Robust Bidding Policies

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We study the decision problem of an auction bidder who has imperfect information about rivals' bids and wants to maximize her worst-case payoff. This information is modeled via an uncertainty set consisting of all possible realizations of rivals' bids. Maximizing the bidder's worst-case payoff over this set yields robust bidding policies that do not depend on distributional assumptions. We study robust bidding policies in auctions with single demand and multiple demand. In these settings, establishing the classical minmax equality yields the construction of optimal robust bidding policies. These robust bidding policies could provide better payoff than truthful bidding. Furthermore, compared to expected-payoff maximizing policies, they could result in less allocation risks and provide higher payoff under adversarial realizations of rivals' bids.

Key words: auction, bidding, imperfect information, robust optimization

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1. Introduction

In the past few decades, there has been a tremendous growth in the use of auctions as transaction procedures for sales and purchases of high-value assets. For example, the Federal Communications Commission has been using spectrum auctions to sell licenses to telecommunication bandwidths and generate billions of dollars in revenue (Gryta and Nagesh 2015). Telecom companies who participate in spectrum auctions often have imperfect information about their rivals' business plans and consequently bids and bidding policies. Moreover, given high-stakes one-time nature of such auctions (in which prices auction winners have to pay could reach multiple billions of dollars), the bidder objective could be to maximize the worst-case payoff (rather than maximizing expected payoff). In this paper we study this kind of a bidder decision problem in detail. More precisely, we study optimal bidding policies for a bidder whose objective is to maximize the worst-case payoff.

We adopt the robust optimization approach (e.g. Ben-Tal and Nemirovski 2002) to model a bidder's preference. Under this approach, the bidder use an *uncertainty set* to describe all possible realization of her rivals' bids. This set could be flexibly specified by equality and inequality constraints on rivals' bids to represent the bidder's knowledge and belief of these bids. The bidder's *robust bidding problem* is to determine a bidding policy that maximizes her *worst-case payoff* with respect to this uncertainty set. Our modeling framework complements the traditional approach of expected utility maximization in that it provides bidding policies that are robust with respect to underlying probabilistic assumptions on rivals' bids. Gilboa and Schmeidler (1989) proposed a related preference model in which an agent maximizes her worst-case expected payoff with respect to a family of priors. However, such an approach still relies on some probabilistic assessment of rivals' bids, whereas our approach is completely distribution-free.

The robust bidding problem is straightforward for the second-price auction and its Vickrey-Clarke-Groves (VCG) auction generalization. These auction formats satisfy the *incentive compatibility* condition, i.e., truthful-bidding (bidding one's valuation) is a weakly-dominant strategy for each bidder (e.g. Krishna 2009). As a result, truthful bidding is a straightforward solution to the robust bidding problem since bidders maximize their worst-case payoff by bidding truthfully. On the other hand, when incentive compatibility does not hold, bidding truthfully is not necessarily a robust bidding policy, and the robust bidding problem is nontrivial.

We study the robust bidding problem in two auction formats: discriminatory and core-selecting auctions. These auction formats are widely used in practice. Discriminatory auction, a generalization of the first-price auction for a multi-item setting, is used, e.g., in electricity procurement and in sales of U.S. Treasury securities. Core-selecting auction has been recently adopted for sales of bundled items, such as spectrum licenses and airport take-off/landing rights. Two auction formats differ in terms of allocation and payment rules, which are directly related to the manner in which (item) supply and (bidder) demand is treated. Specifically, items could be assumed to be homogenous or item heterogeneity could be handled. Similarly, bidders can be limited to unit demand, i.e., limiting each bidder to winning at most one-item, or could be allowed to have multiple demand and potentially win any number of items. Table 1 summarizes the settings for supply and demand in two auction formats.[1] Analyzing the bidding problem in these auctions allows us to gain insights into the structure and performance of robust bidding policies under various auction settings.

	Discriminatory	**Core-selecting**
Demand	Unit	Multiple
Supply	Homogeneous	Heterogeneous

Table 1 **Summary of settings for studied auction formats**

Our analysis shows that in discriminatory auctions, a robust bidding policy for a particular bidder is to bid the minimal amount that guarantees winning one item, regardless of the realization

[1] Note that these three auctions could be considered as "natural" modifications of a VCG auction in each of the demand/supply settings.

of rivals' bids in the uncertainty set (provided positive payoff in the worst case). In core-selecting auctions, if the bidder is *single-minded* (i.e., she has positive valuation for only one bundle) then a robust bidding policy is of similar characteristic: the bidder bids the minimal amount that guarantees winning her target bundle, regardless of the realization of rivals' bids in the uncertainty set. (This is true assuming that the bidder's valuation for her target bundle is high enough so that by bidding truthfully she still wins her target bundle regardless of the realization of rivals' bids in the uncertainty set; otherwise, an optimal policy is simply bidding zero.)

When a bidder has positive valuation for two or more bundles, the robust bidding problem in core-selecting auctions is more involved. We show an example analysis of the robust bidding policy for the case of a *double-minded* bidder (i.e., one with distinct valuations for two inclusive bundles). In this case, except for scenarios where the bidder wins either of her target bundles under truthful bidding and this allocation does not change for any realization of rivals' bids in the uncertainty set, bidding the minimal amount to guarantee winning either of the target bundles is not necessarily a robust policy. In fact, we demonstrate that a robust policy corresponds to bids that maximize the minimum of bundle-specific worst-case payoff functions. For both cases of single-minded and double-minded bidders, establishing a minimax type of an equality is critical in demonstrating and verifying the optimality of the presented robust bidding policies. However, such an approach has limitations. In fact, we show in an example of a core-selecting auction with a *triple-minded* bidder (i.e., one with distinct valuations for a chain of three bundles) that minimax equality does not hold, so demonstrating optimality of candidate bidding policies for general cases would require a different approach and the search for robust policies could become more challenging.

Throughout our analysis, we evaluate the performance of robust bidding policies by comparing them to a couple of benchmarks. In particular, we compare payoffs under robust bidding policies with payoffs under other bidding policies such as truthful bidding and expected-payoff maximizing policies, assuming some known distributions of rivals' bids over the uncertainty set. Our results show that for non-trivial[2] robust bidding policies, the bidder's payoff is at least as large as her payoff under truthful bidding, for all realizations of rivals' bids in the uncertainty set. In addition, for discriminatory and core-selecting auctions, robust bidding policies improve upon expected-payoff maximizing policies (assuming some known distributions of rivals' bids), as they reduce the risk of not winning the target items. In particular, robust policies yields a higher payoff compared to expected-payoff maximizing policies under adversarial realizations of rivals' bids.

In obtaining the aforementioned results, we aim to shed some light on the robust bidding policies of a single bidder, rather than pursuing an analysis of equilibrium bidding behaviors, as it often the

[2] i.e., positive bid for at least one item/bundle

case in the auction literature. We rationalize this choice with some motivating examples of first-price auctions with two bidders and one item. When both bidders are worst-case payoff maximizers and their rationality and uncertainty sets are mutual knowledge, a wide range of bidding profiles, including a rather counter-intuitive one with each bidder bidding truthfully, constitutes bidding equilibria. Multiple equilibria are also observed in other settings of bidders' rationality, such as when one bidder is a worst-case payoff maximizer and the other is an expected payoff maximizer (with respect to some distributional belief). Furthermore, when moving away from the simple first-price auction toward a more general setting of combinatorial auctions with heterogeneous items, an equilibrium analysis could become more technically challenging. Uninformative predictions of equilibrium behaviors and analytical tractability issues refrain us from pursuing an equilibrium analysis of robust bidding behaviors in this paper.

The rest of the paper proceeds as follows. Section 1.1 discusses related works in the literature. Section 2 follows with descriptions of the auction models, assumptions and some motivating examples. In § 3, we provide robust bidding policies for unit demand bidders in discriminatory auctions. Section 4 discusses robust bidding for multiple demand bidders in core-selecting auctions. Finally, in § 5, we conclude and provide some possible directions for future research. Proofs are relegated to Appendix A.

1.1. Related literature

Uncertainty aversion in auctions Traditionally, in the economic literature, risk aversion has been a fundamental concept when agents face uncertainties. For the case of auctions, numerous works have been carried out to study risk aversion (Maskin and Riley 1984, Matthews 1987, Cox et al. 1985, Smith and Levin 1996, Goeree et al. 2002, Campo et al. 2011). However, it has been argued that risk aversion alone is not sufficient in explaining observed bidding behaviour (Harrison 1990, Kagel and Levin 1985). On the other hand, there is substantive evidence of uncertainty aversion (also known as "ambiguity aversion") in decision making (Ellsberg 1961, Camerer and Weber 1992, Sarin and Weber 1993). In auction settings, uncertainty aversion is relevant since the probability of winning an auction not only depends on the joint distribution of private values, but also on the rivals' unknown bidding strategies. Most of the previous studies on auctions with uncertainty aversion focused on analysing equilibrium bidding behaviour (Salo and Weber 1995, Lo 1998, Bose et al. 2006). In this paper, we adopt the decision analysis approach and focus on the bidding problem of a bidder who is averse to uncertainty about rivals' bids. Example applications of the decision analysis approach to practical bidding problems were documented in Capen et al. (1971) and Keefer et al. (1991). Rothkopf (2007) discussed the effectiveness of such an approach over the game theoretic approach.

Auctions with incentive compatibility issues Our study is related to the classical topic of bidding in auctions with incentive issues(e.g. Stark and Rothkopf 1979, Milgrom and Weber 1982). Ausubel and Cramton (2002) studied incentive to reduce demand in discriminatory auctions with a divisible item. Day and Milgrom (2008) provided an optimal bid shading policy in core-selecting auctions under perfect information. Beck and Ott (2013) showed that bidder may have incentives to overbid in core-selecting auctions as well. Hoffman and Menon (2010) discussed incentive issues in designing a centralized combinatorial exchange.

Robust optimization The modeling approach that we use lies under the framework of robust optimization. It has been recently developed as a decision tool to deal with decision making when the input parameters are uncertain (e.g. Ben-Tal and Nemirovski 2002, Bertsimas and Sim 2004). Under this framework, the decision maker does not know the exact distribution of the uncertain parameters. Instead, it is assumed that these uncertain parameters belong to an uncertainty set that can be constructed based on historical data or an expert's belief. The decision maker wants to maximize her worst-case payoff with respect to this uncertainty set. There are close connections between maximizing worst-case objective over an uncertainty set and uncertainty aversion (Bertsimas and Brown 2009). Recently, Bandi and Bertsimas (2014) adopt the robust optimization framework to study the optimal design problem for multi-item auctions, in the spirit of an earlier work by Myerson (1981). In our work, we focus on a particular bidder's decision problem.

2. Model

We consider auctions that allocate m indivisible items from the set $M = \{1, 2, \ldots, m\}$ among n bidders in $N = \{1, 2, \ldots, n\}$. The monopolistic seller is indexed by 0. Items in M can be homogeneous (i.e., identical) or heterogeneous (i.e., distinct). A *bundle* is a set of items. A bidder is said to have *unit demand* if she has positive valuation for only one item. Similarly, a bidder is said to have *multiple demand* if she has positive valuation for one or more bundles of items. For each bidder $j \in N$, we use $v_j(S)$ and $b_j(S)$ to denote her non-negative truthful and reported valuation for a bundle $S \subseteq M$, respectively. When bidder j has unit demand and the items are homogeneous, we abuse the notation and simply write v_j and b_j to denote her unique valuation/bid. The auctioneer uses an *allocation rule* to determine the set of items S_j to be allocated to bidder j (S_j is an empty set if bidder j does not obtain any item). Similarly, the auctioneer uses a *payment rule* to determine the amount p_j that bidder j has to pay. We assume that bidders have quasi-linear preferences, i.e., bidder j's payoff is $\pi_j = v_j(S_j) - p_j$ for $j \in N$. For convenience, we write $b_{-j} = (b_1, b_2, \ldots, b_{j-1}, b_{j+1}, \ldots, b_n)$ to denote the profile of bidder j's rivals' bids. When all b_j are scalar, the k^{th} highest bid in b_{-j} is denoted by $b_{-j}^{(k)}$. We choose to analyze the decision problem of bidder 1. It turns out that only $b_{-1}^{(k)}$ are relevant to our analysis. Thus, for simplicity, we omit the subscript and write $b^{(k)}$ to denote the k^{th} highest bid in b_{-1}. We use $\mathbf{1}_A$ to denote the indicator function of expression A, i.e., it has value one if A is true and zero otherwise. For any $x \in \mathbb{R}$, we denote $x^+ = \max(x, 0)$.

2.1. Studied auction formats

Discriminatory auction In discriminatory auctions, items are homogeneous. Each bidder has unit demand and submits a scalar bid b_j. We assume that there are more bidders than auctioned items, so $n \geq m$. The m highest bidders get allocated one item each and pay their bids b_j. When there is a tie, bidder 1 is favored, i.e., if $b_1 = b^{(m)}$ then bidder 1 wins an item. Thus, bidder 1 wins an item when her bid is no less than $b^{(m)}$, in which case she receives a payoff of $v_1 - b_1$. Bidder 1's payoff function is given by:

$$\pi_1(b_1, b_{-1}) = (v_1 - b_1)\mathbf{1}_{b^{(m)} \leq b_1}. \tag{1}$$

Core-selecting auctions In core-selecting auctions, items are heterogeneous. Each bidder has multiple demand and submits bids $b_j(S)$ for bundles $S \subseteq M$ of her interest. When $S = M$, we refer to it as the *global bundle*. The auctioneer decides the allocation outcome by solving the *winner determination problem*:

$$\begin{aligned}
\max_x \quad & \sum_{j \in N} \sum_{S \subseteq M} b_j(S) \cdot x_j(S) && (\mathscr{W}) \\
\text{s.t.} \quad & \sum_{S \supset \{i\}} \sum_{j \in N} x_j(S) \leq 1, \quad \forall i \in M, \\
& \sum_{S \subseteq M} x_j(S) \leq 1, \quad \forall j \in N, \\
& x_j(S) \in \{0, 1\}, \quad \forall (S, j) \text{ s.t. bid } b_j(S) \text{ was submitted.}
\end{aligned}$$

Let $w_b(N, M)$ be the objective function of ($\mathscr{W}$). In that problem, the auctioneer maximizes the total reported value of bundles. The first constraint ensures that each item is only allocated once. The second constraint implies that the auctioneer only accepts at most one submitted bid from a bidder.[3] Thus, each binary variable $x_j(S)$ equals to one if and only if bidder j is awarded bundle $S \subseteq M$. It is possible that the problem ($\mathscr{W}$) has multiple optimal solutions. In such cases, we assume a tie-breaking rule in favor of bidder 1 winning a pre-specified bundle. For an arbitrary set of bidders $C \subseteq N$ and an arbitrary set of items $S \subseteq M$, the function $w_b(C, S)$ is defined as the maximum surplus generated by allocating items in S among bidders in C given the reported bids b. We refer to the function w_b as the *coalition value function*. When $S = M$, we abuse the notation and write $w_b(C)$ instead of $w_b(C, M)$.

[3] This bidding rule is referred to as the "XOR" bidding language and is commonly used in practice for spectrum auctions.

Next, we define *the core* to be the set of non-negative payoff vectors $\{\pi_j\}_{j \in N \cup 0}$ satisfying the *core constraints*:

$$\sum_{j \in C \cup 0} \pi_j \geq w_b(C), \quad \forall C \subseteq N. \tag{2}$$

The right hand side of (2) is the maximum surplus generated by allocating the items among members in the coalition C (hereafter referred to as a *blocking coalition*). Thus, the constraints (2) guarantee that no group of bidders can ensure better payoff for themselves by excluding others from participation, i.e., there is no incentive to form a blocking coalition in the auction. The core constraints (38) can be written succinctly as linear constraints on the payment vector $\{p_j\}_{j \in N}$ of the form:

$$pA \geq \beta, \tag{3}$$

where A is a $n \times 2^n$ matrix and β is a vector in $\mathbb{R}^{2^n}$ (see detail descriptions in Appendix A.6). A *core-selecting payment rule* is a payment rule that selects a vector $\{p_j\}_{j \in N}$ satisfying the core constraint (3) and the individual rationality constraint $p \leq b$. For specificity, we use the quadratic core-selecting payment rule proposed by Day and Cramton (2012), which selects a payment vector in the core that minimizes the Euclidean distance to a reference payment vector.[4] A related payment rule is the *VCG rule* Vickrey (1961), Clarke (1971), Groves (1973) . Under this payment rule, each winner j pays the opportunity cost that she imposes on other bidders:

$$p_j^{VCG} = w_b(N \setminus j, M) - w_b(N \setminus j, M \setminus S_j). \tag{4}$$

The *VCG payoff* of bidder j is then simply $\pi_j^{VCG} = v_j(S_j) - p_j^{VCG}$. It is known that VCG payment rule satisfies incentive compatibility but may result in low seller's revenue and is vulnerable to collusive bidding Ausubel and Milgrom (2002). In contrast, core-selecting rules are robust to collusion but may have incentive compatibility issues, as we will see in later sections. When the quadratic rule is used with the reference payment being the vector of VCG payments, we refer to such rule as the *nearest-VCG rule.*

2.2. Robust optimization formulation

In each of the aforementioned auction formats, we consider the decision problem of a particular bidder, who we choose to be bidder 1, without loss of generality. We assume that bidder 1 has a belief that her rivals' bids belong to an *uncertainty set* U_{-1}. Such an uncertainty set can be constructed from historical data or an expert's assessments.[5] For discriminatory auctions, we have $U_{-1} \subset \mathbb{R}_+^{n-1}$, while for core-selecting auctions, we have $U_{-1} \subset \mathbb{R}_+^{(n-1)2^m}$. For simplicity, we assume

[4] For other core-selecting payment rules, see e.g., Ausubel and Baranov (2010).

[5] Bandi and Bertsimas (2014) discussed several ways to construct the uncertainty set based on historical data.

that the uncertainty set U_{-1} is a convex polytope, i.e., it can be specified by some linear constraints on b_{-1}:

$$U_{-1} = \{b_{-1} \mid Pb_{-1} \leq q\}, \tag{5}$$

where P and q are matrix and vector with appropriate dimensions. For example, U_{-1} can be a box set (i.e., a hyperrectangle) that corresponds to the case where bids b_j for $j \neq 1$ are independent and belong to some known intervals $[\underline{b}_j, \bar{b}_j]$:

$$U_{-1} = \{b_{-1} \mid \underline{b}_j \leq b_j \leq \bar{b}_j, \forall j \neq 1\}. \tag{6}$$

Given an uncertainty set U_{-1}, bidder 1's objective is to maximize her worst-case payoff with respect to this uncertainty set. In other words, she needs to solve the following robust optimization problem:

$$\pi_1^{MAXMIN} = \sup_{b_1 \in U_1} \inf_{b_{-1} \in U_{-1}} \pi_1(b_1, b_{-1}). \tag{$\mathscr{P}$}$$

In the above problem, we use U_1 to refer to bidder 1's feasible policy space. We assume no restrictions, except for the non-negative condition, so $U_1 = \mathbb{R}_+$ for discriminatory auctions and $U_1 = \mathbb{R}_+^{2^m}$ for core-selecting auctions. We call bidding policy b_1 a *robust policy* if it is an optimal solution to ($\mathscr{P}$).

Let π_1^{MINMAX} be the minimum value over U_{-1} of bidder 1's maximum *ex post* payoff, i.e., payoff that she can achieve using an ex post optimal policy. Thus, π_1^{MINMAX} is given by:

$$\pi_1^{MINMAX} = \inf_{b_{-1} \in U_{-1}} \sup_{b_1 \in U_1} \pi_1(b_1, b_{-1}). \tag{7}$$

By minimax inequality (e.g. Boyd and Vandenberghe 2004), one has that

$$\pi_1^{MAXMIN} \leq \pi_1^{MINMAX}. \tag{8}$$

Thus, π_1^{MINMAX} provides an upper bound for the optimal objective of the robust optimization problem ($\mathscr{P}$). If (8) holds with equality, i.e.,

$$\pi_1^{MAXMIN} = \pi_1^{MINMAX}, \tag{9}$$

then there are two implications. First, the robust policy can be viewed as an ex post optimal policy applied for a particular worst-case bid $b_{-1} \in U_{-1}$. Second, if under a policy $b_1 \in U_1$, bidder 1's worst-case payoff is the same with π_1^{MINMAX}, then such policy must be a robust policy. The later observation is particularly useful in proving the optimality of bidding policies in core-selecting auctions, as we will see in § 4.

3. Bidding with unit demand

3.1. Robust bidding in discriminatory auctions

We consider in this section the robust bidding problem ($\mathscr{P}$) in discriminatory auctions. Let $u(k)$ be the maximum value of $b^{(k)}$ over the uncertainty set U_{-1}, i.e.,

$$\begin{aligned} u(k) = \max \quad & b^{(k)} \\ \text{s.t.} \quad & b_{-1} \in U_{-1}. \end{aligned} \tag{10}$$

When bidder 1 knows her rivals' bids, she can best respond by bidding exactly $b^{(m)}$, the m^{th} highest bid among her rivals, if $b^{(m)} \leq v_1$. The following proposition characterizes bidder 1's robust policy under imperfect information.

Proposition 1. *In discriminatory auctions, a robust policy for bidder 1 is $b_1^{RO} = u(m)\boldsymbol{1}_{u(m)\leq v_1}$. The optimal payoff is $\pi_1^{MAXMIN} = (v_1 - u(m))^+$.*

Since truthful bidding always gives zero payoff, robust bidding results in a strictly better worst-case payoff when $u(m) < v_1$. The next example provides a comparison between robust bidding policy and expected-payoff maximizing policy under uniform distribution assumption.

Example 1. Consider a discriminatory auction with $n = 2$ bidders and $m = 1$ item. In this case, the discriminatory auction is a first-price auction. Bidder 1 has a belief that $b_2 \in [c, d]$ for some non-negative constants c and d. According to Proposition 1, bidder 1's robust policy is $b_1^{RO} = d\mathbf{1}_{d\leq v_1}$. If bidder 1's belief is such that b_2 is uniformly distributed in $[c, d]$ and her objective is to maximize the expected payoff, the optimal bid is:[6]

$$b_1^{EM} = \begin{cases} d & \text{if } 2d - c < v_1 \\ \frac{1}{2}(v_1 + c) & \text{if } c \leq v_1 \leq 2d - c \\ 0 & \text{otherwise.} \end{cases}$$

Figure 1 shows a plot of b_1^{RO} and b_1^{EM} for the case where $c = 0$ and $d = 5$. There are two main differences between these bidding policies. First, when $v_1 < 5$, bidder 1 bids zero under b_1^{RO}, whereas under b_1^{EM}, she bids a positive amount. Second, when $v_1 \geq 5$, we have $b_1^{RO} = 5$ while $b_1^{EM} = \frac{1}{2}v_1$ is an increasing function in v_1. When $v_1 \geq 10$, the two bidding policies are the same. In Figure 2, we show a comparison of bidder 1's payoffs (as functions of b_2) under b_1^{RO} and b_1^{EM} for different values of v_1. For $v_1 = 3$, since $v_1 < 5$, according to Proposition 1, bidder 1's robust policy is $b_1^{RO} = 0$. As a result, her payoff is $\pi_1^{RO} = 0$ for all realization of $b_2 \in [0, 5]$. On the other hand, to maximize her expected payoff, bidder 1 bids $b_1^{EM} = 1.5$. Thus, she still gains a positive payoff of $\pi_1^{EM} = 1.5$ for $b_2 \in [0, 1.5]$. For $v_1 = 8$, we have $b_1^{RO} = 5$ and $b_1^{EM} = 4$. If $b_2 \in [0, 4)$ then bidder 1's payoffs under b_1^{RO} and b_1^{EM} are 3 and 4, respectively. However, when $b_2 \in [4, 5]$, bidder 1 does not win the item if she bids b_1^{EM} so her corresponding payoff is $\pi_1^{EM} = 0$, while her payoff when bidding b_1^{RO} is still $\pi_1^{RO} = 3$. □

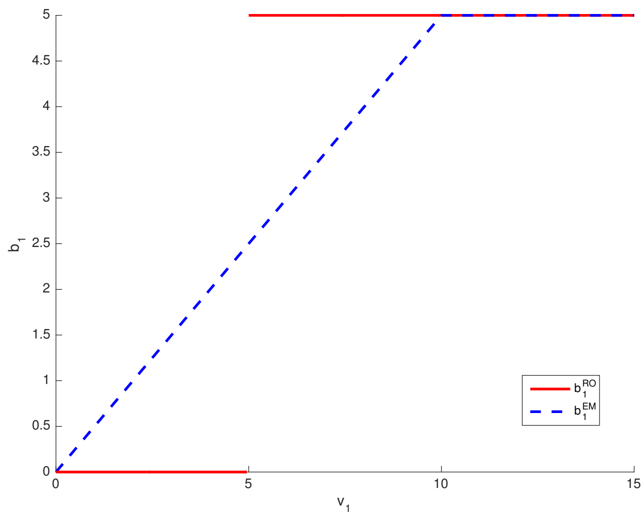

Figure 1 **Illustration of Example 1 – Comparison of b_1^{RO} and b_1^{EM}**

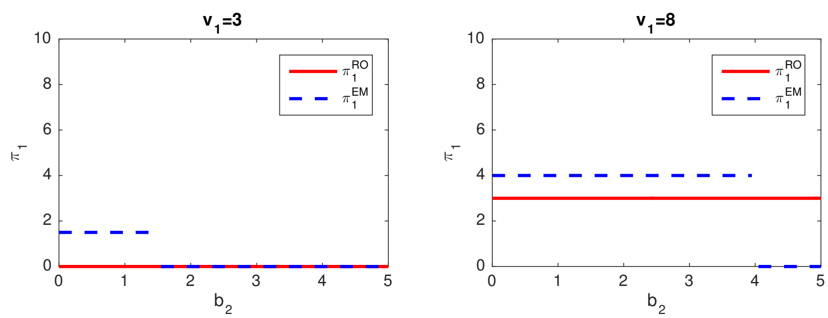

Figure 2 **Illustration of Example 1 – Bidder 1's payoff function under b_1^{RO} and b_1^{EM}**

Example 1 illustrates the fact that under robust policy b_1^{RO}, bidder 1 bids the minimal amount that guarantees winning an item, if such bidding policy is profitable. As we can see, this bidding policy is beneficial when rivals bid adversarially. Similar observations can be made for bidding policies in multiple demand settings, as we will see in § 4.

3.2. Intermezzo: equilibrium bidding under different bidders' rationality assumptions

We provide several examples to illustrate potential issues as one attempts to pursue an equilibrium analysis with different bidders' rationality assumption. Consider a discriminatory auction with $n = 2$ bidders and $m = 1$ item, which is a first-price sealed-bid auction. Each bidder j has valuation v_j for the item and v_j is private information of bidder j. We assume that $v_j \in [0, c]$ for some positive constant c that is common knowledge to both bidders. After learning her private valuation v_j, each

[6] See § A.1

bidder bids $b_j(v_j)$ for the item. Note that $b_j(v_j) \le v_j$ since it is not profitable for bidder j to bid above her own valuation in a first-price auction. A bidder either maximizes her expected payoff, given some belief about the distribution of the rival bidder's valuation, or maximizes her worst-case payoff across all the possible realization of the rival bidder's valuation in $[0, c]$. Bidder j's payoff is given by[7]

$$\pi_j(b_1, b_2, v_1, v_2) = (v_j - b_j)\mathbf{1}_{b_j \ge b_{j'}}, \quad j \neq j' \text{ and } j, j' \in \{1, 2\}. \tag{11}$$

From the classical auction theory, if both bidders are expected payoff maximizers, then the equilibrium bidding profile can be determined (e.g. Krishna 2009). For example, if each bidder believes that her rival's valuation is uniformly distributed over $[0, c]$ and bidders' rationality is common knowledge, then the equilibrium bidding profile is

$$b_j^*(v) = v/2, \quad v \in [0, c], \quad j \in \{1, 2\}. \tag{12}$$

The following example illustrates the equilibrium bidding profiles when both bidders are worst-case payoff maximizer.

Example 2 (Both bidders are worst-case payoff maximizers). Assume that both bidders believe that their rival's valuation is within $[0, c]$ and want to maximize their worst-case payoff across all realization of rival's valuation in this set. Furthermore, bidders' rationality is common knowledge. The equilibrium bidding functions $b_j^*(v_j)$ for $j \in \{1, 2\}$ are defined by:

$$b_1^*(v_1) = \arg\max_{b_1(.)} \min_{v_2 \in [0,c]} \pi_1(b_1, b_2^*(v_2), v_1, v_2) \tag{13}$$

$$b_2^*(v_2) = \arg\max_{b_2(.)} \min_{v_1 \in [0,c]} \pi_2(b_1^*(v_1), b_2, v_1, v_2) \tag{14}$$

We can see that any pair of bidding functions (b_1^*, b_2^*) satisfying $b_j^*(c) = c$ and $b_j^*(v_j) \le v_j$ for $v_j \in [0, c)$ is an equilibrium profile. In fact, if $b_2^*(c) = c$ then for any $v_1 \in [0, c]$, there is a possibility that bidder 1 loses the auction since bidder 2 could bid c when her valuation is realized to be c. Hence, bidder 1's worst-case payoff is always zero and any $b_1(v_1) \le v_1$ is her best response. Note that this result implies both bidders bidding truthfully is also an equilibrium strategy profile. □

The next example shows that one can also obtain multiple equilibria when a bidder is a worst-case payoff maximizer and the other bidder is an expected payoff maximizer (with respect to some distributional belief).

Example 3 (Worst-case payoff maximizer facing expected payoff maximizer). Assume that bidder 1 is a worst-case payoff maximizer while bidder 2 is an expected payoff

[7] The notations $b_j(v_j)$ and $\pi_j(b_1, b_2, v_1, v_2)$ are only used within this subsection.

maximizer who believes that bidder 1's valuation is uniform on $[0,c]$. Furthermore, bidders' rationality is common knowledge. The system of equations for equilibrium bidding functions is:

$$b_1^*(v_1) = \arg\max_{b_1(.)} \min_{v_2\in[0,c]} \pi_1(b_1, b_2^*(v_2), v_1, v_2) \tag{15}$$

$$b_2^*(v_2) = \arg\max_{b_2(.)} \mathbb{E}_{v_1}\pi_2(b_1^*(v_1), b_2, v_1, v_2) \tag{16}$$

Similar to the previous example, there does not exist a unique equilibrium bidding profile. For example, the following bidding policies constitute equilibrium profiles:

$$b_1^*(v_1) \text{ is increasing and convex and } b_1^*(v_1) \leq v_1,\ \forall v_1 \in [0,c], \tag{17}$$

$$b_2^*(v_2) = \max\left\{ v_2 - \frac{\int_0^{b_2} g(b)db}{g(b_2)}, 0\right\},\ \forall v_2 \in [0,c], \tag{18}$$

$$b_1^*(c) = b_2^*(c), \tag{19}$$

where $g(b) = \frac{1}{(b_1^*)'((b_1^*)^{-1}(b))}$ (see § A.2). □

As we can see, even in a relatively simple strategic environment like a first-price auction, having different assumptions on each bidder's rationality can significantly change the bidding behaviour. We therefore choose to focus on a single bidder's decision problem rather than an equilibrium analysis of bidding behaviour.

4. Bidding with multiple demand

We consider in this section the bidding problem of bidder 1 in core-selecting auctions. Section 4.1 starts with a discussion of the incentive to *misreport*, i.e., bidding different from true valuation, in core-selecting auctions. Section 4.2 follows with the results of bidder 1's optimal bidding policies under perfect information. Finally, Section 4.3 provides results of bidder 1's robust policies when she has imperfect information of rivals' bids.

4.1. Relationship between VCG and the core

In core-selecting auctions, the relationship between VCG payoffs and the core provides us useful information about whether bidders have incentive to misreport. We say *VCG is in the core* if VCG payoffs satisfies core constraints (2) or, equivalently, the VCG payments satisfy core constraints (3). If VCG is in the core then it is the unique *bidder-optimal* point in the core, i.e., the point in the core where bidders' total payment is minimized, and no bidder has the incentive to misreport.[8] A sufficient condition for VCG being in the core is the *bidder-submodularity* property of the coalition value function w_b, defined as follows.

[8] See Ausubel and Milgrom (2002).

Definition 1. The coalition value function w_b is *bidder-submodular* if for all $j \in N$ and all coalitions C and C' satisfying $0 \in C \subseteq C'$, one has that $w_b(C \cup \{j\}) - w_b(C) \geq w_b(C' \cup \{j\}) - w_b(C')$.

The bidder-submodularity property of coalition value function is closely related to the submodularity and supermodularity properties of bid functions $b_j(S)$. Thus, it is useful to review these definitions.

Definition 2. A set function $f : 2^M \to \mathbb{R}$ is said to be *submodular* if for every $S, S' \subseteq M$ with $S \subseteq S'$ and every $i \in M$ we have that

$$f(S \cup \{i\}) - f(S) \geq f(S' \cup \{i\}) - f(S'). \tag{20}$$

Definition 3. A set function $f : 2^M \to \mathbb{R}$ is said to be *supermodular* if for every $S, S' \subseteq M$ with $S \subseteq S'$ and every $i \in M$ we have that

$$f(S \cup \{i\}) - f(S) \leq f(S' \cup \{i\}) - f(S'). \tag{21}$$

If b_j is submodular for some $j \in N$ then bidder j's reported valuation for having an extra item $i \in M$ decreases as S increases. Therefore, all goods are *substitutes* for bidder j (with respect to her reported valuation). Similarly, if b_j is supermodular then all goods are *complements* for bidder j (with respect to her reported valuation). The following result establishes the relationship between the bidder-submodularity property of coalition valuation and the submodularity property of bid functions.

Proposition 2. *Ausubel and Milgrom (2006) If b_j is a submodular set function for all $j \in N$, the corresponding coalition value function w_b is bidder-submodular and VCG is in the core.*

Interestingly, if all bid functions are supermodular, the coalition value function is also bidder-submodular and thus VCG is also in the core. The following proposition establishes this result.

Proposition 3. *If b_j is a supermodular set function for all $j \in N$, the corresponding coalition value function w_b is bidder-submodular and VCG is in the core.*

When goods are substitutes, the marginal value of an extra bidder to a coalition is the difference between that new bidder's value for his assigned bundle and the opportunity cost of the coalition for the bundle. Since this opportunity cost increases with the coalition size under substitution, the marginal value of an extra bidder decreases as coalition size increases, which is the definition for bidder-submodular. When goods are complements, the marginal value of an extra bidder is the difference between his valuation for all goods and the coalition valuation for all goods, if such difference is positive. Thus, as the coalition size increases, the valuation of the coalition for all goods increases so the marginal value of the extra bidder decreases, which establishes the bidder-submodularity property of the coalition value function.

As we can see, bidders have incentive to misreport only when there are goods that are substitutes for some bidders and (possibly different) goods that are complements for (possibly different) bidders. These results extend to imperfect information as well.

PROPOSITION 4. *If the uncertainty set U_{-1} is such that for all $b_{-1} \in U_{-1}$ the coalition value function $w_{v_1,b_{-1}}$ is bidder-submodular then truthful bidding is the optimal solution of $(\mathscr{P})$.*

COROLLARY 1. *If v_1 is submodular (supermodular) and $\{b_j\}_{j\neq 1}$ are submodular (supermodular) for all $b_{-1} \in U_{-1}$, then truthful reporting is the optimal robust policy.*

In general, the bidder-submodularity property of $w_{v_1,b_{-1}}$ is not guaranteed so truthful bidding may not be an optimal solution to $(\mathscr{P})$. Our subsequent analysis focuses on these cases.

4.2. Optimal bidding policies under perfect information

If bidder 1 has perfect information about rivals' bids b_{-1}, her VCG payoff is the maximum payoff that she can achieve.[9] There are multiple policies that bidder 1 can use to achieve such payoff. In the next proposition, we show one such policy, which has direct generalization to imperfect information case.[10] For convenience, we assume a tie-breaking rule in which bidder 1 winning bundle S_1, the bundle that bidder 1 would win if she bids truthfully (hereafter referred to as *truthful bundle*), is favored.

PROPOSITION 5. *The following policy is optimal for bidder 1:*

$$b_1^{PI}(S) = \begin{cases} 0 & \textit{if } S \subsetneq S_1 \\ v_1(S_1) - \pi_1^{VCG} & \textit{if } S_1 \subseteq S \subsetneq M \\ w_{b_{-1}}(N \setminus 1) & \textit{if } S = M \end{cases} \tag{22}$$

REMARK 1. Policy (22) is also optimal for bidder 1 even if she uses multiple identities, also known as *shills*.[11]

Under policy (22), bidder 1 shades (underbids) her valuation on truthful bundle S_1 and on any other bundles that contain it, except the global bundle M. In addition, she misreports her valuation for the global bundle and bids $w_v(N \setminus 1)$ for it. This bidding policy has two main effects. First, it guarantees bidder 1 still wins bundle S_1 under the assumed tie-breaking rule. Second, by bidding high on the global bundle, bidder 1 effectively inflates her demand for other bundles. As a result, other bidders are forced to pay for the high opportunity cost they impose on bidder 1. Since a winner's payment decreases if other winners' payments increase under a bidder-optimal payment

[9] See Day and Milgrom (2008).

[10] For other optimal policies under perfect information, see Day and Milgrom (2008) and Beck and Ott (2013).

[11] For more discussion on shill bidding in combinatorial auctions, see e.g., Ausubel and Milgrom (2006)

rule, bidder 1 payment decreases. Notice that policy (22) can be modified to accommodate for different tie-breaking rules. Specifically, if bidder 1 winning bundle S_1 is not favored, bidder 1 can change her bid for S where $S_1 \subseteq S \subsetneq M$ to $b_1(S) = v_1(S_1) - \pi_1^{VCG} + \epsilon$ for some $\epsilon > 0$ arbitrarily small. In such case, bidder 1 still wins S_1 and achieves a payoff that is arbitrarily close to her VCG payoff.

4.3. Single-minded bidder

We consider in this section the robust bidding problem ($\mathscr{P}$) in which bidder 1 is *single-minded*, i.e., she has positive valuation if she wins a particular bundle S_1 and zero valuation otherwise:[12]

$$\begin{aligned} v_1(S) &= a > 0 \quad \text{if } S \supseteq S_1, \\ v_1(S) &= 0 \quad \text{if } S \not\supseteq S_1. \end{aligned} \tag{23}$$

Let $\bar{p}^{VCG}$ be the maximum value over the uncertainty set U_{-1} of bidder 1's VCG payment if she wins S_1, i.e.,

$$\bar{p}^{VCG} = \max_{b_{-1} \in U_{-1}} \quad w_{b_{-1}}(N \setminus 1, M) - w_{b_{-1}}(N \setminus 1, M \setminus S_1). \tag{24}$$

If $v_1(S_1) \leq \bar{p}^{VCG}$ then there exists $b^*_{-1} \in U_{-1}$ such that bidder 1 does not win S_1 by bidding truthfully. Bidder 1's payoff is thus zero in that case. According to Proposition 5, there exists an optimal bidding policy that yields the same allocation outcome for bidder 1. Thus, given b^*_{-1}, bidder 1's payoff is still at most zero if she bids non-truthfully. Therefore, zero is an upperbound on bidder 1's worst-case payoff and bidding zero is a trivial robust policy. Consequently, without loss of generality, we assume throughout the remainder of our analysis in this section that $v_1(S_1) > \bar{p}^{VCG}$, i.e., bidder 1's valuation for bundle S_1 is high enough so that she always wins bundle S_1 by bidding truthfully, regardless of the realization of her rivals' bids $b_{-1} \in U_{-1}$. For convenience, we also assume a tie-breaking rule such that bidder 1 winning S_1 is favored. Notice that this assumption does not affect our results, by a similar reason as in the perfect-information case. The following proposition gives a robust policy for bidder 1.

PROPOSITION 6. *If bidder 1 is single-minded and $v_1(S_1) > \bar{p}^{VCG}$ then a robust policy for bidder 1 is:*

$$\begin{aligned} b_1^{RO}(S) &= \bar{p}^{VCG}, \quad \text{if } S_1 \subseteq S \subsetneq M, \\ b_1^{RO}(M) &= \bar{p}^{VCG} + \min_{b_{-1} \in U_{-1}} w_{b_{-1}}(N \setminus 1, M \setminus S_1), \\ b_1^{RO}(S) &= 0, \quad \text{otherwise.} \end{aligned} \tag{25}$$

The optimal worst-case payoff is $\pi_1^{MAXMIN} = v_1(S_1) - \bar{p}^{VCG}$.

[12] Here we assume free disposal, so bidder 1's valuation for winning $S \supsetneq S_1$ is the same with her valuation for winning S_1

Remark 2. Policy (25) is not uniquely optimal. For example, if $b_1(M)$ is any value in the interval $[0, \bar{p}^{VCG} + \min_{b_{-1}\in U_{-1}} w_{b_{-1}}(N\setminus 1, M\setminus S_1)]$, then the resulting policy is also optimal. However, bidding a higher value of $b_1(M)$ weakly increases the payoff of bidder 1 for any realization of $b_{-1}\in U_{-1}$, since it would increase the payment of other bidders and thus reduce bidder 1's payment.

L\L\G valuation structure. We analyze the performance of the robust policy given by Proposition 6 in an auction with $n=3$ bidders and $m=2$ identical items. In particular, we consider the Local-Local-Global (L\L\G) valuation structure in which bidder 1 is a *local* bidder who is interested in only one item while bidders 2 and 3 are *local* and *global* bidders who are interested in winning one and two items, respectively.[13] Table 2 summarizes notation: a, b, c are problem parameters while x and y are decision variables. Note that in this particular section, we abuse the notation and use b to refer to bidder 2's bid on one and two items, rather than the bid profile of all bidders.

# items	v_1	b_1	b_2	b_3
1	a	x	b	0
2	a	y	b	c

Table 2 Bidder valuations under the L\L\G valuation structure

We consider the simple box-type uncertainty set:

$$U_{-1} = \{(b_2,b_3) \mid \bar{b}-\epsilon_b \le b \le \bar{b}+\epsilon_b, \bar{c}-\epsilon_c \le c \le \bar{c}+\epsilon_c\},$$

where $\epsilon_b < \bar{b}$ and $\epsilon_c < \bar{c}$. Notice that in this setting the coalition value function $w_{v_1,b_{-1}}$ need not to be bidder-submodular. Bidder 1's worst-case VCG payment defined in (24) in this case is simply $\bar{p}^{VCG} = \max_{b_{-1}\in U_{-1}}(c-b)^+$. If $\bar{p}^{VCG} < a$ then according to Proposition 6, a robust bidding policy for bidder 1 is:

$$(x^*, y^*) = (\bar{p}^{VCG}, \bar{p}^{VCG} + \bar{b} - \epsilon_b). \tag{26}$$

Remark 3. Given the nearest-VCG payment rule, bidder 1's payoff under robust policy (26) is greater than her payoff under truthful bidding for any realization of b_{-1} in U_{-1} (see details in § A.10).

The following numerical examples further illustrate the performance of robust bidding policy within the L\L\G structure.

[13] Similar valuation structure has been used in equilibrium analysis of core-selecting auctions (Goeree and Lien 2016, e.g.)

EXAMPLE 4. Consider a core-selecting auction with $n=3$ bidders and $m=2$ items. Bidders have L\L\G valuation structure (see Table 2). Let $v_1=(10,10)$, and $U_{-1}=\{(b_2,b_3)\,|\,7\le b\le 13, 7\le c\le 13\}$. According to Proposition 6, a robust bidding policy is $b_1^{RO,1}=(6,13)$. However, as noted earlier in Remark 2, bidding any value of $y\in[6,13]$ will not change the worst-case payoff, so policy such as $b_1^{RO,2}=(6,6)$ also yields the same worst-case payoff. For a concrete scenario of rivals' bids defined by $b=10$ and $c=10$, the perfect-information policy (22) is $b_1^{PI}=(0,10)$.

Figure 3 shows a comparison of bidder 1's payoffs under the robust policy $b_1^{RO,1}=(6,13)$ and truthful policy $b_1^{TR}=(10,10)$. We can see that for any realization of $b_1\in U_{-1}$, bidder 1 receives a larger payoff by bidding according to the robust policy $b_1^{RO,1}$ instead of reporting her true valuation. This agrees with our earlier observation in Remark 3.

Figure 4 shows a comparison of robust policies $b_1^{RO,1}$ and $b_1^{RO,2}$. Note that in this case the worst-case realization of the rivals' bids is $b=7$ an $c=13$, i.e., bidder 2 bids lowest and bidder 3 bids highest. Both policies $b_1^{RO,1}$ and $b_1^{RO,2}$ yield the same worst-case payoff. However, policy $b_1^{RO,1}$ gives larger payoff at other realizations of b_{-1} in the uncertainty set since bidder 1 bids higher on the global bundle in this policy (recall Remark 2).

In Figure 5, we show a comparison between the robust policy $b_1^{RO,1}$ and the perfect-information policy b_1^{PI}. Under b_1^{PI}, if bidder 2 and 3 bid such that $b\ge c$ and $b\ge 10$, then bidder 1 and 2 win an item each. Since bidder 1 bids zero for one item, bidder 2 has to incur the entire payment burden and bidder 1 is a free rider. Thus, bidder 1 extracts the entire surplus and obtains maximum payoff in this case. However, by bidding b_1^{PI}, bidder 1 also faces the risks of winning the unnecessary extra item or not winning any item. Specifically, if bidder 2 and 3 bid $b<c$ and $c\ge 10$, then bidder 3 wins both items and bidder 1 receives zero payoff. Similarly, if bidder 2 and 3 bid $b<10$ and $c<10$ then bidder 1 wins both item and pays a high payment $p_1=\max(b,c)$. In both cases, bidder 1's payoff is reduced significantly. The robust policy $b_1^{RO,1}$ avoids these risks by ensuring that bidder 1 always wins an item, regardless of how much bidder 2 and 3 bid. □

REMARK 4. In Example 4, if bidder 1 uses a perfect-information optimal policy (22) with respect to the worst-case rivals' bids $b=7$ and $c=13$ then she will recover robust policy $b_1^{RO,1}=(6,13)$. This is because in the single-minded setting, minimax equality (9) holds.

EXAMPLE 5. Consider a core-selecting auction with similar settings as in Example 4. However, in this case, we allow bidder 1's valuation to vary and compare bidder 1's payoff under the robust policy (26) to her payoff under an expected-payoff maximizing policy. To compute the later, we consider the case where bidder 1 has a belief that her rivals' bids are such that b and c are independent and the corresponding probability density functions are:

$$f_b(b)=\begin{cases}\frac{1}{18}(b-7) & \text{if } 7\le b\le 13\\ 0 & \text{otherwise}\end{cases} \tag{27}$$

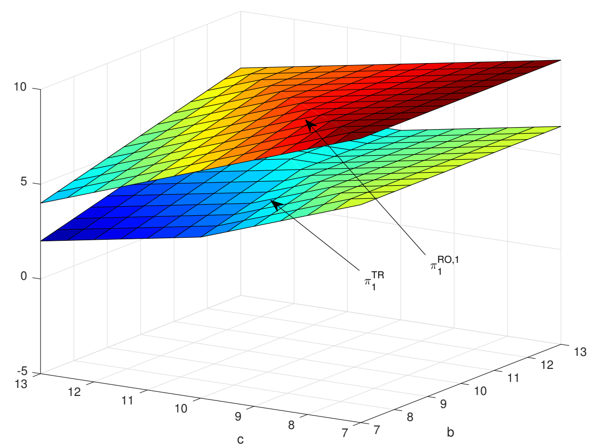

Figure 3 **Illustration of Example 4 – Comparison of bidder 1's payoffs under the robust policy** $b_1^{RO,1} = (6, 13)$ **and the truthful policy** $b_1^{TR} = (10, 10)$

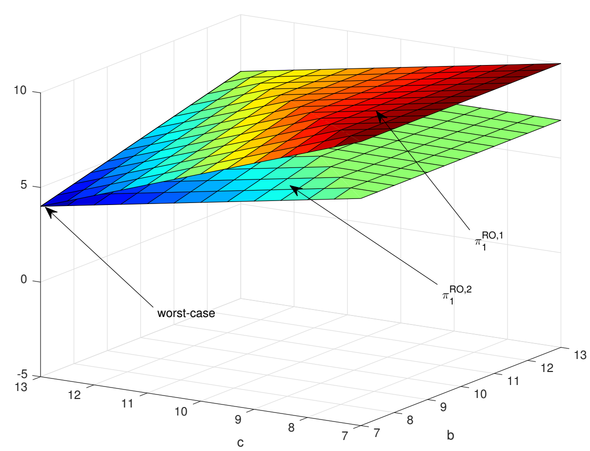

Figure 4 **Illustration of Example 4 – Comparison of bidder 1's payoffs under robust policies** $b_1^{RO,1} = (6, 13)$ **and** $b_1^{RO,2} = (6, 6)$

and

$$f_c(c) = \begin{cases} \frac{1}{18}(13 - c) & \text{if } 7 \leq c \leq 13 \\ 0 & \text{otherwise.} \end{cases} \tag{28}$$

Note that if b and c are uniformly distributed then it turns out that bidder 1's expected-payoff maximizing policy is the same with her robust policy (26) and the comparison is trivial. Thus, for illustration purpose, we consider instead the distributions f_b and f_c described above. Our observations are qualitatively the same for other choices of f_b and f_c.

When $v_1 = (4, 4)$, bidder 1's robust policy and expected-payoff maximizing policies are $b_1^{RO} = b_1^{EM} = (0, 0)$. Thus, bidder 1's payoff is the same under the two policies. When $v_1 = (10, 10)$, we

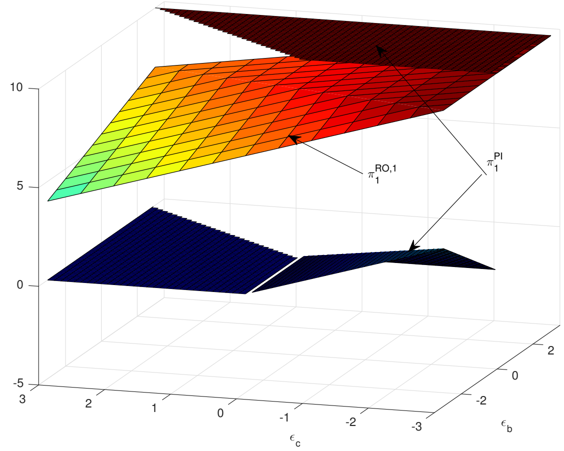

Figure 5 **Illustration of Example 4 – Comparison of bidder 1's payoffs under the robust policy $b_1^{RO,1} = (6, 13)$ and the perfect-information policy $b_1^{PI} = (0, 10)$**

have $b_1^{RO} = (6, 13)$ and $b_1^{EM} = (4, 12)$ (see Appendix A.11 for details). Notice that f_b is increasing in b for $b \in [7, 13]$ and f_c is decreasing in c for $c \in [7, 13]$, so under these distributional assumptions, bidder 2 are more likely to bid high and bidder 3 are more likely to bid low, relative to the support ranges given by U_{-1}. As a result, under b_1^{EM}, bidder 1 bids low for one item in anticipation that she would win one item with high probability and pay less as a result of her low bid. However, this policy exposes her to the risks of not winning any item or winning both items. Figure 6 shows a comparison of bidder 1's payoff under b_1^{RO} and b_1^{EM} when $v_1 = (10, 10)$. As we can see, bidder 1's payoff under b_1^{RO} is significantly greater than her payoff under b_1^{EM} at realizations of rivals' bids in U_{-1} such that bidder 1 wins both items or does not win any items by bidding b_1^{EM}. Figure 7 shows a comparison of bidder 1's payoff under b_1^{RO} and b_1^{EM} when $v_1 = (20, 20)$. Similar observations can be made regarding the payoff functions under the two policies. Note that b_1^{RO} is the same as in the case of $v_1 = (10, 10)$ since the policy only depends on parameters of U_{-1} as long as $a > \bar{p}^{VCG}$. On the other hand, $b_1^{EM} = (3, 11)$ when $v_1 = (20, 20)$, i.e., bidder 1 shades more on her bids compared to the case of $v_1 = (10, 10)$. □

4.4. Double-minded bidder

We now extend our analysis to the case where bidder 1 is *double-minded*, i.e., she has positive valuation for two bundles. In particular, we assume that bidder 1 is interested in two bundles S_1 and S_2 where $\emptyset \subsetneq S_1 \subsetneq S_2 \subseteq M$:

$$\begin{aligned} v_1(S) &= a > 0 \quad \text{if } S_1 \subseteq S \subsetneq S_2, \\ v_1(S) &= a' \geq a \quad \text{if } S_2 \subseteq S, \\ v_1(S) &= 0 \quad \text{otherwise.} \end{aligned} \tag{29}$$

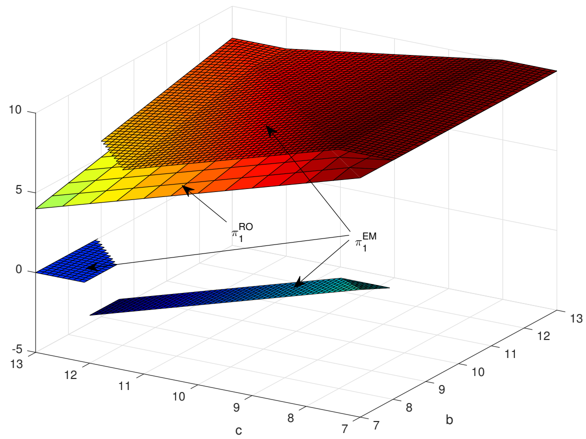

Figure 6 **Illustration of Example 5 – Comparison of bidder 1's payoffs under the robust policy $b_1^{RO,1} = (6, 13)$ and the expected-payoff maximizing policy $b_1^{EM} = (4, 12)$ when $v_1 = (10, 10)$**

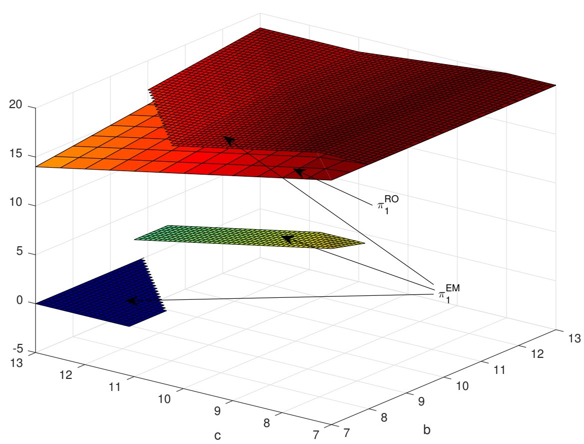

Figure 7 **Illustration of Example 5 – Comparison of bidder 1's payoffs under the robust policy $b_1^{RO,1} = (6, 13)$ and the expected-payoff maximizing policy $b_1^{EM} = (3, 11)$ when $v_1 = (20, 20)$**

Let

$$\bar{p}_1^{VCG} = \max_{b_{-1} \in U_{-1}} \big(w_{b_{-1}}(N \setminus 1, M) - w_{b_{-1}}(N \setminus 1, M \setminus S_1)\big) \tag{30}$$

and

$$\bar{p}_2^{VCG} = \max_{b_{-1} \in U_{-1}} \big(w_{b_{-1}}(N \setminus 1, M) - w_{b_{-1}}(N \setminus 1, M \setminus S_2)\big) \tag{31}$$

be the maximum values of bidder 1's VCG payment over the uncertainty set U_{-1} when she wins bundle S_1 and S_2, respectively. As in the single-minded bidder case, if $v_1(S_1) < \bar{p}_1^{VCG}$ then bidder 1's worst-case payoff is bounded above by zero, so bidding zero is a robust policy. Thus, without loss of generality, we assume that $v_1(S_1) \geq \bar{p}_1^{VCG}$, i.e., bidder 1's valuation for S_1 is high enough so that she always wins either S_1 or S_2 under truthful reporting.

A bundle is said to be the *unique truthful allocation* for bidder 1 if by bidding truthfully bidder 1 always wins that bundle regardless of the realization of her rivals' bids in the uncertainty set. The following proposition shows robust policies for bidder 1 if either S_1 or S_2 is her unique truthful allocation.

PROPOSITION 7. *Let bidder 1 be double-minded with* $0 < \bar{p}_1^{VCG} < v_1(S_1) \leq v_1(S_2)$*:*

(a) If S_1 is bidder 1's unique truthful allocation then a robust policy is:

$$\begin{aligned} b_1^{RO}(S) &= \bar{p}_1^{VCG} \quad \textit{if } S_1 \subseteq S \subsetneq M, \\ b_1^{RO}(M) &= \bar{p}_1^{VCG} + \min_{b_{-1} \in U_{-1}} w_{b_{-1}}(N \setminus 1, M \setminus S_1), \\ b_1^{RO}(S) &= 0 \quad \textit{otherwise.} \end{aligned} \tag{32}$$

The optimal worst-case payoff is $\pi_1^{MAXMIN} = v_1(S_1) - \bar{p}_1^{VCG}$.

(b) If S_2 is bidder 1's unique truthful allocation then a robust policy is:

$$\begin{aligned} b_1^{RO}(S) &= \bar{p}_2^{VCG} \quad \textit{if } S_2 \subseteq S \subsetneq M, \\ b_1^{RO}(M) &= \bar{p}_2^{VCG} + \min_{b_{-1} \in U_{-1}} w_{b_{-1}}(N \setminus 1, M \setminus S_2), \\ b_1^{RO}(S) &= 0 \quad \textit{otherwise.} \end{aligned} \tag{33}$$

The optimal worst-case payoff is $\pi_1^{MAXMIN} = v_1(S_2) - \bar{p}_2^{VCG}$.

REMARK 5. Note that robust policies (32) and (33) are similar to the robust policy (25) in the single-minded case. In fact, results from Proposition 7 can be extended to the case where bidder 1 has positive valuation for a collection of bundles $\{S_k\}_{k=1}^K$ that satisfies $\emptyset \subsetneq S_1 \subsetneq S_2 \subsetneq \ldots \subsetneq S_K$. In such case, if S_k is the unique truthful allocation for bidder 1 then a robust policy is:

$$\begin{aligned} b_1^{RO}(S) &= \bar{p}_k^{VCG} \quad \text{if } S_k \subseteq S \subsetneq M \\ b_1^{RO}(M) &= \bar{p}_k^{VCG} + \min_{b_{-1} \in U_{-1}} w_{b_{-1}}(N \setminus 1, M \setminus S_k) \\ b_1^{RO}(S) &= 0 \quad \text{otherwise.} \end{aligned}$$

We now examine the case where neither S_1 nor S_2 is the unique truthful allocation of bidder 1. For analytical tractability, we study this situation in an example auction setting similar to that of the single-minded case.

LG\L\G valuation structure. Consider a core-selecting auction with $n = 3$ bidders and $m = 2$ homogeneous items. We extend the setting of § 4.3 by making bidder 1 double-minded, i.e., bidder 1 is now interested in winning either one or two items. We call this valuation structure the LG\L\G structure. Table 3 summarizes the notation for this case (notice that bidder 1's valuation for both items is now $a' \geq a$ instead of just a as in the L\L\G case).

# items	v_1	b_1	b_2	b_3
1	a	x	b	0
2	a'	y	b	c

Table 3 Bidder valuations under the LG\L\G valuation structure

The quantities $\bar{p}_1^{VCG}$ and $\bar{p}_2^{VCG}$ defined in (30) and (31) specialized for this case are:

$$\bar{p}_1^{VCG} = \max_{b_{-1}\in U_{-1}} (c-b)^+,$$
$$\bar{p}_2^{VCG} = \max_{b_{-1}\in U_{-1}} \max(b,c).$$

.

We assume that $\bar{p}_1^{VCG} < a$, so bidder 1 wins either one or two items under truthful bidding. According to Proposition 7, if $a' \le a + \bar{b} - \epsilon_b$ then since winning one item is the unique truthful allocation for bidder 1, her robust bidding policy is $b_1^{RO} = (\bar{p}_1^{VCG}, \bar{p}_1^{VCG} + \bar{b} - \epsilon_b)$. Similarly, if $a + \bar{b} + \epsilon_b < a'$ then winning both items is bidder 1's unique truthful allocation. As a result, her robust bidding policy is $b_1^{RO} = (0, \bar{p}_2^{VCG})$. The optimal worst-case payoff is thus $\pi_1^{MAXMIN} = a - \bar{p}_1^{VCG}$ if $a' \le a + \bar{b} - \epsilon_b$ and $\pi_1^{MAXMIN} = a' - \bar{p}_2^{VCG}$ if $a + \bar{b} + \epsilon_b < a'$.

Now consider the case when $a' \in (a + \bar{b} - \epsilon_b, a + \bar{b} + \epsilon_b]$. In this case, bidder 1 can win either one item or both items under truthful bidding. To find the robust bidding policy for bidder 1, we first restrict our policy space to the set $U_1' = \{(x,y) \in \mathbb{R}_+^2 \mid x = \bar{p}_1^{VCG}, y \ge x\}$. We will show that by choosing the optimal value for y in this reduced policy space U_1', bidder 1 can obtain the highest possible worst-case payoff given by minimax inequality, so such policy is also optimal in the original policy space $U_1 = \mathbb{R}_+^2$.

For any $(x,y) \in U_1'$, let $\pi_1^{WO,1}(y)$ and $\pi_1^{WO,2}(y)$ be the worst-case payoff function when bidder 1 bids (x,y) and wins one and two items, respectively. We have:

$$\inf_{b_{-1}\in U'_{-1}} \pi_1(b_1, b_{-1}) = \min\left(\pi_1^{WO,1}(y), \pi_1^{WO,2}(y)\right)$$

The following lemma provides some properties of $\pi_1^{WO,1}(y)$ and $\pi_1^{WO,2}(y)$.

LEMMA 1. *The worst-case payoff functions $\pi_1^{WO,1}(y)$ and $\pi_1^{WO,2}(y)$ satisfy:*
(a) $\pi_1^{WO,1}(y)$ is piece-wise linear and increasing in y. Furthermore,

$$\pi_1^{WO,1}(y) = a \quad \text{if} \quad y \ge \bar{p}_1^{VCG} + \bar{c} + \epsilon_c.$$

(b) $\pi_1^{WO,2}(y)$ is piece-wise linear and decreasing in y. Furthermore,

$$\pi_1^{WO,2}(y) = a' - \bar{c} - \epsilon_c \quad \text{if} \quad y \le \bar{p}_1^{VCG} + \bar{c} + \epsilon_c.$$

In the next lemma, we show the relationship between these worst-case payoff functions and bidders' valuations.

Lemma 2. *We have:*

(a) $\pi_1^{WO,2}(\bar{p}_1^{VCG}+\bar{b}-\epsilon_b)\le\pi_1^{WO,1}(\bar{p}_1^{VCG}+\bar{b}-\epsilon_b)$ *if and only if* $a'\le a+\bar{b}-\epsilon_b$,

(b) $\pi_1^{WO,2}(\bar{p}_1^{VCG}+\bar{b}+\epsilon_b)\le\pi_1^{WO,1}(\bar{p}_1^{VCG}+\bar{b}+\epsilon_b)$ *if and only if* $a'\le a+\bar{b}+\epsilon_b$.

Figure 8 shows these worst-case payoff function under different valuation scenarios. When either winning one item or winning two items is the unique truthful allocation for bidder 1, the worst-case payoff function corresponding to that unique allocation dominates the other (Figures 8a and 8b). When this is not the case, the two worst-case payoff functions intersect (Figures 8c and 8d).

We now characterize of the robust bidding policy under the LG\L\G setting when $a'\in(a+\bar{b}-\epsilon_b,a+\bar{b}+\epsilon_b]$.

Proposition 8. *Consider LG\L\G setting. If* $a'\in(a+\bar{b}-\epsilon_b,a+\bar{b}+\epsilon_b]$ *then a robust bidding policy for bidder 1 is* $b_1^{RO}=(x^*,y^*)$ *with* $x^*=\bar{p}_1^{VCG}$ *and* y^* *is a solution of the equation* $\pi_1^{WO,1}(y)=\pi_1^{WO,2}(y)$. *The worst-case payoff is* $\pi_1^{MAXMIN}=\min(a,a'-\bar{c}-\epsilon_c)$.

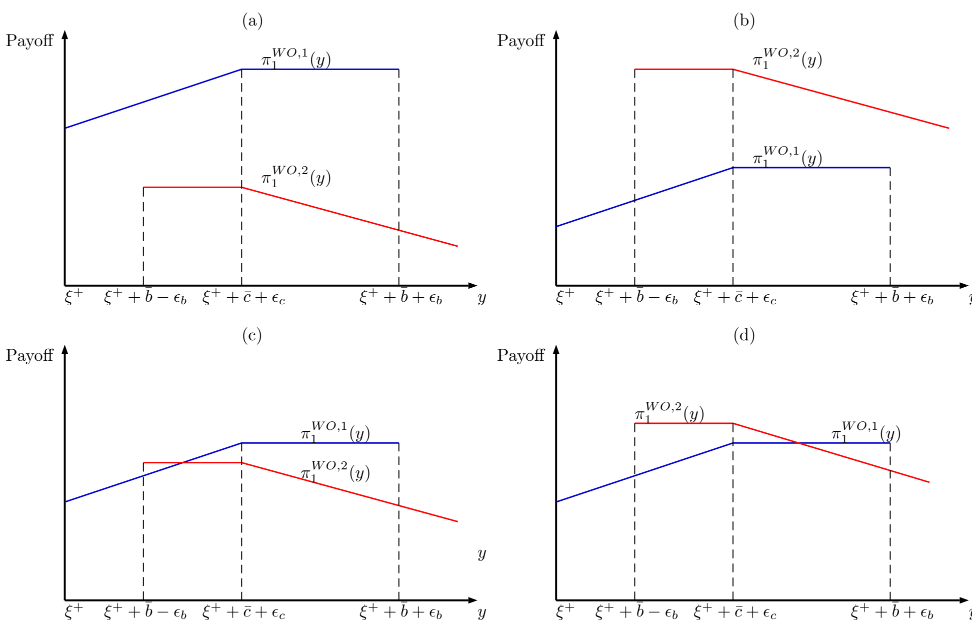

Figure 8 **Worst-case payoff functions in different cases under LG\L\G setting. (a)** $a'\le a+\bar{b}-\epsilon_b$**, winning one item always yields better worst-case payoff. (b)** $a+\bar{b}+\epsilon_b<a'$**, winning two items always yields better worst-case payoff. (c) and (d)** $a+\bar{b}-\epsilon_b<a'\le a+\bar{b}+\epsilon_b$**, two worst-case payoff functions intersects.**

Remark 6. It turns out that in this LG\L\G setting bidding truthfully is also a robust policy as long as there exists $b_{-1}\in U_{-1}$ such that bidder 1 wins the global bundle under truthful bidding, i.e., $a+\bar{b}-\epsilon_b<a'$ (see Appendix A.16). This may not be true in general when S_2 is not the global bundle.

4.5. Beyond double-minded bidders: the role of minimax equality

Demonstrating that the minimax equality (9) holds was central to our establishing of a robust bidding policy for the double-minded bidder (Proposition 8). Thus, identifying other settings in which minimax equality (9) holds would indicate a possibility for our approach to yield a robust bidding policy in such settings. For example, note that if there exists a unique truthful allocation for bidder 1, one can readily obtain minimax equality (9). Indeed, when there is a unique truthful allocation, bidder 1's ex post optimal policy is to bid the minimal amount to win this allocation. Since there is only one allocation outcome for the bidder, the resulting worst-case payoff is the same with that of robust bidding policies, so minimax equality (9) must hold. The following proposition summarizes this for the two auction formats we analyzed.

Proposition 9. *Minimax equality* (9) *holds in:*

(a) Discriminatory auctions,

(b) Core-selecting auctions with single-minded bidders.

In contrast, when bidder 1 receives different allocations for different realizations of rivals' bids in the uncertainty set under truthful bidding, minimax equality (9) may not hold. The condition that minimax equality does not hold is closely related to the convexity of level sets of bidder 1's payoff function (Sion 1958). We next discuss the applicability of Sion (1958) level-set based sufficient conditions for the minimax equality. Specifically, for any $\lambda \in R$, the lower level set and upper level set of $f(x,y)$ are defined as follows:

$$LE(x,\lambda) = \{y : y \in Y, f(x,y) \leq \lambda\},$$
$$GE(\lambda,y) = \{x : x \in X, f(x,y) \geq \lambda\}.$$

Let X be a convex subset of a linear topological space, Y be a compact convex subset of a linear topological space, and $f : X \times Y \to \mathbb{R}$ be upper semicontinuous on X and lower semicontinuous on Y. Suppose that

$$GE(\lambda,y) \text{ is convex for all } y \in Y \text{ and } \lambda \in \mathbb{R}, \tag{34}$$

and,

$$LE(x,\lambda) \text{ is convex for all } x \in X \text{ and } \lambda \in \mathbb{R}. \tag{35}$$

Then we have Sion (1958):

$$\min_Y \sup_X f = \sup_X \min_Y f. \tag{36}$$

The following example shows that condition (34) on the convexity of upper level sets in b_1 of bidder 1's payoff function $\pi_1(b_1, b_{-1})$ is violated in a core-selecting auction with multiple demand settings.

EXAMPLE 6. Consider a core-selecting auction with $n=3$ bidders and $m=3$ homogeneous items. Bidder 1's valuation vector is $v_1 = (5, 15, 16.5)$, bidder 2's bid vector is $b_2 = (5, 8.5, 14.5)$ and bidder 3's bid vector is $b_3 = (3, 6, 12)$. Figure 9 shows the upper level set in b_1 of bidder 1's payoff function $\pi_1(b_1, b_{-1})$ corresponding to level $\lambda = 1$. In this case, the upper level set is not connected, so condition (34) is violated. $\square$

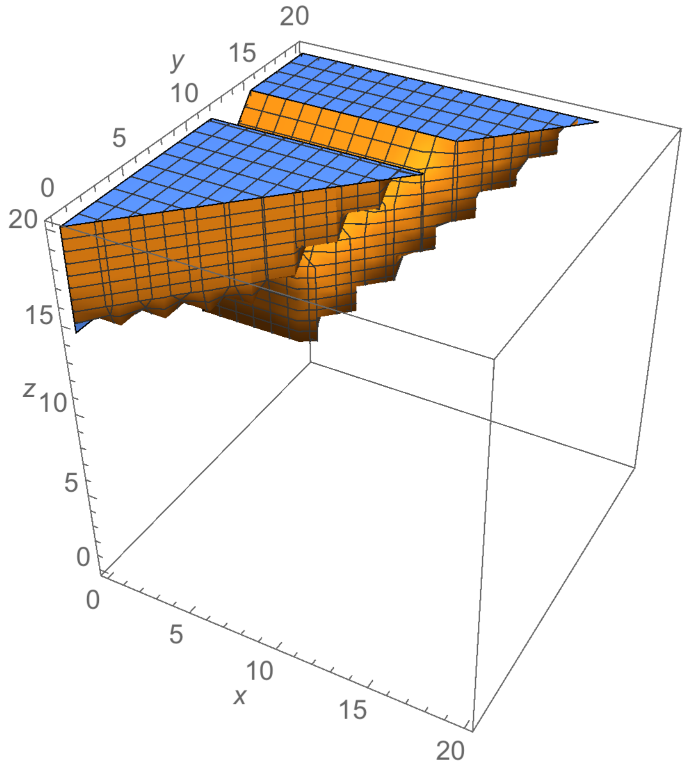

Figure 9 **Illustration of Example 6: Non-convexity of the upper level set of in b_1 of $\pi_1(b_1, b_{-1})$ with $\lambda = 1$**

Note that the non-convexity of level sets of the bidder's payoff function does not exclude the possibility of minimax equality (9) holding. However, in the next example, we show that minimax equality (9) does not hold in a core-selecting auction where bidder 1 is *triple-minded*. (Recall that the bidder is triple minded if she has three distinct positive valuations $v(S_1) < v(S_2) < v(S_3)$ for some bundles $S_1 \subset S_2 \subset S_3$.)

EXAMPLE 7. Consider a core-selecting auction with $n=3$ bidders and $m=3$ homogeneous items. Bidder 1 has valuation for one, two and three items. Bidder 2 has valuation for one and two items, but does not have an extra value for winning the third item. Bidder 3 is only interested in winning all items and has zero valuation for either one or two items. Table 4 summarizes the bidder valuation structure.

# items	v_1	b_1	b_2	b_3
1	a	x	b	0
2	a'	y	c	0
3	a''	z	c	d

Table 4 Bidder valuations, Example 7

We assume that the uncertainty set U_{-1} is of box-type and is given by:

$$U_{-1} = \{(b_2, b_3) \mid \bar{b} - \epsilon_b \leq b \leq \bar{b} + \epsilon_b, \bar{c} - \epsilon_c \leq c \leq \bar{c} + \epsilon_c, \bar{d} - \epsilon_d \leq d \leq \bar{d} + \epsilon_d\}.$$

The numerical values we consider here are $v_1 = (a, a', a'') = (7, 13, 13.4)$, $\bar{b}_{-1} = (\bar{b}, \bar{c}, \bar{d}) = (4, 8, 10)'$, $\epsilon = (\epsilon_b, \epsilon_c, \epsilon_d) = 0.23\ \bar{b}_{-1}$. It is straightforward to establish that, with these parameters, bidder 1 is guaranteed to win at least one item if she bids her true valuations (and consequently bidder 3 never wins). Bidder 1's worst-case profit functions corresponding to winning one, two and three items, respectively, are:

$$\pi_1^{WO,1}(x,y,z) = \min\left(a, a - \frac{1}{2}x - \bar{d} - \epsilon_d + \frac{1}{2}c^*(x,y,z) + \frac{1}{2}\max(\bar{d} + \epsilon_d, z)\right),$$
$$\pi_1^{WO,2}(x,y,z) = \min\left(a', a' - \bar{d} - \epsilon_d + \frac{1}{2}b^*(x,y,z) - \frac{1}{2}y + \frac{1}{2}\max(\bar{d} + \epsilon_d, z)\right),$$
$$\pi_1^{WO,3}(x,y,z) = a'' - \bar{d} - \epsilon_d,$$

where b^* and c^* are given by:

$$b^*(x,y,z) = \max(\bar{b} - \epsilon_b, x + \bar{c} - \epsilon_c - y, z - y),$$
$$c^*(x,y,z) = \max(\bar{c} - \epsilon_c, \max(y + \bar{b} - \epsilon_b - x, z - x).$$

The feasible region for $\pi_1^{WO,i}(x,y,z)$ is $U_1^{(i)}$, for $i \in \{1,2,3\}$. These feasible regions are given by:

$$U_1^{(1)} = \{(x,y,z) \in \mathbb{R}_+^3 \mid x \geq \bar{d} - \bar{c} + \epsilon_c + \epsilon_d,$$
$$x + \bar{c} + \epsilon_c - \bar{b} + \epsilon_b \geq y, x + \bar{c} + \epsilon_c \geq z\},$$
$$U_1^{(2)} = \{(x,y,z) \in \mathbb{R}_+^3 \mid x \geq \bar{d} - \bar{c} + \epsilon_c + \epsilon_d,$$
$$y \geq x + \bar{c} - \epsilon_c - \bar{b} - \epsilon_b, y \geq z - \bar{b} - \epsilon_b\},$$
$$U_1^{(3)} = \{(x,y,z) \in \mathbb{R}_+^3 \mid x \geq \bar{d} - \bar{c} + \epsilon_c + \epsilon_d,$$
$$z \geq x + \bar{c} - \epsilon_c, z \geq y + \bar{b} - \epsilon_b\}.$$

Similar to the case of a double-minded bidder (§ 4.4), in order to find the robust policy, it is sufficient for bidder 1 to fix x at $x^* = \bar{d} - \bar{c} + \epsilon_c + \epsilon_d$ so that she wins at least one item regardless of the realization of rivals' bids $b_{-1} \in U_{-1}$, and then optimize over y and z. Figure 10 shows

$\pi_1^{WO,1}(x,y,z)$, $\pi_1^{WO,2}(x,y,z)$ and $\pi_1^{WO,3}(x,y,z)$ when $x = x^*$. We establish (by brute force numerical search) that an optimal robust policy is $b_1^{RO} = (x^*, y^*, z^*) = (6.14, 12.5, 15.5)$, which is at the intersection of $\pi_1^{WO,1}(x,y,z)$ and $\pi_1^{WO,2}(x,y,z)$. The worst-case payoff corresponding to this robust policy is $\pi_1^{MAXMIN} = 3.765$. Note that $\pi_1^{WO,1}(x,y,z)$ and $\pi_1^{WO,3}(x,y,z)$ also intersect, but the resulting bidding policies are sub-optimal. We also have that $\pi_1^{MAXMIN} = 3.765 < 3.78 = \pi_1^{MINMAX}$, so minimax equality (9) does not hold. □

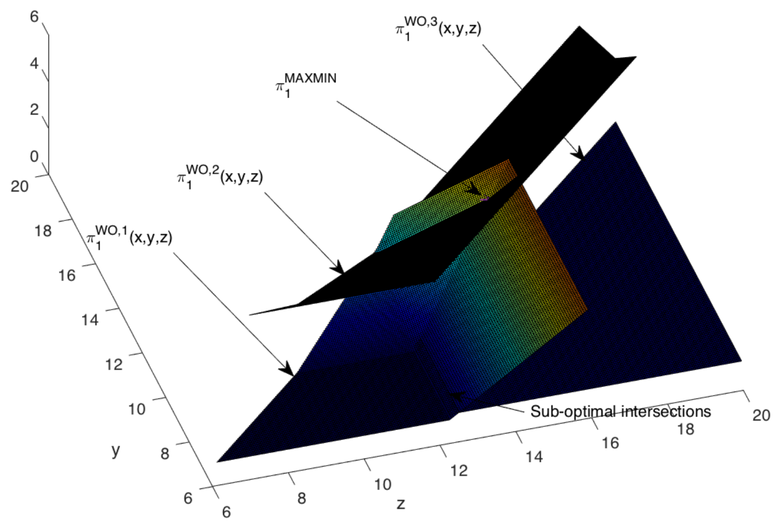

Figure 10 **Illustration of Example 7: Worst-case payoff functions corresponding to winning one, two and three items**

Example 7 illustrates the limitations of the approach used to establish robust policies for double-minded bidders (Proposition 8). We have shown in the setting of double minded bidders that minimax equality (9) could be used to prove the optimality of robust bidding policies. Nevertheless, as established in Example 7, the equality may not hold in more general settings. Thus, finding robust bidding policies might be computationally challenging in complex settings like core-selecting auctions.

5. Concluding remarks

In this paper, we study the bidding problem of an auction bidder who has imperfect information of rivals' bids and wants to maximize her worst-case payoff. We model bidder's information about rivals' bids via an uncertainty set: every point of the uncertainty set is a possible realization of rivals' bids. The bidder's objective is to maximize her worst-case payoff with respect to this uncertainty set. The focus on the worst-case payoff objective requires a different approach from the expected-payoff maximizing bidder models that are prevalent in the auctions literature. Furthermore, unlike those models, our setting allows for distribution-free analysis.

One of the main challenges with our robust optimization approach to the bidding problem is the computational tractability. Our analysis indicates that solving a robust bidding problem involves maximizing a non-concave, discontinuous worst-case profit function. When the profit function is multi-dimensional, finding the robust bidding policy could be a challenging task. Nevertheless, our analysis provides some insights that could apply when devising optimal or heuristic biding policies to maximize worst-case payoff in other auction settings. For one simple example, if there is a unique favorable allocation outcome for the bidder, i.e., one that yields better payoff for the bidder than other allocation outcomes regardless of rivals' bids realization, our results readily apply to quite general settings: bidder's robust policy is to bid the minimal amount that ensures winning such allocation. If there is no such favorable allocation outcome, which may happen in settings with heterogeneous items and/or multiple demand, robust bidding policies correspond to the intersections of allocation-specific worst-case payoff functions. The search for optimal bids would then be restricted to bids at the intersections of those worst-case payoff functions. When dealing with multiple demand settings, to overcome the computational challenge of searching over high-dimensional policy spaces, one could employ variable reduction techniques and restrict the attention to only bids on bundles of interest – e.g., bundles with positive valuations, bundles demanded by competitors, or the global bundle.

Finally, note that in settings we considered, minimax equality (9) holding was important for establishing the existence of and providing descriptions of robust bidding policies. Hence, understanding of the settings in which minimax equality holds could be useful in finding optimal solutions to the robust bidding problem. Similarly, in settings in which minimax equality fails to hold, one needs to be aware of difficulties of finding and establishing such optimal robust policies, as demonstrated in the case of multiple demand core-selecting auctions.

Appendix

A. Proofs

A.1. Example 1

Bidder 1's payoff function is:

$$\pi_1(b_1, b_2) = \begin{cases} v_1 - b_1 & \text{if } b_1 \geq b_2 \\ 0 & \text{otherwise.} \end{cases}$$

Since b_2 is uniform on $[c, d]$, bidder 1's expected payoff as a function of b_1 is:

$$\mathbb{E}[\pi_1(b_1)] = \begin{cases} v_1 - b_1 & \text{if } d < b_1 \\ \frac{(v_1 - b_1)(b_1 - c)}{d - c} & \text{if } c \leq b_1 \leq d \\ 0 & \text{otherwise.} \end{cases}$$

By solving first-order condition, the maximizer of $\mathbb{E}[\pi_1(b_1)]$ is:

$$b_1^{EM} = \begin{cases} d & \text{if } 2d - c < v_1 \\ \frac{1}{2}(v_1 + c) & \text{if } c \le v_1 \le 2d - c \\ 0 & \text{otherwise.} \end{cases}$$

A.2. Example 3

Assume that $b_2^*(c) = B$, then any $b_1^*(v_1)$ satisfying $b_1^*(v_1) = B$ for $v_1 \ge B$ and $b_1^*(v_1) \le v_1$ for $v_1 < B$ is a best response of bidder 1. Now assume that $b_1^*(v_1)$ satisfies those conditions and is given. Let us derive $b_2^*(v_2)$. Note that the objective of problem (16) can be rewritten as:

$$\begin{aligned} \mathbb{E}_{v_1}[v_2 - b_2]\mathbf{1}_{b_2 \ge b_1^*(v_1)} dv_1 &= \frac{1}{c}\int_0^c [v_2 - b_2]\mathbf{1}_{b_2 \ge b_1^*(v_1)} dv_1 \\ &= \frac{1}{c}\int_0^{b_2} [v_2 - b_2]\mathbf{1}_{b_2 \ge b_1} \frac{1}{(b_1^*)'((b_1^*)^{-1}(b_1))} db_1. \end{aligned}$$

Let $g(b) = \frac{1}{(b_1^*)'((b_1^*)^{-1}(b))}$. Also, let

$$f(b_2) = \frac{1}{c}\int_0^{b_2} (v_2 - b_2) g(b_1) db_1.$$

Taking the first and second derivatives:

$$\begin{aligned} f'(b_2) &= (v_2 - b_2)g(b_2) - \int_0^{b_2} g(b_1) db_1, \\ f''(b_2) &= -2g(b_2) + (v_2 - b_2)g'(b_2). \end{aligned}$$

Note that since $b_1^*(.)$ is increasing and convex, $g(b) > 0$ and $g'(b) < 0$ for $b \in [0, c]$. Thus, $f''(b_2) > 0$ and the optimal bid is uniquely determined by first order condition:

$$b_2^*(v_2) = \max\left\{ v_2 - \frac{\int_0^{b_2} g(b) db}{g(b_2)}, 0 \right\}.$$

A.3. Proposition 1

If $b_1 < u(m)$, there exists $b_{-1} \in U_{-1}$ such that $b_1 < b^{(m)}$ so that bidder 1 loses the auction and receives zero payoff. Thus, bidder 1's worst-case payoff is no more than zero if she bids $b_1 < u(m)$. On the other hand, if $b_1 \ge u(m)$ then bidder 1 always wins an item and receives a payoff $\pi_1(b_1, b_{-1}) = v_1 - b_1$. Hence, her optimal policy is to bid exactly $u(m)$ if $u(m) \le v_1$ and bid zero if $v_1 < u(m)$. In other words, a solution to $(\mathscr{P})$ is $b_1^{RO} = u(m)\mathbf{1}_{u(m) \le v_1}$. The optimal worts-case payoff is thus $\pi_1^{MAXMIN} = (v_1 - u(m))^+$.

A.4. Proposition 3

Since VCG is in the core if the coalition value function is bidder-submodular, it suffices to show this bidder-submodularity property. For any coalition S, let $b_S(M) = max_{j \in S} b_j(M)$. Then due to supermodularity, we have $w_b(S) = b_S(M)$ and $w_b(S \cup l) = \max(b_S(M), b_l(M))$. Therefore, $w_b(S \cup l) - w_b(S) = \max(b_S(M), b_l(M)) - b_S(M) = \max(0, b_l(M) - b_S(M))$. As a result, if $0 \in S \subset S'$ then $\max(0, b_l(M) - b_S(M)) \ge \max(0, b_l(M) - b_{S'}(M))$ so that $w_b(S \cup l) - w_b(S) \ge w_b(S' \cup l) - w_b(S')$. Thus, w_b is bidder-submodular.

A.5. Proposition 4

For any $b_1 \in U_1$ and $b_{-1} \in U_{-1}$, since $w_{v_1, b_{-1}}$ is bidder-submodular, we have that $\pi_1(b_1, b_{-1}) \le \pi_1(v_1, b_{-1})$ Ausubel and Milgrom (2006). Thus, $\inf_{b_{-1} \in U_{-1}} \pi_1(b_1, b_{-1}) \le \inf_{b_{-1} \in U_{-1}} \pi_1(v_1, b_{-1})$ which implies

$$\sup_{b_1 \in U_1} \inf_{b_{-1} \in U_{-1}} \pi_1(b_1, b_{-1}) \le \inf_{b_{-1} \in U_{-1}} \pi_1(v_1, b_{-1}).‘$$

Thus, truthful reporting is the optimal solution to $(\mathscr{P})$.

A.6. Quadratic core-selecting payment rule

We can re-write (2) to be constraints on payments as follows. First, recall that S_j is the allocated bundle for bidder j. By substituting $\pi_0 = \sum_{j\in N} p_j$ and $\pi_j = b_j(S_j) - p_j$, we get

$$\sum_{j\in W} p_j \geq w_b(C) - \sum_{j\in C}(b_j(S_j) - p_j), \quad \forall C \subseteq N, \tag{37}$$

where W is the set of bidders who receives nonempty bundles. After rearranging, the above constraints become:

$$\sum_{j\in W\setminus C} p_j \geq w_b(C) - \sum_{j\in C} b_j(S_j), \quad \forall C \subseteq N. \tag{38}$$

Let $\beta_C = w_b(C) - \sum_{j\in C} b_j(S_j)$ and $\beta \in \mathbb{R}^{2^n}$ be the vector of all β_C's. Also, let A be a $n \times 2^n$ matrix comprising columns a_C that has the jth entry equals to zero if bidder j is in coalition C and one otherwise. The constraints (38) are then of the form:

$$pA \geq \beta.$$

. Let p^0 be a reference payment vector. Under this the quadratic core-selecting payment rule, the payment vector p is the optimal solution of the following quadratic program:

$$\begin{aligned} \min_p \quad & (p - p^0)(p - p^0)^T \\ \text{s.t.} \quad & pA \geq \beta, \quad p \leq b, \quad p1 = \mu, \end{aligned} \tag{$\mathscr{Q}$}$$

where μ is defined as

$$\begin{aligned} \mu = \min_p \quad & p1 \\ \text{s.t.} \quad & pA \geq \beta, \quad p \leq b. \end{aligned} \tag{39}$$

The payment vector p determined by ($\mathscr{Q}$) minimizes the Euclidean distance from reference payment vector p^0 to the core. The quantity μ is the minimum value of total payment from bidders. Thus, the constraint $p1 = \mu$ guarantees that the payment rule is bidder-optimal, i.e., the total payment from bidders is minimized. This has the effect of minimizing the bidders' total incentive to deviate, as shown by Day and Milgrom (2008).

A.7. Proposition 5

Let S_1 be the bundle that bidder 1 wins when she bids truthfully. By definition, we have

$$w_{v_1,b_{-1}}(N) = v_1(S_1) + w_{b_{-1}}(N \setminus 1, M \setminus S_1).$$

We observe that by using policy (22), bidder 1 also wins S_1. In fact, the maximum reported valuation generated by allocating S_1 to bidder 1 and $M \setminus S_1$ to the remaining bidders is

$$\begin{aligned} v_1(S_1) - \pi_1^{VCG} + w_{b_{-1}}(N \setminus 1, M \setminus S_1) &= w_{v_1,b_{-1}}(N) - (w_{v_1,b_{-1}}(N) - w_{b_{-1}}(N \setminus 1)) \\ &= w_{b_{-1}}(N \setminus 1), \end{aligned}$$

which is equal to the maximum reported valuation generated by allocating items in M to the remaining bidders. Under the assumed tie-breaking rule, S_1 is her allocation outcome. The VCG payoffs with respect to the reported valuation profile (b_1, b_{-1}) is

$$\pi_j^{VCG} = w_b(N) - w_b(N \setminus j) = w_{b_{-1}}(N \setminus 1) - w_{b_{-1}}(N \setminus 1) = 0,$$

for $j \in N$, and

$$\pi_0^{VCG} = w_b(N) - \sum_{j \in N} \pi_j^{VCG} = w_{b_{-1}}(N \setminus 1).$$

This VCG profile is in the core with respect to the reported bids b so bidder 1 is charged exactly her VCG payment p_1^{VCG}, which is a function of b_{-1} only. As a result, bidder 1 gets

$$\pi_1 = v_1(S_1) - p_1^{VCG} = v_1(S_1) - (v_1(S_1) - \pi_1^{VCG}) = \pi_1^{VCG},$$

which is her VCG payoff with respect to the reported valuation (v_1, b_{-1}). Since π_1^{VCG} is the maximum payoff that bidder 1 can get, policy (22) is optimal.

A.8. Remark 1

Let $C \subseteq N$ be the set of bidders corresponding to bidder 1's shills. Also, given a bid profile b, let S_j be the bundle allocated to bidder j . We have

$$\begin{aligned}
\pi_0 + \sum_{j \in N \setminus C} \pi_i &= \sum_{j \in N} p_j + \sum_{j \in N \setminus C} (b_j(S_j) - p_j) \\
&= \sum_{j \in C} p_j + \sum_{j \in N \setminus C} b_j(S_j) \\
&\leq \sum_{j \in C} p_j + w_b(N \setminus C, M \setminus \cup_{j \in C} S_j),
\end{aligned}$$

where the last inequality follows from the definition of w_b. From the core constraints, we have $w_b(N \setminus S) \leq \pi_0 + \sum_{j \in N \setminus C} \pi_i$. Hence,

$$p_1^{VCG} = w_b(N \setminus C) - w_b(N \setminus C, M \setminus \cup_{j \in C} S_j) \leq \sum_{j \in C} p_j.$$

Thus, bidder 1 pays at least p_1^{VCG} even when she uses shills, which implies that her payoff is at most π_1^{VCG}. Since bidding policy (22) gives bidder 1 this VCG payoff, it is also optimal in the extended policy space in which bidder 1 uses shills.

A.9. Proposition 6

We first show that bidder 1's worst-case payoff under policy (25) is at least $v_1(S_1) - \bar{p}^{VCG}$. By construction, we have that

$$b_1(S_1) + w_{b_{-1}}(N \setminus 1, M \setminus S_1) \geq b_1(M) \quad \forall b_{-1} \in U_{-1}. \tag{40}$$

Furthermore, from the definition of $\bar{p}^{VCG}$, we have

$$b_1(S_1) + w_{b_{-1}}(N \setminus 1, M \setminus S_1) \geq w_{b_{-1}}(N \setminus 1, M) \quad \forall b_{-1} \in U_{-1}. \tag{41}$$

Since $b(S) = 0$ for all $S \not\supseteq S_1$ and $w_{b_{-1}}(N \setminus 1, M \setminus S) \leq w_{b_{-1}}(N \setminus 1, M)$, the above inequality also implies that

$$b_1(S_1) + w_{b_{-1}}(N \setminus 1, M \setminus S_1) > b_1(S) + w_{b_{-1}}(N \setminus 1, M \setminus S), \tag{42}$$

for all $S \not\supseteq S_1$ and $b_{-1} \in U_{-1}$. The inequalities (40), (41) and (42) jointly show that it is always optimal for the seller to allocate bundle S_1 to bidder 1 and the rest of the items to other bidders. Since a bidder's payment can never exceed her bid, bidder 1's payoff under policy (25) is at least $v_1(S_1) - \bar{p}^{VCG}$. Hence, the optimal worst-case payoff under robust bidding satisfies $\pi_1^{MAXMIN} \geq v_1(S_1) - \bar{p}^{VCG}$. On the other hand, recall that we have the minimax inequality (8), so $\pi_1^{MAXMIN} \leq \pi_1^{MINMAX}$. For any realization of $b_{-1} \in U_{-1}$, bidder 1 can response optimally by bidding according to a perfect-information optimal policy, e.g., policy (22). Under such a policy, bidder 1 receives her VCG payoff, so we have

$$\begin{aligned}
\pi_1^{MINMAX} &= \min_{b_{-1} \in U_{-1}} (w_b(N, M) - w_b(N \setminus 1, M)) \\
&= \min_{b_{-1} \in U_{-1}} (v_1(S_1) + w_b(N \setminus 1, M \setminus S_1) - w_b(N \setminus 1, M)) \\
&= v_1(S_1) - \max_{b_{-1} \in U_{-1}} (w_b(N \setminus 1, M) - w_b(N \setminus 1, M \setminus S_1)) \\
&= v_1(S_1) - \bar{p}^{VCG}.
\end{aligned}$$

Since $v_1(S_1) - \bar{p}^{VCG} \leq \pi_1^{MAXMIN} \leq \pi_1^{MINMAX} = v_1(S_1) - \bar{p}^{VCG}$, we must have that $\pi_1^{MAXMIN} = \pi_1^{MINMAX} = v_1(S_1) - \bar{p}^{VCG}$ and the bidding policy (25) is optimal.

A.10. Remark 3

When bidders 1 and 2 win exactly one item each, their VCG payments are $p_1^{VCG} = \max(0, c - b)$ and $p_2^{VCG} = \max(y - x, c - x)$ and the core constraints are $p_1 \leq x$, $p_2 \leq b$ and $p_1 + p_2 \geq c$. The projection of VCG on the core gives us bidder 1's payment:

$$p_1 = \left(x - \frac{1}{2}\min(x, x + b - c) + \frac{1}{2}\min(c - y, 0)\right)^+. \tag{43}$$

The right hand side of (43) corresponds to projecting the VCG payment (p_1^{VCG}, p_2^{VCG}) onto the blocking constraint $p_1 + p_2 \geq c$ created by bidder 3's bid on the global bundle. This term can also be written as

$$\max(0, c - b) + \frac{1}{2}(c - \max(0, c - b) - \max(y - x, c - x)),$$

which is bidder 1's VCG payment plus an extra amount that is half of the total surcharge that bidder 1 and 2 together have to pay to overcome the blocking global bidder.[14] However, such projection does not always yield a non-negative payment, so it is truncated at zero as in (43). By substituting $(x, y) = (\bar{p}^{VCG}, \bar{p}^{VCG} + \bar{b} - \epsilon_b)$ into (43), we get bidder 1's payment under the robust policy:

$$p_1^{RO} = \left(\bar{p}^{VCG} - \frac{1}{2}\min(\bar{p}^{VCG}, \bar{p}^{VCG} + b - c) + \frac{1}{2}\min(c - \bar{p}^{VCG} - \bar{b} + \epsilon_b, 0)\right)^+ \tag{44}$$

Bidder 1's payoff under the robust policy is thus given by

$$\pi_1^{RO} = \min\left(a + \frac{1}{2}\min(0, b - c) - \frac{1}{2}\min(c - \bar{b} + \epsilon_b, \bar{p}^{VCG}), a\right). \tag{45}$$

On the other hand, by bidding truthfully bidder 1 gets

$$\pi_1^{TR} = \min\left(a + \frac{1}{2}\min(0, b - c) - \frac{1}{2}\min(c, a), a\right). \tag{46}$$

Since $\min(c - \bar{b} + \epsilon_b, \bar{p}^{VCG}) = c - \bar{b} + \epsilon_b \leq \min(c, a)$, we have that $\pi_1^{RO} \geq \pi_1^{TR}$ for all realization of $b_{-1} \in U_{-1}$.

[14] See e.g., Goeree and Lien (2016).

A.11. Example 5

Bidder 1's payoff function is:

$$\pi_1(x,y,b,c) = \begin{cases} 0 & \text{if } x+b<c \text{ and } y<c \\ \min\Big(a, a-x+\frac{1}{2}\min(x,x+b-c)-\frac{1}{2}\min(c-y,0)\Big) & \\ & \text{if } c\le x+b \text{ and } y\le x+b \\ a-\max(b,c) & \text{if } x+b<y \text{ and } c\le y \end{cases} \tag{47}$$

Since b and c are independent and distributed according to f_b and f_c given by (27) and (28), we have:

$$\begin{aligned} &\mathbb{E}[\pi_1(x,y,b,c)] \\ &\quad = \frac{1}{324}\int_{\max(y-x,\bar{b}-\epsilon_b)}^{\bar{b}+\epsilon_b}\int_{\bar{c}-\epsilon_c}^{\min(x+b,\bar{c}+\epsilon_c)} \pi_1^1(x,y,b,c)(b-7)(13-c)\;dc\;db \\ &\quad + \frac{1}{324}\int_{\bar{b}-\epsilon_b}^{\min(y-x,\bar{b}+\epsilon_b)}\int_{\bar{c}-\epsilon_c}^{\min(y,\bar{c}+\epsilon_c)} \pi_1^2(b,c)(b-7)(13-c)\;dc\;db, \end{aligned}$$

where $\pi_1^1(x,y,b,c) = \Big(a, a-x+\frac{1}{2}\min(x,x+b-c)-\frac{1}{2}\min(c-y,0)\Big)$ and $\pi_1^2(b,c) = a-\max(b,c)$. The expected-payoff maximizing policy b_1^{EM} can then be obtained by choosing (x,y) that maximizes $\mathbb{E}[\pi_1(x,y,b,c)]$ over $\mathbb{R}_+^2$.

A.12. Proposition 7

(a) Bidder 1 has unique truthful allocation S_1 if and only if, for all $b_{-1}\in U_{-1}$:

$$v_1(S_1)+w_{b_{-1}}(N\setminus 1, M\setminus S_1) \ge v_1(S_2)+w_{b_{-1}}(N\setminus 1, M\setminus S_2), \tag{48}$$

and

$$v_1(S_1)+w_{b_{-1}}(N\setminus 1, M\setminus S_1) \ge w_{b_{-1}}(N\setminus 1, M), \quad \forall b_{-1}\in U_{-1}. \tag{49}$$

We will prove that similar to the single-minded case, the worst-case payoff of bidder 1 under the robust policy b_1^{RO} is at least as large as the upper bound provided by minimax inequality and thus b_1^{RO} must be optimal. First, given policy b_1^{RO}, we can partition uncertainty set U_{-1} into two disjoint subsets $U_{-1}^{(1)}$ and $U_{-1}^{(2)}$ such that bidder 1 always wins S_i on the set $U_{-1}^{(i)}$ for $i\in\{1,2\}$. The worst-case payoff under policy b_1^{RO} is given by

$$\inf_{b_{-1}\in U_{-1}} \pi_1(b_1^{RO}, b_{-1}) = \min\left(\inf_{b_{-1}\in U_{-1}^{(1)}}(v_1(S_1)-p_1^{(1)}), \inf_{b_{-1}\in U_{-1}^{(2)}}(v_1(S_2)-p_1^{(2)})\right)$$

where $p_1^{(1)}$ and $p_1^{(2)}$ denote the corresponding payments for bidder 1. Note that by individual rationality , we have $p_1^{(1)}\le \bar{p}_1^{VCG}$ and $p_1^{(2)}\le \bar{p}_1^{VCG}$. In addition, $v_1(S_1)\le v_1(S_2)$, so we have

$$\inf_{b_{-1}\in U_{-1}} \pi_1(b_1^{RO}, b_{-1}) \ge v_1(S_1)-\bar{p}_1^{VCG}.$$

Since $\pi_1^{MAXMIN}\ge \inf_{b_{-1}\in U_{-1}} \pi_1(b_1^{RO}, b_{-1})$, we have $\pi_1^{MAXMIN}\ge v_1(S_1)-\bar{p}_1^{VCG}$. On the other hand, if we let $p_1^{VCG}(S_1)$ and $p_1^{VCG}(S_2)$ be bidder 1's VCG payment when she wins S_1 and S_2, respectively, then the minmax payoff is

$$\begin{aligned} \pi_1^{MINMAX} &= \inf_{b_{-1}\in U_{-1}} \max\left(v_1(S_1)-p_1^{VCG}(S_1), v_1(S_2)-p_1^{VCG}(S_2)\right) \\ &= v_1(S_1)-\bar{p}_1^{VCG}, \end{aligned}$$

where the last equality follows directly from the condition (48) that guarantees bidder 1 always win bundle S_1 under truthful bidding. Finally, by minimax inequality, we have $\pi_1^{MAXMIN} \leq \pi_1^{MINMAX}$. As a result, the minimax equality (9) must hold and b_1^{RO} is the optimal solution to $(\mathscr{P})$.

(b) We observe that under policy b_1^{RO} defined in (33), bidder 1 always wins S_2. As a result, the optimality follows directly from minimax inequality argument analogous to the proof of Proposition 6.

A.13. Lemma 1

When bidder 1 wins one item, her payment is given by (43). Recall that when $(x,y) \in U_1'$, we have $x = \bar{p}_1^{VCG}$. As a result, $\pi_1^{WO,1}(y)$ is given by

$$\begin{aligned} \pi_1^{WO,1}(y) = \inf \quad & \min\{a, a - \bar{p}_1^{VCG} + \frac{1}{2}\min(\bar{p}_1^{VCG}, \bar{p}_1^{VCG} + b - c) + \frac{1}{2}(y-c)^+\} \\ \text{s.t.} \quad & b_{-1} \in U_{-1} \\ & y \leq \bar{p}_1^{VCG} + b. \end{aligned} \tag{50}$$

The objective function on the right hand side is increasing in b and decreasing in c so the worst-case scenario corresponds to $b^* = \max(\bar{b} - \epsilon_b, y - \bar{p}_1^{VCG})$ and $c^* = \bar{c} + \epsilon_c$. Substituting these values into (50), we have

$$\pi_1^{WO,1}(y) = \min\{a, a - \bar{p}_1^{VCG} + \frac{1}{2}\min\left(\bar{p}_1^{VCG}, (y - \bar{c} - \epsilon_c)^+\right) + \frac{1}{2}(y - \bar{c} - \epsilon_c)^+\}. \tag{51}$$

Similarly, when bidder 1 wins both items, her payment is $\max(b,c)$, so $\pi_1^{WO,2}(y)$ is given by

$$\begin{aligned} \pi_1^{WO,2}(y) = \inf \quad & \{a' - \max(b,c)\} \\ \text{s.t.} \quad & b_{-1} \in U_{-1} \\ & \bar{p}_1^{VCG} + b < y. \end{aligned} \tag{52}$$

In this case, the worst-case bids are $b^* = \min(\bar{b} + \epsilon_b, y - \bar{p}_1^{VCG})$, $c^* = \bar{c} + \epsilon_c$ so we have

$$\pi_1^{WO,2}(y) = a' - \max\{\min(\bar{b} + \epsilon_b, y - \bar{p}_1^{VCG}), \bar{c} + \epsilon_c\}. \tag{53}$$

We can see that $\pi_1^{WO,1}(y)$ is a piece-wise linear increasing function in y while $\pi_1^{WO,2}(y)$ is a piece-wise linear decreasing function in y. Furthermore, one can directly verify that $\pi_1^{WO,1}(y) = a$ for $y \geq \bar{p}_1^{VCG} + \bar{c} + \epsilon_c$ and $\pi_1^{WO,2}(y) = a' - \bar{c} - \epsilon_c$ for $y \leq \bar{p}_1^{VCG} + \bar{c} + \epsilon_c$.

A.14. Lemma 2

From (51) and (53), we have the following identities:

$$\begin{aligned} &\pi_1^{WO,1}(\bar{p}_1^{VCG} + \bar{b} - \epsilon_b) \\ &= \min(a, a - \bar{p}_1^{VCG} + \frac{1}{2}\min(\bar{p}_1^{VCG}, \bar{p}_1^{VCG} - \bar{c} - \bar{b} + \epsilon_b - \epsilon_c) + \frac{1}{2}(\bar{p}_1^{VCG} - \bar{c} - \bar{b} + \epsilon_b - \epsilon_c)) \\ &= \min(a, a - \frac{1}{2}(\bar{p}_1^{VCG} + \bar{c} + \bar{b} - \epsilon_b + \epsilon_c)), \end{aligned}$$

$$\begin{aligned} \pi_1^{WO,2}(\bar{p}_1^{VCG} + \bar{b} - \epsilon_b) &= a' - \max(\min(\bar{b} + \epsilon_b, \bar{b} - \epsilon_b), \bar{c} + \epsilon_c) \\ &= a' - \max(\bar{b} - \epsilon_b, \bar{c} + \epsilon_c), \\ \pi_1^{WO,1}(\bar{p}_1^{VCG} + \bar{b} + \epsilon_b) &= \min\left(a, a + \frac{1}{2}\min(\bar{p}_1^{VCG}, \bar{p}_1^{VCG} - \bar{p}^{VCG} + 2\epsilon_b) - \frac{1}{2}(\bar{p}_1^{VCG} + \bar{p}^{VCG} - 2\epsilon_b)\right), \\ \pi_1^{WO,2}(\bar{p}_1^{VCG} + \bar{b} + \epsilon_b) &= a' - \max(\bar{b} + \epsilon_b, \bar{c} + \epsilon_c). \end{aligned}$$

There are three cases to consider:

1. $\bar{b}+\epsilon_b \leq \bar{c}+\epsilon_c$:

   We have

$$\begin{aligned}
\pi_1^{WO,1}(\bar{p}_1^{VCG}+\bar{b}-\epsilon_b) &= a-(\bar{c}-\bar{b}+\epsilon_b+\epsilon_c),\\
\pi_1^{WO,2}(\bar{p}_1^{VCG}+\bar{b}-\epsilon_b) &= a'-\bar{c}-\epsilon_c,\\
\pi_1^{WO,1}(\bar{p}_1^{VCG}+\bar{b}+\epsilon_b) &= a-(\bar{c}-\bar{b}-\epsilon_b+\epsilon_c),\\
\pi_1^{WO,2}(\bar{p}_1^{VCG}+\bar{b}+\epsilon_b) &= a'-\bar{c}-\epsilon_c.
\end{aligned}$$

2. $\bar{b}-\epsilon_b \leq \bar{c}+\epsilon_c < \bar{b}+\epsilon_b$:

   We have

$$\begin{aligned}
\pi_1^{WO,1}(\bar{p}_1^{VCG}+\bar{b}-\epsilon_b) &= a-(\bar{c}-\bar{b}+\epsilon_b+\epsilon_c),\\
\pi_1^{WO,2}(\bar{p}_1^{VCG}+\bar{b}-\epsilon_b) &= a'-\bar{c}-\epsilon_c,\\
\pi_1^{WO,1}(\bar{p}_1^{VCG}+\bar{b}+\epsilon_b) &= a,\\
\pi_1^{WO,2}(\bar{p}_1^{VCG}+\bar{b}+\epsilon_b) &= a'-\bar{b}-\epsilon_b.
\end{aligned}$$

3. $\bar{c}+\epsilon_c < \bar{b}-\epsilon_b$:

   We have

$$\begin{aligned}
\pi_1^{WO,1}(\bar{p}_1^{VCG}+\bar{b}-\epsilon_b) &= a,\\
\pi_1^{WO,2}(\bar{p}_1^{VCG}+\bar{b}-\epsilon_b) &= a'-\bar{b}+\epsilon_b\\
\pi_1^{WO,1}(\bar{p}_1^{VCG}+\bar{b}+\epsilon_b) &= a\\
\pi_1^{WO,2}(\bar{p}_1^{VCG}+\bar{b}+\epsilon_b) &= a'-\bar{b}-\epsilon_b
\end{aligned}$$

In all three cases, one can verify that $\pi_1^{WO,2}(\bar{p}_1^{VCG}+\bar{b}-\epsilon_b) \leq \pi_1^{WO,1}(\bar{p}_1^{VCG}+\bar{b}-\epsilon_b)$ is equivalent to $a' \leq a+\bar{b}-\epsilon_b$ and $\pi_1^{WO,2}(\bar{p}_1^{VCG}+\bar{b}+\epsilon_b) \leq \pi_1^{WO,1}(\bar{p}_1^{VCG}+\bar{b}+\epsilon_b)$ is equivalent to $a' \leq a+\bar{b}+\epsilon_b$.

A.15. Proposition 8

We first consider the case when bidder 1 bids in the restricted policy space $U_1' = \{(x,y) \in \mathbb{R}_+^2 \mid x = \bar{p}_1^{VCG}, y \geq x\}$. When $a' \in (a+\bar{b}-\epsilon_b, a+\bar{b}+\epsilon_b]$, by Lemma 2, we have $\pi_1^{WO,1}(\bar{p}_1^{VCG}+\bar{b}-\epsilon_b) < \pi_1^{WO,2}(\bar{p}_1^{VCG}+\bar{b}-\epsilon_b)$ and $\pi_1^{WO,1}(\bar{p}_1^{VCG}+\bar{b}+\epsilon_b) \geq \pi_1^{WO,2}(\bar{p}_1^{VCG}+\bar{b}+\epsilon_b)$. Due to the monotonicity of $\pi_1^{WO,1}(y)$ and $\pi_1^{WO,2}(y)$ (Lemma 1), the optimal choice of y (with respect to the restricted space U_1') is at the intersection of $\pi_1^{WO,1}(y)$ and $\pi_1^{WO,2}(y)$. To determine the optimal worst-case payoff with respect to the restricted space U_1', we recall that $\pi_1^{WO,1}(y)$ is constant for $y \geq \bar{p}_1^{VCG}+\bar{c}+\epsilon_c$ and $\pi_1^{WO,2}(y)$ is constant for $y \leq \bar{p}_1^{VCG}+\bar{c}+\epsilon_c$. Thus, at the intersection point of these two payoff functions, bidder 1's payoff is equal to $\min\{\pi_1^{WO,1}(\bar{p}_1^{VCG}+\bar{c}+\epsilon_c), \pi_1^{WO,2}(\bar{p}_1^{VCG}+\bar{c}+\epsilon_c)\} = \min(a, a'-\bar{c}-\epsilon_c)$ (see Figure 8c and 8d). Since this is a lower bound for the optimal worst-case payoff on the original policy space $U_1 = \mathbb{R}_+^2$, one has that

$$\min(a, a'-\bar{c}-\epsilon_c) \leq \pi_1^{MAXMIN} \tag{54}$$

We now show that the above lower bound is the same with the upper bound on π_1^{MAXMIN} provided by the minimax inequality. For any $b_{-1} \in U_{-1}$, bidder 1's best response payoff is her VCG payoff. Thus, $\sup_{b_1 \in U_1} \pi_1(b_1, b_{-1}) = \max(a+b, a') - \max(b, c)$. As a result, we have

$$\begin{aligned}
\pi_1^{MINMAX} &= \inf_{b_{-1} \in U_{-1}} \sup_{b_1 \in U_1} \pi_1(b_1, b_{-1}) \\
&= \inf_{b_{-1} \in U_{-1}} (\max(a+b, a') - \max(b, c)) \\
&= \min \begin{pmatrix} \inf & a' - \max(b, c), & \inf & a - (c-b)^+ \\ \text{s.t.} & b < a' - a & \text{s.t.} & a' - a \le b \\ & b_{-1} \in U_{-1} & & b_{-1} \in U_{-1} \end{pmatrix} \\
&= \min\{a' - \max(a' - a, \bar{c} + \epsilon_c), a - (\bar{c} + \epsilon_c - a' + a)^+\} \\
&= \min(a, a' - \bar{c} - \epsilon_c),
\end{aligned}$$

where the second last equality follows by substituting the minimizing values for b and c. By minimax inequality, we have $\pi_1^{MAXMIN} \le \pi_1^{MINMAX} = \min(a, a' - \bar{c} - \epsilon_c)$. Together with (54), we have $\pi_1^{MAXMIN} = \min(a, a' - \bar{c} - \epsilon_c)$ and the policy (x^*, y^*) is optimal.

A.16. Remark 6

Let $\pi_1^{WO,TR}$ be her worst-case payoff under truthful bidding. We want to show that if $a + \bar{b} - \epsilon_b < a'$ then this truthful worst-case payoff is the same with the optimal worst-case payoff from robust bidding, i.e.,

$$\pi_1^{WO,TR} = \pi_1^{MAXMIN}. \tag{55}$$

First let us consider the case when $a + \bar{b} + \epsilon_b < a'$. By reporting truthfully, bidder 1 always wins two items, so her worst-case payoff is

$$\pi_1^{WO,TR} = a' - \max(\bar{b} + \epsilon_b, \bar{c} + \epsilon_c),$$

On the other hand, according to Proposition 7, we have $\pi_1^{MAXMIN} = a' - \max(\bar{b} + \epsilon_b, \bar{c} + \epsilon_c)$. Hence, $\pi_1^{WO,TR} = \pi_1^{MAXMIN}$.

Now let us assume that $a + \bar{b} - \epsilon_b < a' \le a + \bar{b} + \epsilon_b$. By Proposition 8, $\pi_1^{MAXMIN} = \min(a, a' - \bar{c} - \epsilon_c)$. There are three possible cases to be considered:

1. $\bar{b} + \epsilon_b \le \bar{c} + \epsilon_c$:

   By reporting truthfully, bidder 1's worst-case payoffs when winning one and two items are respectively:

$$\begin{aligned}
\pi_1^{WO1,TR} &= \min(a, \frac{1}{2}\min(a, a' - \bar{c} - \epsilon_c) + \frac{1}{2}(a' - \bar{c} - \epsilon_c)) \\
&= \min(a, a' - \bar{c} - \epsilon_c), \\
\pi_1^{WO2,TR} &= a' - \max(\min(a' - a, \bar{b} + \epsilon_b), \bar{c} + \epsilon_c) \\
&= a' - \bar{c} - \epsilon_c.
\end{aligned}$$

   Hence, bidder 1's worst-case payoff is

$$\pi_1^{WO,TR} = \min(\pi_1^{WO1,TR}, \pi_2^{WO2,TR}) = \min(a, a' - \bar{c} - \epsilon_c)$$

   which is the same with π_1^{MAXMIN}.

2. $\bar{b}-\epsilon_b \leq \bar{c}+\epsilon_c < \bar{b}+\epsilon_b$:

   Similar to the previous case, by reporting truthfully, bidder 1's worst-case payoffs when winning one and two items are respectively:

$$\pi_1^{WO1,TR} = \min(a, a'-\bar{c}-\epsilon_c),$$
$$\pi_1^{WO2,TR} = a' - \max(\min(a'-a, \bar{b}+\epsilon_b), \bar{c}+\epsilon_c).$$

   Now we prove that $\min(\pi_1^{WO1,TR}, \pi_2^{WO2,TR}) = \min(a, a'-\bar{c}-\epsilon_c)$. If $a'-\bar{c}-\epsilon_c < a$ then $\pi_1^{WO1,TR} = \pi_1^{WO1,TR} = a'-\bar{c}-\epsilon_c$ so we indeed have $\min(\pi_1^{WO1,TR}, \pi_2^{WO2,TR}) = \min(a, a'-\bar{c}-\epsilon_c)$. On the other hand, if $a \leq a'-\bar{c}-\epsilon_c$ then $\pi_1^{WO1,TR} = a$ and $\pi_1^{WO2,TR} = a' - \min(a'-a, b+\epsilon_b) \geq a$, so $\min(\pi_1^{WO1,TR}, \pi_2^{WO2,TR}) = a = \min(a, a'-\bar{c}-\epsilon_c)$. Therefore, $\pi_1^{WO,TR} = \pi_1^{MAXMIN}$.

3. $\bar{c}+\epsilon_c < \bar{b}-\epsilon_b$:

   In this case, the worst-case payoffs of bidder 1 when winning one and two items are $\pi_1^{WO1,TR} = \pi_1^{WO2,TR} = a = \min(a, a'-\bar{c}-\epsilon_c)$. Thus, we also have $\pi_1^{WO,TR} = \pi_1^{MAXMIN}$, which completes our proof.

A.17. Proposition 9

(a) For each $b_{-1} \in U_{-1}$, a best response of bidder 1 is to bid

$$b_1^{BEST} = b^{(m)} \mathbf{1}_{b^{(m)} \leq v_1}.$$

The maximum payoff of bidder 1 given b_{-1} is

$$\pi_1(b_1^{BEST}, b_{-1}) = (v_1 - b^{(m)}) \mathbf{1}_{b^{(m)} \leq v_1}. \tag{56}$$

Minimizing the above payoff function over the uncertainty set U_{-1}, one gets that $\pi_1^{MINMAX} = (v_1 - u(m))^+$, which is exactly the same with π_1^{MAXMIN} according to Proposition 1.

(b) Minimax equality (9) follows directly from the proof of Proposition 6.

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