-\alpha_{12}\alpha_{21}}{4-\alpha_{12}\alpha_{21}},$$

$$P_1^{10}(t)=\alpha_{12}y(t)+\frac{2-\alpha_{12}}{2-\alpha_{21}}y(t) \text{ and } P_2^{10}(t)=\alpha_{21}x(t)+\frac{2-\alpha_{21}}{2-\alpha_{12}}x(t),$$

where

$$\frac{y(t)}{x(t)}=\frac{2-\alpha_{21}}{2-\alpha_{12}},\ y(0)=\frac{2-\alpha_{21}}{4-\alpha_{12}\alpha_{21}},\ y(T)=\frac{1}{2-\alpha_{12}},\text{ and } y(T)=y(0)+\frac{y(T)-y(0)}{T}t,$$

and the final prices are

$$P_1^{11}=\frac{1}{2-\alpha_{21}} \text{ and } P_2^{11}=\frac{1}{2-\alpha_{12}}.$$

Without the loss of generality, we consider that only buyer 1 accepts the offer at $t=0$. Since buyer 2 always has zero utility, it must truthfully report its valuation. Buyer 1 prefers the exclusive allocation iff

$$x+\alpha_{12}y\geq\frac{1}{2}+\frac{1}{2}\alpha_{12}y,$$

which is $x\geq 1/2-\alpha_{12}y/2=x_0(y)$. Moreover, when $x(t)=1/2-\alpha_{12}y(t)/2$, asking for exclusivity only brings zero utility. A similar argument works on the case where only buyer 2 accepts the offer at $t=0$.

If the auction does not end at time $t=0$, we will prove the following equilibrium: Buyer 1 does not quit until $x(t)=x$ and buyer 2 does not quit until $y(t)=y$.

We first consider the case when prices reach $\{P_1^{10}(T),P_2^{10}(T)\}$. This indicates that $x\geq x(T)$ and $y\geq y(T)$. Note that buyer 1 does not have incentive to deviate from the proposed equilibrium. Buyer 1 gets positive utility by asking for exclusivity if $x\geq x(T)$. Therefore, buyer 1 with $x\geq x(T)$ will not quit before time T, which only gives it zero utility. A similar argument works on buyer 2.

Consider $x(0)\leq x\leq x(T)$ and $y(0)\leq y\leq y(T)$. Without the loss of generality, we show that buyer 1 does not have incentive to deviate from the proposed equilibrium.

1) Buyer 1 with $x(t)$ will not postpone quitting. If buyer 1 asks for exclusivity until $x(\tau)$ for $\tau\geq t.$, the expected utility is

$$\begin{aligned}&\int_{y(t)}^{y(\tau)}\frac{1}{1-y(t)}\left(x(t)+\alpha_{12}y(\tilde{t})-P_1^{10}(\tilde{t})\right)dy(\tilde{t})+\int_{y(\tau)}^{y(T)}0dy(\tilde{t})+\int_{y(T)}^{1}0dy(\tilde{t})\\&=\int_{y(t)}^{y(\tau)}\frac{1}{1-y(t)}\left(x(t)+\alpha_{12}y(\tilde{t})-\alpha_{12}y(\tilde{t})-\frac{2-\alpha_{12}}{2-\alpha_{21}}y(\tilde{t})\right)dy(\tilde{t}).\end{aligned}$$

If buyer 1 quits at time t, the expected utility is 0. Then we need to verify

$$x(t)\leq\frac{1}{2}\frac{2-\alpha_{12}}{2-\alpha_{21}}(y(t)+y(\tau)).$$

Since $x(t)=y(t)(2-\alpha_{12})/(2-\alpha_{21})$ and $y(\tau)\geq y(t)$, the above condition holds. Hence, buyer 1 will quit at time t if $x=x(t)$.

2) Buyer 1 with $x(\tau)$ for $\tau>t$ will ask for exclusivity at t. By the above proof, buyer 1 asks for exclusivity until $x(\tau)$. By asking for exclusivity, it gets

$$\begin{aligned}&\int_{y(t)}^{y(\tau)}\frac{1}{1-y(t)}\left(x(\tau)+\alpha_{12}y(\tilde{t})-P_1^{10}(\tilde{t})\right)dy(\tilde{t})\\&=\int_{y(t)}^{y(\tau)}\frac{1}{1-y(t)}\left(x(\tau)+\alpha_{12}y(\tilde{t})-\alpha_{12}y(\tilde{t})-\frac{2-\alpha_{12}}{2-\alpha_{21}}y(\tilde{t})\right)dy(\tilde{t}).\end{aligned}$$

If buyer 1 quits, the expected utility is 0. Then we need to verify

$$x(\tau)\geq\frac{1}{2}\frac{2-\alpha_{12}}{2-\alpha_{21}}(y(t)+y(\tau)).$$

Since $x(\tau) = y(\tau)(2-\alpha_{12})/(2-\alpha_{21})$ and $y(\tau) > y(t)$, the above condition holds. Hence, buyer 1 with $x(\tau)$ for $\tau > t$ will ask for exclusivity at t.

The argument works on buyer 2 as well.

At time 0, buyer 1 with $x < P_1^{10}(0)$ does not have incentive to deviate from quitting. Buyer 2 with $y < P_2^{10}(0)$ does not have incentive to deviate, either.

Since the price system is consistent with the payment rule in the optimal mechanism and the construction of the above procedure is consistent with the allocation rule, the hybrid auction-pricing procedure implements the optimal mechanism. $\square$

Proof of Proposition 4. Let N^1 be the set of buyers who get items and N^0 be the set of buyers who do not get items.

If an item is allocated to buyer i, the seller can charge buyer i at most $v_i + \sum_{j\in S(i)} \alpha_{ij} v_j 1_{\{S(i)\subseteq N^0\}}$. To see this upper bound, we consider two different cases.

Case 1: $S(i) \cap N^0 = S(i)$. This indicates that buyer i gets an item exclusively and thus the seller can raise $v_i + \sum_{j\in S(i)} \alpha_{ij} v_j$ from buyer i.

Case 2: $S(i) \cap N^0 \neq S(i)$. This indicates that at least one of buyer i's neighbors gets an item and thus the seller can raise v_i from buyer i.

If an item is not allocated to buyer i, the seller can charge buyer i at most 0.

Let $S^{-1}(j)$ denote the set of buyers for whom buyer j is in its neighborhood, i.e., $S^{-1}(j) = \{i : j \in S(i)\}$. Hence, the total payments from the system is bounded above by

$$\begin{aligned}
&\sum_{i\in N^1} \left(v_i + \sum_{j\in S(i)} \alpha_{ij} v_j 1_{\{S(i)\subseteq N^0\}} \right) \\
&= \sum_{i\in N^1} v_i + \sum_{i\in N} \sum_{j\in N} \alpha_{ij} v_j 1_{\{i\in N^1, S(i)\subseteq N^0, j\in S(i)\}} \\
&\leq \sum_{i\in N^1} v_i + \sum_{j\in N} \sum_{i\in N} \alpha_{ij} v_j 1_{\{j\in N^0, S^{-1}(j)\nsubseteq N^0, i\in S^{-1}(j)\}} \\
&= \sum_{i\in N^1} v_i + \sum_{j\in N^0} v_j 1_{\{S^{-1}(j)\nsubseteq N^0\}} \sum_{i\in S^{-1}(j)} \alpha_{ij},
\end{aligned} \tag{21}$$

where the inequality comes from the fact that

$$\{(i,j) : i \in N^1, S(i) \subseteq N^0, j \in S(i)\} \subseteq \{(i,j) : j \in N^0, S^{-1}(j) \nsubseteq N^0, i \in S^{-1}(j)\}.$$

With BLLE valuations, equation (21) is smaller than

$$\sum_{i\in N^1} v_i + \sum_{j\in N^0} v_j 1_{\{S^{-1}(j)\nsubseteq N^0\}} \leq \sum_{i\in N^1} v_i + \sum_{i\in N^0} v_i = \sum_{i\in N} v_i,$$

which is the total payment when the seller allocates an item to every buyer. Hence, the optimal allocation is to allocate an item to every buyer and the optimal payment is to charge v_i to buyer i. $\square$

Proof of Proposition 5. Let everybody have the same exclusivity premium, i.e., $\gamma = \sum_{j\in S(i)} \alpha_{ij} v_j$ for all i. Note that such instance occurs when $v_i = v$ for all i and $\alpha_{ij} = \frac{c}{|S(i)|}$, where c is a constant. Suppose there are exactly $k-1$ exclusive allocations. Then the seller can get at most $(k-1)(\overline{v}+\gamma)$ from these $k-1$ exclusive allocations, plus at most $(n-k+1)\overline{v}$ from $n-k+1$ non-exclusive allocations.

Next, suppose there are exactly k exclusive allocations. Then the seller can get at least $k(\underline{v}+\gamma)$. Therefore, the revenue for k exclusive allocations dominate that for $k-1$ exclusive allocations if

$$\gamma \geq (n-k+1)\overline{v} + (k-1)\overline{v} - k\underline{v}. \tag{22}$$

Note that γ is increasing in α_{ij}. Hence, if α_{ij} is sufficiently large such that condition (22) holds, then finding the optimal allocation also results in determining the size of the largest independent set. $\square$

Proof of Proposition 6. By the definition of virtual valuation, $v_i \geq \psi_i(v_i)$ holds. Furthermore, there always exists $\underline{\psi}$, $\overline{\psi}$, and v_i such that $0 \leq \underline{\psi} \leq \psi_i(v_i) \leq \overline{\psi} < v_i$, where $\overline{\psi}$ ($\underline{\psi}$) is an upper (lower) bound on the virtual valuation. Since $\overline{\psi} < v_i$, there also exists $\{v_i\}_{i \in N}$ such that

$$0 \leq \overline{\psi} \leq \left(\sum\nolimits_{j \in S(i)} \alpha_{ij} v_j\right)/n,$$

even when (BLLE) holds. Note that such instance occurs when $v_i = v$ and $\overline{\psi} = v\left(\sum_{j \in S(i)} \alpha_{ij}\right)/n$.

Consider the realization $\{v_i\}_{i \in N}$ such that $v_i > 0$, $\underline{\psi} \leq \psi_i(v_i) \leq \overline{\psi}$ and $0 \leq \underline{\psi} \leq \overline{\psi} \leq \left(\sum_{j \in S(i)} \alpha_{ij} v_j\right)/n$ for all i. Also let everybody have the same exclusivity premium, i.e., $\gamma = \sum_{j \in S(i)} \alpha_{ij} v_j$. By equation (14) and condition (22) in the proof of Proposition 5, the revenue for k exclusive allocations dominate that for $k-1$ exclusive allocations if

$$\gamma \geq (n-k+1)\overline{\psi} + (k-1)\overline{\psi} - k\underline{\psi}. \tag{23}$$

It is clear that this condition holds, since $0 \leq \underline{\psi} \leq \overline{\psi} \leq \left(\sum_{j \in S(i)} \alpha_{ij} v_j\right)/n$. Then finding a deterministic optimal solution to the (SB-RMP) problem also results in determining the size of the largest independent set. $\square$

Proof of Example 10. Consider $F_1 = F_2 = \mathbf{U}[0,1]$. Let $x = 0.01 + 2/3$, $y = 2/3$, and $\alpha_{12} = \alpha_{21} = 0.4$. Then, in an optimal solution, items are allocated to both buyers, since $x = 0.01 + 2/3 > 1/(2-\alpha_{21}) = 1/1.6$ and $y = 2/3 > 1/(2-\alpha_{21}) = 1/1.6$. Buyer 1 pays $m_1 = 1/1.6$ and buyer 2 also pays $m_2 = 1/1.6$.

Next, consider an increase in α_{12} to $\widetilde{\alpha}_{12} = 0.9$. An item is allocated exclusively to buyer 1, since $y = 2/3 < 1/(2-\widetilde{\alpha}_{12}) = 1/1.1$, $x = 0.01 + 2/3 > y(2-\widetilde{\alpha}_{12})/(2-\alpha_{21}) = \frac{2}{3}\frac{1.1}{1.6}$, and $2x + \widetilde{\alpha}_{12}y > 1$. Then the payment is $\widetilde{m}_1 = \widetilde{\alpha}_{12}y + y(2-\widetilde{\alpha}_{12})/(2-\alpha_{21}) = 0.6 + \frac{2}{3}\frac{1.1}{1.6} \simeq 1.058$ (since $y = 2/3 \geq (2-\alpha_{21})/(4-\widetilde{\alpha}_{12}\alpha_{21}) = 1.6/3.64$) and $\widetilde{m}_2 = 0$. Therefore, $m_1 + m_2 > \widetilde{m}_1 + \widetilde{m}_2$.

Also, consider an increase in both α_{12} and α_{21} to 0.7. An item is still allocated exclusively to buyer 1, since $y < 1/(2-\widehat{\alpha}_{12})$, $x = 0.01 + 2/3 > y = 2/3$, and $2x + \widehat{\alpha}_{12}y > 1$. Then the payment is $\widehat{m}_1 = \widehat{\alpha}_{12}y + y(2-\widehat{\alpha}_{12})/(2-\widehat{\alpha}_{21}) = 1.7\frac{2}{3} \simeq 1.133$ (since $y = 2/3 \geq (2-\widehat{\alpha}_{21})/(4-\widehat{\alpha}_{12}\widehat{\alpha}_{21}) = 1.3/3.51$) and $\widehat{m}_2 = 0$. Therefore, $m_1 + m_2 > \widehat{m}_1 + \widehat{m}_2$. $\square$

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