Computation of Optimal Dynamic Mechanism with Participation Requirements

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We consider dynamic mechanism design problems in which a principal procures up to one unit of a product/service in every period from an agent who is privately informed about its marginal production cost in each period, which are i.i.d. random variables. The model incorporates *participation requirements*: continued participation by the agent is costly (e.g., the agent consumes a fixed amount in each period to continue the partnership), and in each period the agent's production cost must be fully paid for. In this setting, "static mechanisms" are no longer optimal and closed-form solutions are not available, which makes the development of tractable computational methods crucial.

Our main results provide regularity conditions on the distribution of the private information such that the optimal contracts demonstrate "state-dependent-threshold" structures. Under our conditions, in any period, the optimal procurement level is the whole unit if the marginal production cost is below a threshold, and is a fraction between 0 and 1 if the marginal production cost is above this threshold. The threshold and the fractional procurement amount depend on the state of the system, which characterize the agent's total expected future utility. These results allow dramatic reduction of the action space in the dynamic optimization model for the mechanism design problem. We demonstrate the impact of our structural results in computation time in a numerical study. Our results are based on the theory of infinite-dimensional conic optimization and show that "ironing" can occur even with well behaved distribution, such as uniform, in this dynamic setting. The analysis focuses on what we call the "dynamic virtual valuation," a generalization of the Myersonian virtual valuation from the static setting, and is potentially relevant to other dynamic mechanism design problems.

To illustrate the applicability of our framework in other settings, we apply the methodology to a model where no promise of future payment is allowed but the agent can accumulate gains to cover its future participation costs. This problem is non-concave, due to strategic termination of the partnership from the principal. Despite this difficulty we establish similar structural properties for the optimal contract, which imply tractable computational methods for this model as well.

Key words: dynamic mechanism design, optimization, dynamic programming, dynamic virtual valuation

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1. Introduction

In his seminal work, Myerson (1981) analyzed what is arguably the most fundamental mechanism design model, in which a seller (principal) aims at selling one unit of an item to one of several potential buyers (agents) with private information on the item's valuation. Under mild regularity conditions concerning the distribution of the item's private valuation, it can be shown that when there is only one potential buyer, a take-it-or-leave-it offer (a single selling price) characterizes the optimal mechanism. The selling price is characterized by the point at which the so-called "virtual valuation" crosses zero. The virtual valuation summarizes the marginal value the trade brings to the principal for each agent type, adjusted to incorporate the agent's incentives. The regularity conditions guarantee that the virtual valuation is monotonically increasing in the agent's private valuation, which guarantees a unique point at which the virtual valuation is zero. A direct consequence is the easy computation of the optimal contract. This seminal work initiated a large literature on mechanism design and auctions (see, for example, McAfee and McMillan 1987, Krishna 2009).

Our paper focuses on analyzing a dynamic version of this fundamental model in procurement settings (rather than purchasing). In particular, we consider a dynamic model in which a principal repeatedly procures up to one unit of product/service from an agent whose variable production costs over time are commonly known i.i.d. random variables. The principal has the credibility of issuing long term contract with the agent (i.e. commitment power). However, the model incorporates two additional features that are common in practice:[1] continued participation by the agent is costly (the agent needs to consume a fixed amount in each period for the partnership to continue) and in each period the agent's production cost must be reimbursed (which can be seen as a liquidity constraint of the agent).

Such features create non-trivial inter-temporal dependence. Indeed, the liquidity constraint, which is typically binding in each period, rules out the possibility that the agent pays an lump sum in the beginning and buy out the principal. In particular, repeated "static mechanisms" are no longer optimal, and closed-form solutions are not available. Therefore, it is crucial to develop tractable computational methods. Broadly speaking, the goal of this paper is to provide a methodology to establish new structural properties for the optimal dynamic mechanisms that can be exploited by algorithms. In essence, we advocate for the development of computational methods that fully exploit the structure of the problem at hand.

[1] In fact the model allows for endogenous termination of the partnership, which reflects a key trade-off for the principal that only occurs in dynamic settings.

In a similar spirit to Myerson (1981), we provide conditions on the distribution of the private information under which the optimal procurement quantity in each period depends on a single threshold on the revealed marginal production cost in the period. The difference in the optimal structure, compared with the static model, is that while the principal procures the full unit in a period if the marginal production cost is lower than the threshold, the procurement amount may be a positive constant if the marginal production cost is greater than the threshold. The optimal policy in each period, therefore, is determined by at most two (state dependent) parameters: the *threshold*, and the *order quantity* if the marginal production cost is above the threshold. Such a structure generalizes the simple take-it-or-leave-it contract structure in Myerson (1981). Fundamentally, the policy structure arises from the so-called "ironing" effect when the virtual valuation is no longer monotonically decreasing.[2] These features highlight the additional complexities of optimal mechanism design in dynamic settings.

A direct consequence of this structural result is dramatically reduced computational burden to numerically solve the problem. Indeed, we can formulate our dynamic mechanism design problem a dynamic program on a single dimensional space. The main difficulty lies in the space of decision variables. The decision in each period is a function: procurement quantities as a function of marginal production costs. Our structural results reduce the decision space from the space of monotone functions to a two dimensional space, representing the two policy parameters of the optimal mechanism. Furthermore, structure properties of the value function can also be used to improve algorithms. For example, it is shown that the value function is concave and nondecreasing, and its value at certain points are readily available. Therefore we can use piecewise linear approximations in computation.

Our proof strategy relies on a new concept we propose, called *dynamic virtual valuations*, which generalizes virtual valuations in the static case. The dynamic virtual valuation consists of two components: one from the current period's profits, resembling virtual valuation in the static model, and the other based on the principal's value function to account for future profits. We characterize the dynamic virtual valuation as a functional derivative. Since the value function is defined through a Bellman equation, establishing the structural properties of the dynamic virtual valuation is much more involved than in the corresponding basic single-period setting. Similarly to Myerson's approach, our regularity conditions on input parameters still guarantee that the

[2] We note that in the static model, when the regularity conditions do not hold, Myerson further showed that the optimal mechanism is slightly more complex, but can still be characterised by various levels of purchase quantities, determined by the shape of the virtual valuation. The technique is often referred to as "ironing" in the literature (see, for example, Toikka 2011) and view as an exception. In contrast, in the dynamic setting with participation requirements, our analysis shows that ironing seems to be much more prevalent even for well behaved distributions.

dynamic virtual valuations have certain structures. For example, one set of sufficient conditions guarantees that the dynamic virtual valuation is monotonically non-increasing. In this case the optimal procurement quantity follows a simple single threshold structure, similar to the static model. For uniformly distributed marginal production cost, on the other hand, we show that the dynamic virtual valuation is convex, following arguments in infinite dimensional optimization. The convex dynamic virtual valuation yields a potentially positive order quantity when the marginal production cost is above a threshold.

We believe the concept of dynamic virtual valuation is of independent interest and will play a central role in the theoretical analysis of other dynamic mechanism design settings. Furthermore, the methodology proposed here – to establish structural properties and new computational methods – can be applied to other dynamic mechanism design problems. To illustrate that, in the last section of the paper, we provide analogous results for a different dynamic mechanism design problem where no promise of future payment is allowed but the agent can accumulate gains to cover its future participation costs. Therefore, the principal has to provide incentives in each period by cash payment. The agent has access to a bank account, which, in each period, either cumulates surplus beyond a fixed consumption S, or fund this consumption if the net payment from the principal is below S. The principal can terminate the partnership by not replenishing the account when the balance is blow this fixed consumption level.[3] We note that the dynamic programming problem induced by this setting is non-standard because of a "boundary condition" with important economic interpretations. Such a structure can induce discontinuities and non-concavities that need to be considered in the analysis. Despite such non-concavities of the value function (due to strategic termination of the partnership from the principal) we establish similar structural properties for the optimal contract (based on the dynamic virtual valuations) and propose corresponding tractable computational methods.[4]

Our paper contributes to the growing literature concerning revenue-maximizing dynamic mechanism design problems initiated by Baron and Besanko (1984), in which the authors consider a principal with full commitment power and an agent that privately observes costs over two periods. Dynamic contracting and dynamic mechanism design have attracted much more attention in recent

[3] A by product of our analysis based on the dynamic virtual valuation reveals interesting dynamics associated with the optimal mechanism in this model. For example, it might be optimal for the principal to terminate interactions with the agent despite the fact that the expected gain from each game stage is positive. Furthermore, we show that the principal's procurement amount is always higher compared with the single period model because of the incentives generated by potential future gains.

[4] Again the structural results of the optimal procurement function allow to drastically reduce the computation effort, since the set of optimal decisions at each state can be characterized by at most three parameters in either problem setting, compared with infinite dimensional set of monotone functions without our structural results.

years. Battaglini (2005), for example, considers revenue-maximizing long-term contracts from a monopolist in a model with an infinite time horizon when the valuation of the buyer changes in a Markovian fashion over time. There is another recent stream of literature concerning dynamic mechanism design that focuses on socially efficient allocation (see, for example, Bergemann and Välimäki 2010, and references therein). Our model is based on Krishna et al. (2013) where there is no participation costs. Moreover, our paper differs from Krishna et al. (2013) by deriving results on the structure of the optimal contract based on the ironing technique and its use on computational procedures. In contrast, Krishna et al. (2013) characterizes interesting dynamics, which can be used to interpret the so-called "sweat equity" phenomenon. Practical examples of their model are drawn from the venture capital market, where founders launch a business based on their personal expertise but do not possess sufficient financial resources, from franchising situations, where an owner wants to expand into a specific market but lacks idiosyncratic knowledge about local factors, and from other areas. A recent paper, Pavan et al. (2014), proposes a general framework for dynamic mechanism design that allows for multiple agents with serially correlated private information, as long as utilities are quasi-linear and each agent's private information is unidimensional in each period. The paper provides envelope-theory-type results for the derivative of an agent's expected equilibrium utility with respect to private information, and describes how to construct optimal payments given an allocation rule. Although our model, along with many other existing models of dynamic adverse selection, is a special case of the modeling framework of Pavan et al. (2014), our analysis and results are new and have a different focus.

Methodologically, the recursive formulations of our dynamic mechanism design model with a dynamic agent date back to Thomas and Worrall (1990). That paper reveals, in a specific dynamic adverse selection problem, that repeated principal-agent interactions can be modeled as a dynamic optimization that maximizes the principal's utility subject to the agent's incentive compatibility constraint. Interestingly, the agent's total future utility serves as a state variable in the principal's dynamic optimization problem. Such a recursive formulation is also valid in our model. Fernandes and Phelan (2000) extends the problem studied in Thomas and Worrall (1990) to an endowment process where the private information is serially correlated over time, and demonstrates that the incentive compatibility constraints can still be formulated in a recursive manner. This, again, allows for dynamic programming representations of the problem.

Dynamic contracting problems have also received considerable attention recently in the Operations Research and Management Science literature. Li et al. (2013), for example, study a long-term moral hazard problem for a firm seeking to induce efforts from competing suppliers, while Zhang et al. (2010) and Lobel and Xiao (2013) study dynamic adverse selection models that seek to efficiently manage inventory systems. In particular, the modeling framework of Li et al. (2013) also contains promised utility, which captures agents' value functions under the optimal contract. Zhang (2012a,b) study theoretical properties and solution approaches to dynamic adverse selection problems with serially correlated private information.

The paper is organized as follows. The dynamic model with participation requirements is discussed in Section 2. Sections 3 provides the analysis of the structural properties of the optimal contract as well as of the value function that motivates more efficient numerical algorithms. Numerical comparisons between algorithms are given in Section 4. We also provide a detailed numerical example to illustrate the optimal policy structure. Section 5 illustrate the applicability of the ideas developed here to yet another dynamic model which has non-concave value function. Additional technical results and all proofs are given in Appendices A and C.

2. Dynamic Mechanism Design under Participation Requirements

In this dynamic model a principal designs a direct mechanism to procure up to one unit of a good from an agent at each (discrete) time period over an infinite time horizon. Each unit of the good is worth p to the principal. In each period t, the agent is privately informed about the marginal production cost c. The production cost follows a commonly known distribution F with support $[\underline{c}, \bar{c}]$, which is assumed to be i.i.d. over time periods.

The principal has the commitment power and the agent is forward looking. In particular, the principal commits to a dynamic mechanism, which determines the procurement quantity q_t and payment m_t in each period t, depending on the entire history of the reported private information. Correspondingly, the agent decides what information to reveal, taking into consideration its impact on current as well as future payments. In each period, in addition to the income m_t and variable production cost $c_t q_t$, the agent also bears a cost S, representing, for example, consumption or rent and salary payments. Overall, the agent's utility is represented by the discounted summation of single period profits $m_t - c_t q_t - S$ over the entire future, which may be affected by the current period's report through the mechanism. Similar to Thomas and Worrall (1990), the repeated strategic interactions between the principal and the agent can be modeled as a dynamic program in which the agent's utility (promised value) serves as a summary statistic of the history, and therefore is the state variable.

Formally, denote $h_t = \{c_0, \dots, c_t\}$ to represent the production costs from the beginning of the time horizon to period t, that is observable only by the agent. Consider "direct mechanisms" under which the agent reports (truthfully or not) the production cost in each period. Denote $h'_t = \{c'_0, \ldots, c'_t\}$ to represent the reported history. The procurement quantity q_t and monetary payment m_t in each period depend on the entire reported history h'_t. In general, the agent's utility depends on both the true history h_t and reported history h'_t, as well as the agent's reporting strategy, denoted as ι, that maps (h_t, h'_{t-1}) into a report c'_t in each period t. Therefore, the agent's total discounted utility can be represented as

$$w_0(\iota) = \mathbb{E}\left[\sum_{t=0}^{\infty} \beta^t \big(m_t(h'_t) - c_t q_\tau(h'_t) - S\big)\right] ,$$

where β is the discount factor, in which for any given true history h_t and reported h'_{t-1}, the new reported history is

$$h'_t = \big(h'_{t-1}, \iota(h_t, h'_{t-1})\big) .$$

Following the "revelation principle," without loss of generality, we can focus on direct mechanisms, under which the agent truthfully reveal the marginal production cost c_t in each period. Therefore, the principal can maximize over mechanisms such that any history-dependent reporting strategy must not outperform the one that always reports the truth. Denote $\bar{\iota}$ to represent a "truthful reporting strategies," such that if truth has been reviewed so far, strategy $\bar{\iota}$ keeps reporting the truth. Formally, $\bar{\iota}_t(\{h_{t-1}, c_t\}, h_{t-1}) = c_t$. The "revelation principle" states in any equilibrium of the aforementioned Bayesian game, there exists a payoff equivalent mechanism under which the truth revelation strategy $\bar{\iota}$ is the best response by the agent. Therefore, the principal can maximize over mechanisms $(m_t, p_t)_{t=1,\ldots,\infty}$ that satisfy the following condition

$$w_0(\bar{\iota}) \geq w_0(\iota) , \quad \forall \iota . \tag{1}$$

Following an argument similar to Lemma 2.1 in Fernandes and Phelan (2000), one can represent the so-called "incentive compatibility" constraint (1) in a recursive form, which requires that if an agent has been reporting true costs, he or she is better off by continuing to tell the truth.

Lemma 1 *Mechanism $\{m_t, q_t\}_t$ satisfies incentive compatibility constraints (1) if and only if it satisfies the following recursive formulation.*

$$\begin{aligned} & m_t(h_{t-1}, c) - c q_t(h_{t-1}, c) + \beta w_t(h_{t-1}, c) \geq \\ & \qquad m_t(h_{t-1}, c') - c q_t(h_{t-1}, c') + \beta w_t(h_{t-1}, c') , \quad \forall h_{t-1}, c, c', \text{ in which} \\ & w_t(h_t) = \mathbb{E}\left[m_{t+1}(h_t, \tilde{c}) - \tilde{c} q_{t+1}(h_t, \tilde{c}) + \beta w_{t+1}(h_t, \tilde{c})\right] . \end{aligned} \tag{2}$$

Focus on the principal's decision in period t. In order to ensure incentive compatibility, the principal also needs to commit to future payments summarized by the agent's value function

w_t. As a result, in the beginning of the subsequent period $t+1$ (and therefore in each period), the committed future payment w_t has an impact on the principal's current period decision, and therefore needs to be in the state space of the principal's optimization problem. The principal, whose time discount factor is also β, faces the following dynamic optimization problem. Here we use $\mathcal{J}_t(h_{t-1}, v)$ to represent the principal's value function, $v = w_{t-1}(h_{t-1})$ to represent the agent's value function at the beginning of the period, and w to replace w_t.

$$\mathcal{J}_t(h_{t-1}, v) = \max_{q,m,w} \mathbb{E}\left[pq(c) - m(c) + \beta \mathcal{J}_{t+1}(\{h_{t-1}, c\}, w(c))\right]$$

subject to the following constraints, in light of (2),

$$m(c) - cq(c) + \beta w(c) \geq m(c') - cq(c') + \beta w(c') \ , \quad \forall c, c' \ , \tag{IC}$$

and

$$v = \mathbb{E}\left[m(c) - cq(c) + \beta w(c)\right] - S \ . \tag{PK}$$

Constraint (IC) is called "incentive compatibility" constraint, while constraint (PK) is commonly referred to in the literature as the "promise keeping" constraint. Here we omit the decision variables' dependence upon history in the expressions because they already are implied from the state variable h_{t-1}. It can also be verified that the optimal value function $\mathcal{J}_t$ is constant over its first argument h_{t-1} for a given v. That is, the principal's value function and the optimal dynamic mechanism depends on history only through the agent's value function v in the beginning of the period.

Furthermore, we enforce the following "individual rationality" constraint.

$$m(c) - cq(c) \geq 0 \ , \forall c \in [\underline{c}, \bar{c}] \ , \tag{IR}$$

such that variable production costs need to be reimbursed on each period. Note that we do not require the fixed cost to be paid in cash in each period under (IR), so it is assumed that the agent is able to fund the fixed cost. (In an alternative setting, one can replace (IR) with $m(c) - cq(c) - S \geq 0$, which is a tighter constraint, and therefore yields less value for the principal. Our analysis demonstrates that the dynamics of the system in our model setting are different, and, arguably, more interesting, than in this alternative setting.)

Because the agent's future value function $w(c)$ (often referred to as the "promised value" Ljungqvist and Sargent (2004)) is one of the decision variables, it is important to include some additional constraints to ensure economic reality as well as the model being bounded mathematically. First, the agent is only willing to continue the collaboration with the principal if the agent's value function is non-negative, i.e., $w(c) \geq 0$. Furthermore, the principal cannot promise a total utility from future payments that is higher than the maximal societal profit that can be generated over time. In particular, the highest value that can be generated from the system is through the so-called "first best" production policy $\bar{q}(c)$, defined as

$$\bar{q}(c) = \begin{cases} 1 , & c \leq p , \\ 0 , & c > p . \end{cases} \tag{3}$$

Therefore, the first best profit in each period is $\mathbb{E}[(p-c)\bar{q}(c)] - S$ for the system. As a result, the agent's value function $w(c)$ cannot exceed the present value of first best cash flow,

$$\bar{v} = \frac{\mathbb{E}[(p-c)\bar{q}(c)] - S}{1-\beta} . \tag{4}$$

We also assume that $\bar{v} > 0$ (otherwise the principal would not participate in the game). Equivalently, this implies the following condition on the fixed cost S

$$S < \mathbb{E}[(p-c)\bar{q}(c)] . \tag{5}$$

To summarize, we have the following box constraint for the promised value $w(c)$,

$$0 \leq w(c) \leq \bar{v} \quad \forall c \in [\underline{c}, \bar{c}]. \tag{BW}$$

Overall, the dynamic mechanism design problem considered in this section can be modeled as the following dynamic optimization problem, in which J represents the principal's value function, whose time discount factor is also β.

$$\begin{aligned} J(v) = \max\nolimits_{q,m,w} \mathbb{E}\left[pq(c) - m(c) + \beta J(w(c))\right] \\ \text{s.t.} \quad \text{(IC), (IR), (PK), and (BW)} . \end{aligned} \tag{P}$$

3. Optimal policy and Structural Properties

In this section we analyze the model (P) introduced in the last section. We begin with Lemma 2, which removes the decision variables m in the direct mechanism from the optimization formulation (P), following standard treatment of the (IC) and (IR) constraints (see, for example, Chapter 5 in Krishna 2009).

Lemma 2 *Model (P) is equivalent to*

$$J(v) = \max_{(q,w)} \mathbb{E}\left[(p-c)q(c)\right] + \beta\mathbb{E}\left[w(c)\right] - v - S + \beta\mathbb{E}\left[J\Big(w(c)\Big)\right] \tag{6}$$

for $v \in [0, \bar{v})$, in which (q, w) are subject to the following constraints.

$$0 \leq q(c) \leq 1 , \quad q(c) \text{ is non-increasing,} \quad \mathbb{E}[Q(c)] \leq v + S , \tag{7}$$

$$\text{and} \quad 0 \leq w(c) \leq \min\left\{\left(v + S - \mathbb{E}[Q(c)] + Q(c)\right)/\beta, \bar{v}\right\} , \tag{8}$$

where $Q(c) := \int_c^{\bar{c}} q(t)dt$ is the "information rent" for an agent with type c.

It is convenient to study the total value function of the system that combines the principal's and agent's value functions, $V(v) = J(v) + v$. It follows that solving (6) is equivalent to solving $V = \Gamma V$, in which the dynamic programming operator Γ is defined as

$$(\Gamma V)(v) := \max_{(q,w)} \mathbb{E}\left[(p-c)q(c)\right] - S + \beta\mathbb{E}\left[V\Big(w(c)\Big)\right], \text{ subjec to (7) and (8) .} \tag{9}$$

Next we define the agent's cumulative expected surplus v^* associated with the first-best allocation $\bar{q}$,

$$v^* = \frac{\mathbb{E}\bar{Q}(c) - S}{1-\beta} = \frac{\mathbb{E}[(p \wedge \bar{c} - c)\bar{q}(c)] - S}{1-\beta} \le \bar{v} \ , \tag{10}$$

where $\bar{q}$ is defined in (3), and $\bar{Q}(c)$ is given by $\bar{Q}(c) = \int_c^{\bar{c}} \bar{q}(t)dt$. The following result slightly generalizes Krishna et al. (2013) to cases with $S \ge 0$ and $c \in [\underline{c}, \bar{c}]$.

Proposition 1 (Structure of Value Function) *The following properties hold:*

(i) V is an increasing and concave function on $[0, \bar{v}]$.

(ii) $V(v) = \bar{v}$ for all $v \ge v^*$, and $(q(c) = \bar{q}(c), w(c) = v^*)$ is an optimal solution to $(\Gamma V)(v^*)$.

(iii) For any feasible policy q in (9) at a given $v \in [0, \bar{v}]$, there is an optimal promised utility function w that satisfies

$$w(c) = \min\{(v + S - \mathbb{E}[Q(c)] + Q(c))/\beta,\ v^*\} \ . \tag{11}$$

Furthermore, the optimal net payment $m(c) - cq(c) = \max\left\{\big(v + S - \mathbb{E}[Q(c)] + Q(c)\big)/\beta - v^*, 0\right\}$.

Proposition 1(iii) implies that the principal pays a fixed salary to reimburse the agent's variable production cost only to secure the agent's participation. Rewards for better types (lower marginal production costs) are in the form of future payments, or promised value, as much as possible.

Proposition 1(ii) further states that v^* is an absorbing state of promised value, following this specific optimal mechanism. At this point, the principal repeatedly procures according to the first-best allocation. It is worth noting that the optimal mechanism is not unique. An alternative optimal mechanism allows the promised value to increase to a different absorbing state, $\bar{v}$, at which point the principal's value function $J(\bar{v})$ remains at 0. This means that the principal essentially operates as if the enterprise belongs to the agent. This is consistent with the "sweat equity" interpretation provided in Krishna et al. (2013).

Proposition 1 allows us to remove the promised value decision $w(c)$ from the optimization problem (9) with (11). Furthermore, because function $V(v)$ is already a constant on $v \in [v^*, \bar{v}]$, if we extend the domain of V such that $V(v) = \bar{v}$ for any $v > \bar{v}$, we can further simplify (9) by replacing $w(c)$ with $g(c)$, defined as

$$g(c) = \frac{v + S - \mathbb{E}[Q(c)] + Q(c)}{\beta} \ . \tag{12}$$

Therefore, the optimization on the right hand side of dynamic program (9) can be expressed as $\max_q G(q, v)$ subject to constraint (7), where we define

$$G(q, v) = \mathbb{E}\left[(p - c)q(c)\right] - S + \beta \mathbb{E}\left[V\left(g(c)\right)\right] \ . \tag{13}$$

At this point, we can see some of the impact of including a positive fixed cost S in the model. In the case of $S = 0$, if in a period the optimal promised value v becomes zero, then constraint $\mathbb{E}[Q(c)] \leq v$ implies that the future procurement amount and promised value will remain zero. Therefore, state $v = 0$ is an absorbing state, at which point the collaboration terminates. When $S > 0$, on the other hand, even if $v = 0$, the (PK) constraint guarantees that $w(c) > 0$ for some $c \in [\underline{c}, \bar{c}]$. Therefore, there is always a chance that the collaboration will revive, as long as the agent can finance the fixed cost S.[5]

3.1. Dynamic Virtual Valuation and Optimal Procurement Structure

Before we introduce the concepts of this section, it is instructive to recall the basic static version of our model in a single period, in which the mechanism design problem is also an optimization problem with decision variables being the allocation decision q, which has to be a monotone function of c. Following Myerson (1981), under a regularity condition on the distribution of the private information c, the so called "virtual valuation," $p - c - F(c)/f(c)$, is monotone in c. Under this condition the optimization problem becomes separable, and the optimal allocation q is a step function, taking value 1 when the virtual valuation is above 0, and 0 otherwise.

Here, for model (P), we derive a quantity similar to Myerson's virtual valuation, which we call the "dynamic virtual valuation." This generalization of the virtual valuation relies on the functional derivative of the objective function $G(q, v)$ for maximization over q.

Proposition 2 *For any measurable function $h : [\underline{c}, \bar{c}] \to \mathbb{R}$, we have*

$$\frac{d}{d\delta} G(q + \delta h, v)\bigg|_{\delta=0} = \mathbb{E}\left[h(c)\xi(c)\right] \ ,$$

[5] The alternative setting of the (IR) constraint in the paper, where each period's payment has to cover the fixed cost S, yields the same dynamics as the $S = 0$ case. The only difference is that each period's payment m is increased by S.

where the dynamic virtual valuation ξ *is defined as*

$$\xi(c) = p - c - \frac{F(c)}{f(c)}\left(\int_{\underline{c}}^{\bar{c}} V'(g(\tau))f(\tau)d\tau\right) + \frac{1}{f(c)}\left(\int_{\underline{c}}^{c} V'(g(\tau))f(\tau)d\tau\right) .$$

Unlike the (static) virtual valuation $p - c - F(c)/f(c)$, the dynamic virtual valuation $\xi(c)$ contains additional terms that depend on the value function V. The dynamic virtual valuation $\xi(c)$ therefore captures the value of future partnerships with the agent when the current marginal production cost is c. In the static model, the simple assumption of $F(c)/f(c)$ non-decreasing in c implies single crossing as mentioned before. The monotonicity of dynamic virtual valuation $\xi(c)$, however, requires a different condition and is harder to establish. Even if it is monotonic, there is an additional constraint, $\mathbb{E}[Q(c)] \le v + S$, which prevents the optimization problem from being separable in c as in the static model.

The following result demonstrates that we are still able to characterize the structures of the dynamic virtual valuation under mild conditions on the distribution of the marginal production cost. This result is essential in showing the optimality of the corresponding structures of the optimal procurement levels.

Theorem 1 (Properties of Dynamic Virtual Valuations, Model (P))

(1) Assume that the production cost distribution satisfies the following sufficient condition,

$$\frac{1 - F(c)}{f(c)} \text{ is non-decreasing in } c. \tag{Suff}$$

Then the dynamic virtual valuation $\xi(c)$ is decreasing in $[\underline{c}, \bar{c}]$.

(2) Assume that the production cost distribution follows a uniform distribution, which does not satisfy condition (Suff).[6] *The dynamic virtual valuation $\xi(c)$ is convex in c on $[\underline{c}, \bar{c}]$.*

It is worth noting that exponential distributions and Pareto distributions satisfy condition (Suff).

In the spirit of Myerson's approach, Theorem 1 provides conditions on the probability distributions of private information such that the dynamic virtual valuations have certain structures. In turn, these structures allow us to establish the structure of optimal procurement policies which constitutes the main theoretical result for this model.

Theorem 2 (Structure of Optimal Procurement, Model (P))

[6] In the proof we provide a sufficient condition, beyond the uniform distribution, for the result to hold.

(1) Under condition (Suff), the optimal procurement policy q^* follows a single threshold $\widehat{c}(v)$ policy structure, that is, there is a threshold $\widehat{c}(v)$ such that it is optimal to procure a whole unit when $c < \widehat{c}(v)$, and 0 unit otherwise.

(2) If the production cost follows a uniform distribution, the optimal procurement policy q^* follows a $\big(\widehat{c}(v), \gamma(v)\big)$ policy structure, under which it is optimal to procure a whole unit when $c < \widehat{c}(v)$, and a partial $\gamma(v) \in [0,1]$ unit otherwise.

The structure revealed in Theorem 2(1) is similar to the structure of the static model, except that the threshold $\widehat{c}(v)$ depends on the state v of the dynamic program in each period. The structure in Theorem 2(2), which can be different from the single threshold structure in the static model, remains simple. The conditions that guarantee these simple structures, however, are different from the regularity condition presented in Myerson (1981). This highlights the difference between static and dynamic models.

The simple structures revealed in Theorem 2 can be used directly in computation. For example, in each step of the dynamic programming algorithm, we can restrict the search of the optimal allocation function $q(\cdot)$ to the search of at most two parameters, $\widehat{c}$ and γ if the distribution of the marginal production cost is either uniform or satisfies condition (Suff).

Remark 3.1 (Ironing in Dynamic Settings) *The proof of Theorem 2(2), detailed in the Appendix, requires the use of "ironing," which can be traced back to Myerson (1981). Ironing in our setting, however, is due to the dynamic nature of the problem, which is new to the literature. (In contrast, ironing arises because of irregular probability distributions of private information in the classic work of Myerson (1981).) Furthermore, our proof techniques rely on conic duality, which avoids differentiability assumptions.*

In the next subsection, we present a numerical example for the uniform distribution case to illustrate this relatively more complex structure described above.

3.2. A detailed numerical example

Here we present a numerical example for the optimal mechanism and the value function associated with model (P), for which the marginal production cost follows a uniform distribution, which, according to Theorem 1, yields a slightly more complex policy structure compared with distributions that satisfy condition (Suff).The model parameters in the examples are $p = 1.5$, $S = 0.1$ and $\beta = 0.85$.

Figure 1 depicts the value function $V(v)$, which is increasing concave and converging to $\bar{v} = 6$ following Proposition 1. The figure also depicts the corresponding principal's value function $J(v)$. Function $J(v)$ is maximized at $v = 0.4$. Therefore, at the beginning of the time horizon, the principal proposes promised utility $v = 0.4$ as the starting point of the mechanism. Furthermore, in this setting $v^* = 2.6667$, even though function $V(v)$ appears essentially constant when v is less than v^*.

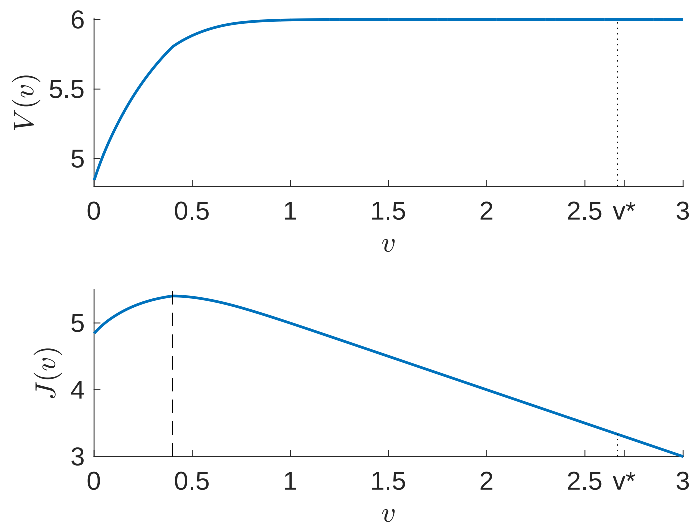

Figure 1 **Optimal value function $V(v)$ and $J(v)$, with $p = 1.5$, $S = 0.1$, $\beta = 0.85$, $\underline{c} = 0$, $\bar{c} = 1$, and c uniformly distributed.**

Figures 2(a) and 2(b) depict the optimal threshold $\widehat{c}(v)$ and quantity $\gamma(v)$, respectively. In particular, the solid curve in 2(a) depicts the optimal threshold $\widehat{c}(v)$, while the dashed curve depicts the optimal threshold if we restrict the policy to be within the class of single threshold policies with $\gamma(v)$ fixed at 0. We observe from the figure that when v takes values between about 0.12 and 0.4, it is indeed optimal for the principal to procure a positive quantity if the marginal production cost is above the optimal threshold, as described in Theorem 2. If we restrict the procurement to be zero when the marginal production cost is above a threshold, the corresponding best threshold is different from the optimal one.

Next, we focus on a particular value of $v = 0.25$ and demonstrates the optimal procurement policy and dynamic virtual valuation in Figures 2(c) and 2(d), respectively. In particular, Figure 2(c) shows the optimal procurement quantity $q(c)$, which has a two-level structure as described in Theorem 2, with $\widehat{c}$ and γ values consistent with Figure 2 at $v = 0.25$. Figure 2(d), on the other hand, shows the dynamic virtual valuation $\xi(c)$, which is convex, as described in Theorem 1(2).

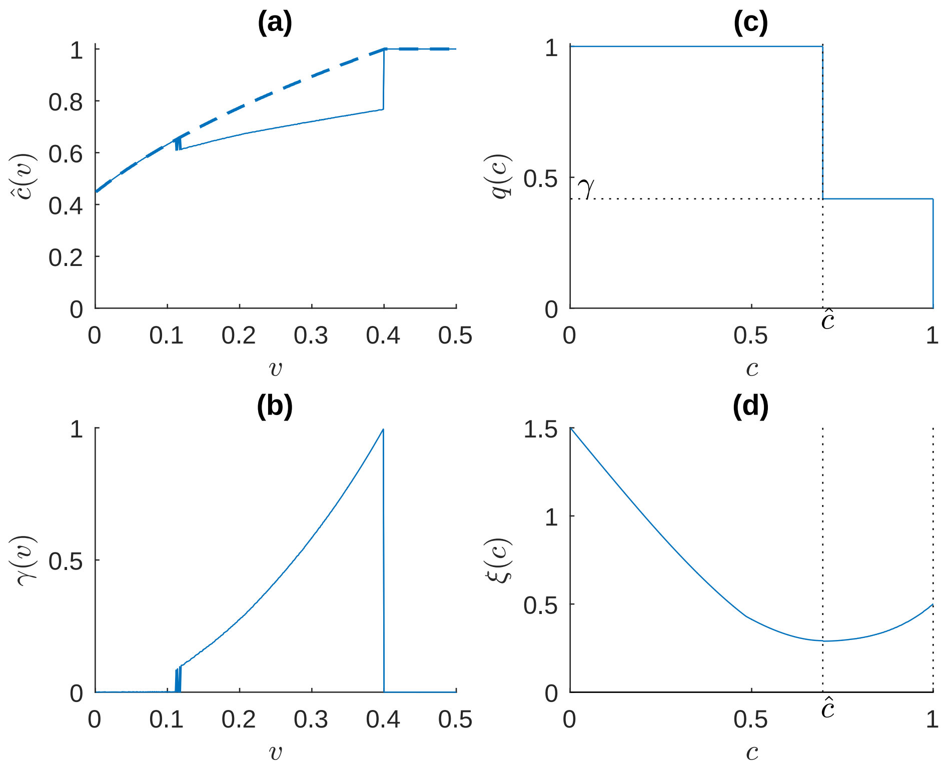

Figure 2 **The left column (plots (a) and (b)) displays the optimal policies, characterized by the threshold $\widehat{c}(v)$ and quantity level $\gamma(v)$, as a function of the promise utility v. The right column (plots (c) and (d)) display the procurement quantity $q(c)$ and dynamic virtual valuation $\xi(c)$ for a fix level of promise $v = 0.25$. The parameters used for the model were: $p = 1.5$, $S = 0.1$, $\beta = 0.85$, $\underline{c} = 0$, $\bar{c} = 1$, and c uniformly distributed.**

4. Computational Methods and Numerical Experiments

In this section, we describe an algorithm based the structure illustrated in Theorem 2. (We call the algorithm the "Structure-Based-Algorithm.") We also present a "Cutting-Plane-Algorithm" that takes advantage of the concavity of the value function as illustrated in Proposition 1 and linear programming solvers, but not the specific structure of the optimal allocation decisions.

In either algorithm, we discretize the interval $[\underline{c}, \bar{c}]$ into a grid of M points, $c_j = \underline{c} + (j - 1/2)\delta$ for $j = 1, \ldots, M$, in which $\delta = (\bar{c} - \underline{c})/M$. Similarly, we discretize the one dimensional state space on $[0, v^*]$ into a grid of $1 + \lceil v^*/\delta \rceil$ points $v_i = i\delta$ for $i = 0, \ldots, \lceil v^*/\delta \rceil - 1$ and $v_{\lceil v^*/\delta \rceil} = v^*$. Both algorithms proceed through value iteration. That is, in iteration k, equipped with a value function V^k, we need to solve an optimization $(\Gamma V^k)(v_i)$ at each state v_i, where the dynamic programming operator Γ is defined in (9). The starting point is $V^1(v_i) = \bar{v}$ for all points v_i in the grid. Following Proposition 1, we replace decision variables $w(c)$ as functions of $q(c)$ according to (11). Therefore, we reduce decision variables of the optimization to only include allocation decisions $q_j := q(c_j)$.

More specifically, for the Structure-Based-Algorithm, if the marginal production cost follows a distribution that satisfies (Suff), the optimization is simplified to the search of the optimal threshold $\widehat{j}$ such that $q_j = 1$ for $j \leq \widehat{j}$, and $q_j = 0$ for $j > \widehat{j}$. For uniformly distributed marginal production cost, we also need to determine the optimal γ such that $q_j = \gamma$ when $j > \widehat{j}$. Furthermore, we can speed up the algorithm by restricting the search to a single threshold $\widehat{j}$ in the first number of value iterations, before transiting to searching for both $\widehat{j}$ and γ towards the last few value iterations. The algorithm then proceeds with $V^{k+1} = \Gamma V^k$.

The Cutting-Plane-Algorithm is based on approximating the value function V^k as a piece-wise-linear function with k pieces. That is,

$$V^k(v) = \min_{\ell=1,\dots,k} \{a_\ell v + b_\ell\} \ . \tag{14}$$

The starting point of the algorithm, $V^1(v_i) = \bar{v}$ for all $i = 0, \dots, \lceil v^*/\delta \rceil$, corresponds to $a_1 = 0$ and $b_1 = \bar{v}$. In each iteration k, we solve $(\Gamma V^k)(v_i)$ for each state v_i via the following linear program.

$$\begin{aligned}
(\Gamma V^k)(v_i) = \max_{q_j, z_j, i=0\dots,M; Q, EQ} & \ \delta \sum_{j=1}^{M} [(p - c_j) q_j + z_j] f(c_j) - S \\
\text{s.t.} \quad & 1 \geq q_1 \geq \dots \geq q_M \geq 0 \\
& EQ = \delta^2 \sum_{j=1}^{M} f(c_j) \sum_{j'=j}^{M-1} q_{j'} \\
& EQ \leq v_i + S \\
& z_j \leq a_\ell \left(v_i + S - EQ + \delta \sum_{j'=j}^{M-1} q_{j'} \right) + \beta b_\ell \ , \quad \forall \ell = 1, \dots, k; j = 0, \dots, M \\
& z_j \leq \beta (a_\ell v^* + b_\ell) \ , \quad \forall \ell = 1, \dots, k; j = 0, \dots, M
\end{aligned}$$

The above linear program has $2M + 1$ decision variables and $M(2k+1) + 3$ constraints, which can be solved relatively easily using off-the-shelf linear programming solvers. Because we need to repeatedly solve this linear program, we can take advantage of "warm start" of simplex method by using the optimal solution from $(\Gamma V^k)(v_i)$ as the initial solution for $(\Gamma V^{k+1})(v_i)$.

Following the monotonicity of the dynamic programming operator (see, for example, page 60 of Bertsekas 2005), we have $\Gamma V^k \leq V^k$. Take

$$\widehat{i} = \arg \max_{i=0,\dots,\lceil v^*/\delta \rceil} V^k(v_i) - (\Gamma V^k)(v_i)$$

which marks the point where ΓV^k decrease V^k the most. Define the new cutting plane for V^{k+1} with

$$a_{k+1} = [(\Gamma V^k)(v_{\widehat{i}+1}) - (\Gamma V^k)(v_{\widehat{i}})] / (v_{\widehat{i}+1} - v_{\widehat{i}}) \text{ and } b_{k+1} = (\Gamma V^k)(v_{\widehat{i}}) - a_{k+1} v_{\widehat{i}} \ .$$

Therefore, we obtain the new value function V^{k+1}, following the definition (14), which satisfies that $\Gamma V^k \leq V^{k+1} \leq V^k$.

Next we discuss the a numerical study on the performance of both methods. We fix the distribution of c to be a uniform $[0,1]$ distribution, and vary the value of p from the set $\{1, 1.5, 2, 2.5, 3\}$, S from $\{0.1, 0.3, 0.5, \ldots, 1.9\}$, and β from $\{0.7, 0.75, \ldots, 0.95\}$. The production cost is assumed to be in the interval $[0,1]$. We restrict to cases that satisfy condition (5), which leaves us with 204 cases.

For each case, we set $\delta = 0.01$ and we set the stopping criteria for the value iteration to be $\epsilon = 0.001\bar{v}$. In particular, the Cutting-Plane-Algorithm stops if $\max_{i=0,\ldots,\lceil v^*/\delta \rceil} V^k(v_i) - V^{k+1}(v_i) < \epsilon$. For the Structure-Based-Algorithm, on the other hand, we significantly speed up the algorithm by first running the value iteration algorithm restricting the policy to be of the single threshold threshold structure (that is, ignoring the possibility of positive $\gamma(v)$). (We call this Phase I of the algorithm.) After the value functions converge (the maximum difference between value functions of two consecutive iterations is less than ϵ) we start running value iterations searching through both the threshold $\widehat{c}(v_i)$ and $\gamma(v_i)$. (We call this Phase II of the algorithm.)

We program the Structure-Based-Algorithm in Matlab on a server with equipped with Intel(R) Xeon(R) 2.4GHz CPU.

For the Structure-Based-Algorithm, in every single one of the 204 cases, we only need to run one value iteration in Phase II to reach optimality, following the convergence (according to our stopping criteria) of value iteration in the much simpler Phase I. In fact, in 144 out of the 204 cases, the simple $\widehat{c}(v)$ policy structure already achieves optimality. The one value iteration step in Phase II does not yield an improved value function. Consequently, the running time for each of these cases is well below 0.1 second. These cases correspond to model parameter $S \geq 0.5$.

The remaining $60 (= 204 - 144)$ cases, in which the optimal $\gamma(v)$ for some v is strictly between 0 and 1, correspond to model parameters $S \in \{0.1, 0.3\}$, and all 5 variations of p and 6 variations of β values. These cases take relatively longer time to solve. Still, Phase II converges in two value iterations in each of these cases. The case with the longest running time correspond to $p = 3$, $S = 0.1$, and $\beta = 0.95$. The CPU time for this case is 188 seconds, over 17 iterations in Phase I and 1 Phase II iteration. Table 1 reports the CPU time for all these 60 cases.

For the 60 cases involving $S \in \{0.1, 0.3\}$, we also implement the Cutting-Plane-Algorithm in AMPL, calling CPLEX as the linear optimization solver, on the same computer. The overall CPU time is much longer, as reported in Table 2. We attribute part of the much longer running time

	S=0.1						S=0.3					
	β											
p	0.7	0.75	0.8	0.85	0.9	0.95	0.7	0.75	0.8	0.85	0.9	0.95
1	21.7	26.8	34.5	47.6	79.0	153.7	12.5	13.4	14.8	17.4	21.5	29.8
1.5	25.3	30.8	39.2	53.4	87.0	177.6	13.7	14.6	15.9	18.2	22.6	36.0
2	25.3	30.8	39.3	53.4	87.0	188.3	13.7	14.6	15.9	18.1	22.6	36.0
2.5	25.3	30.8	39.3	53.4	86.8	188.1	13.6	14.5	15.9	18.2	22.6	36.0
3	25.3	30.8	39.3	53.4	86.9	188.2	13.7	14.5	15.9	18.2	22.6	36.0

Table 1 Total CPU time (seconds) for the Structure-Based-Algorithm.

	S=0.1						S=0.3					
	β											
p	0.7	0.75	0.8	0.85	0.9	0.95	0.7	0.75	0.8	0.85	0.9	0.95
1	2442.4	2882.6	5122.5	8598.5	16233.6	20722.2	57.1	82.5	75.2	99.3	50.6	62.9
1.5	2330.9	3153.4	4860.0	10046.9	18394.7	35391.0	64.3	61.5	77.3	71.7	109.4	104.0
2	2357.2	3368.9	5090.3	9825.6	19034.8	33703.8	51.0	57.2	72.8	72.0	106.3	59.7
2.5	2362.6	3189.2	5104.1	9893.2	17783.5	38038.6	48.8	43.9	55.3	71.8	75.4	99.0
3	2500.7	3225.3	4841.1	9604.9	17933.9	37495.7	47.7	43.7	53.5	71.4	74.4	94.5

Table 2 Total CPU time (seconds) on AMPL and CPLEX for the Cutting-Plane-Algorithm.

	S=0.1						S=0.3					
	β											
p	0.7	0.75	0.8	0.85	0.9	0.95	0.7	0.75	0.8	0.85	0.9	0.95
1	38	38	45	50	54	47	11	11	9	9	6	5
1.5	37	40	44	54	58	58	10	9	9	8	8	6
2	37	41	45	54	59	57	9	9	9	8	8	5
2.5	37	40	45	54	56	60	9	8	8	8	7	6
3	38	40	44	53	56	59	9	8	8	8	7	6

Table 3 Number of iterations in the Cutting-Plane-Algorithm.

to the relatively inefficient of loops in AMPL. (The advantage of using AMPL is the automatic "warm-start" in solving linear optimization models. This means that the optimal solution from the solver is retained as the initial solution the next time the same (or slightly revised) optimization model is solved.) Consequently, we also record the total CPU time used by CPLEX on solving the sequence of linear optimization models for each case, which serves as a lower bound of the computation time of the Cutting-Plane-Algorithm (Table 4 in the Appendix). As we can observe from both Tables 2 and 4, it takes much longer to solve problem with $S=0.1$ than with $S=0.3$. This is because the number of iterations required to reach the accuracy is much higher for the case $S=0.1$ than for $S=0.3$, as depicted in Table 3.

Finally, Figure 3 compares the total CPU times of the Structure-Based-Algorithm with the CPU times spent on CPLEX for solving the linear optimization models in the Cutting-Plane-Algorithm. As we observe from the left side figure, when $S=0.1$, the Structure-Based-Algorithm is much faster, even compared with the only the time spent on solving linear optimization in the Cutting-Plane-Algorithm. (Note the log scales in the y-axis for this case.) When $S = 0.3$, on the other hand, the computation times between the two algorithms are more comparable to each other. This comparison clearly demonstrates the advantage of the Structure-Based-Algorithm over algorithms that do not take advantage of the structure of the optimal solution identified in our paper. We also tested cases with $S = 0$. The differences between the computational time become even more dramatic, with the Structure-Based-Algorithm finishing in minutes while the Cutting-Plane-Algorithm in weeks.

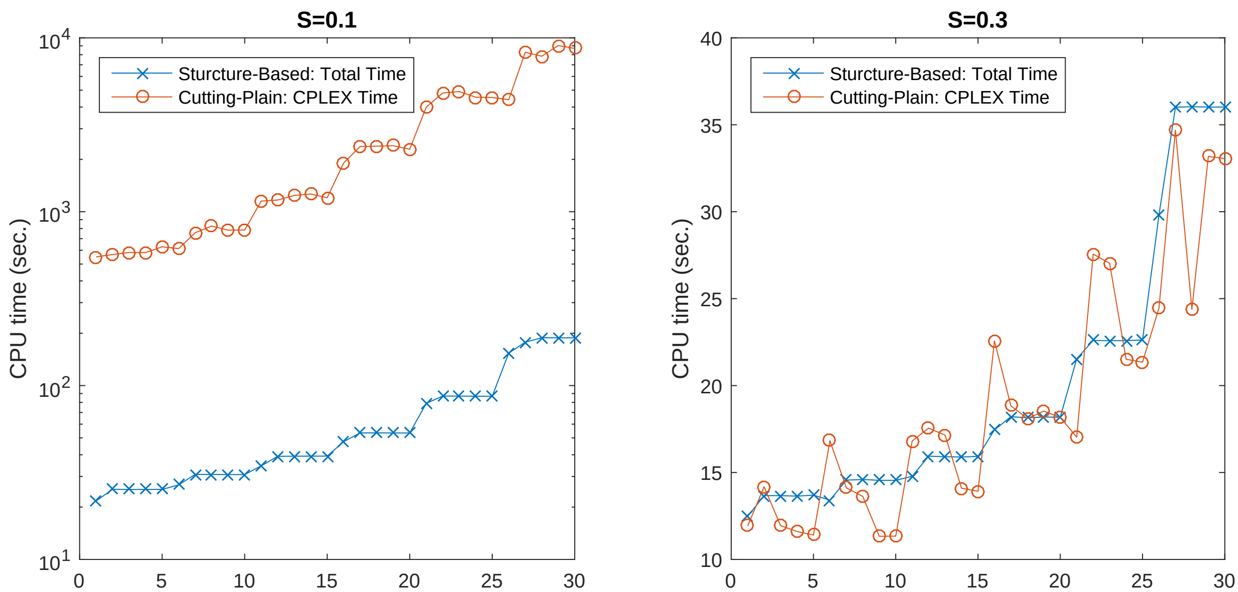

Figure 3 **Comparisons of CPU times between the two algorithms. (The graph for $S = 0.1$ is in log scale.)**

To summarize, the computational study indicates that when the fixed cost S is large, the problem is fairly easy to solve with either the Structure-Based-Algorithm or the Cutting-Plane-Algorithm. The advantage of the Structure-Based-Algorithm over the Cutting-Plane-Algorithm becomes more and more prominent as S becomes smaller.

5. Additional Results and Applicability

To illustrate the applicability of the proposed approach to other settings, we consider an alternative model. (Here we provide a concise exposition of the results and refer the reader to Appendix C for a detailed description of the results and the properties of the underlying dynamic virtual valuations.) In this model the agent perceives the collaboration with the principal myopically. That is, in each period, the agent only aims at maximizing the profit of the current period. Furthermore, the profit from each period is cumulated in a cash account, which is used to fund a fixed consumption

S per period. If the payment in a period exceeds the production cost and the consumption S, the cash account balance increases. Otherwise, the cash account is depleted. We assume that the initial balance of the cash account is common knowledge. According to the revelation principle, the principal always knows the account balance following a direct mechanism. We denote x to represent the account balance. In the beginning of each period, depending on x, the principal announces a mechanism that includes the quantity-payment pair, $q(c)$ and $m(c)$, that satisfy the following incentive compatibility constraint as in a static model.

$$m(c) - cq(c) \geq m(c') - cq(c') \tag{ICs}$$

We also assume that the mechanism has to satisfy the same (IR) constraint, so that the agent cannot be forced into using the cash account to fund the variable production cost. In addition, before knowing each period's marginal production cost c, we assume that the agent is willing to participate only if the expected profit from production exceeds the fixed cost S', that is,

$$\mathbb{E}\left[m(c) - cq(c)\right] \geq S' \ . \tag{ES}$$

When $S' = S$, the expected profit needs to exceed the fixed cost. When $S' = 0$, on the other hand, the constraint is automatically satisfied with (IR), and therefore is removed. More general S' values in (ES) allow for other potential contract requirements.[7]

Thus, given the current period cash account balance x, and the marginal production cost c, the *account balance* in the beginning of the next period becomes

$$x^+(c) = (x + m(c) - cq(c) - S)/\beta \ , \quad \forall c \in [\underline{c}, \bar{c}]. \tag{AB}$$

in which $\beta \in (0,1)$ reflects the interest payment to the cash account. The strategic interaction between the principal and agent continues only if $x^+(c) \geq 0$.

The principal's objective is to maximize the total future payoff with a discount factor β. In a recursive form, denote $J_{\mathbb{C}} : \mathbb{R} \to \mathbb{R}$ to represent the value function of the principal that depends on the cash account balance x. The mechanism design problem can be expressed as the following dynamic optimization model. For $x < 0$, we have $J_{\mathbb{C}}(x) = 0$. For non-negative x, we have

$$\begin{aligned} J_{\mathbb{C}}(x) := \max\{ \ & \max_{q,m} \mathbb{E}\left[pq(c) - m(c) + \beta J_{\mathbb{C}}(x^+(c))\right], \ 0 \ \} \\ & \text{s.t.} \quad (\text{ICs}), (\text{IR}), (\text{ES}) \text{ and } (\text{AB}), \end{aligned} \tag{C}$$

[7] In certain applications such a constrain may not be required. For example, a retiree may be willing to work on a part-time job that does not pay to fully cover consumption. Therefore, we assume the (ES) for a generic parameter S' that represents a required compensation.

where the outer maximization reflects that the principal may terminate the strategic interaction whenever it becomes too costly to provide an expected profit exceeding S. Alternatively, the principal can also terminate the interaction with the agent by not replenishing the cash account so that the fixed consumption S drives the cash account balance to below zero. Therefore, our model studies the optimal way to dynamically adjust the time horizon of the collaboration. Similar to condition (5), here we assume that the required compensation S' does not offset gains of trade,[8] namely

$$S' \leq \mathbb{E}[(p-c)\bar{q}(c)] \ . \tag{15}$$

It follows that the value function of the dynamic optimization problem (C) is equivalent to:

$$\begin{array}{ll} \text{for } x<0, & J_{\mathbb{C}}(x)=0; \\ \text{for } x\geq 0, & J_{\mathbb{C}}(x)=\max_{q,\bar{u}}\ E\left[\{p-c-F(c)/f(c)\}q(c)\right]+\beta E\left[J_{\mathbb{C}}\Big(\big(x+\bar{u}-S+Q(c)\big)/\beta\Big)\right]-\bar{u} \\ & \qquad \text{s.t.} \\ & \qquad\qquad \bar{u}\geq 0;\ q \text{ non-increasing};\ 0\leq q(c)\leq 1;\ \mathbb{E}[Q(c)]\geq S'-\bar{u}\ , \end{array} \tag{16}$$

Despite possible non-concavity induced by the discontinuity at zero, we establish several structural properties of the value function and of the optimal contract. The following theorem summarizes the main results (see Appendix C for more detailed results).

Theorem 3 (Structural Properties, Model (C)) *For Model (C) we have that:*
(i) For $x>x'\geq 0$, we have $0\leq J_{\mathbb{C}}(x)-J_{\mathbb{C}}(x')\leq x-x'$.
(ii) Suppose that $f(c)$ and $F(c)/f(c)$ are non-decreasing in c. Then, for $x\in[0,S)$, there are two thresholds $c_{(1)}\leq c_{(2)}$ and a level $\gamma\in[0,1]$ such that the optimal procured function q^* that solves (16) satisfies

$$q^*(c)=\begin{cases} 1\ ,\ c\in\big[\underline{c},c_{(1)}\big) \\ \gamma\ ,\ c\in\big[c_{(1)},c_{(2)}\big) \\ 0\ ,\ c\in\big[c_{(2)},\bar{c}\big] \end{cases} . \tag{17}$$

Moreover, for $x\in[S,\infty)$, there exists a single threshold $\widehat{c}\in[\underline{c},\bar{c}]$ such that the optimal procurement $q^*(c)=1$ if $c\leq\widehat{c}$ and $q^*(c)=0$ otherwise.
(iii) Suppose that $f(c)$ is non-decreasing in c. Then the optimal solution $(q^*,\bar{u}^*)$ satisfies

$$\bar{u}^*=\begin{cases} \max\big\{0,S'-\mathbb{E}\big[Q^*(c)\big]\big\}\,, & x\geq\min\{S,S-S'+\mathbb{E}[Q^*(c)]\}; \\ S-x \ \textit{or}\ \max\big\{0,S'-\mathbb{E}\big[Q^*(c)\big]\big\}\,, & x<\min\{S,S-S'+\mathbb{E}[Q^*(c)]\}\ . \end{cases}$$

The condition of $f(c)$ and $F(c)/f(c)$ non-decreasing in c allows for a variety of distributions, including, for example, the class of distribution $F(c)=(c-\underline{c})^k/(\bar{c}-\underline{c})^k$ for any $k\geq 1$. (Note that for $k=1$ we have the uniform distribution over $[\underline{c},\bar{c}]$.)

[8] In Section C.1, we show that under this assumption the outer "max" in (C) can be removed.

These structural properties can be used to reduce the search space of optimal mechanism to derive a more efficient computational algorithm. Take, for example, the case of $S' = 0$. When $x \geq S$, we only need to search within a family of mechanisms parameterized by one parameter. We then search through three parameters ($c_{(1)}$, $c_{(2)}$ and γ) only when $x < S$. At each state x, the decision variable $\bar{u}$ can take at most two possible values. This is particularly relevant in this case, because the non-conconcavity of the value function requires proper discretization of state and action spaces. Fortunately, the Lipschitz property of the value function yields sufficient numerical stability for such approaches to be reliable. Figure 4 depicts the value function associated with a particular numerical example. We refer readers to Section C.3 in the appendix for more details of this numerical example.

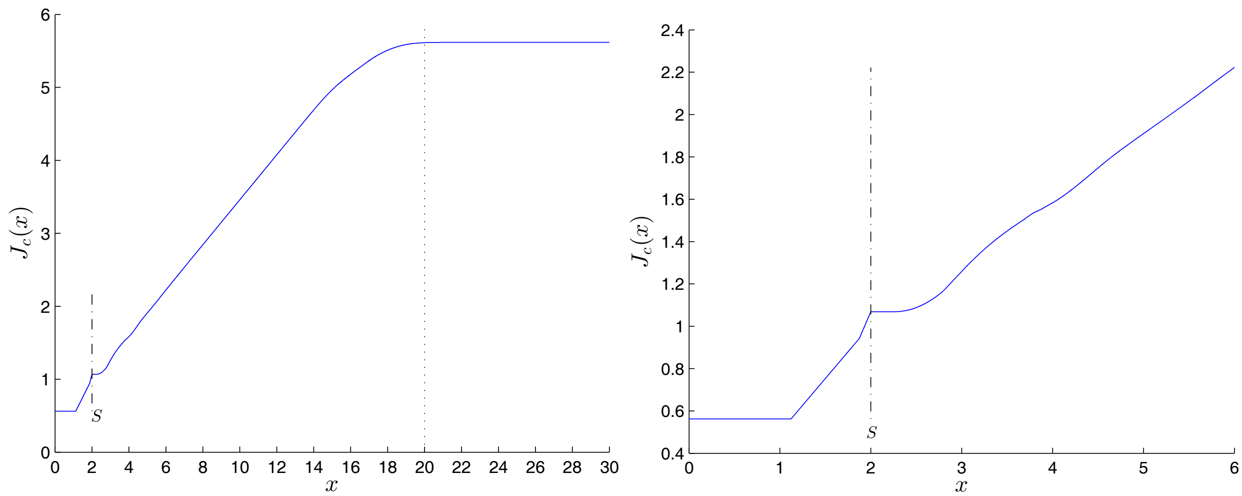

Figure 4 Value function of a cash payment model, with $p = 1.5$, $S = 2$, $\beta = 0.9$, $\underline{c} = 0$, $\bar{c} = 1$, and c uniformly distributed. (The plot on the right is a zoomed in version of the one on the left.)

6. Conclusion

In this paper, we study a basic dynamic mechanism design model, and identify conditions under which the optimal allocation decision takes simple forms. Computation study verifies that our theoretical results are important for efficient computation.

Acknowledgments

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Appendix

A. Proofs for results in Section 3

Proof of Lemma 2

Denote function $u(c) = m(c) - cq(c) + \beta w(c)$. The (IC$_d$) constraint implies that $u(c) = \max_{\widehat{c}} m(\widehat{c}) - cq(\widehat{c}) + \beta w(\widehat{c})$, and therefore it is convex as it is the upper envelope of a linear functions. Moreover, by the envelope theorem, $-q(c)$ is an element of the subgradient of u at c. Convexity of $u(c)$ also implies that $-q(c)$ is non-decreasing. Denote $\bar{u} = m(\bar{c}) - \bar{c}q(\bar{c}) + \beta w(\bar{c})$, we have

$$m(c) - cq(c) + \beta w(c) = u(c) = \bar{u} + \int_c^{\bar{c}} q(\tau) d\tau = \bar{u} + Q(c) \ .$$

(PK) implies $\bar{u} = v + S - \mathbb{E}[Q(c)]$. So

$$m(c) - cq(c) = v + S - \mathbb{E}[Q] + Q(c) - \beta w(c).$$

(IR) then implies that

$$w(c) \leq \big(v + S - \mathbb{E}[Q] + Q(c)\big)/\beta \ .$$

Also, $w(\bar{c}) \geq 0$ implies that

$$0 \leq v + S - \mathbb{E}[Q] + Q(c) \quad \text{for all } c \in [\underline{c}, \bar{c}] \Rightarrow \ \mathbb{E}[Q] \leq v + S \ .$$

Finally we replace the $m(c)$ in the objective function with $cq(c) + v + S - \mathbb{E}[Q] + Q(c) - \beta w(c)$.

■

Proof of Proposition 1

First, argument v does not appear in the objective function in the optimization problem $\Gamma\widehat{V}$ for any function $\widehat{V} : [0, \bar{v}] \to R$. And the constraint set $\Pi(v)$ is such that $\Pi(v) \subseteq \Pi(v')$ for $v \leq v'$. Therefore $(\Gamma\widehat{V})(v) \leq (\Gamma\widehat{V})(v')$ for $v \leq v'$, or, $\Gamma\widehat{V}$ is increasing for any function $\widehat{V}$. In particular, since $V = \Gamma V$, we must have V is increasing.

To show the concavity of V, we consider the value iteration algorithm $\Gamma^k\widehat{V} = \Gamma(\Gamma^{k-1}\widehat{V})$, $k = 1, 2, \ldots$ starting from a concave increasing function $\widehat{V}$. For any $\lambda \in (0,1)$ and $v_i \in [0, \bar{v}]$ for $i = 1, 2$, denote (q_i, w_i) to be the optimal solution in the optimization $(\Gamma\widehat{V})(v_i)$. Obviously, $(\lambda q_1 + (1-\lambda)q_2, \lambda w_1 + (1-\lambda)w_2)$ is feasible to the optimization problem $(\Gamma\widehat{V})\big(\lambda v_1 + (1-\lambda)v_2\big)$, and generates an objective function that is at least as good as $\lambda(\Gamma\widehat{V})(v_1) + (1-\lambda)(\Gamma\widehat{V})(v_2)$ due to concavity of $\widehat{V}$. As a result,

$$(\Gamma\widehat{V})\big(\lambda v_1 + (1-\lambda)v_2\big) \geq \lambda(\Gamma\widehat{V})(v_1) + (1-\lambda)(\Gamma\widehat{V})(v_2) \ .$$

This implies that $V = \lim_{k\to\infty} T^k \widehat{V}$ is concave.

For part (ii), when $v \geq v^*$, the first best production policy $q(c) = \bar{q}(c)$ is a feasible solution together with $w(c) = v^*$, which means the profit in each period for the system is exactly the first best profit. That is, $V(v) = \bar{v}$, and $(\bar{q}(c), v^*)$ is an optimal solution to $(\Gamma V)(v^*)$.

Part (iii) follows from the monotonicity of V.

■

Proof of Proposition 2

For a measurable function $h : [\underline{c}, \bar{c}] \to \mathbb{R}$, let $H(c) = \int_c^{\bar{c}} h(t)dt$. Recall that V is concave so that V' exists almost surely. Therefore, we have

$$\begin{aligned}
&\frac{d}{d\delta} G(q+\delta h)|_{\delta=0} \\
&= \mathbb{E}[(p-c)h(c)] + \beta \frac{d}{d\delta}\int_{\underline{c}}^{\bar{c}} V\Big(\big(v + S - \mathbb{E}[(Q+\delta H)(c)] + Q(c) + \delta H(c)\big)/\beta\Big) f(c)dc \\
&= \mathbb{E}[(p-c)h(c)] + \int_{\underline{c}}^{\bar{c}} (H(c) - \mathbb{E}[H]) V'\Big(\big(v + S - \mathbb{E}[Q(c)] + Q(c)\big)/\beta\Big) f(c)dc \\
&= \mathbb{E}[(p-c)h(c)] + \int_{\underline{c}}^{\bar{c}} \left(\int_c^{\bar{c}} h(\tau)d\tau - \int_{\underline{c}}^{\bar{c}} h(t)F(t)dt\right) V'(g(c)) f(c)dc \\
&= \mathbb{E}[(p-c)h(c)] + \int_{\underline{c}}^{\bar{c}} h(c)\left(\int_{\underline{c}}^{c} V'(g(\tau))f(\tau)d\tau\right) dc - \int_{\underline{c}}^{\bar{c}} h(c)F(c)\left(\int_{\underline{c}}^{\bar{c}} V'(g(\tau))f(\tau)d\tau\right) dc
\end{aligned}$$

where $g(c)$ is defined in (12).

■

Proof of Theorem 1

Recall that $V' \geq 0$, $V'' \leq 0$, $g' \leq 0$. (For simplicity of exposition, we assume these functions are twice continuously differentiable. If otherwise, we can resort to standard approximation arguments and subsequently taking limits as V is concave.) The dynamic virtual valuation is

$$\xi(c) = p - c - \frac{F(c)}{f(c)}\left(\int_{\underline{c}}^{\bar{c}} V'(g(\tau))f(\tau)d\tau\right) + \frac{1}{f(c)}\left(\int_{\underline{c}}^{c} V'(g(\tau))f(\tau)d\tau\right) \tag{18}$$

Define $A(c) := \int_c^{\bar{c}} V'(g(\tau))f(\tau)d\tau \geq 0$, so that $A'(c) = -V'(g(c))f(c)$.

$$\begin{aligned}
\xi(c) &= p - c + \frac{1-F(c)}{f(c)}A(\underline{c}) - \frac{A(c)}{f(c)} = p - c + \frac{1-F(c)}{f(c)}\left(A(\underline{c}) - \frac{A(c)}{1-F(c)}\right) \\
&= p - c - \frac{1-F(c)}{f(c)}\underbrace{\left(\frac{1}{1-F(c)}A(c) - A(\underline{c})\right)}_{B(c)}
\end{aligned}$$

$$B'(c) = \frac{f(c)A(c)}{\big(1-F(c)\big)^2} - \frac{V'(g(c))f(c)}{1-F(c)} = \frac{f(c)}{\big(1-F(c)\big)^2}\underbrace{\big[A(c) - V'(g(c))\big(1-F(c)\big)\big]}_{D(c)}$$

Since $D'(c) = A'(c) - V''(g(c))g'(c)(1-F(c)) + V'(g(c))f(c) = -V''(g(c))g'(c)(1-F(c)) \le 0$ and $D(\bar{c}) = 0$, we have $D(c) \ge 0$, and, therefore, $B'(c) \ge 0$. Further,

$$B(\underline{c}) = \frac{1}{1-F(\underline{c})} A(\underline{c}) - A(\underline{c}) = 0 \ .$$

Therefore $B(c) \ge 0$ for all $c \in [\underline{c}, \bar{c}]$, and

$$\xi'(c) = -1 - \frac{d}{dc}\left(\frac{1-F(c)}{f(c)}\right) B(c) - \frac{1-F(c)}{f(c)} B'(c) < 0,$$

where the last inequality follows from $\frac{1-F(c)}{f(c)}$ being non-decreasing, according to (Suff). Therefore ξ is also decreasing in $[\underline{c}, \bar{c}]$, and hence (1).

Now consider the (2). We show the result under the following sufficient condition

$$f(c) \text{ is non-increasing in } c, \text{ and } \frac{f'(c)F(c)}{f^2(c)} \text{ is non-decreasing in } c \tag{Suff2}$$

which is satisfied by the uniform distribution. We can write

$$\begin{aligned} \xi(c) &= p - c - \frac{F(c)}{f(c)} \left(\int_{\underline{c}}^{\bar{c}} V'(g(\tau)) f(\tau) d\tau - \frac{1}{F(c)} \int_{\underline{c}}^{c} V'(g(\tau)) f(\tau) d\tau \right) \\ &= p - c - \frac{F(c)}{f(c)} \left(A(\underline{c}) - \frac{1}{F(c)} \int_{\underline{c}}^{c} V'(g(\tau)) f(\tau) d\tau \right). \end{aligned}$$

So that

$$\begin{aligned} \xi'(c) &= -1 - \left(\frac{F(c)}{f(c)}\right)' \left(A(\underline{c}) - \frac{1}{F(c)} \int_{\underline{c}}^{c} V'(g(\tau)) f(\tau) d\tau \right) - \frac{F(c)}{f(c)} \left(-\frac{1}{F(c)} A(\Phi(q)) \right)' \\ &= -1 - \left(\frac{F(c)}{f(c)}\right)' \left(A(\underline{c}) - \frac{1}{F(c)} \textstyle\int_{\underline{c}}^{c} V'(g(\tau)) f(\tau) d\tau \right) + \frac{F(c)}{f(c)} \left(-\frac{f(c)}{F^2(c)} A(\underline{c}) + \frac{f(c)}{F(c)} V'(g(c)) \right) \\ &= -1 - \left(\frac{f^2(c) - F(c)f'(c)}{f^2(c)}\right) \left(A(\underline{c}) - \frac{1}{F(c)} \textstyle\int_{\underline{c}}^{c} V'(g(\tau)) f(\tau) d\tau \right) \\ &\quad - \frac{1}{F(c)} \textstyle\int_{\underline{c}}^{c} V'(g(\tau)) f(\tau) d\tau + V'(g(c)) \\ &= -1 - A(\underline{c}) + \frac{F(c)f'(c)}{f^2(c)} \left(A(\underline{c}) - \frac{1}{F(c)} \textstyle\int_{\underline{c}}^{c} V'(g(\tau)) f(\tau) d\tau \right) + V'(g(c)) \end{aligned}$$

Note that because $V' \ge 0$, we have $0 \le A(c) \le A(\underline{c})$.

$$\begin{aligned} \xi''(c) &= \left(\frac{f'(c)F(c)}{f^2(c)}\right)' \underbrace{\left(A(\underline{c}) - \frac{1}{F(c)} \int_{\underline{c}}^{c} V'(g(\tau)) f(\tau) d\tau \right)}_{\bar{B}(c)} \\ &\quad - \frac{f'(c)}{f(c)F(c)} \underbrace{\left(V'(g(c))F(c) - \int_{\underline{c}}^{c} V'(g(\tau)) f(\tau) d\tau \right)}_{\bar{D}(c)} - \frac{q(c)V''(c)}{\beta} \end{aligned}$$

Direct calculations yield $\bar{B}'(c) = -\frac{f(c)}{F^2(c)} \bar{D}(c)$ and $\bar{D}'(c) = -\frac{q(c)V''(c)}{\beta} \ge 0$. Therefore, since $\bar{D}'(\underline{c}) = 0$, we have that $\bar{D}(c) \ge 0$ for all $c \in [\underline{c}, \bar{c}]$. Furthermore, $\bar{B}'(c) \le 0$ and $\bar{B}(\bar{c}) = 0$ implies $\bar{B}(c) \ge 0$. Therefore (Suff2) implies that $\xi(c)$ is convex in $[\underline{c}, \bar{c}]$.

■

Proof of Theorem 2

The (infinite dimensional) optimization problem can be formulated as

$$\begin{array}{lll} \max_q \ G(q,v) & \equiv \max\limits_{q} \min\limits_{\lambda,\eta_m,\eta_1} & \mathcal{L}(\lambda,\eta_m,\eta_1;q) := G(q,v) - \lambda(\int F(c)q(c)dc - v - S) \\ s.t. \ \int F(c)q(c)dc \leq v + S & & + \int \eta_m(c)q(c)dc - \int \eta_1(c)\{q(c)-1\}dc \\ q(c) \leq 1, q \in \mathcal{M}_+ & & s.t. \ \eta_m \in \{\mathcal{M}_+\}^*, \eta_1 \geq 0, \lambda \geq 0 \end{array}$$

where $\mathcal{L}$ is the Lagrangian function

$$\mathcal{L}(\lambda,\eta_m,\eta_1;q) = G(q,v) + \int \{\eta_m(c) - \eta_1(c) - \lambda F(c)\}q(c)dc + \lambda(v+S) + \int \eta_1(c)dc \ ,$$

cone of functions $\mathcal{M}_+ = \{r : [\underline{c},\bar{c}] \to \mathbb{R} : r$ non-increasing and non-negative function$\}$, and $\mathcal{M}^*$ is its dual, $\lambda \geq 0$ is a scalar, and η_1 is a non-negative function. The first-order conditions requires that at an optimal primal-dual pair $(\lambda,\eta_m,\eta_1;q)$, we have that for any (functional) direction $h : [\underline{c},\bar{c}] \to \mathbb{R}$

$$0 = \frac{d}{d\delta}\mathcal{L}(\lambda,\eta_m,\eta_1;q+\delta h)\Big|_{\delta=0} = \int_{\underline{c}}^{\bar{c}} \Big[f(c)\xi(q,c) - \lambda F(c) + \eta_m(c) - \eta_1(c)\Big]h(c)dc$$

where $\xi(q,c)$ is the dynamic virtual valuation. Thus, optimal primal-dual pairs satisfy

$$\eta_1(c) - \eta_m(c) = f(c)\{\xi(q,c) - \lambda F(c)/f(c)\}.$$

The dual cone of non-increasing and non-negative functions is characterized by $(\mathcal{M}_+)^* = \{\eta_m : \int_{\underline{c}}^{c} \eta_m(c)dc \geq 0, \text{ for all } c \in [\underline{c},\bar{c}]\}$, and changes on q occurs at c only if $\int_{\underline{c}}^{c} \eta_m(c)dc = 0$.

Define also $\check{c}_1 := \sup\{c : q(c) = 1\}$. We can assume that $\check{c}_1 < \bar{c}$ and that $\liminf_{c\to\check{c}_1} q(c) > 0$, otherwise the result holds. Complementary slackness implies that $\eta_1(c) = 0$ for $c > \check{c}_1$, and therefore

$$\begin{array}{rll} \eta_1(c) - \eta_m(c) & = f(c)\{\xi(q,c) - \lambda F(c)/f(c)\} & \text{for } c < \check{c}_1 \\ \eta_m(c) & = -f(c)\{\xi(q,c) - \lambda F(c)/f(c)\} & \text{for } c > \check{c}_1 \end{array}$$

First we prove (1). For any choice of q, under (Suff) we have that $f(c)$ and $\{\xi(q,c) - \lambda F(c)/f(c)\}$ are non-increasing functions. (Indeed, by Theorem 1(1) we have $\xi(q,c)$ decreasing and $-\lambda F(c)/f(c)$ is decreasing by (Suff).) Therefore $f(c)\{\xi(q,c) - \lambda F(c)/f(c)\}$ is decreasing and it crosses zero at most once say at $\widehat{c}_1$.

We claim that $\check{c}_1 \geq \widehat{c}_1$. Suppose $\check{c}_1 < \widehat{c}_1$. Since neither monotonicity nor nonnegativity is binding at $\check{c}_1$, we have $\int_{\underline{c}}^{\check{c}_1} \eta_m(c)dc = 0$. Because $f(c)\{\xi(q,c) - \lambda F(c)/f(c)\} > 0$ for any $\check{c}_1 < c < \widehat{c}_1$, we have $\int_{\underline{c}}^{c} \eta_m(t)dt = \int_{\check{c}_1}^{c} \eta_m(t)dt < 0$, violating dual feasibility.

Furthermore, because $f(c)\{\xi(q,c) - \lambda F(c)/f(c)\} < 0$ for all $c > \widehat{c}_1$, the optimal allocation satisfies $q(c) = 0$ for $c > \widehat{c}_1$. (Otherwise setting $h(c) = -1\{c > \widehat{c}_1\}q(c)$ we obtain a direction that yields improvement. This can also be seen by complementary slackness condition.) This implies that $\check{c}_1 = \widehat{c}_1$ and the result follows.

Next we proceed to show (2). We assume that condition (Suff2) holds (which is implied by the uniform distribution). Then $\xi(q,c)$ is convex by Theorem 1(2) (in fact also decreasing on $[\underline{c},\Phi(q)]$). Condition (Suff2)

	S=0.1						S=0.3					
	β											
p	0.7	0.75	0.8	0.85	0.9	0.95	0.7	0.75	0.8	0.85	0.9	0.95
1	550.0	613.3	1151.0	1908.2	3994.4	4409.1	11.9	16.8	16.8	22.6	17.0	24.5
1.5	567.0	756.7	1167.0	2371.1	4814.8	8263.6	14.2	14.1	17.6	18.8	27.6	34.7
2	582.7	826.7	1240.0	2382.0	4910.2	7840.1	11.9	13.6	17.1	18.1	27.0	24.4
2.5	582.0	783.4	1264.3	2396.8	4553.3	9027.1	11.6	11.3	14.1	18.5	21.5	33.2
3	628.5	784.8	1203.2	2289.0	4537.1	8695.2	11.4	11.3	13.9	18.2	21.3	33.1

Table 4 Total CPLEX CPU time (seconds) for the Cutting-Plane-Algorithm. This provides a lower bound for the Cutting-Plane-Algorithm running time.

also implies that $\lambda F(c)/f(c)$ is increasing and concave. Indeed, this is because $f(c)$ being non-increasing implies that $F(c)/f(c)$ is non-decreasing. Further note that

$$\frac{d^2}{dc^2}\frac{F(c)}{f(c)} = \frac{d}{dc}\left\{1 - \frac{f'(c)F(c)}{f^2(c)}\right\} = -\frac{d}{dc}\left\{\frac{f'(c)F(c)}{f^2(c)}\right\} \leq 0,$$

since $f'(c)F(c)/f^2(c)$ is assumed to be non-decreasing in c in condition (Suff2). Combining these observations we have that $\{\xi(q,c) - \lambda F(c)/f(c)\}$ is convex and also non-increasing on $[\underline{c}, \Phi(q)]$. Therefore there are at most two thresholds $\widehat{c}_1, \widehat{c}_2$ with $\underline{c} \leq \widehat{c}_1 \leq \widehat{c}_2 \leq \bar{c}$, such that $\{\xi(q,c) - \lambda F(c)/f(c)\}$ crosses zero.

Following similar argument as before, we know that $\check{c}_1 = \widehat{c}_1$.

Note that $\eta_m(c) < 0$ for $c > \widehat{c}_2$ and $\eta_m(c) < 0$ for $\widehat{c}_1 < c < \widehat{c}_2$. Since $\int_{\underline{c}}^{\check{c}_1} \eta_m(c)dc = 0$, it follows that $\int_{\underline{c}}^{c} \eta_m(t)dt = \int_{\check{c}_1}^{c} \eta_m(t)dt > 0$ for any $\check{c}_1 < c < \bar{c}$. In turn, q can only change value at $\check{c}_1$ and we have $q(c) = \gamma$ for $c > \check{c}_1$.

■

B. Table 4

C. More details and proofs for Section 5

For completeness in this section we provide a detailed analysis of the alternative model (C).

C.1. Optimal Policy for Model (C)

Similar to Lemma 2 for model (P), we have the following revenue equivalence result for model (C).

Lemma 3 *Under assumption (15), the dynamic optimization problem (C) is equivalent to:* $J_{\mathbb{C}}(x) = 0$ *for* $x < 0$*, and for* $x \geq 0$

$$\begin{aligned} J_{\mathbb{C}}(x) = \max_{q,\bar{u}} \; & E\left[\rho(c)q(c)\right] + \beta E\left[J_{\mathbb{C}}\Big(\big(x + \bar{u} - S + Q(c)\big)/\beta\Big)\right] - \bar{u} \\ s.t. \; & \\ & \bar{u} \geq 0;\; q \text{ non-increasing};\; 0 \leq q(c) \leq 1;\; \mathbb{E}[Q(c)] \geq S' - \bar{u}\ , \end{aligned} \tag{19}$$

where $\rho(c) = p - c - F(c)/f(c)$*. Furthermore, the net payment associated with the optimal solution* $(q^*, \bar{u}^*)$ *can be expressed as* $m^*(c) - cq^*(c) = \bar{u}^* + Q^*(c)$.

Proof: Consider the first best allocation $\bar{q}$ and payment $\bar{m}(c) = p\bar{q}(c)$. It is easy to verify that $(\bar{q}, \bar{m})$ satisfies constraints (ICs), (IR) and (ES). Therefore, the inner maximization in (C) must yield a non-negative value. Therefore, the outer maximization in model (C) can be removed under assumption (15). The remaining results in the Lemma follows the same derivation as in the proof of Lemma 2.

■

Here, the decision variable $\bar{u}$ represents the net payment when the agent's marginal production cost is at the highest level $\bar{c}$.

If $S = 0$, the next period cash account balance, $(x + \bar{u} + Q(c))/\beta$, remains non-negative from any non-negative x. As a result, the boundary condition $J_{\mathbb{C}}(x) = 0$ when $x < 0$ never plays a role in the model. Furthermore, if S' is also equal to 0, the (ES) constraint is always satisfied. In this case, one can verify through induction that the optimal value function $J_{\mathbb{C}}$ is a constant with respect to state variable x. Therefore, the model reduces to a standard one period procurement auction problem applied to each period. That is, it is optimal to set $q^*(c) = 1$ when the virtual valuation $\rho(c) = p - c - F(c)/f(c) \geq 0$, and $q^*(c) = 0$ otherwise. The same is true when $S > 0$ and $x \geq \bar{x}$, as long as $S \leq \mathbb{E}[m^*(c) - cq^*(c)]$.

Consider another case, in which the cash account balance is high enough such that $x \geq \bar{x} := S/(1-\beta)$. Constraints (AB) and (IR) imply that the next period cash balance $x^+ \geq x > S$. This, essentially, removes constraints on participation due to fixed costs. Therefore, similar to model (P), in which promised value v is defined on the interval $[0, \bar{v}]$, interesting dynamics occur in model (C) only when x falls in the interval $[0, \bar{x})$, and when there is uncertainty whether the partnership will eventually dissolve $(x < 0)$ or become permanent $x \geq \bar{x}$.[9]

In contrast, when $S > 0$ and $x < \bar{x} = S/(1-\beta)$, the model becomes much harder to analyze. Due to the boundary condition, the value function $J_{\mathbb{C}}(x)$ may not be continuous at $x = 0$. As a result, its integral on the right-hand side of the Bellman equation implies that the $J_{\mathbb{C}}$ function lacks concavity properties that are often handy in maximization problems. In a numerical example that we will show later in Section C.3, it is clear that even for uniformly distributed private information, the $J_{\mathbb{C}}$ function is indeed discontinuous at $x = 0$ and non-concave.

Despite the discontinuity at zero and potential non-concavity, the value function still enjoys some important properties, summarized in the following three lemmas. First, the value function $J_{\mathbb{C}}$ is monotone.

Lemma 4 *Value function $J_{\mathbb{C}}(x)$ is non-decreasing in x.*

[9] Reaching the region of $x \geq \bar{x}$ can be perceived as the agent gaining permanent employment status. To avoid confusion about x approaching infinity after $x > \bar{x}$, we can revise the model to allow the agent to keep the cash account balance at $\bar{x}$ and consume the rest in each period. Such a revision is equivalent to the original model.

Proof: For any non-decreasing function $J(x)$ such that $J(x)=0$ for $x<0$ and $\lim_{x\to\infty} J(x)<+\infty$, denote operator T such that

$$(TJ)(x):=\begin{cases}\max\{\ \max_{q,m:(ICs),(IR),(ES),(AB)} \mathbb{E}\left[pq(c)-m(c)+\beta J(x^+(c))\right],\ 0\}, & x\geq 0,\\ 0, & x<0.\end{cases}$$

Following convergence of dynamic programming algorithms (Proposition 9.16 in Bertsekas and Shreve 1996), we have $J_{\mathbb{C}}=T^{\infty}J$.

Consider any x_1 and x_2 such that $x_1\leq x_2$. If $x_1<0$, we have $0=(TJ)(x_1)\leq(TJ)(x_2)$. If $x_1\geq 0$, denote q_1 and m_1 to represent the optimal solution of the inner maximization problem in $(TJ)(x_1)$, and $x_1^+(c)$ and $x_2^+(c)$ to represent the corresponding account balance under mechanism q_1 and m_1, following initial account balances x_1 and x_2, respectively. Following (AB), $x_1^+(c)\leq x_2^+(c)$, which implies that $J(x_1^+(c))\leq J(x_2^+(c))$. Furthermore, q_1 and m_1 are feasible to the inner optimization in $(TJ)(x_2)$. Therefore $(TJ)(x_1)\leq(TJ)(x_2)$.

The monotonicity of TJ implies monotonicity of $J_{\mathbb{C}}$.

■

Echoing discussions above, when the cash balance x is large enough, the optimal value function $J_{\mathbb{C}}(x)$ is a constant, which is realized by implementing the optimal static mechanism repeatedly. In particular, if $S'=0$, $J_{\mathbb{C}}(x)=J_S/(1-\beta)$ for all $x\geq\bar{x}$.

Lemma 5 *For any $x\geq S/(1-\beta)$,*

$$J_{\mathbb{C}}(x)=\frac{1}{1-\beta}\max\ \mathbb{E}[pq(c)-m(c)]\ \text{s.t.}\ (ICs),(IR),(ES);$$

Proof: For any $x\geq 0$, $J_{\mathbb{C}}(x)\leq\frac{1}{1-\beta}\max\ \mathbb{E}[pq(c)-m(c)]$ s.t. $(ICs),(IR),(ES)$ since the value function will be no less than itself with one constraint removed. To establish the reverse inequality for any $x\geq S/(1-\beta)$, note that for m,q satisfying (IR),

$$x^+(c)\geq\frac{1}{\beta}\left(\frac{S}{1-\beta}-S+m(c)-cq(c)\right)=\frac{S}{1-\beta}+\frac{1}{\beta}(m(c)-cq(c))\geq\frac{S}{1-\beta}.$$

Therefore,

$$J_{\mathbb{C}}(x)\geq J_{\mathbb{C}}\left(\frac{S}{1-\beta}\right)=\max\{\ \max_{q,m}\mathbb{E}\left[pq(c)-m(c)+\beta J_{\mathbb{C}}(x^+(c))\right],\ 0\ \}\ \text{s.t.}\ \ (\text{ICs}),\ (\text{IR}),\ (\text{ES})\text{ and }(\text{AB}),$$

$$\geq(\max\ \mathbb{E}[pq(c)-m(c)]\ \text{s.t.}\ (\text{ICs}),\ (\text{IR}),\ (\text{ES}))+\beta J_{\mathbb{C}}\left(\frac{S}{1-\beta}\right),$$

which implies

$$J_{\mathbb{C}}(x)\geq J_{\mathbb{C}}\left(\frac{S}{1-\beta}\right)\geq\frac{1}{1-\beta}\max\ \mathbb{E}[pq(c)-m(c)]\ \text{s.t.}\ (ICs),(IR),(ES).$$

■

Finally, the value function $J_{\mathbb{C}}$ is also a Lipschitz function with constant 1 when x is positive, as summarized in the following result, which establishes Theorem 3(i).

Lemma 6 *For any $x > x' \geq 0$, we have* $\frac{J_{\mathbb{C}}(x) - J_{\mathbb{C}}(x')}{x - x'} \leq 1$.

Proof: Monotonicity of $J_{\mathbb{C}}$ implies the first inequality. Denote $(q^*, \bar{u}^*)$ to be an optimal solutions of $J_{\mathbb{C}}(x)$. Clearly $(q^*, \bar{u}^* + x - x')$ is a feasible solution to $J_{\mathbb{C}}(x')$ given that $x - x' > 0$.

$$J_{\mathbb{C}}(x) - J_{\mathbb{C}}(x') \leq E[p^*(c)q(c)] + \beta E\left[J_{\mathbb{C}}\Big(\big(x + \bar{u}^* - S + Q^*(c)\big)/\beta\Big)\right] - \bar{u}^*$$

$$-E[p^*(c)q(c)] - \beta E\left[J_{\mathbb{C}}\Big(\big(x' + \bar{u}^* + x - x' - S + Q^*(c)\big)\Big)\right] - \bar{u}^* + x - x' = x - x' \ .$$

■

In what follows it is convenient to represent the value function as the sum of a step function and a Lipschitz function: $J_{\mathbb{C}}(x) = \tilde{J}_{\mathbb{C}}(x) + J_{\mathbb{C}}(0)\mathbf{1}_{\{x \geq 0\}}$. When writing derivative $J'_{\mathbb{C}}$ we refer to $\tilde{J}_{\mathbb{C}}{}'$, which is well defined almost everywhere.

C.2. Dynamic Virtual Valuation and Optimal Procurement Structure

Similar to the analysis in the previous section, we focus on deriving the function derivative of the objective function in the maximization problem (19). Specifically, the maximization on the right-hand side of the Bellman equation (19) can be expressed as $\max_{q,\bar{u}} G_{\mathbb{C}}(q, u)$ subject to corresponding constraints on $\bar{u}$ and q, where

$$G_{\mathbb{C}}(q, \bar{u}) = E\left[\rho(c)q(c)\right] + \beta E\left[\tilde{J}_{\mathbb{C}}\Big(\big(x + \bar{u} - S + Q(c)\big)/\beta\Big)\right] + \beta J_{\mathbb{C}}(0)\mathbb{E}\left[\mathbf{1}_{\{x + \bar{u} - S + Q(c) \geq 0\}}\right] - \bar{u} \ .$$

Here we define the following threshold $\Psi(q)$ on the marginal production cost c, above which the next period's cash account would be negative, and therefore the collaboration between the principal and agent terminates.

$$\Psi(q) = \max\{c : x + \bar{u} - S + Q(c) \geq 0, c \in [\underline{c}, \bar{c}]\} \ . \tag{20}$$

It follows that if $\Psi(q) \in (\underline{c}, \bar{c})$, we have $x + \bar{u} - S + Q(\Psi(q)) = 0$.

Proposition 3 *For any measurable function $h : [\underline{c}, \bar{c}] \to \mathbb{R}$, we have*

$$\frac{d}{d\delta} G_{\mathbb{C}}(q + \delta h, \bar{u})\Big|_{\delta = 0} = \mathbb{E}\left[\xi_{\mathbb{C}}(c)h(c)\right]$$

where the dynamic virtual valuation $\xi_{\mathbb{C}}(c)$ *for model (C) is defined, for any feasible q function, as*

$$\xi_{\mathbb{C}}(c) = \rho(c) + \frac{1}{f(c)} \int_{\underline{c}}^{\min\{c, \Psi(q)\}} J'_{\mathbb{C}}\Big(\big(x + \bar{u} - S + Q(t)\big)/\beta\Big) f(t) dt + \frac{\beta J_{\mathbb{C}}(0) f(\Psi(q))}{q(\Psi(q)) f(c)} \mathbf{1}_{\{c \geq \Psi(q)\}} \ . \tag{21}$$

Proof: Express $G_{\mathbb{C}}(q, \bar{u})$ as

$$G_{\mathbb{C}}(q, \bar{u}) = \mathbb{E}[\rho(c)q(c)] + \beta \int_{\underline{c}}^{\Psi(q)} \tilde{J}_{\mathbb{C}}\Big(\big(x + \bar{u} - S + Q(c)\big)/\beta\Big) f(c) dc + \beta J_{\mathbb{C}}(0) \int_{\underline{c}}^{\Psi(q)} f(c) dc - \bar{u} \ .$$

Similar to the definition of $Q(c)$, we define $H(c)=\int_c^{\bar{c}} h(t)dt$. Following the Newton-Leibniz rule for continuous function $\tilde{J}_{\mathbb{C}}\big(x+\bar{u}-S+Q(c)\big)$, we have the following.

$$\frac{d}{d\delta}G_{\mathbb{C}}(q+\delta h,\bar{u})\Big|_{\delta=0}=\mathbb{E}[p(c)h(c)]+\int_{\underline{c}}^{\Psi(q)}\tilde{J}_{\mathbb{C}}{}'\Big(\big(x+\bar{u}-S+Q(c)\big)/\beta\Big)H(c)f(c)dc$$
$$+\beta\,\frac{d}{d\delta}\Psi(q+\delta h)\Big|_{\delta=0}\tilde{J}_{\mathbb{C}}\Big(\big(x+\bar{u}-S+Q(\Psi(q))\big)/\beta\Big)f(\Psi(q))+\beta\,\frac{d}{d\delta}\Psi(q+\delta h)\Big|_{\delta=0}J_{\mathbb{C}}\big(0\big)f(\Psi(q))$$

When $x+\bar{u}-S+Q(\Psi(q))=0$ we have $\tilde{J}_{\mathbb{C}}\Big(\big(x+\bar{u}-S+Q(\Psi(q))\big)/\beta\Big)=0$. Further,

$$\begin{aligned}\int_{\underline{c}}^{\Psi(q)}\tilde{J}_{\mathbb{C}}{}'\Big(\big(x+\bar{u}-S+Q(c)\big)/\beta\Big)H(c)f(c)dc &= \int_{\underline{c}}^{\Psi(q)}\tilde{J}_{\mathbb{C}}{}'\Big(\big(x+\bar{u}-S+Q(c)\big)/\beta\Big)\int_c^{\bar{c}}h(t)dtf(c)dc\\ &= \int_{\underline{c}}^{\Psi(q)}h(t)\int_{\underline{c}}^{t}\tilde{J}_{\mathbb{C}}{}'\Big(\big(x+\bar{u}-S+Q(c)\big)/\beta\Big)f(c)dcdt\\ &\quad+\int_{\Psi(q)}^{\bar{c}}h(t)\int_{\underline{c}}^{\Psi(q)}\tilde{J}_{\mathbb{C}}{}'\Big(\big(x+\bar{u}-S+Q(c)\big)/\beta\Big)f(c)dcdt\\ &= \mathbb{E}\left[\tfrac{h(c)}{f(c)}\int_{\underline{c}}^{\min\{c,\Psi(q)\}}\tilde{J}_{\mathbb{C}}{}'\Big(\big(x+\bar{u}-S+Q(c)\big)/\beta\Big)f(t)dt\right]\ ,\end{aligned}$$

By definition $x+\bar{u}-S+Q(\Psi(q+\delta h))+\delta H(\Psi(q+\delta H))=0$, and we have

$$\frac{d}{d\delta}\Psi(q+\delta h)\Big|_{\delta=0}Q'(\Psi(q))+H(\Psi(q))=0\ .$$

So

$$\tfrac{d}{d\delta}\Psi(q+\delta h)\big|_{\delta=0}=\tfrac{1}{-Q'(\Psi(q))}H(\Psi(q))=\tfrac{1}{q(\Psi(q))}\int_{\Psi(q)}^{\bar{c}}h(c)dc=\mathbb{E}\left[\tfrac{1}{q(\Psi(q))}h(c)\tfrac{1}{f(c)}\mathbf{1}_{\{c\ge\Psi(q)\}}\right]$$

Finally, we replaced $\tilde{J}_{\mathbb{C}}{}'$ with $J'_{\mathbb{C}}$ since $\tilde{J}_{\mathbb{C}}{}'(c)=J'_{\mathbb{C}}(c)$ when $c\ge 0$.

■

We note that the term $\int_{\underline{c}}^{c}J'_{\mathbb{C}}\big((x+\bar{u}-S+Q(t))/\beta\big)f(t)dt$ is a well behaved function since $J_{\mathbb{C}}$ is monotonic and a Lipzchitz function with constant 1, according to Lemma 6. Compared with the dynamic virtual valuation ξ for model (P), which is continuous, here $\xi_{\mathbb{C}}$ may have a "vertical jump" at $\Psi(q)$ if $\Psi(q)$ is in the interior of the support $[\underline{c},\bar{c}]$.

Remark C.1 (Dynamics Reduce Inefficiencies) *We note that $J_{\mathbb{C}}\ge 0$ and its derivative $J'_{\mathbb{C}}\ge 0$, for any density function f. Therefore, we have $\xi_{\mathbb{C}}(c)\ge\rho(c)$ for all $c\in[\underline{c},\bar{c}]$, following (21). In turn, this shows that, under the dynamic model, the optimal procurement quantity is larger than under the static case for any given realized private information. This is a consequence of the incentives from potential future gains. Therefore, the dynamic aspect of the model reduces inefficiencies due to private information compared with the static model.*

Parallel to Theorem 1, we provide a sufficient condition on the probability distribution that yields certain structures of the dynamic virtual valuation.

Theorem 4 (Properties of Dynamic Virtual Valuation, Model (C)) *Consider the following condition on the probability distribution.*

$$\textit{Both } f(c) \textit{ and } \frac{F(c)}{f(c)} \textit{ are non-decreasing in } c\ . \tag{22}$$

For any $x\in[0,S)$, feasible allocation function q, and $\lambda\in[0,1]$, $\xi_{\mathbb{C}}(c)+\lambda F(c)/f(c)$ is non-increasing on $[\underline{c},\Psi(q))$ and $[\Psi(q),\bar{c}]$.

Proof: Denote $g(c;x,J_{\mathbb{C}}) = \int_{\underline{c}}^{c} J'_{\mathbb{C}}\big((x+\bar{u}-S+Q(t))/\beta\big)f(t)dt \geq 0$ and recall that $\Psi(q) = \max\{c : x+\bar{u}-S+Q(c)\geq 0, c\in[\underline{c},\bar{c}]\}$. For $c<\Psi(q)$, we have

$$\frac{d}{dc}\left(\xi_{\mathbb{C}}(c)+\lambda\frac{F(c)}{f(c)}\right) = \underbrace{J'_{\mathbb{C}}\big((x+\bar{u}-S+Q(c))/\beta\big)}_{\leq 1} -1-(1-\lambda)\underbrace{\frac{d}{dc}\left(\frac{F(c)}{f(c)}\right)}_{\geq 0} - \underbrace{\frac{f'(c)g(c;x,J_{\mathbb{C}})}{f(c)^2}}_{\geq 0} \leq 0$$

where $J'_{\mathbb{C}}\big((x+\bar{u}-S+Q(c))/\beta\big)\leq 1$ follows Lemma 6, and $\frac{d}{dc}\left(\frac{F(c)}{f(c)}\right)\geq 0$ and $f'(c)\geq 0$ following (22).

For $c\geq\Psi(q)$, following similar arguments we have

$$\frac{d}{dc}\left(\xi_{\mathbb{C}}(c)+\lambda\frac{F(c)}{f(c)}\right) = -1-(1-\lambda)\left(1-\frac{f'(c)F(c)}{f(c)^2}\right) - \frac{\beta f(\Psi(q))J_{\mathbb{C}}(0)f'(c)}{q(\Psi(q))f(c)^2} < 0\ .$$

■

Theorem 4 provides conditions under which the dynamic virtual valuation is well behaved. In particular, the class of distribution $F(c) = (c-\underline{c})^k/(\bar{c}-\underline{c})^k$ for any $k\geq 1$ satisfies conditions (22) (Note that for $k=1$ we have the uniform distribution over $[\underline{c},\bar{c}]$.)

Theorem 4 immediately implies the following structure, which is used directly in the proof of Theorem 3.

Corollary 1 *Under condition (22), for any $x\in[0,S)$, feasible q and $\lambda\in[0,1]$, there exist two thresholds $c_{(1)}\in[\underline{c},\Psi(q)]$ and $c_{(2)}\in[\Psi(q),\bar{c}]$ such that*

i. $\xi_{\mathbb{C}}(c)f(c)+\lambda F(c)>0$ *for* $c\in\big(\underline{c},c_{(1)}\big)$*;*

ii. $\xi_{\mathbb{C}}(c)f(c)+\lambda F(c)<0$ *for* $c\in\big(c_{(1)},\Psi(q)\big)$*;*

iii. $\xi_{\mathbb{C}}(c)f(c)+\lambda F(c)>0$ *for* $c\in\big(\Psi(q),c_{(2)}\big)$*;*

iv. $\xi_{\mathbb{C}}(c)f(c)+\lambda F(c)<0$ *for* $c\in\big(c_{(2)},\bar{c}\big)$*.*

Now we are ready to prove Theorem 3(ii).

Proof: [Theorem 3(ii)] The (infinite dimensional) optimization problem can be formulated as

$$\begin{array}{lll} \max_{q,\bar{u}} & G_{\mathbb{C}}(q,\bar{u}) & \equiv \max\limits_{q,\bar{u}}\ \min\limits_{\lambda_S,\lambda_u,\eta_m,\eta_1}\ \mathcal{L}(\lambda_S,\lambda_u,\eta_m,\eta_1;q,\bar{u}) \\ s.t. & \int F(c)q(c)dc \geq S'-\bar{u} & \qquad s.t.\ \eta_m\in(\mathcal{M}_+)^*, \eta_1\geq 0, \lambda_u\geq 0, \lambda_S\geq 0, \\ & q(c)\leq 1, q\in\mathcal{M}_+, \bar{u}\geq 0 & \end{array}$$

where $\mathcal{L}$ is the Lagrangian function

$$\mathcal{L}(\lambda_S,\lambda_u,\eta_m,\eta_1;q,\bar{u}) := G_{\mathbb{C}}(q,\bar{u}) + \int\{\lambda_S F(c)-\eta_1(c)+\eta_m(c)\}q(c)dc + \int\eta_1(c)dc + (\lambda_u+\lambda_S)\bar{u} - \lambda_S S'\ ,$$

set $\mathcal{M}_+ = \{r:[\underline{c},\bar{c}]\to\mathbb{R} : r \text{ non-increasing, non-negative function}\}$ is the cone of non-increasing non-negative functions, and $(\mathcal{M}_+)^*$ is its dual, λ_S,λ_u are scalars, and η_1 is a non-negative functions. The first-order conditions require that at an optimal primal-dual pair $(\lambda_S,\lambda_u,\eta_m,\eta_1;q,\bar{u})$, we have that for any (functional) direction $h:[\underline{c},\bar{c}]\to\mathbb{R}$

$$0 = \frac{d}{d\delta}\mathcal{L}(\lambda_S,\lambda_u,\eta_m,\eta_1;q+h,\bar{u})\Big|_{\delta=0} = \int_{\underline{c}}^{\bar{c}}\Big[f(c)\xi_{\mathbb{C}}(c)+\lambda_S F(c)-\eta_1(c)+\eta_m(c)\Big]h(c)dc$$

where $\xi_{\mathbb{C}}(c)$ is the dynamic virtual valuation. Thus, optimal primal-dual pairs satisfy (a.s. in $c \in [\underline{c}, \bar{c}]$)

$$\eta_1(c) - \eta_m(c) = f(c)\xi_{\mathbb{C}}(c) + \lambda_S F(c).$$

The dual cone of non-increasing and non-negative functions is characterized by $(\mathcal{M}_+)^* = \{\eta_m : \int_{\underline{c}}^{c} \eta_m(c)dc \geq 0, \text{ for all } c \in [\underline{c}, \bar{c}]\}$, and changes in q occurs at c only if $\int_{\underline{c}}^{c} \eta_m(c)dc = 0$.

Define also $\check{c}_1 := \sup\{c : q(c) = 1\}$. We can assume that $\check{c}_1 < \bar{c}$ and that $\liminf_{c \to \check{c}_1} q(c) > 0$, otherwise the result holds. Complementary slackness implies that $\eta_1(c) = 0$ for $c > \check{c}_1$. Therefore we have

$$\begin{aligned} \eta_1(c) - \eta_m(c) &= \quad f(c)\xi_{\mathbb{C}}(c) + \lambda_S F(c) \quad \text{for } c < \check{c}_1, \\ \eta_m(c) &= -\{f(c)\xi_{\mathbb{C}}(c) + \lambda_S F(c)\} \quad \text{for } c > \check{c}_1. \end{aligned}$$

Moreover, the optimality condition for $\bar{u}$ yields

$$0 = \frac{d}{d\bar{u}}\mathcal{L}(\lambda_S, \lambda_u, \eta_m, \eta_1; q, \bar{u}) = -1 + \lambda_S + \lambda_u + \int \tilde{J}_c'\left(\frac{x + \bar{u} - S + Q(c)}{\beta}\right) f(c)dc + \beta J_c(0)\frac{f(\Psi(q))}{q(\Psi(q))}$$

which implies that $\lambda_S \leq 1$. Following Corollary 1 with $\lambda = \lambda_S \leq 1$, there are at most three values $c_{(1)}, \Psi(q), \widehat{c}_2$ with $\underline{c} \leq c_{(1)} \leq \Psi(q) \leq c_{(2)} \leq \bar{c}$, such that function $\{\xi_{\mathbb{C}}(c)f(c) + \lambda_S F(c)\}$ crosses zero. We argue that $\check{c}_1 \geq c_{(1)}$. Suppose otherwise. Since neither monotonicity nor nonnegativity is binding at $\check{c}_1$, we have $\int_{\underline{c}}^{\check{c}_1} \eta_m(c)dc = 0$. Because $f(c)\xi_{\mathbb{C}}(c) + \lambda_S F(c) > 0$ for $\check{c}_1 < c < c_{(1)}$ by Corollary 1(i), which implies $\int_{\underline{c}}^{c} \eta_m(t)dt = \int_{\check{c}_1}^{c} \eta_m(t)dt < 0$, violating dual feasibility.

Because $\int_{\underline{c}}^{\check{c}_1} \eta_m(c)dc = 0$, for any $\check{c}_1 < c < \bar{c}$ we have $\int_{\underline{c}}^{c} \eta_m(t)dt = \int_{\check{c}_1}^{c} \eta_m(c)dc \geq 0$. Note that $\eta_m(c) > 0$ for $c_{(1)} \leq \check{c}_1 < c < \Psi(q)$ by Corollary 1(ii), while $\eta_m(c) < 0$ for $\Psi(q) < c < c_{(2)}$ by Corollary 1(iii), and $\eta_m(c) > 0$ for $c_{(2)} < c < \bar{c}$ by Corollary 1(iv). Therefore, besides $c = \check{c}_1$, $\int_{\check{c}_1}^{c} \eta_m(c)dc = 0$ can only occur at a single value $\check{c}_2 \in (\Psi(q), c_{(2)}]$, and at a single value $\check{c}_3 \in [c_{(2)}, \bar{c}]$. However, dual feasibility requires that $\int_{\check{c}_2}^{c_{(2)}} \eta_m(c)dc \geq 0$, and $\eta_m(c) < 0$ for $c \in (\check{c}_2, c_{(2)})$, it follows that $\check{c}_2 = c_{(2)} = \check{c}_3$.

Finally, for any given $x \in [S, \infty)$, we have $\Psi(q) = \bar{c}$, and therefore $\xi_{\mathbb{C}}(c)$ is non-increasing on $[\underline{c}, \bar{c}]$, which implies the result.

■

Note that Theorems 4 and 3(ii) continue to hold even if we remove the (ES) constraint from model (C). The only change in the results is to set $\lambda = 0$.

Theorem 3(ii) implies a indirect mechanism in which the agent does not have to report the exact production cost in each period in the model (C) setting. A single threshold corresponds to a take-it-or-leave it offer. In the two threshold case, the principal offers price $c_{(2)}$ to procure up to $\gamma(x)$ units, and price $c_{(1)}$ for the remaining $1 - \gamma(x)$. Note that in the model presented in the previous section, we have to implement the direct mechanism because the current period payment is to reimburse the production cost, which depends on a specific report of the marginal production cost.

To complete the description of the optimal mechanism, the following theorem studies optimal choices for $\bar{u}$, and encompasses Theorem 3(iii). In contrast to the static case where the optimal $\bar{u}$ is set to zero, in the dynamic setting, $\bar{u}$, the net payment to the highest cost type, may be positive for certain x values to avoid termination of the game for future trading opportunities. And this result simplifies the search for the optimal $\bar{u}$ in computation.

Theorem 5 (Optimal $\bar{u}$, Model (C)) *Assume that $f(c)$ is non-decreasing. At state x, the optimal solution $(q^*,\bar{u}^*)$ satisfies*

$$\bar{u}^* = \begin{cases} \max\left\{0, S' - \mathbb{E}\left[Q^*(c)\right]\right\}, & x \geq \min\{S, S - S' + \mathbb{E}[Q^*(c)]\}; \\ S - x \ \ or \ \max\left\{0, S' - \mathbb{E}\left[Q^*(c)\right]\right\}, & x < \min\{S, S - S' + \mathbb{E}[Q^*(c)]\} \ . \end{cases}$$

Furthermore, when $S' = 0$, there exists a threshold $\widehat{S} \in [0, S)$ such that the optimal net payment to the highest cost type, $\bar{u}^*$, has the following structure

$$\bar{u}^* = \begin{cases} 0, & x \geq S \\ S - x, & \widehat{S} \leq x < S \\ 0, & x < \widehat{S} \end{cases}$$

and the two-threshold structure described in Theorem 3(ii) may only occur when $x < \widehat{S}$.

Proof: Step 1. (Main Step) Define $y = x + \bar{u}$ and $\underline{y}(q,x) = x + \max\{0, S' - \mathbb{E}[Q(c)]\}$ for any feasible q. With a slight abuse of notation, we have

$$G_{\mathbb{C}}(q,y,x) = \mathbb{E}\left[\rho(c)q(c)\right] + \beta\mathbb{E}\left[\tilde{J}_{\mathbb{C}}\Big(\big(y - S + Q(c)\big)/\beta\Big)\right] + \beta J_{\mathbb{C}}(0)\mathbb{E}\left[\mathbf{1}_{\{y-S+Q(c)\geq 0\}}\right] - y + x \ .$$

so that our problem can be expressed as

$$\max_{q:0\leq q(c)\leq 1, q\searrow} \ \max_{y:y\geq \underline{y}(q,x)} G_{\mathbb{C}}(q,y,x) \ . \tag{23}$$

Fix an arbitrary x and a feasible allocation function q. We first consider the case that $\underline{y}(q,x) > S$ which implies $\mathbf{1}_{\{y-S+Q(c)\geq 0\}} = 1$ for all feasible y. In this case,

$$\frac{\partial G_{\mathbb{C}}}{\partial y} = \int_{\underline{c}}^{\bar{c}} \tilde{J}_{\mathbb{C}}{}'\big((y - S + Q(c))/\beta\big) f(c) dc - 1 \ .$$

Following $0 \leq \tilde{J}_{\mathbb{C}}{}' \leq 1$ from Lemmas 4 and 6, we have

$$-1 \leq \frac{\partial G_{\mathbb{C}}}{\partial y} \leq 0 \ . \tag{24}$$

Therefore it is optimal to have y^* at the lower bound $\underline{y}(q,x)$, so that $\bar{u} = \max\{0, S - \mathbb{E}[Q(c)]\}$.

Next we consider the case that $\underline{y}(q,x) \leq S$, or, $x \leq \bar{x}(q) := \min\{S, S - S' + \mathbb{E}[Q(c)]\}$. By **Step 2** below we have that $G_{\mathbb{C}}$ is convex in $y \in [\underline{y}(q,x), S]$. In turn, maximization of a convex function over $y \in [\underline{y}(q,x), S]$ is achieved at the boundary values. Those corresponds to either $\bar{u} = \max\{0, S' - \mathbb{E}[Q(c)]\}$ or $\bar{u} = S - x$.

For the case of $S' = 0$, denote

$$G^q_{\mathbb{C}}(x; \bar{u} = 0) = \max_{q:0\leq q\leq 1; q\searrow} \mathbb{E}[\rho(c)q(c)] + \beta\mathbb{E}\left[J_{\mathbb{C}}\Big(\big(x - S + Q(c)\big)/\beta\Big)\right] \ ,$$

$$G^q_{\mathbb{C}}(x; \bar{u} = x - S) = \max_{q:0\leq q\leq 1; q\searrow} \mathbb{E}[\rho(c)q(c)] + \beta\mathbb{E}\left[J_{\mathbb{C}}\big(Q(c)/\beta\big)\right] + x - S \ .$$

It follows that the function $G^q_{\mathbb{C}}(x;\bar{u}=x-S)$ is linear in x. As an upper envelope of convex functions, function $G^q_{\mathbb{C}}(x;\bar{u}=0)$ is still convex in $x\in[0,S]$. As a result, functions $G^q_{\mathbb{C}}(x;\bar{u}=0)$ and $G^q_{\mathbb{C}}(x;\bar{u}=x-S)$, as functions of x, intersect at most in two values of x. Obviously one of them is $x=S$. The other potential intersect, less than S, is denoted as $\widehat{S}$. When $x\in(\widehat{S},S)$, clearly, $G^q_{\mathbb{C}}(x;\bar{u}=x-S)>G^q_{\mathbb{C}}(x;\bar{u}=0)$. Finally, when $\bar{u}=S-x$, $\Psi(q)=\bar{c}$. Therefore there is no "vertical jump" in $\xi_{\mathbb{C}}$, or two-threshold structure in q.

Step 2. (Auxiliary Calculations) In this step we show that $G_{\mathbb{C}}(q,y,x)$ is convex in $y\in[\underline{y}(q,x),S]$. We have

$$\frac{\partial G_{\mathbb{C}}}{\partial y}=\int_{\underline{c}}^{\Psi}\tilde{J}_{\mathbb{C}}{}'\Big(\big(y-S+Q(c)\big)/\beta\Big)f(c)dc+\beta J_{\mathbb{C}}(0)\frac{\partial\Psi}{\partial y}f(\Psi)-1 \tag{25}$$

and we will show (25) is non-decreasing in y.

Consider the second term of the right hand side of (25). By the definition, Ψ satisfies $y-S+Q(\Psi)=0$. Therefore we have $1+Q'(\Psi)\frac{\partial\Psi}{\partial y}=0$ so that

$$\frac{\partial\Psi}{\partial y}=\frac{1}{q(\Psi)}>0 \tag{26}$$

which establishes that Ψ is increasing in y. Therefore, since q is non-increasing itself, $\frac{\partial\Psi}{\partial y}$ is non-decreasing in y. Furthermore, $f(\Psi)$ is also non-decreasing in y as Ψ is non-decreasing by (26) and f is non-decreasing by **C1**.

Next we turn to the first term of the right hand side of (25). First note that for any $t\in[0,\big(y-S+Q(\underline{c})\big)/\beta]$, denote $\Phi(t)$ so solve $y-S+Q(\Phi(t))=\beta t$ which implies that $\frac{\partial\Phi(t)}{\partial y}=\frac{1}{q(\Phi(t))}>0$. By a change of variables we can write

$$\int_{\underline{c}}^{\Psi}\tilde{J}_{\mathbb{C}}{}'\Big(\big(y-S+Q(c)\big)/\beta\Big)f(c)dc=\beta\int_0^{\frac{y-S+Q(\underline{c})}{\beta}}\tilde{J}_{\mathbb{C}}{}'(t)\frac{f(\Phi(t))}{q(\Phi(t))}dt.$$

Since Φ is non-decreasing in y and q is non-increasing, the relation above is non-decreasing in y as $\tilde{J}_{\mathbb{C}}{}'\geq 0$ and f is non-decreasing by **C1**. Thus the first term of the right hand side of (25) is also increasing in y.

■

The implication of Theorem 5 is that when the cash balance is higher than S, the agent with the highest marginal production cost receives non-negative net payment, $m(\bar{c})-\bar{c}q(\bar{c})=\bar{u}\geq 0$. Better (lower) cost induces better payment. Therefore, in this case, the game continues regardless of the cost level c because the agent is able to afford the participation cost S funded by the cash account.

When cash balance x becomes less than S, the highest marginal production cost induces two possible net payment. In the first case, a net payment of $S-x$ guarantees that the worst type agent has S in the account, and therefore is able to continue working with the principal into the next period. This is an interesting possibility that arise because future collaboration is profitable, which justifies paying extra in the current period. In the other case, the agent with the highest marginal production cost will not be able to afford the consumption S to continue into the next period. In this situation, the interaction continues into the next period only if the agent's marginal production cost c is low enough such that $\bar{u}^*+Q^*(c)\geq S$.

Finally, the optimal structures derived in Theorem 3 can also substantially simplify computation by restricting the search over incentive compatible mechanisms to those following the corresponding structures. Take, for example, the case of $S' = 0$. When $x \geq S$, we only need to search within a family of mechanisms parameterized by one parameter. We then search through three parameters ($c_{(1)}$, $c_{(2)}$ and γ) only when $x < S$. At each state x, the decision variable $\bar{u}$ can take at most two possible values. These structures will be very helpful in the computational study we report in the next Section.

C.3. A detailed numerical example

Similar to the study of model (P), we have tested a large set of model parameters, with production cost c taking uniform $[0,1]$ distributions. Interestingly, we never observe the two-threshold policy as presented in Theorem 3 despite our extensive numerical study. That is, the optimal procurement policy turns out to always follow single threshold structures for any state, cash account balance x, in all our examples. In fact, we suspect that the two-threshold structure may only occur on a measure zero set for the cash model, although we have not been able to formally prove such a result even for the uniform distribution case. In this section we present two particular examples to illustrate our thinking. In the first example, we take $p = 1.5$, $S = 2$, $S' = 0$ and $\beta = 0.9$.

The value function $J_{\mathbb{C}}(x)$ is, in general, not concave, as illustrated in Figure 4. It is also evident that function $J_{\mathbb{C}}(x) > 0$ for all $x \geq 0$, and therefore is not continuous at $x = 0$. In fact, global concavity is only ensured if gains from trade are such that no type is ever discontinued from the game. On the other hand, Figure 4 also confirms that value function $J_{\mathbb{C}}(x)$ is increasing and Lipschitz continuous with constant 1, as stated in Lemma 6. We also observe that $J_{\mathbb{C}}(x)$ converges to $J_S/(1-\beta)$ for $x \geq \bar{x} = S/(1-\beta) = 20$, following Lemma 5.

Furthermore, Figure 5 illustrates that the net payment $\bar{u}$ when the agent's cost is highest equals $S - x$ when $x \in [\widehat{S}, S)$, and equals 0 when x is outside the interval. (Here $\widehat{S} = 1.8736$.) This is consistent with Theorem 5. The bottom plot of Figure 5 demonstrates that the thresholds $c_{(1)}$ and $c_{(2)}$ are, in fact, the same for all x values, despite the fact that they are allowed to be different when $x < \widehat{S}$ in the optimization algorithm. As a result, next we investigate the dynamic virtual valuations.

Following Theorem 5, we focus on $x < \widehat{S} = 1.8736$. In particular, Figure 6 demonstrates the optimal procurement quantities $q(c)$ (up) and dynamic virtual valuations $\xi_{\mathbb{C}}(c)$ (down) when x equals 1.7 (left) and 1.8 (right), respectively. As we observe, for either x value, the dynamic virtual valuation function contains a vertical upward "jump"at the corresponding $\Psi(q)$'s. For the case $x = 1.7$, the dynamic virtual valuation is always positive. Therefore the corresponding procurement quantities are set at the highest value 1 for all c. For the case $x = 1.8$, on the other hand, the dynamic virtual valuation $\xi_{\mathbb{C}}(c)$ crosses zero once on the left of $\Psi(q)$, at $c_{(1)}$, and once on the right of it at $c_{(2)} = \bar{c} = 1$. It is clear that the optimal procurement quantity $q(c)$ should be set at its upper bound 1 when $c < c_{(1)}$, because the derivative $\xi_{\mathbb{C}}(c) > 0$ in that region. When $c \in (c_{(1)}, \Psi(q))$, on the other hand, derivative $\xi_{\mathbb{C}}(c) < 0$. However, since the procurement function $q(c)$ needs

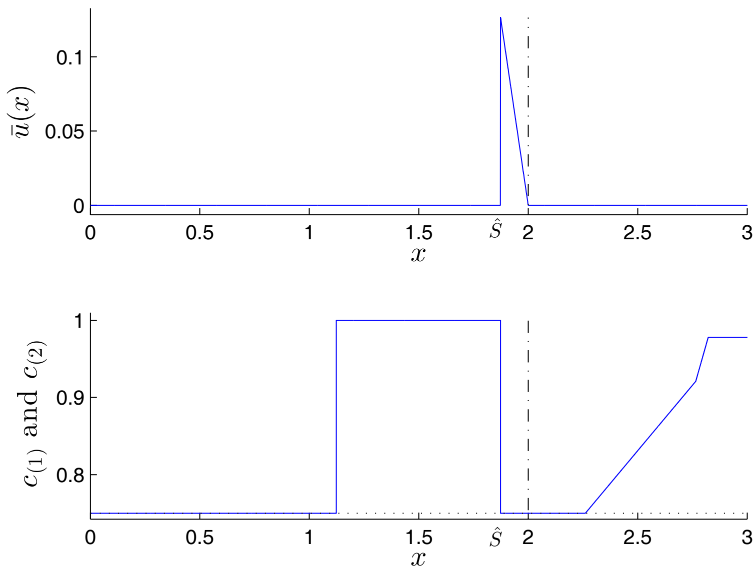

Figure 5 **Optimal $\bar{u}$ and thresholds $c_{(1)}$ and $c_{(2)}$ as functions of cash balance x, with $p=1.5$, $S=2$, $\beta=0.9$, $\underline{c}=0$, $\bar{c}=1$, and c uniformly distributed.**

to be non-increasing, and the integral of $\xi_{\mathbb{C}}(c)$ between $c_{(1)}$ and $c_{(2)}$ (the total areas beneath $\xi_{\mathbb{C}}$) is clearly positive, the optimal procurement quantities $q(c)$ should still be set as high as possible, and therefore at the upper bound 1.

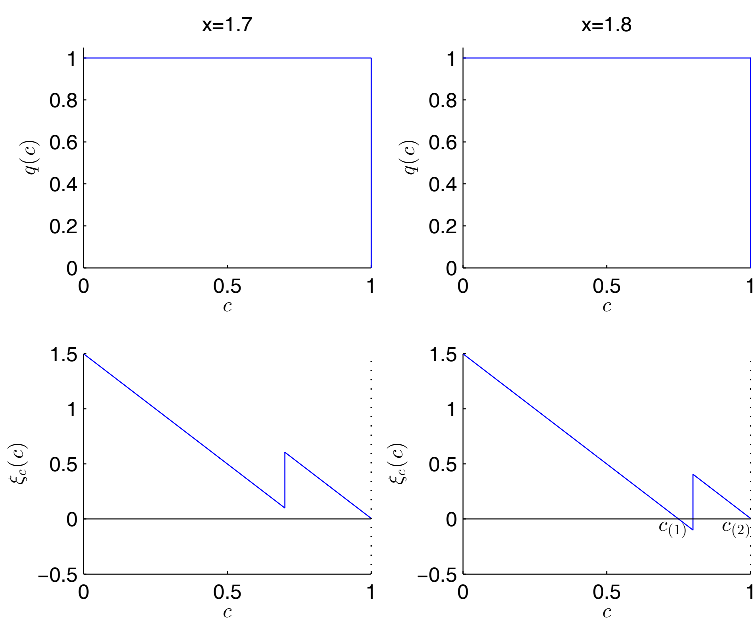

Figure 6 **Optimal procurement quantities and dynamic virtual valuations for cash balance $x=1.7$ and $x=1.8$, with $p=1.5$, $S=2$, $\beta=0.9$, $\underline{c}=0$, $\bar{c}=1$, and c uniformly distributed.**

As we increase the x value to 1.8735, very close to $\bar{S}$, the vertical jump $\Psi(q)$ moves to the right (see Figure 8, left). As a result, the integral of $\xi_{\mathbb{C}}$ between $c_{(1)}$ and $c_{(2)}$ decreases to a value very close to zero, but still positive. The corresponding optimal procurement policy remains the same. When we increase x to 1.8736, however, x becomes $\widehat{S}$. As a result, the single price structure becomes optimal, following Theorem 5. The corresponding policy and dynamic virtual valuations are presented on the right of Figure 8. Therefore, the two-threshold structure of the optimal procurement policy does not arise in this example.

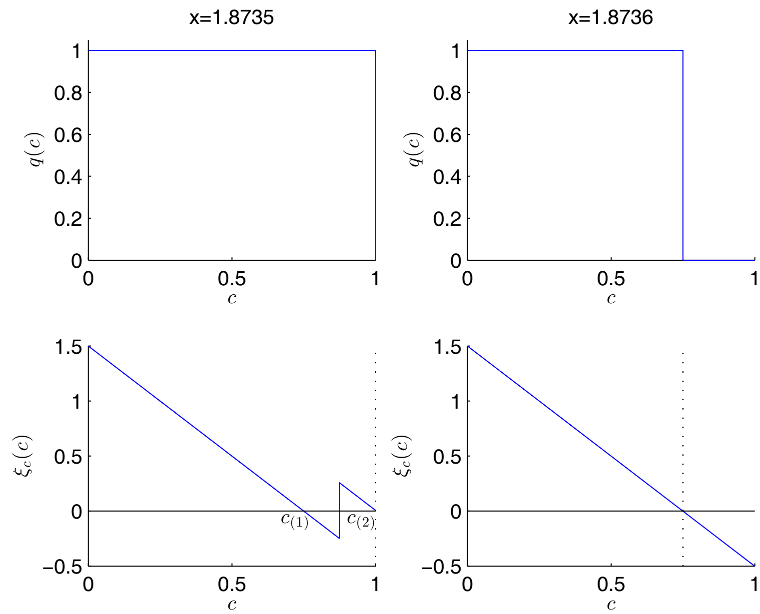

Figure 7 **Optimal procurement quantities and dynamic virtual valuations for cash balance** $x = 1.8735$ **and** $x = 1.8736$**, with** $p = 1.5$**,** $S = 2$**,** $\beta = 0.9$**,** $\underline{c} = 0$**,** $\bar{c} = 1$**, and** c **uniformly distributed.**

In fact, we have not been able to construct a convincing numerical example showing that the two-threshold policy structure exists in optimal policies, at least when private information follows a uniform distribution. The aforementioned numerical example is quite representative of the cases where the two-threshold policy structure "almost" emerges, in the sense that the dynamic virtual valuation contains a vertical jump and crosses zero multiple times.

When we enforce the constraint (ES) with $S' = S$, on the other hand, the two-threshold policy structure appears even less likely to emerge. Figure 8, for example, illustrates such a case. As we increase the x value to 0.3225, the vertical jump at $\Psi(q)$ moves to the right. A further increase of x to 0.3230 (the smallest increase according to our discretization), however, moves $\Psi(q)$ to $\bar{c}$. As a result, the single-threshold structure is optimal for all x.

On the other hand, we also cannot prove that the optimal policy is indeed a single price policy in general. We suspect that the two-threshold structure only occurs, if at all, in a measure zero point of x, as illustrated in the first example in this section. We leave it as an open question for future research.

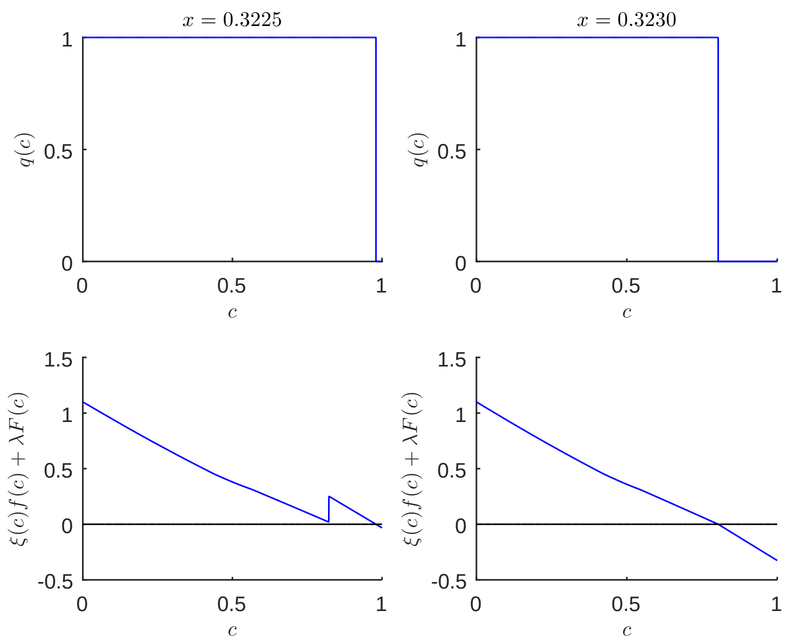

Figure 8 **Optimal procurement quantities and dynamic virtual valuations for cash balance** $x = 0.3225$ **and** $x = 0.3230$**, with** $p = 1.1$**,** $S = S' = 0.48$**,** $\beta = 0.8$**,** $\underline{c} = 0$**,** $\bar{c} = 1$**, and** c **uniformly distributed.**

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