Management Science

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MANAGEMENT SCIENCE
Vol. 64, No. 9, September 2018, pp. 4193–4217
ISSN 0025-1909 (print), ISSN 1526-5501 (online)

Optimal Contract to Induce Continued Effort

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Received: November 3, 2016
Revised: January 29, 2017; March 27, 2017
Accepted: April 8, 2017
Published Online in Articles in Advance: September 22, 2017

https://doi.org/10.1287/mnsc.2017.2826

Abstract. We consider a basic model of a risk-neutral principal incentivizing a risk-neutral agent to exert effort to raise the arrival rate of a Poisson process. The effort is costly to the agent, is unobservable to the principal, and affects the instantaneous arrival rate. Each arrival yields a constant revenue to the principal. The principal, therefore, devises a mechanism involving payments and a potential stopping time to motivate the agent to always exert effort. We formulate this problem as a stochastic optimal control model with an incentive constraint in continuous time over an infinite horizon. Although we allow payments to take general forms contingent on past arrival times, the optimal contract has a simple and intuitive structure, which depends on whether the agent is as patient as or less patient than the principal toward future income.

History: Accepted by Manel Baucells, decision analysis.

Keywords: dynamic • moral hazard • optimal control • jump process • principal–agent model • continuous time • Poisson

1. Introduction

The problem of managing incentives when agents' actions are unobservable frequently arises in the private and public sectors. A firm needs to continuously motivate its sales force or research and development (R&D) team to work hard; a research university's reputation hinges on output of its faculty's productive effort; government agencies need to be held accountable for achieving progress milestones; fundraising groups are responsible for turning up donors over time for political campaigns, charity organizations, or academic institutions. These diverse settings share some common features: incentives need to be managed over a long period of time; observable consequences from agents' efforts are uncertain; therefore, contracts need to be based on realized consequences over time.

In this paper, we study a stylized incentive management model over an infinite time horizon. In particular, we consider a risk-neutral principal optimally incentivizing a risk-neutral agent to increase the arrival rate of a Poisson process. Arrivals may correspond to new customers, R&D results, academic publications, successes in achieving public policy milestones, new donations, etc. Each arrival yields a certain "revenue" to the principal. The agent, if exerting effort, is able to increase the arrival rate from a base level to a high level. Because the agent bears the cost of effort, which is unobservable to the principal, the two players' incentives are misaligned. Therefore, the principal needs to design a contract to optimally induce and compensate for the agent's efforts.

We allow the compensation to take general forms, as long as at any point in time it is contingent on information available to the principal—that is, past arrival times. In particular, we allow payments to be in the form of instantaneous amounts at various points in time (bonuses) or a flow with a time-varying rate (salaries). Besides the "carrot" of payments, we also allow the principal to use the "stick" of terminating the contract. The threat of termination, therefore, is also contingent on past arrival times. The principal designs and commits to a contract that combines payments and a potential termination time to induce effort from the agent. In an extension of the basic model, we allow the principal to find a replacement agent at a fixed cost on terminating the focal agent, which does not change the nature of our results.

We formulate the problem as a dynamic optimization model with an incentive constraint, and demonstrate that the optimal contract takes a simple form, despite the generality of contract design allowed. Specifically, first consider the case that the principal and the agent have the same patience level toward future payoffs. Under an optimal contract, the principal should keep track of a "performance score," which is essentially the agent's total *promised future utility* (Spear and Srivastava 1987). Before reaching a fixed upper threshold, the performance score takes an upward jump of a fixed magnitude on each arrival and keeps decreasing between arrivals, while no payment is made. If the score decreases to zero before it could jump to the threshold, the contract terminates. If the performance score reaches the upper threshold with a jump less than the fixed magnitude of earlier jumps, then the clipped jump is the amount of the first payment. From that point on, the performance score is maintained at the threshold level, while each future arrival yields an instantaneous payment that equals the magnitudes of earlier jumps. Therefore, the initial phase of the contract resembles an "internship" period with uncertain length, during which arrivals may occur but the agent is not paid. See Figure 1 in Section 3.1 for a depiction of sample paths following this optimal contract.

This contract is simple and bears intuitive interpretations. First of all, it is not surprising that the optimal contract provides a reward for each arrival. Second, the starting time of payments depends on the timing of earlier arrivals. The longer interarrival times are, the later payments may start. Furthermore, there are two absorbing states following this contract. With luck, the performance score reaches the upper threshold, which is an absorbing state, and ensures the agent to be paid for all future arrivals. If the interarrival time becomes too long, the system falls into the other absorbing state, the performance score decreases to zero, and the contract terminates.

If the agent is less patient than the principal, on the other hand, the optimal contract is somewhat different. The performance score follows a very similar dynamic as in the equal patience case with jumps on and decreases between arrivals. The upper threshold on which payment starts is lower than the level that would ensure performance score not to decrease. Such a lower upper threshold allows the impatient agent to start getting paid earlier, at the cost of the performance score always decreasing after each upward jump, including those that yield payments. Consequently, the agent is always subject to the threat of termination. In this case, unlike the equal time discount case, there is not a clear "initial internship" period, because there could be periods of payments on arrival followed by no payment on arrival. See Figure 5 in Section 4.3 for a sample path for this case.

There has been a proliferation of continuous-time moral hazard models in recent years since the seminal paper, Sannikov (2008), that proposes a martingale representation of the incentive compatibility constraint. In his model, the uncertain outcome of an agent's effort follows a Brownian motion, which is a natural modeling choice for problems in areas such as corporate finance (see, e.g., DeMarzo and Sannikov 2006, Biais et al. 2007, Shi 2015, to name a few). While our absorbing states are similar to the corresponding results on retirement in Sannikov (2008), the structure of our payment contract is quite different and, arguably, simpler and cleaner to describe and control, due to the obvious distinction between Brownian motions and Poisson processes.

Biais et al. (2010) apply similar techniques on a model in which the agent's effort influences the rate of a Poisson process, rather than the drift of a Brownian motion. In their model, arrivals are "bad" outcomes (large risks). The corresponding contract structure, therefore, rewards the agent only if the interarrival time is longer than a threshold, which is different from ours, in which arrivals are "good." Myerson (2015) studies a model in a political economy setting, in which arrivals are also bad. Note that the distinction between outcomes being good and bad in this problem is far from superficial. In fact, the optimal contract structures are quite different. To incentivize longer interarrival times between bad outcomes, the optimal contract starts a *flow* of payment only if the interarrival time is longer than a certain threshold. Therefore, the longer the interarrival time is, the more the agent is paid. Such a contract structure is clearly different from the one described above for our case.

Besides differences in problem settings and some additional features of the model in Myerson (2015), another fundamental distinction with Biais et al. (2010) is that Myerson (2015) assumes that the principal and agent have the same patience level. In this case, to avoid unboundedness of the optimal solution, Myerson (2015) has to introduce an upper bound on the credibility of the principal's promise. The optimal contract, in turn, critically depends on this upper bound. Such a problem does not arise in Biais et al. (2010), which assumes that the agent is less patient. In our setting, where arrivals are "good" outcomes, we do not need to introduce an arbitrary upper bound. And we study both equal and less patient agent cases.

Various recent papers also consider moral hazard models with "good" Poisson arrivals, although the numbers of arrivals in these models are often finite. Mason and Välimäki (2015), for example, consider a model with a single Poisson arrival, which represents the completion of a project. Varas (2017) also considers a single Poisson arrival, which is unobservable to the principal. Therefore, the model contains both moral hazard and adverse selection issues. The arrival rate is affected by both the agent's hidden effort and imperfectly observable quality of the project. The agent can fake the arrival, which has to be taken into consideration in the contract. The model of Green and Taylor (2016) contains two stages (arrivals) to complete a project, which are also unobservable to the principal. The contract needs to induce effort as well as truthful revelation of the arrivals. Shan (2017) also considers arrivals as progress milestones for a project, which are publicly observable. The number of such milestones is not limited to two but is finite and exogenously determined. The paper also considers cases with multiple agents, whose efforts could either substitute or complement other agents'. Our model differs from the single agent model in Shan (2017) on two aspects. First, we consider an infinite horizon Poisson process, instead of a finite number of arrivals. Second, while Shan (2017) assumes that the principal and agent share the same time discount, we also study the case in which the agent is less patient than the principal. In Hidir (2017), arrivals represent experimental results which could only occur if the agent exerts effort, are unobservable to the principal, and could be either good or bad signaling potential quality of a project. The first good signal indicates the project is of high quality, at which point no more experimentation effort from the agent is needed. Cumulation of bad signals decreases the prior probability of high quality, and eventually also induces termination of the contract. Therefore, the number of arrivals allowed before termination is also finite.

More generally, the core idea of formulating dynamic mechanism design problems recursively as discrete time stochastic dynamic programming models with agents' promised utility as state variables is rooted in Abreu et al. (1990) for repeated games and Spear and Srivastava (1987) for moral hazard, further developed in Thomas and Worrall (1990) and Fernandes and Phelan (2000) for adverse selection problems, and widely adopted in the economics and finance literature.

The dynamic contracting problem has also been a subject of recent studies in the OR/MS literature (see, e.g., Zhang 2012a, b; Li et al. 2012; Belloni et al. 2016). The model that is related to ours, in terms of moral hazard problem with Poisson arrivals, is Plambeck and Zenios (2003). They study a more sophisticated make-to-stock queuing system with risk averse players in a finite time horizon. Although the model also includes contingent instantaneous and flow payment schemes, the focus of the paper is on the second-best production decisions. Therefore, the paper does not contain an explicit expression of the dynamic payment scheme beyond in the first-best setting.

2. Model

Consider a continuous time setting, in which a principal receives a stream of Poisson arrivals with rate $\underline{\mu}$ in the base case. Each arrival generates a fixed revenue R to the principal. The principal can choose to contract an agent to raise the arrival rate to a higher level μ, only if the agent exerts effort. The effort costs the agent a rate of c per unit of time and is unobservable by the principal. We denote the effort process by the agent as $\nu = \{\nu_t\}_{t\in[0,\infty)}$, with $\nu_t \in \{\mu, \underline{\mu}\}$. Both the principal and the agent are assumed to be risk-neutral and discount future cash flows. The time discount rates are r and ρ for the principal and agent, respectively. We assume that $\rho \geq r > 0$. That is, the principal is no less patient than the agent.

Define the ratio between the cost of effort c and the difference in arrival rate $\Delta\mu := \mu - \underline{\mu}$,

$$\beta := \frac{c}{\Delta\mu}. \tag{1}$$

The value β is significant in the optimal contract. In fact, if the principal pays a bonus β to the agent for each arrival, at any point in time, the higher expected value of the bonus during the next δ time units, $\beta\Delta\mu\delta$, offsets the agent's cost of exerting effort, $c\delta$. Therefore, paying such a bonus provides the incentive for the agent to exert effort.

We assume that

$$\beta \leq R, \tag{2}$$

which ensures that exerting effort is socially optimal.

We assume that the principal has the power to commit to a long term payment contract based on all of the information accessible to the principal at any point in time. Such information includes all of the arrival times. The arrival rate is determined by the effort process ν. And the agent can adjust the effort level according to the arrival process. More formally, at any time $t \in [0,\infty)$, we denote $N = \{N_t\}_{t\geq 0}$ to represent the counting process that represents the number of arrives up to and including time t. We also let $\mathscr{F}^N$ be the filtration generated by the process N. And the effort process ν is $\mathscr{F}^N$-predictable.

Next, we let $L = \{L_t\}_{t\geq 0}$ be a $\mathscr{F}^N$-adapted process that tracks the principal's cumulative payment to the agent. In particular, at any time t, the payment can be an instantaneous payment I_t, or a payment rate l_t, such that $dL_t = I_t + l_t dt$. We assume that the agent has limited liability, so that payments from the principal to the agent have to be nonnegative, i.e., $I_t \geq 0$ and $l_t \geq 0$. This is not a strong assumption, because in many practical settings the agent is liquidity constrained. In fact, without such a limited liability assumption, it is well known that the dynamic contracting problem becomes trivial—the principal simply sells out the enterprise to the agent up front, which allows the agent to implement the first-best control policy afterward.

As part of the contract, the principal can also stop the interaction with the agent at any time, τ that we assume to be an $\mathscr{F}^N$-stopping time. If $\tau < \infty$, the principal terminates the collaboration with the agent, while if $\tau = \infty$ the collaboration continues throughout the infinite time horizon.

2.1. Agent Utility and Incentive-Compatible Contracts

Given a dynamic contract $\Gamma = (L, \tau)$ and an effort process ν, the expected discounted utility of the agent is

$$u(\Gamma, \nu) = \mathbb{E}^{\nu}\left[\int_0^{\tau} e^{-\rho t}\,(dL_t - c\,\mathbb{1}\{\nu_t = \mu\}dt)\right], \tag{3}$$

in which the expectation $\mathbb{E}^{\nu}$ is taken with respect to probabilities generated from the effort process ν.

We focus on *incentive-compatible* contracts that induce the agent to always maintain a high arrival rate $\bar{\nu} := \{\nu_t = \mu\}_{t\geq 0}$. That is, a contract Γ is called *incentive compatible* (IC) if

$$u(\Gamma, \bar{\nu}) \geq u(\Gamma, \nu), \quad \forall\, \nu. \tag{IC}$$

We also define the agent's continuation utility at time t as the following:

$$W_t(\Gamma,\nu) = \mathbb{E}^{\nu}\left[\int_t^{\tau} e^{-\rho(s-t)}\,(dL_s - c\,\mathbb{1}\{\nu_s = \mu\}ds)\,\middle|\,\mathcal{F}_t^N\right]\mathbb{1}\{t<\tau\}. \quad (4)$$

Therefore, $W_0(\Gamma,\nu) = u(\Gamma,\nu)$ and the agent's continuation utility is zero after termination. Note that because τ is an $\mathcal{F}^N$ stopping time, the event $\{t < \tau\}$ is $\mathcal{F}_t^N$-measurable. So is random variable W_t.

It is clear that if a contract Γ satisfies $W_t(\Gamma,\bar{\nu}) \geq W_t(\Gamma,\nu)$ for any effort process ν and time t, constraint (IC) is satisfied. In fact, the (IC) constraint is equivalent to the seemingly more restrictive condition that the agent prefers exerting effort at any time and under any contingency, as stated in the following lemma.

Lemma 1. *Any contract Γ that satisfies the constraint (IC) must also satisfy that at any time t, $W_t(\Gamma,\bar{\nu}) \geq W_t(\Gamma,\nu)$ almost surely for any effort process ν.*

2.2. Principal Utility

In this paper, we focus on finding optimal incentive-compatible contracts, so that the agent prefers the effort process $\bar{\nu}$. To that end, we denote $U(\Gamma)$ to represent the expected discounted profit of the principal following an incentive-compatible contract Γ and the corresponding agent's effort process $\bar{\nu}$. On terminating the agent, the principal is left with only the baseline revenue,

$$\underline{v} := \frac{\underline{\mu}R}{r}, \quad (5)$$

from the low rate arrivals. In Section 5.2, we extend the model to allow the principal to contract with another identical agent at a fixed cost. The structure of the optimal contract is the same as the basic model.

Overall, the principal's utility under an incentive-compatible contract Γ is

$$U(\Gamma) = \mathbb{E}^{\bar{\nu}}\left[\int_0^{\tau} e^{-rt}\,(R\,dN_t - dL_t) + e^{-r\tau}\underline{v}\right]. \quad (6)$$

For the rest of the paper, we omit superscript $\bar{\nu}$ in the expectation under incentive-compatible contracts.

2.3. A Simple Incentive-Compatible Contract

Before we turn to the optimal contract, it is worth studying a simple incentive-compatible contract, $\bar{\Gamma}$. According to contract $\bar{\Gamma}$, the agent receives an instantaneous payment β after each arrival, while the flow payment is kept at zero (i.e., $dL_t = \beta\, dN_t$); and the principal never terminates the contract ($\tau = \infty$). It is easy to verify that contract $\bar{\Gamma}$ satisfies (IC). In fact, the (IC) constraint is binding under this contract, because the agent is always indifferent between exerting or not exerting effort. Following this contract, the agent's continuation utility at any time t is

$$W_t(\bar{\Gamma},\bar{\nu}) = \bar{w} := \frac{\underline{\mu}\beta - c}{\rho}. \quad (7)$$

It is worth noting that under the contract $\bar{\Gamma}$, the agent is indifferent between exerting effort or not, but still receives a nonnegative utility $\bar{w}$ because of the base-case arrival rate $\underline{\mu}$. In the special case where $\underline{\mu} = 0$, we have $\mu\beta = c$, and, therefore, $\bar{w} = 0$.

The principal's utility under contract $\bar{\Gamma}$ is, therefore,

$$U(\bar{\Gamma}) = \bar{U} := \frac{\mu(R-\beta)}{r}. \quad (8)$$

It is clear that because the contract $\bar{\Gamma}$ always induces full effort from the agent, it achieves the first-best outcome. However, it is not clear, at this point, if contract $\bar{\Gamma}$ is the optimal dynamic contract. Starting from the next section, we propose a different dynamic incentive-compatible contract that outperforms $\bar{\Gamma}$ and, in fact, is the optimal contract for the principal.

3. Equal Time Discount

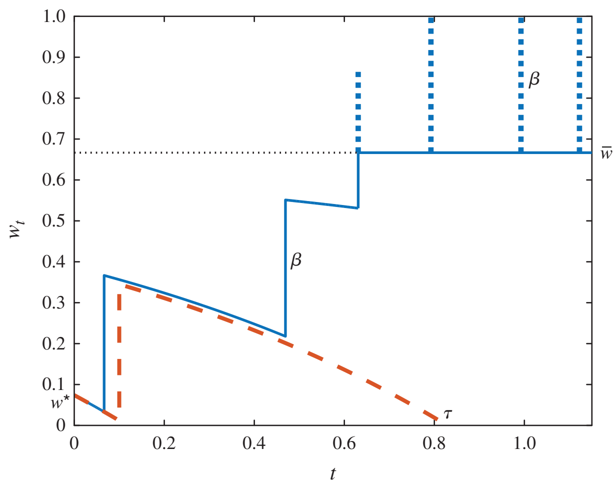
In this section, we assume that the principal and agent's time discount rates are the same. We use r to represent the time discount rate. We first propose a particular contract, Γ^*, and show that it is incentive compatible and optimal. All proofs for this section are presented in Appendix B.

3.1. Optimal Contract Γ^*

We first provide a heuristic description of contract Γ^* before providing a rigorous mathematical definition. This contract keeps track of a "performance score" w_t over time for the agent. (The reason that we use the letter w here is because under the optimal contract, this performance score corresponds to the agent's continuation utility.) Starting from an initial performance score $w_0 < \bar{w}$, this performance score keeps decreasing (the exact form to be specified later) until either there is an arrival or the score reaches zero. Whenever the score w_t reaches zero, the contract terminates. If an arrival occurs at time t with $w_t > 0$, on the other hand, the performance score w_t takes an upward jump of $\min\{\beta, \bar{w} - w_t\}$, and the agent receives an instantaneous payment of $(w_t + \beta - \bar{w})^+$. If $w_t + \beta \geq \bar{w}$, the performance score remains at $\bar{w}$; otherwise, the performance score again decreases, as described above.

Figure 1 depicts sample trajectories of such a policy. In particular, the solid curve represents one sample trajectory, which starts the performance score at $w_0 = w^*$. The performance score keeps decreasing between arrivals. Each of the first two arrivals induces an upward jump of β in the performance score. The performance score jumps to $\bar{w}$ with third arrival, which

Figure 1. (Color online) Sample Trajectories of Performance Score w_t with $r = 1$, $c = 1$, $R = 0.4$, $\underline{\mu} = 2$, and $\mu = 5$



Notes. In this case, $\bar{w} = 0.667$ and $\beta = 0.333$. The policy starts from $w_0 = w^* = 0.074$. The solid curve depicts a sample trajectory under which the agent's performance score reaches $\bar{w}$. The dashed curve depicts another sample trajectory under which the performance score reaches zero, when the collaboration between the principal and the agent dissolves. The vertical dotted lines depicts the payments according to the first sample trajectory.

also yields the first payment to the agent, at a level that equals to β minus the height of the jump. After this point in time, the performance score remains at $\bar{w}$. Each later arrival yields a payment of β to the agent. The figure depicts three such arrivals along this sample trajectory.

The dashed curve in Figure 1 represents another sample trajectory. The performance score also starts at the same w^*. There is only one arrival before the performance score decreases to zero, at time τ.

Similar to contract $\bar{\Gamma}$, according to contract Γ^*, each payment occurs at the time of an arrival, and all positive payments, except for the first one, are instantaneous and fixed at level β. Positive payments only start after the performance score reaches $\bar{w}$. Before the performance score may reach $\bar{w}$ due to the upward jumps, however, it could decrease to zero, which terminates the interaction between the principal and the agent.

In the special case when the base case arrival rate $\underline{\mu} = 0$, because the agent's utility $\bar{w} = 0$, the initial dynamic phase described above does not appear any more. Therefore, contract Γ^* reduces to $\bar{\Gamma}$. If the base-case arrival rate $\underline{\mu}$ is positive, the principal cannot distinguish whether an arrival is due to the agent's effort. Only in this case the agent receives positive expected utility, w^*.

Now, we specify how the performance score w_t (which is also the agent's continuation utility) decreases over time between arrivals. We first derive the expression pretending that the principal observes the effort process. The result turns out to not require the principal to observe the effort process anyway.

To this end, we consider a small time interval $[t, t+\delta)$. Because the principal is assumed, for the moment, to observe the effort level ν_t, with probability $\nu_t\delta + o(\delta)$ we have one Poisson arrival. This yields a payment $(w_t + \beta - \bar{w})^+$ and a continuation performance score $\min\{w_t + \beta, \bar{w}\}$ at the end of the time interval. With probability $1 - \nu_t\delta + o(\delta)$, on the other hand, the performance score becomes $w_{t+\delta}$. Also considering the cost of effort $c\delta$ during the time interval and the time discount factor r, we have the following discrete time approximation of the continuation value for the agent:

$$\begin{aligned} w_t = {} & -c\,\mathbb{1}_{\nu_t=\mu}\,\delta + [\nu_t\delta + o(\delta)] \\ & \cdot [(w_t + \beta - \bar{w})^+ + \min\{w_t + \beta, \bar{w}\}] \\ & + [1 - (r + \nu_t)\delta + o(\delta)] w_{t+\delta}. \end{aligned} \tag{9}$$

Following the standard procedure of dividing both sides of the above equation with δ, assuming that w_t is differentiable in t, and taking the limit of δ to zero, Equation (9) becomes the following differential equation:

$$\frac{dw_t}{dt} = rw_t + c\,\mathbb{1}_{\nu_t=\mu} - \nu_t\beta.$$

It can be verified that

$$c\,\mathbb{1}_{\nu_t=\mu} - \nu_t\beta \equiv c - \mu\beta = -r\bar{w}. \tag{10}$$

Therefore, if there is no arrival at time t, the performance score changes according to

$$\frac{dw_t}{dt} = r(w_t - \bar{w}). \tag{11}$$

If there is an arrival at time t—that is, $dN_t = 1$—the performance score w_t takes an upward jump of $\min\{\bar{w} - w_t, \beta\}$, as mentioned earlier. Together with (11), and the fact that w_t is kept at zero whenever it reaches zero, we obtain the following expression, which describes how w_t changes over time when $w_t \in [0, \bar{w}]$.[1]

$$dw_t = [r(w_t - \bar{w})dt + \min\{\bar{w} - w_t, \beta\}dN_t]\,\mathbb{1}_{w_t>0}. \tag{DW}$$

We call this Equation (DW), which stands for "dynamics of w" under the optimal contract. Note that if $w_t = \bar{w}$ at some time t, (DW) guarantees that for any future time $s > t$, w_s is maintained at $\bar{w}$. When the performance score w_t is lower than $\bar{w}$, however, it keeps decreasing before the next arrival. This smooth decrease balances the potential upward jump whenever there is an arrival, such that the performance score keeps track of the agent's total future utility, which is often referred to as the "promised utility" or "promised value" in the literature (see, e.g., Ljungqvist and Sargent 2004).

Remark 1. Note that managing the performance score w_t according to (DW) is quite easy. At each point in time t, w_t remains a constant if it equals $\bar{w}$. For $w_t < \bar{w}$, if there is no arrival, the slope at which w_t changes is $r(w_t - \bar{w})$, which is always negative, and uniquely determined at any level of w_t.

Finally, note that process w_t following (DW) depends on the effort process only through the arrival process N, which is observable by the principal. Therefore, (DW), together with an initial value w_0, uniquely specifies a performance score process that can be legitimately maintained by a principal observing the arrivals but not the effort process for the contract.

Now, we provide a formal definition of the contract Γ^*.

Definition 1. Contract $\Gamma^* = (L^*, \tau^*)$ is generated from a process $\{w_t\}_{t\geq 0}$ following (DW) with a given $w_0 \in [0, \bar{w}]$, such that $dL_t^* = (w_t + \beta - \bar{w})^+ dN_t$ and $\tau^* = \min\{t\colon\ w_t = 0\}$, in which the counting process N in (DW) and dL_t^* is generated from the agent's effort process ν. When necessary in the context, we use notations $\Gamma^*(w_0)$ to highlight input w_0.

We now formally show that the performance score process w_t as defined in Definition 1, which is an $\mathcal{F}_t^N$-adapted random variable, is, in fact, the agent's continuation utility under contract Γ^*. And this is true regardless of the effort process ν, because agent's efforts are exactly compensated by level β bonuses, either in the form of payments or upward jumps of promised utility.

Lemma 2. *For any $w_0 \in [0, \bar{w}]$, effort process ν, and time t, $W_t(\Gamma^*, \nu) = w_t$ almost surely.*

Lemma 2 implies that $u(\Gamma^*(w), \nu) = w$ at time zero. That is, contract Γ^* starting with an initial performance score w delivers the agent a utility w. This also implies that $u(\Gamma^*(w), \nu) = u(\Gamma^*(w), \bar{\nu})$, regardless of the effort process ν. Therefore, contract Γ^* is incentive compatible. (In fact, constraint (IC) is binding according to contract Γ^*.)

3.2. Principal's Value Function $F(w)$

Similar to the previous section, we first derive, heuristically, a differential equation for the principal's value function following contract Γ^* before formal mathematical statements. Specifically, we denote a time homogeneous function $F(w)$ to represent the principal's value function, as a function of the agent's continuation value (performance score) w.

Consider a small time interval $[t, t+\delta]$. Under the incentive-compatible contract Γ^*, the agent exerts effort, which yields an arrival rate μ. Therefore, with probability $\mu\delta + o(\delta)$, there is an arrival, which brings a revenue R and incurs a payment $(w_t + \beta - \bar{w})^+$ in this time interval. Correspondingly, the performance score increases to $\min\{w_t + \beta, \bar{w}\}$. If there is no arrival in the time period, on the other hand, the net utility in the period is zero and the performance score evolves to $w_{t+\delta}$. Overall, we have the following discrete time approximation for the principal's value function:

$$\begin{aligned} F(w_t) = {} & [\mu\delta + o(\delta)] \\ & \cdot [R - (w_t + \beta - \bar{w})^+ + F(\min\{w_t + \beta, \bar{w}\})] \\ & + [1 - (\mu + r)\delta + o(\delta)] F(w_{t+\delta}). \end{aligned}$$

Following the standard derivation of assuming $F(w)$ to be differentiable, taking its Taylor expansion, dividing both sides by δ, taking the limit of δ approaching zero, and, finally, replacing dw_t/dt with $r(w_t - \bar{w})$ following (DW) when there is no jump at time t, we obtain the following differential equation for $F(w)$:

$$\begin{aligned} (c - \mu\beta + rw)F'(w) = {} & (\mu + r)F(w) - \mu[R - (w + \beta - \bar{w})^+ \\ & + F(\min\{w + \beta, \bar{w}\})]. \end{aligned} \tag{12}$$

The boundary condition is $F(0) = \underline{v}$, in which $\underline{v}$ is defined in (15), reflecting that the principal faces a base-case arrival stream with rate $\underline{\mu}$ without an agent. (This boundary condition is revised in the extension studied in Section 5.2, where we allow the principal to replace the agent with a new one at a fixed cost.)

It is informative to consider the total value function of principal and agent, $V(w) = F(w) + w$. Following (12), we obtain a differential equation for $V(w)$:

$$\begin{aligned} 0 = {} & (\mu + r)V(w) - \mu V(\min\{w + \beta, \bar{w}\}) \\ & + r(\bar{w} - w)V'(w) + (c - \mu R). \end{aligned} \tag{13}$$

Lemma 3. *Differential equation (13) with boundary condition $V(0) = \underline{v}$ has a unique solution $V(w)$ on $[0, \bar{w}]$, which is increasing and strictly concave. Furthermore,*

$$V(w) = \bar{V} := \frac{\mu R - c}{r}, \quad \forall\, w \geq \bar{w}. \tag{14}$$

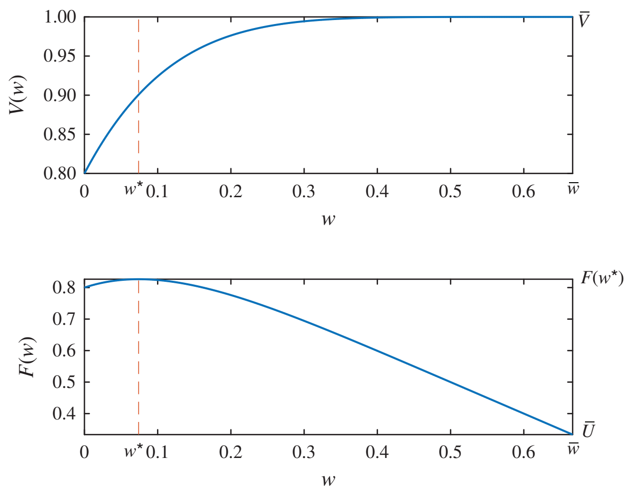
Consequently, differential equation (12) also has a unique solution $F(w)$, which is concave on $[0, \bar{w}]$.

Finally, we formally show that function $F(w)$ is indeed the value function of the principal under contract $\Gamma(w)$ starting from performance score w.

Proposition 1. *Starting from any $w_0 \in [0, \bar{w}]$, we have $U(\Gamma^*) = F(w_0)$.*

Figure 2 depicts the societal value function $V(w)$ and the principal's value function $F(w)$, with the same model parameters as behind Figure 1. To implement the contract, the principal needs to designate an initial performance score w_0. Clearly, this initial performance score should be set at w^*, the maximizer of the principal's value function $F(w)$, as depicted in Figure 2. The fact that the initial performance score w^*, which is also the agent's expected utility of entering the contract, is

Figure 2. (Color online) Value Functions with $r = 1$, $c = 1$, $R = 0.4$, $\underline{\mu} = 2$, and $\mu = 5$



Note. In this case, $\bar{w} = 0.667$, $w^* = 0.074$, $\bar{V} = 1$, $\underline{v} = F(0) = V(0) = 0.8$, $F(w^*) = 0.8266$, and $\bar{U} = 0.333$.

positive, guarantees that the agent is willing to participate. And, clearly, value w^* corresponds to the agent's information rent.

In the example behind Figure 2, if the principal implements contract $\bar{\Gamma}$ from the very beginning, the principal losses $\underline{v} - \bar{U} = 0.467$, compared with not hiring the agent at all ($\underline{v}$ is defined in (5) and $\bar{U}$ in (8)). Contract $\Gamma^*(w^*)$, on the other hand, yields an expected benefit of $F(w^*) - \underline{v} = 0.0266$ to the principal. In the next subsection, we prove that contract Γ^* not only improves on $\bar{\Gamma}$ but is, in fact, optimal.

Remark 2. Figure 2 confirms Lemma 3 that both functions V and F are concave. Concavity is crucial in proving optimality of contract Γ^*, which comes in the next subsection. Concavity of F allows us to argue that the incentive compatibility constraint is binding at optimality. That is, on each arrival, the upward jump of promised utility or the payment can be set exactly at β, and no more. Although it may appear intuitive that upward jumps/payments should be kept as low as possible, as long as they provide the incentive for effort, without concavity this may not hold. Hypothetically, the principal may want to take on an upward jump higher than β to raise the agent's promised utility further away from zero to avoid terminating the contract too soon. Should this happen, the optimal contract structure would almost certainly be much more complex. Our result shows that, fortunately, under condition (2), the value function is indeed concave and the optimal contract structure simple.

3.3. Optimality

In this subsection, we show that the contract Γ^* proposed in the previous section is indeed optimal. In the next proposition, we show that function F introduced in the last section is an upper bound for the principal's utility under any incentive-compatible contract Γ. Similar to Biais et al. (2010), our proof relies on first summarizing the dynamics of promised utilities under any incentive-compatible contracts, as presented in Lemma 6 in Appendix B.5. The inequality in Proposition 2 then follows from concavity of function F.

Proposition 2. *For any contract Γ that satisfies (IC), we have $F(u(\Gamma, \bar{v})) \geq U(\Gamma)$.*

Proposition 2 implies that if the agent's initial continuation utility under an incentive-compatible contract Γ is w, then function $F(w)$ is an upper bound to the principal's utility $U(\Gamma)$ under contract Γ. This, combined with Proposition 1, implies that contract Γ^* that we proposed in the last section is optimal, as stated in the following theorem.

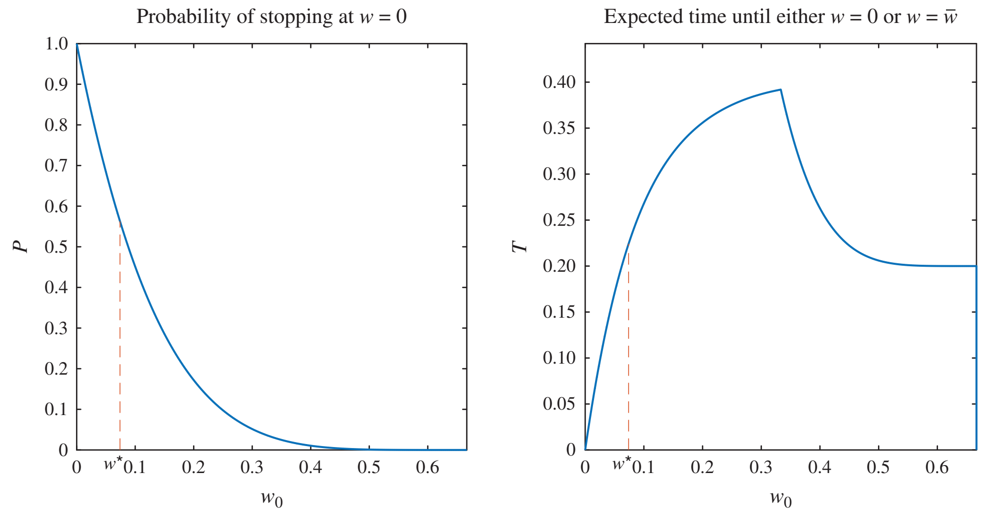
Theorem 1. *Assume that $\rho = r$. Let strictly concave function $F(w)$ be the unique solution to the differential equation (12) on $[0, \bar{w}]$ with boundary condition $F(0) = \underline{\mu}R/r$, and let w^* be the unique maximizer of $F(w)$ on $[0, \bar{w}]$. Then, $\Gamma^*(w^*)$ is an optimal incentive-compatible dynamic contract. That is, $U(\Gamma^*(w^*)) \geq U(\Gamma)$ for any contract Γ that satisfies (IC).*

Similar to general mechanism design settings, the optimal contract Γ^* is not efficient. This is because there is a chance that the agent is terminated, leaving the system with only the base-case low arrival rate $\underline{\mu}$. Therefore, compared with the contract $\bar{\Gamma}$, the optimal contract not only shifts some of the agent's surplus to the principal, but also introduces some "waste" to the system in the form of firing the agent. The only case that the optimal contract achieves the first-best solution is when $\underline{\mu} = 0$. In this case, the payment β for each arrival exactly compensates the effort cost.

It is worth pointing out that when the two players' time discounts are the same, the optimal contract is not unique. In fact, the principal can delay a payment to a later time, as long as corresponding interests are paid. It is intuitive that because the two players discount future income in the same way, the principal's benefit from the delay exactly offsets the interest the agent would demand.

In theory, all payments can be delayed into the future infinitely, and the corresponding magnitudes of the payment also becomes infinite. This is similar to, however not exactly the same as, the so-called "infinite back-loading" issue discussed in Myerson (2015) for equal time discount models. Myerson (2015) has to introduce an upper bound on the agent's promised utility for the model to be sound. In turn, the optimal contract in Myerson (2015) critically depends on this upper bound. We do not need to introduce such

Figure 3. (Color online) Probability of the Performance Score Reaching Zero and the Expected Time of the Performance Score Reaching Either $\bar{w}$ or zero, Starting from Any Initial Score w_0



Note. Here, $r = 1$, $c = 1$, $R = 0.4$, $\underline{\mu} = 2$, and $\mu = 5$. In this case, $\bar{w} = 0.667$ and $w^* = 0.074$.

an upper bound in our setting, where the arrivals are good outcomes.

Our specific contract Γ^* is provably optimal and has the advantage of being simple. When the agent is less patient than the principal, however, the agent demands higher interests for delayed payments. Intuitively, it would cost the principal more to delay payments, which mitigates the infinite back-loading issue. In the next section, we study this case in more details.

Before we proceed, we present some results from numerical computation to gain further insights into the optimal contract. The (delayed) differential equation (13) can be solved as a system of linear equations on a grid. We leave the detailed procedures to Appendix G.1.

Figure 3 demonstrates the probability of the collaboration stopping at $w = 0$ from any initial performance score w_0, as well as the expected time until the performance score reaching either zero or $\bar{w}$. As we can see from the figure, the higher the initial performance score w_0, the higher the probability of the score eventually reaching $\bar{w}$ (the lower the probability of the score ending at zero).

The expected time until reaching either absorbing state, however, is not monotone in the initial performance score w_0. This is intuitive because if the initial performance score is very high, it takes little time to reach the upper bound $\bar{w}$. If it is very low, on the other hand, it also takes little to reach the lower bound zero. The expected time is not smooth in the initial score, reflecting the change between likely reaching the upper bound and the lower bound when w_0 changes.

Furthermore, the expected time is not even continuous at the upper bound $\bar{w}$. This is because if the promised utility is already at $\bar{w}$, the expected time to reach it is zero. If the promised utility is at any point below $\bar{w}$, however, it requires at least one arrival for the promised utility to reach $\bar{w}$. Therefore, in the limit as the promised utility approaches $\bar{w}$, the expected time to reach $\bar{w}$ is exactly the expected value of a Poisson arrival, $1/\mu$.

4. Different Time Discount

In this section, we generalize the model to allow the agent's time discount rate ρ to be strictly larger than the principal's discount rate r. That is, the principal is more patient than the agent. Intuitively, a less patient agent prefers earlier payment. This implies that the optimal contract may lower the payment threshold from $\bar{w}$ so that the agent starts receiving payment earlier. In this section, our approach reveals that the optimal payment threshold, which is denoted as $\hat{w}$, is indeed lower than $\bar{w}$. Furthermore, the threshold $\hat{w}$ is no longer an absorbing state when $r < \rho$.

For this case, however, we are not able to use the "guess and verify" approach described in the last section to show the result. In fact, even if one has a correct guess of the optimal contract structure and obtain its value function through a Hamilton–Jacobi–Bellman (HJB) differential equation, we are not able to directly establish that such a function is concave, as in Lemma 3. And, in our setting, concavity is crucial for establishing optimality of simple contract structure, as argued in Remark 2. Therefore, in this section, we have to rely on a different approach, based on constructing a discrete time approximation and its upper concave envelope. Proofs of all results in this section are presented in Appendices C–E.

4.1. Discrete Time Approximation

In the discrete time approximation, we consider a very small time interval δ. Denote $\gamma = e^{-r\delta}$ and $\varrho = e^{-\rho\delta}$ to represent the principal and agent's discount factors, respectively. Therefore $\gamma \geq \varrho$. In each time period, if the agent exerts effort, it costs the agent $c\delta$ and there is a probability $\mu\delta$ that an arrival occurs in a period. With no effort from the agent, the probability of an arrival is $\underline{\mu}\delta$.

We can represent the contract design problem as the following infinite horizon discrete time dynamic program. In particular, denote $F_\delta(w)$ to represent the principal's value function, as a function of the agent's utility w. In each time period, the principal first decides whether to pay off the promised utility w to the agent and terminate the contract. The corresponding principal's utility is $\underline{v}_\delta - w$, in which $\underline{v}_\delta$ is the discrete time counterpart of $\underline{v}$, the principal's baseline revenue with no effort from the agent,

$$\underline{v}_\delta := \frac{\underline{\mu} R\delta}{1-\gamma}, \quad \text{with } \lim_{\delta\downarrow 0} \underline{v}_\delta = \underline{v}, \tag{15}$$

in which $\underline{v}$ is defined in (5). (Again, this boundary condition is revised if the principal can replace the agent with a new one at a fixed cost, to be discussed in Section 5.2.)

If, on the other hand, the strategic interaction is to continue, then the principal decides on payment I if an arrival occurs, and on payment l if there is no arrival, as well as the corresponding promised continuation utilities to the agent, w_A and w_N, respectively. The principal's Bellman equation is, therefore,

$$F_\delta(w) = \max\Big\{\underline{v}_\delta - w, \max_{(I,l,w_A,w_N)\in\Pi_w} \mu\delta[R - I + \gamma F_\delta(w_A)] + (1-\mu\delta)[-l + \gamma F_\delta(w_N)]\Big\}, \tag{16}$$

in which the feasible region Π_w is defined as

$$-c\delta + \mu\delta(I + \varrho w_A) + (1-\mu\delta)(l + \varrho w_N) \geq \underline{\mu}\delta(I + \varrho w_A) + (1-\underline{\mu}\delta)(l + \varrho w_N), \tag{IC$_\delta$}$$

$$-c\delta + \mu\delta(I + \varrho w_A) + (1-\mu\delta)(l + \varrho w_N) = w, \tag{PK$_\delta$}$$

$$I,\ l,\ w_A,\ w_N \geq 0. \tag{IR$_\delta$}$$

Here, the incentive compatibility constraint (IC$_\delta$) is a discrete time counterpart of the recursive representation of the continuous time incentive-compatible constraint as described in Lemma 1. It can be equivalently stated as

$$I + \varrho w_A - l - \varrho w_N \geq \beta.$$

A binding (IC$_\delta$) constraint, together with the promise-keeping constraint (PK$_\delta$), correspond to the dynamic (DW) in the continuous time model. Finally, the outer maximization in (16) represents the principal's decision on whether to pay off the promised utility to the agent and terminate the contract.

Note that the feasible region Π_w is nonempty only for $w \geq \underline{w}_\delta$, in which

$$\underline{w}_\delta := \delta(\mu\beta - c), \quad \text{with } \lim_{\delta\downarrow 0} \underline{w}_\delta = 0. \tag{17}$$

Therefore, we must have $F_\delta(w) = \underline{v}_\delta - w$ for $w \in [0, \underline{w}_\delta)$. That is, if the promised utility becomes too low, the principal can only pay off the promised utility to the agent and terminates the contract.

Similar to the treatment of the continuous time model, we further define the societal value function $V_\delta(w) = F_\delta(w) + w$. Value function V_δ satisfies the Bellman equation $V_\delta = \mathrm{T}_\delta V_\delta$, in which the operator T_δ is defined for any bounded function J to be $(\mathrm{T}_\delta J)(w) = \underline{v}_\delta$ for $w \in [0, \underline{w}_\delta)$, and, for $w \geq \underline{w}_\delta$,

$$(\mathrm{T}_\delta J)(w) = -c\delta + \max_{(I,l,w_A,w_N)\in\Pi_w}\{\mu\delta[R + \gamma J(w_A) - (\gamma-\varrho)w_A] + (1-\mu\delta)[\gamma J(w_N) - (\gamma-\varrho)w_N]\}. \tag{18}$$

To this end, we first present the following characterization of the optimal solution of the optimization in the T_δ operator, if function J is concave.

Proposition 3. *For any concave and nondecreasing function J on $[0,\infty)$, such that $J(0) = \underline{v}_\delta$, and $J(w)$ remains a constant for w large enough, define quantity $\hat{w}_\delta(J)$ to be the maximum point at which $(\gamma-\varrho)/\gamma$ is a superderivative of J. Also define quantity $\bar{w}_\delta(J) = \varrho\hat{w}_\delta(J) + \underline{w}_\delta$.*

We have the following characterization of the optimal solution $(I_\delta^*, l_\delta^*, w_{A\delta}^*, w_{N\delta}^*)_J(w)$ to $(\mathrm{T}_\delta J)(w)$:

- *For $w \geq \bar{w}_\delta(J)$,*

$$(I_\delta^*, l_\delta^*, w_{A\delta}^*, w_{N\delta}^*)_J(w) = (\beta + (w - \bar{w}_\delta(J)), (w - \bar{w}_\delta(J)), \hat{w}_\delta(J), \hat{w}_\delta(J)). \tag{19}$$

- *For $w \leq \bar{w}_\delta(J)$,*

$$(I_\delta^*, l_\delta^*, w_{A\delta}^*, w_{N\delta}^*)_J(w) = \left((w - \underline{w}_\delta + \beta - \varrho\hat{w}_\delta(J))^+, 0, \min\left\{\frac{w - \underline{w}_\delta + \beta}{\varrho}, \hat{w}_\delta(J)\right\}, \frac{w - \underline{w}_\delta}{\varrho}\right). \tag{20}$$

Furthermore, $(\mathrm{T}_\delta J)(w)$ takes a constant value for all $w \geq \bar{w}_\delta(J)$.

Proposition 3 provides an important building block for the optimal contract and optimal value function in the continuous time, and deserves elaboration. First, the threshold $\hat{w}_\delta(J)$ is the point at which the slope of function J is at $(\gamma-\varrho)/\gamma$, which approaches zero as δ approaches zero. Therefore, in the limit, $\hat{w}_\delta(J)$ approaches to the point at which function J becomes "flat." Furthermore, because ϱ approaches one and $\underline{w}_\delta$ approaches zero as δ approaches zero, $\hat{w}_\delta(J)$ and $\bar{w}_\delta(J)$ converge to the same quantity, call it $\hat{w}$, in the limit. This is very similar to $\bar{w}$'s role in the equal discount factor case. However, a key difference that we will formally demonstrate later is that $\hat{w}$ is, in fact, smaller than $\bar{w}$. Putting this difference aside, in the limit as δ approaches zero, it is clear that the optimal solution in Proposition 3 resembles the structure of contract Γ^*.

Furthermore, it is worth investigating intuitions behind the formulas of the optimal solution. To maximize social welfare, Equation (18) implies that, ideally, w_N and w_A should satisfy $J'(w) = (\gamma - \varrho)/\gamma$—i.e., $w^*_{N\delta} = w^*_{A\delta} = \hat{w}_\delta(J)$. This explains the solution in (19). To provide the right incentive to the agent, however, constraint (PK_δ) and binding (IC_δ) imply that $l + \varrho w_N = w - \underline{w}_\delta$ and $I + \varrho w_A = \beta + w - \underline{w}_\delta$. Intuitively, current payments I and l and discounted future payments ϱw_A and ϱw_N are equally effective in providing incentives to the agent, as long as the difference in the total payments is kept at β. From the perspective of the social welfare, however, it is more beneficial to differentiate current payment and future ones, as seen in the optimal solution (19). A problem occurs when w is so low that the resulting payment l would be negative, violating the agent's limited liability. In this case, the principal has to set l^*_δ to be zero, and let $\varrho w^*_{N\delta} = w - \underline{w}_\delta$, as in (20), and, if needed, truncate I^*_δ at zero as well. The requirement that $\varrho w^*_{N\delta} \geq 0$ is consistent with our earlier claim that only promised utility $w \geq \underline{w}_\delta$ can be supported by any incentive-compatible contract.

The following lemma presents some further properties of the operator T_δ.

Lemma 4. *For any $\delta > 0$ and bounded, nondecreasing and concave function J, function $\mathrm{T}_\delta J$ is nondecreasing on $[0, \infty)$, concave on $[\underline{w}_\delta, \infty)$, while taking value $\underline{v}_\delta$ on $[0, \underline{w}_\delta)$.*

Therefore, function $\mathrm{T}_\delta J$ is, in general, not concave on $[0, \infty)$, which implies that the optimal value function V_δ in this discrete time model is not concave. To proceed, we propose the following "concavification" operator Λ, for any bounded function J that is nondecreasing on $[0, \infty)$, concave on $[\underline{w}_\delta, \infty)$, and satisfies $J(w) = \underline{v}_\delta$ for $w \in [0, \underline{w}_\delta)$:

$$(\Lambda J)(w) = \min_{a,\, b:\, ax+b \geq J(x),\, \forall x \geq 0} \{aw + b\}, \quad \forall\, w \geq 0.$$

Therefore, function ΛJ is the concave upper envelope of function J on $[0, \infty)$. Further define operator $\Upsilon_\delta = \Lambda \circ \mathrm{T}_\delta$, which takes the concave upper envelope after a step of value iteration T_δ. Lemma 7 in the appendix verifies that operator Υ_δ has a fixed point, which is clearly a concave function. As we will show later in the paper, in the limit as δ approaches zero, the optimal value function V_δ uniformly converges to the fixed point of operator Υ_δ, which is concave. (This concavification approach corresponds to an approach in game theory that allows the principal to use a mixed strategy that randomizes between decisions for two state values as long as the expectation of the two states is w.)

Let concave function J_δ be the fixed point of the concavified operator Υ_δ, which, due to its being an upper envelope, must be an upper bound of the optimal value function. Use J_δ in place of J, Proposition 3 establishes a feasible policy for $w \geq \underline{w}_\delta$. Repeatedly using this policy yields a lower bound for the optimal value function for the principal. In the next subsection, we establish that the upper bound and lower bound uniformly converge to each other as the time step δ approaches zero, which implies that the continuous time optimal value function is indeed concave, even though the discrete time optimal value function is not.

4.2. Converging to Continuous Time: Optimal Value Function

First, we provide a heuristic derivation for the HJB equation for the optimal value function. To this end, consider policy $(I^*_\delta, l^*_\delta, w_{A\delta}^*, w_{N\delta}^*)_{J_\delta}$ derived from the concave upper envelope J_δ following Proposition 3. In particular, consider the value function ψ_δ from repeated using this policy over an infinite horizon for $w \geq \underline{w}_\delta$. That is, $\psi_\delta(w) = \underline{v}_\delta$ for $w \in [0, \underline{w}_\delta]$ and $\psi_\delta(w) = (\Xi_\delta \psi_\delta)(w)$ for $w \geq \underline{w}_\delta$, in which the dynamic programming operator Ξ_δ for any function J is defined as

$$\begin{aligned}(\Xi_\delta J)(w) = \delta(\mu R - c) &+ \mu\delta[\gamma J(w_{A\delta}^*) - (\gamma - \varrho) w_{A\delta}^*] \\ &+ (1 - \mu\delta)[\gamma J(w_{N\delta}^*) - (\gamma - \varrho) w_{N\delta}^*]. \qquad (21)\end{aligned}$$

Note that since the function J satisfy the conditions of Proposition 3, the operator Ξ_δ corresponds to the operator T_δ defined in (18), with the max operator replaced with the optimal policy. However, because $T_\delta J$ is no longer concave, the fixed point V_δ of operator T_δ is different from the fixed point ψ_δ of operator Ξ_δ. In fact, because Ξ_δ corresponds to a fixed policy while T_δ contains maximization, function ψ_δ is a lower bound of V_δ.

Following the standard heuristic procedure of dividing both sides of the Bellman equation $\psi_\delta = \Xi_\delta \psi_\delta$ with δ and letting δ approach zero, while assuming (to be verified later in the proof) that the value function ψ_δ converges to a differentiable function V_d and $\hat{w}_\delta(\psi_\delta)$ converges to a quantity $\hat{w}$, we obtain the following stochastic differential equation for the optimal value function.

$$\begin{aligned} 0 = (r + \mu) V_d(w) &- \mu V_d(\min\{w + \beta, \hat{w}\}) \\ &+ \rho(\bar{w} - w) V_d'(w) + (c - \mu R) + (\rho - r) w, \qquad (22) \\ &\text{with boundary conditions } V_d(0) = \underline{v}, \\ &\text{and } V_d(\hat{w}) = \bar{V}_d := \bar{V} + \frac{r - \rho}{r} \hat{w}, \qquad (23) \end{aligned}$$

in which $\bar{V}$ is defined in (14). It is clear that if $\rho = r$ and $\hat{w} = \bar{w}$, then Equation (22) reduces to (13).

The next proposition formally establishes the existence and uniqueness of its solution.

Proposition 4. *If $r < \rho$, for any $\tilde{w} \in [0, \bar{w})$, there exists a unique function $\tilde{V}_{\tilde{w}}$ that solves the differential equation (22) on $[0, \tilde{w}]$ with boundary condition $\tilde{V}_{\tilde{w}}(\tilde{w}) = \bar{V} + \tilde{w}(r-\rho)/r$. Furthermore, $\tilde{V}_{\tilde{w}}(0)$ is monotonically decreasing in $\tilde{w}$, and the derivative of function $\tilde{V}_{\tilde{w}}$ at point $\tilde{w}$ is zero.*

Therefore, there exists a unique value $\hat{w}$ in $[0, \bar{w})$ and a unique function V_d that satisfy the differential equation (22) on $[0, \hat{w}]$ with boundary conditions (23). Furthermore, $V_d'(\hat{w}) = 0$.

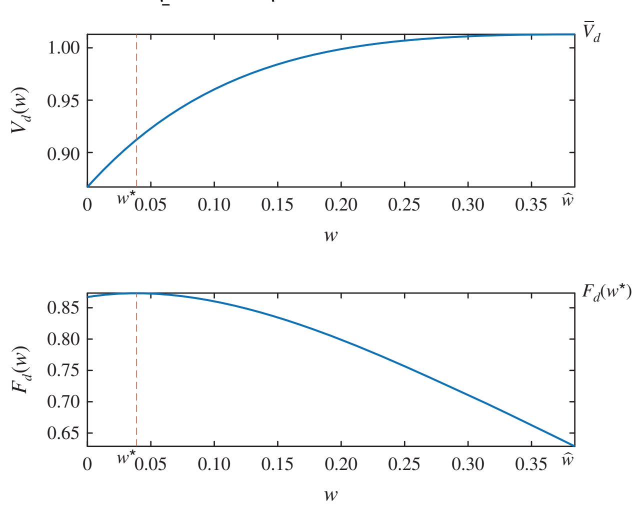
Note that from the differential equation (22), we cannot directly establish that the solution V_d is concave. However, recall that function J_δ, which is the fixed point of the concavified operator Υ_δ, and, obviously, an upper bound of V_δ, is concave. The following Proposition 5 states that function V_δ's upper bound, J_δ, and lower bound, ψ_δ, uniformly converge to each other as δ approaches zero. Consequently, the limit of V_δ converges to a concave function. Furthermore, function J_δ also uniformly converges to V_d. Therefore, function V_d is the limit of V_δ, and is concave. This is a crucial step to show that V_d is the optimal value function. Arguments in Biais et al. (2007) are useful in proving the uniform convergence result, which is presented in Appendix D.2.

Proposition 5. *We have*

$$\lim_{\delta \downarrow 0} \|\psi_\delta - J_\delta\| = \lim_{\delta \downarrow 0} \|V_d - J_\delta\| = 0.$$

Figure 4 depicts function $V_d(w)$, as well as function $F_d(w) := V_d(w) - w$, similar to Figure 2. Both functions are concave, as implied by Proposition 5. The detailed procedure to obtain the function V_d and threshold $\hat{w}$ is presented in Appendix G.2.

Figure 4. (Color online) Value Functions with $r = 0.9$, $\rho = 1$, $c = 1$, $R = 0.39$, $\underline{\mu} = 2$, and $\mu = 5$



Note. In this case, $\bar{w} = 0.667$ (not depicted in the figure), $\hat{w} = 0.384$, $w_d^* = 0.039$, $\bar{V}_d = 1.0129$, $\underline{v} = V_d(0) = F_d(0) = 0.8679$, and $F_d(w_d^*) = 0.8735$.

As δ approaches zero, the optimal solution (20) for the promised utilities also converges to a stochastic differential equation, which is similar to (DW). We formally establishes this in the next subsection.

4.3. Continuous Time Limit: Promised Utility Process and Optimal Contract

Similar to before, we first provide a heuristic derivation of the promised utility expression in continuous time—again, following Proposition 3—before formally establishing its optimality. Denote w_t and $w_{t+\delta}$ to represent the agent's promised utilities at times t and $t+\delta$, respectively, such that $w_t \le \bar{w}_\delta(J_\delta)$ and $w_{t+\delta}$ is either $w_{A\delta}^*$ or $w_{N\delta}^*$ following $(I_\delta^*, l_\delta^*, w_{A\delta}^*, w_{N\delta}^*)_{J_\delta}(w_t)$. If there is no arrival, then $w_{t+\delta} = w_{N\delta}^*$. It is clear that if $w_t \le \bar{w}_\delta(J_\delta)$, we have $w_{t+\delta} < w_t$, and,

$$\frac{dw_t}{dt} = \lim_{\delta \downarrow 0} \frac{w_{t+\delta} - w_t}{\delta} = \lim_{\delta \downarrow 0} \frac{1}{\delta}\left(\frac{w_t - \underline{w}_\delta}{\varrho} - w_t\right)$$
$$= \rho w_t - (\mu\beta - c) = \rho(w_t - \bar{w}).$$

If there is an arrival, then $w_{t+\delta} = w_{A\delta}^*$, and, correspondingly,

$$dw_t = \lim_{\delta \downarrow 0}(w_{t+\delta} - w_t) = \min\{\beta, \hat{w} - w_t\}.$$

Therefore, the agent's utility process w_t, following the policy described in Proposition 3, converges to the following process:

$$dw_t = [\rho(w_t - \bar{w})\, dt + \min\{\beta, \hat{w} - w_t\}\, dN_t]\, \mathbb{1}_{w_t > 0}, \tag{DWd}$$

in which $\hat{w} < \bar{w}$ for $r < \rho$.[2]

Similar to the contract Γ^* defined for the same time discount case, we propose the following contract Γ_d^*.

Definition 2. Contract $\Gamma_d^*(w_0) = (L^*, \tau^*)$ is generated from a process $\{w_t\}_{t \ge 0}$ following (DWd) with a given $w_0 \in [0, \hat{w}]$, in which $\hat{w}$ is determined according to Proposition 4, $dL_t^* = (w_t + \beta - \hat{w})^+ dN_t$ and $\tau^* = \min\{t: w_t = 0\}$. Here, the counting process N in (DWd) and dL_t^* is generated from the agent's effort process ν.

Figure 5 depicts a sample trajectory of the performance score process w_t under the optimal contract $\Gamma_d^*(w_d^*)$. In this sample path, the contract lasts $\tau = 2.24$ time units before being terminated when the agent's utility (performance score) decreases to zero. During the time period when the agent is hired, there are seven arrivals, which yield five payments at various amounts. The sooner an arrival occurs after the previous one, the higher the corresponding payment.

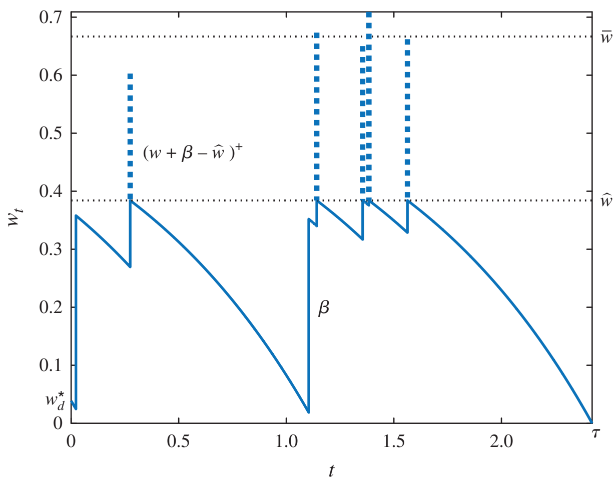
The proof of Lemma 2 already establishes the following result.

Lemma 5. *For any $w_0 \in [0, \hat{w}]$, effort process ν, and time t, $W_t(\Gamma_d^*(w_0), \nu) = w_t$ almost surely.*

In light of Lemma 1, Lemma 5 implies that the contract Γ_d^* is incentive compatible in this setting.

Now, we are ready to establish the proof of optimality.

Figure 5. (Color online) A Sample Trajectory of Performance Score w_t According to Γ_d^* with $r = 0.9$, $\rho = 1$, $c = 1$, $R = 0.39$, $\underline{\mu} = 2$, and $\mu = 5$



Notes. In this case, $\bar{w} = 0.667$, $\hat{w} = 0.384$, and $\beta = 0.333$. The policy starts from $w_0 = w_d^* = 0.039$. The solid curve depicts a sample trajectory of the performance score. The dotted lines depicts the payments.

4.4. Optimality for the Continuous Time Model

Define concave function $F_d(w) = V_d(w) - w$, in which V_d is the unique solution of (22)–(23). Similar to Propositions 1 and 2, we have the following results, which establish that $F_d(w)$ is indeed the principal's value function under the optimal contract.

Proposition 6. 1. *Starting from any* $w_0 \in [0, \hat{w}]$, *we have* $U(\Gamma_d^*(w_0)) = F_d(w_0)$.

2. *For any contract* Γ *that satisfies* (IC), *we have* $F_d(u(\Gamma, \bar{\nu})) \geq U(\Gamma)$.

Again, the proof for the inequality in Item 2 of Proposition 6 relies critically on the concavity of value function $F_d(w)$.

Summarizing the results above, we have the following main result for the paper.

Theorem 2. *Assume that* $\rho \geq r$. *Let concave function* $V_d(w)$ *and* $\hat{w} \in [0, \bar{w}]$ *be uniquely determined by* (22)–(23). *Further define* $F_d(w) = V_d(w) - w$ *and* $w_d^* \in \arg\max_{w \in [0, \hat{w}]} F_d(w)$. *Then,* $\Gamma_d^*(w_d^*)$ *according to Definition 2 is an optimal incentive-compatible dynamic contract. That is,* $U(\Gamma_d^*(w_d^*)) \geq U(\Gamma)$ *for any contract* Γ *that satisfies* (IC).

It is clear that there are three main distinctions between the optimal contract Γ_d^* when $\rho > r$, compared with Γ^* when $\rho = r$.

First, the threshold $\hat{w}$ at which payments are made is lower than $\bar{w}$, which implies that the agent is getting paid earlier. As discussed in the beginning of this section, this is intuitive because the agent is less patient and, therefore, values earlier payments.

Second, in contrast with the equal discount case, the threshold $\hat{w}$ is no longer an absorbing state. This is because under the promise-keeping constraint (DWd), the promised utility w_t keeps decreasing (with derivative $\rho(w_t - \bar{w})$) as long as w_t is lower than $\bar{w}$. When $\rho > r$, we have $\hat{w} < \bar{w}$, and the promised utility w_t never goes above $\hat{w}$. Therefore, the promised utility cannot stay at $\hat{w}$ and always decreases between arrivals. When $r = \rho$, in which case $\hat{w} = \bar{w}$, however, the dynamics (DW) implies that if w_t reaches the upper threshold $\bar{w}$, the slope $\rho(w_t - \bar{w}) = 0$. Therefore, the promised utility always stays at $\bar{w}$, which is an absorbing state.

Third, because of the previous discussion, in the case of $\rho > r$, the contract terminates within finite time with probability one. The exact number of arrivals before termination, however, is random and unbounded.

Finally, it is worth mentioning that in the equal discount case, condition (2) guarantees that the principal always wants to induce effort. However, when the time discounts between the two players are different, it may be optimal for the principal to allow the agent to shirk.

A rigorous description of the optimal contract that allows shirking appears to be quite intricate. Zhu (2013) derives such an optimal contract for the Brownian motion model. It is unclear if an exact description of the optimal contract in Zhu (2013), which involves frequently starting and stopping the effort process, is managerially relevant, or is true for the Poisson process model. Therefore, we leave studies of optimal or close to optimal contracts allowing shirking to future research.

5. Generalizations and Extensions

In this section, we present a number of generalizations and extensions of our model and results.

5.1. Alternative Interpretations of Contract Termination

In the basic interpretation of the current model, the principal permanently terminates the contract with the agent when the promised utility reaches zero. In fact, alternative interpretations allow our model to represent a wider range of real-life situations, so that the model is not as limited as it may appear at the first sight.

Terminating the contract with an agent is equivalent to keeping the agent working while not inducing costly effort. Therefore, our model can be slightly extended for an agent to have two levels of effort. The lower effort level is observable, incurs a cost rate $\underline{c}$, and yields a lower arrival rate $\underline{\mu}$; the higher effort level, on the other hand, is not observable, incurs a cost $\underline{c} + c$, and yields a higher arrival rate μ. In this case, the principal always pays a flow of salary at rate $\underline{c}$, as long as $\underline{c} < R\underline{\mu}$. According to our results, any additional payments are instantaneous (bonus) and after arrivals. Therefore, the corresponding optimal contract follows the same structure as Γ^* and Γ_d^* for the equal and different time discount rate cases, respectively, with the additional flow payment $\underline{c}$.

In this case, the principal never "terminates" the contract with the agent. Instead, if the performance score reaches zero, the agent reverts to working at the lower effort level and loses any future opportunity of earning bonuses.

This new interpretation of our slightly generalized model is somewhat related to salary-plus-commission sales compensation plans and cost-plus-reimbursement contracts that are commonly seen in practice.

The key reason for committing to potentially terminating the contract with an agent is to provide a threat for bad performances. In practice, the principal may have access to a pool of agents and, therefore, can replace a terminated agent with a new one. We discuss this extension of our model in more details in the next subsection.

Finally, if we do not allow the principal to ever terminate the contract with the agent, contract $\bar{\Gamma}$, which pays a bonus β for each arrival, is, in fact, optimal. (The proof is in Appendix F.1.)

Proposition 7. *Contract $\bar{\Gamma}$ maximizes $U(\Gamma)$ among incentive-compatible contracts Γ with $\tau = \infty$.*

That is, if the agent cannot be terminated, then the principal can do no better than start paying bonuses from the very beginning.

5.2. Replacing the Agent on Termination

Having the opportunity of replacing the focal agent with a new one is beneficial to the principal. This is because instead of facing the lower arrival rate $\underline{\mu}$ after terminating the previous agent, the principal can continuously enjoy the high effort level and arrival rate μ.

With only a single agent, always achieving this high arrival rate μ requires that the principal rely on contract $\bar{\Gamma}$, as indicated in Proposition 7. Having access to replacements, however, allows the principal to do better with dynamic contracts.

Consider an extension of our current model, which allows the principal to replace an agent with a new one at a fixed cost k. That is, the principal may choose to incur this replacement cost, or not if k is too high. Formally, at the stopping time τ, the principal chooses between replacing the agent with a new contract, which yields utility $U(\Gamma) - k$, and terminating the contract without replacement, which yields utility $\underline{v}$. Therefore, this extension departs from the basic model introduced in Section 2 only in the definition of the principal's utility (6):

$$U(\Gamma) = \mathbb{E}\left[\int_0^{\tau} e^{-rt}(R\,dN_t - dL_t) + e^{-r\tau}\max\{U(\Gamma) - k, \underline{v}\}\right]. \tag{24}$$

If the fixed cost k is high enough, such that $F_d(w_d^*) - k < \underline{v}$, in which F_d is the principal's value function described in Theorem 2, and w_d^* is its maximizer, then it is clear that the principal never takes this replacement option. For example, for the model parameters behind Figure 4, as long as the fixed cost k of replacing an agent is higher than $F_d(w_d^*) - \underline{v} = 0.0056$, the principal has no incentive to replace a terminated agent.

If k is not so high, replacement makes sense. The corresponding analysis in Section 3 for the equal discount case remains largely intact. We only need to revise the boundary conditions for the HJB equations (12) and (13) accordingly. Specifically, we can establish an updated version of Lemma 3, which states that there exists a unique function V that satisfies the HJB equation (13) and $V(0) = \max\{V(w^*) - w^* - k, \underline{v}\}$, in which $w^* \in \arg\max_w V(w) - w$ is the initial promised utility of a new contract. (See Proposition 9 in Appendix F.2.) Furthermore, function V is increasing and strictly concave on $[0, \bar{w}]$. This allows us to prove all of the other results for the equal time discount case. Consequently, the corresponding optimal contract retains the same structure in this extension. The only change in the optimal contract is the initial promised utility level w^* and, therefore, the length of the initial internship period for each agent.

The update for the different time discount case of Section 4 is more substantial. We need to start with defining the HJB equation (22) with the new boundary condition $V_d(0) = \max\{V_d(w_d^*) - w_d^* - k, \underline{v}\}$, in which $w_d^* \in \arg\max_w V_d(w) - w$, and show that its solution exists and is unique. (See Proposition 10 in Appendix F.2.) After this, all of the dynamic programming operators and their fixed points in this section are defined with the boundary condition values $\underline{v}$ or $\underline{v}_\delta$ replaced with $V_d(0)$. Because all current results in this section do not depend specifically on the boundary condition value (due to the free parameter $\underline{\mu}$ in the base case), they still hold with this new boundary condition. Therefore, function V_d is concave, and the fixed point of the discrete time approximation dynamic programming operator Υ_δ uniformly converges to V_d. The concavity of function V_d further implies its optimality. This proof strategy allows us to establish that strategy Γ_d^* defined in Definition 2 is still optimal for the continuous time model with the appropriate initial promised utility. Here, not only the initial promised utility w^*, but also the upper bound $\hat{w}$, which is obtained together with function V_d from the differential equation (22) with the new initial condition, increases with the replacement cost k.

Technicalities aside, it is important to consider the intuitions behind. First, our basic model formulation does not rule out the possibility that the principal pays off the promised utility w_t in a lump-sum payment to terminate or replace an agent. Our result shows, however, that this is never optimal. That is, the principal is always better off running the promised utility down to zero before terminating the agent, with or without replacement.

Second, if k is not too high, the principal is able to always enjoy the high arrival rate μ forever while replacing agents along the way. Therefore, replacement opportunities allow the principal to impose a threat to induce continuous effort from agents without resorting to contract $\bar{\Gamma}$—i.e., paying β for each arrival. Specifically, consider the equal discount rate case. According to the optimal dynamic contract structure, the principal starts payment only after an agent is lucky enough to drive the promised utility to the threshold $\bar{w}$. All previous arrivals, including all arrivals under previous agents, are not paid for.

5.3. Agent More Patient Than Principal

In most of the dynamic contract literature that we cited, the principal is assumed to be either more patient than or as patient as the agent. This is because, in practice, the principal, who is the contract designer, often enjoys a stronger financial position than the agent.

Our paper is one of the few that compare the cases between $r = \rho$ and $r < \rho$. To complete the discussion, it is worth considering what happens if the agent is more patient, or $r > \rho$.

In this case, for any payment that may occur at a time t, the principal is always better off delaying it to a future time, while paying an interest according to the agent's time discount rate. The principal is better off because the additional interest payment in the future is discounted more heavily by the impatient principal. Such a "time arbitrage" opportunity implies that the principal should never pay the agent while pushing all potential payments into the future.

The objective function in (18) helps us seeing this through in the discrete time approximation. Because $\gamma - \varrho < 0$ now, the ideal future promised utility w_N and w_A should be set to positive infinity. Limited liability, again, dictates that current period payments I and l should be set at zero, while (discounted) future promise ϱw_N at $w - \underline{w}_\delta$ and ϱw_A at $w - \underline{w}_\delta + \beta$, following ($\text{PK}_\delta$) and binding ($\text{IC}_\delta$).

In the continuous time, following (DW), the agent's promised utility w_t should change according to the following condition,

$$dw_t = \rho(w_t - \bar{w})\,dt + \beta\,dN_t,$$

which keeps increasing after w_t reaches $\bar{w}$.

Such a model clearly does not reflect reality. To make the model more useful, we may follow the approach of Myerson (2015) and introduce an arbitrary upper bound, call it $\bar{\mathcal{W}}$, on the promised utility w_t. After the promised utility w_t reaches $\bar{\mathcal{W}}$, the principal has to pay to keep w_t at the upper bound. In this case, payments include not only an instantaneous payment (bonus) β for each arrival, but also a flow payment (salary) to maintain the agent's promised utility at $\bar{\mathcal{W}}$. The salary level, l, must satisfy

$$\frac{l + \mu\beta - c}{\rho} = \bar{\mathcal{W}},$$

which implies that $l = \rho(\bar{\mathcal{W}} - \bar{w})$. The salary, therefore, depends on the modeling choice of the upper bound $\bar{\mathcal{W}}$. The upper bound $w = \bar{\mathcal{W}}$ and $w = 0$ are the two absorbing states in this case.

6. Concluding Remarks

We study a basic dynamic moral hazard problem in continuous time over an infinite time horizon, in which the agent's effort increases the arrival rate of a Poisson process. The optimal contract structure is simple and intuitive, and depends on whether the agent is as patient as or less patient than the principal. In particular, the agent's promised utility (performance score) is a sufficient statistic of the entire history of arrival times. Although we allow general payment structures, the optimal contract only involves compensating the agent with bonuses on arrivals. This makes intuitive sense because bonuses on arrival motivate the agent to exert effort for higher frequency of compensations. The bonus is set at a level that makes the agent indifferent between exerting effort or not—that is, the incentive compatibility constraint is binding at optimality. This is consistent with long-held intuitions from linear optimization. Furthermore, the optimal contract delays actual payments until an arrival carries the agent's promised utility above a threshold. This delay, clearly, is beneficial to the principal. It does not dilute the agent's incentive to exert effort, because the agent's future promised utility takes an upward jump at each arrival.

Our analysis reveals a key difference between optimal contracts for good outcomes, where efforts increase the arrival rate, versus bad outcome, where efforts decrease the arrival rate. With bad outcomes, to maintain incentive compatibility, a bad outcome requires the agent's promised utility to take a downward drop. However, limited liability prevents such a drop when the agent's promised utility is already lower than the magnitude of the drop. Therefore, even in continuous time limit, such as in Biais et al. (2010) and Myerson (2015), the incentive-compatibility constraint and promise-keeping constraint are not simultaneously satisfied when the agent's promised utility is lower than a threshold corresponding to our β. This lack of severe punishment significantly undermines the effectiveness of adjusting payments to induce effort in this region. As a result, when the promised utility is running low, the optimal contract in Myerson (2015) involves randomizing between terminating the agent and continuation at a higher promised utility. In Biais et al. (2010), smaller promised utility is achieved by allowing the principal to adjust the size of the firm. In fact, without randomization/downsizing the firm in the model, the principal's value function may not be concave. Consequently, the incentive compatibility constraint may not be binding at optimality, rendering the optimal contract structure no longer tractable or interesting. In our continuous time model with good arrivals, the principal never needs such randomization. In fact, the concavification approach in our discrete time approximation of Section 4 is equivalent to randomization. We show that in the limit as the discrete time intervals approach zero, randomization/downsizing disappears from the optimal contract. This finding also highlights the importance of continuous time modeling in our setting: despite more involved analysis, abstracting the model into continuous time reveals simpler and cleaner results.

Our results and analysis also shed lights on some commonly used contract forms in practice that motivate higher arrivals rates over time. For instance, unobservable efforts are often evaluated by counting the cumulative number of arrivals during a fixed time window, instead of the random time window that we mentioned in the introduction. For example, a sales person's annual bonus level is often based on the number of clients brought in during the year. Our analysis and results shed some light on why these contract structures may not be optimal. The key issue is that a fixed time window may make it harder to incentivize an agent to fully exert effort toward the end of the window. Instead, toward the end of the time window, the agent has an incentive to continue exerting effort for more arrivals only if the number of the arrivals that have already occurred is "hanging in the balance." This may not be in the best interest of the principal.

Our model serves as a foundation for many dynamic contract design problems that involve time epochs in more general managerial settings. For example, in an equipment maintenance/repair setting, a principal needs to incentivize an agent to exert effort to repair whenever the equipment is down, so that episodes of equipment downtime are kept as short as possible. In this case, "good arrivals" correspond to the machine coming back online. More generally, in a vendor managed inventory (VMI) setting similar to the one described in Plambeck and Zenios (2003), a principal hires an agent to run a production process that can be modeled as a queuing control system. Each "arrival" in our paper corresponds to a "departure" in such a queuing system. According to an optimal contract, the agent's compensation depends on not only past departure times, but also queue lengths. We believe that the results and insights obtained in our paper shed light on designing the optimal payment contract schemes for these more general settings.

Finally, it is also interesting to study the case where the agent is risk averse, similar to considerations in Thomas and Worrall (1990) and Sannikov (2008).

Acknowledgments
The authors gratefully acknowledge contributions from Alessandro Arlotto to this project, Yongbo Xiao for pointing out several typos in an earlier version of the paper, and valuable comments and suggestions for improvements from the review team.

Appendix A. Summary of Notations

Model Parameters

R: revenue to the principal for each arrival.
$\underline{\mu}$ and μ: base case and high arrival rates, respectively.
c: cost of effort per unit of time.
ν and $\bar{\nu}$: generic and full effort process.
r and ρ: principal and agent's discount rates.

Contracts and Utilities

I and l: instantaneous and flow payments, respectively.
L: payment process $dL_t = I_t + l_t dt$.
τ: stopping time.
Γ: generic contract, $\Gamma = (L, \tau)$.
$\bar{\Gamma}$: a contract that pays β for each arrival.
Γ^* and Γ_d^*: optimal contracts for equal and different discount cases, respectively.
u and U: agent's and principal's utilities, respectively.
w_t and W_t: agent's performance score following Γ^* and continuation utility, respectively.
w_A and w_N: future promised utility if there is an arrival and no arrival, respectively, for the discrete time approximation.

Derived Quantities

β: defined in (1).
$\underline{v}$: defined in (5).
$\bar{w}$ and $\bar{U}$: defined in (7) and (8), respectively.
$\bar{V}$ and $\bar{V}_d$: defined in (14) and (23), respectively.
w^* and w_d^*: maximizers of function $F(w)$ and $F_d(w)$, respectively.
$\hat{w}$: determined in Proposition 4.

Value Functions

J: a generic function.
F: unique solution to differential equation (12) with boundary condition $F(0) = \underline{v}$.
V: societal value function for equal time discount, unique solution to differential equation (13) with boundary condition $V(0) = \underline{v}$.
V_d: societal value function for different time discount, unique solution to differential equation (22) with boundary condition $V_d(0) = \underline{v}$ and $V_d(\hat{w}) = \bar{V}_d$.
F_d: principal's value function for different time discount, $F_d(w) = V_d(w) - w$.

Discrete Time Approximation

δ: time interval.
γ and ϱ: principal's and agent's discount factors, respectively.
F_δ: principal's value function, defined in (16).

$\hat{w}_\delta$ and $\bar{w}_\delta$: thresholds defined in Proposition 3.
T_δ and Ξ_δ: dynamic programming operators defined in (18), and with a fixed policy from Proposition 3, respectively.
V_δ and ψ_δ: fixed point functions of operators T_δ and Ξ_δ, respectively.
Λ: concavification operator.
Υ_δ: convolution of the dynamic programming operator T_δ and concavification operator Λ, $\Upsilon_\delta = \Lambda \circ T_\delta$.
J_δ: fixed point function of operator Υ_δ.

Appendix B. Proofs in Section 3

B.1. Proof of Lemma 1

Suppose otherwise. There must exist a process $\hat{\nu} := \{\hat{\nu}_s\}_{s\ge 0}$, a time t, and a positive measure event $A \in \mathscr{F}_t^N$ such that $W_t(\Gamma, \bar{\nu}) < W_t(\Gamma, \hat{\nu})$. Construct a new effort process $\check{\nu} = \{\check{\nu}_s\}_{s\ge 0}$, such that $\check{\nu}_s = \bar{\nu}_s = \mu$ for $s < t$. At time t, if the state of the world is in A, then $\check{\nu}_s = \hat{\nu}_s$ for $s \ge t$; otherwise, $\check{\nu}_s = \bar{\nu}_s = \mu$ for $s \ge t$.

Note that, generally,

$$u(\Gamma, \nu) = \mathbb{E}[B_t(\Gamma, \nu) + e^{-\rho t} W_t(\Gamma, \nu)],$$

in which $B_t(\Gamma, \nu) = \int_0^{t\wedge\tau} e^{-\rho s}(dL_s - c\,\mathbb{1}\{\nu_s = \mu\}ds)$, which is $\mathscr{F}_t^N$-measurable.

From the construction of $\check{\nu}$, we have $B_t(\Gamma, \check{\nu}) = B_t(\Gamma, \bar{\nu})$ almost everywhere with respect to $\mathscr{F}_t^N$.

Therefore,

$$\begin{aligned} u(\Gamma, \check{\nu}) &= \mathbb{E}[B_t(\Gamma, \check{\nu})] + e^{-\rho t}\{\mathbb{P}(A)\mathbb{E}[W_t(\Gamma, \hat{\nu}) \mid A] \\ &\qquad + (1 - \mathbb{P}(A))\mathbb{E}[W_t(\Gamma, \bar{\nu}) \mid \neg A]\} \\ &> \mathbb{E}[B_t(\Gamma, \bar{\nu})] + e^{-\rho t}\{\mathbb{P}(A)\mathbb{E}[W_t(\Gamma, \bar{\nu}) \mid A] \\ &\qquad + (1 - \mathbb{P}(A))\mathbb{E}[W_t(\Gamma, \bar{\nu}) \mid \neg A]\} = u(\Gamma, \bar{\nu}). \end{aligned}$$

Therefore, Γ cannot satisfy (IC). Q.E.D.

B.2. Proof of Lemma 2

Here, we show a more general result for a contract $\Gamma' = (L', \tau')$ that is generated from a process $\{w_t\}_{t\ge 0}$ as follows:

$$dw_t = [\rho(w_t - \tilde{w})\,dt + \min\{\bar{w} - w_t, \beta\}dN_t]\mathbb{1}_{w_t > 0}, \qquad \text{(DW')}$$

in which $\tilde{w} \le \bar{w}$, and $w_0 \in [0, \bar{w}]$. The payment process $dL_t' = (w_t + \beta - \bar{w})^+ dN_t$ and the stopping time $\tau' = \min\{t: w_t = 0\}$. The counting process N in (DW') and dL_t' is generated form an effort process ν.

For any $t \in [0, \tau']$, we have

$$\begin{aligned} e^{-\rho t} w_t &= w_0 e^{\rho 0} + \int_0^t d(e^{-\rho s} w_s) \\ &= w_0 + \int_0^t w_s de^{-\rho s} + \int_0^t e^{-\rho s} dw_s \\ &= w_0 + \int_0^t w_s(-\rho)e^{-\rho s}\,ds \\ &\quad + \int_0^t e^{-\rho s}[\rho(w_s - \tilde{w})\,ds + \min\{\bar{w} - w_s, \beta\}dN_s] \\ &= w_0 + \int_0^t e^{-\rho s}[c\,ds + \beta(dN_s - \mu\,ds) - dL_s'], \end{aligned}$$

in which the third equality follows from (DW') and the fourth equality from the definition of L'.

Because w_t is bounded in $[0, \bar{w}]$, and $w_{\tau'} = 0$ if $\tau' < \infty$, we have $e^{-r\tau'} w_{\tau'} = 0$, and

$$e^{-\rho\tau'} w_{\tau'} = w_0 + \int_0^\tau e^{-\rho s}[c\,ds + \beta(dN_s - \mu\,ds) - dL_s'].$$

Therefore,

$$e^{-\rho t} w_t = \int_t^{\tau'} e^{-\rho s}[dL_s' - c\,ds - \beta(dN_s - \mu\,ds)].$$

Taking conditional expectation on both side, and noting that w_t is $\mathscr{F}_t^N$-adapted, we obtain

$$\begin{aligned} e^{-\rho t} w_t &= \mathbb{E}\left[\int_t^{\tau'} e^{-\rho s}[dL_s' - c\,ds - \beta(dN_s - \mu\,ds)] \,\middle|\, \mathscr{F}_t^N\right] \\ &= \mathbb{E}\left[\int_t^{\tau'} e^{-\rho s}[dL_s' - c\,\mathbb{1}_{\nu_s=\mu}\,ds - \beta(dN_s - \nu_s ds)] \,\middle|\, \mathscr{F}_t^N\right] \\ &= \mathbb{E}\left[\int_t^{\tau'} e^{-\rho s}(dL_s' - c\,\mathbb{1}_{\nu_s=\mu}\,ds) \,\middle|\, \mathscr{F}_t^N\right] = e^{-\rho t} W_t(\Gamma', \nu), \end{aligned}$$

in which the second equality follows from the identity (10) and the third equality from the (potentially nonhomogeneous) Poisson process, which implies that

$$\mathbb{E}[N_{\tau'} - N_t \mid \mathscr{F}_t^N] = \mathbb{E}\left[\int_t^{\tau'} \nu_s\,ds \,\middle|\, \mathscr{F}_t^N\right]. \quad \text{Q.E.D.}$$

B.3. Proof of Lemma 3

Recall that function V satisfies differential equation (13).

Case 1: $\bar{w} \le \beta$ ($\underline{\mu} \le r$). Rearrange Equation (13) as

$$(r + \mu)V(w) - rV'(w)(w - \bar{w}) + c - \mu R = \mu V(\bar{w}). \qquad \text{(B.1)}$$

Consider the above equation in $[0, \bar{w})$, it is a linear ordinary differential equation with boundary condition. The solution is

$$V(w) = V(\bar{w}) + c_1(\bar{w} - w)^{(r+\mu)/r} \quad \text{if } w \in [0, \bar{w}],$$

with $c_1 = (\underline{v} - \bar{V})\bar{w}^{-(r+\mu)/r} < 0$. (Our initial assumption of $R > \beta$ is equivalent to $\underline{v} < \bar{V}$.)

Then, with $V'(w) = -c_1(r + \mu)(\bar{w} - w)^{\mu/r}/r > 0$, $V''(w) = c_1(r + \mu)\mu(\bar{w} - w)^{(\mu - r)/r}/r^2 < 0$ for $w \in [0, \bar{w}]$. Hence, V is increasing and strictly concave in $[0, \bar{w}]$. Furthermore, it can be verified that $V(\bar{w}) = \bar{V}$, $V'(\bar{w}-) = 0$, and $V(w) = \bar{V}$ for $w \in [\bar{w}, \infty)$ solves (13).

Case 2: $\bar{w} > \beta$ ($\underline{\mu} > r$). Rearrange Equation (13) as

$$\begin{aligned} (r + \mu)V(w) - rV'(w)(w - \bar{w}) + c - \mu R &= \mu V(\bar{w}) \\ &\text{for } w \in [\bar{w} - \beta, \infty), \qquad \text{(B.2)} \\ (r + \mu)V(w) - rV'(w)(w - \bar{w}) + c - \mu R &= \mu V(w + \beta) \\ &\text{for } w \in (0, \bar{w} - \beta). \qquad \text{(B.3)} \end{aligned}$$

We then show the result according to the following steps.

1. Demonstrate the solution of (B.2) as a parametric function V_b, with parameter b.
2. Show that the solution of (B.3) is unique and twice continuously differentiable for any b, also called V_b.
3. Show that the V_b is convex and decreasing for $b > 0$ and concave and increasing for $b < 0$.

4. Show that $V_b(0)$ is increasing in b for $b<0$, which implies that the boundary condition $V_b(0)=\underline{v}$ uniquely determines b, and therefore the solution of the original differential equation.

Step 1. The solution to the linear ordinary differential equation (B.2) on $[\bar{w}-\beta,\bar{w})$ must have the following form, for any scalar b.

$$V_b(w)=\bar{V}+b(\bar{w}-w)^{(r+\mu)/r} \quad \text{for } w\in[\bar{w}-\beta,\bar{w}). \tag{B.4}$$

Also, define $V_b(w)=\bar{V}$ for $w\in[\bar{w},\infty)$, which satisfies (B.2), so that V_b is continuously differentiable on $[\bar{w}-\beta,\infty)$.

Step 2. Using (B.4) as the boundary condition, we show that differential equation (B.3) has a unique solution (also called $V_b(w)$, on $(0,\bar{w}-\beta)$), which is continuously differentiable. In fact, differential equation (B.3) is equivalent to a sequence of initial value problems over the intervals $[\bar{w}-(k+1)\beta,\bar{w}-k\beta)$, $k=1,2,\ldots$. This sequence of initial value problems satisfy the Cauchy–Lipschitz theorem and, therefore, bear unique solutions. Also, computing $V_b'(\bar{w}-\beta)$ from (B.4), and comparing it with (B.3), we see that V_b is continuously differentiable at $\bar{w}-\beta$, and therefore on $[0,\infty)$.

Further, deriving the expressions for $V_b''(w)$ following (B.3) and (B.4), respectively, confirms that V_b is twice continuously differentiable on $([0,\infty))$. In particular, (B.3) implies that

$$V_b''(w)=\frac{\mu[V_b'(w+\beta)-V_b'(w)]}{r(\bar{w}-w)}. \tag{B.5}$$

Step 3. Next, we argue that to satisfy the boundary condition $V_b(0)=\underline{v}$, we must have $b<0$. Equivalently, we show that if $b>0$, V_b must be convex and decreasing, which violates $V_b(0)=\underline{v}<\bar{V}=V_b(\bar{w})$. In fact, if $b>0$, (B.4) implies that V_b is decreasing and convex on $[\bar{w}-\beta,\bar{w})$, and therefore $V_b''(w)>0$ on this interval. Assume that there exists $\tilde{w}\in[0,\bar{w}-\beta)$, such that $V_b''(\tilde{w})\le 0$, then V_b twice continuously differentiable implies that there must $\tilde{w}=\max\{w\in[0,\bar{w}-\beta)\mid V_b''(w)=0\}$, and $V_b''(w)>0$, $\forall w>\tilde{w}$. Equation (B.5) implies that $V_b'(\tilde{w}+\beta)=V_b'(\tilde{w})$. However, it contradicts with

$$V_b'(\tilde{w}+\beta)=V_b'(w)+\int_0^\beta V_b''(\tilde{w}+x)\,dx>V_b'(\tilde{w}).$$

Therefore, we must have $V_b''(w)>0$, and V_b is decreasing on $[0,\bar{w})$ if $b>0$. In case $b=0$, $V_b(w)$ is a constant following (B.3) and (B.4), which also contradicts the boundary condition. Therefore, we must have $b<0$.

The same logic implies that for $b<0$, V_b must best be increasing and strictly concave on $[0,\bar{w})$.

Step 4. Finally, we show that $V_b(0)$ is strictly increasing in b for $b<0$, which allows us to uniquely determine b that satisfies $V_b(0)=\underline{v}$. For any $b_1<b_2<0$, it can be verified that $V_{b_1}(w)<V_{b_2}(w)$, $V_{b_1}'(w)>V_{b_2}'(w)$, for $w\in[\bar{w}-\beta,\bar{w})$ from (B.4). We claim that $V_{b_1}'>V_{b_2}'$ $\forall w\in[0,\bar{w}]$. Otherwise, because $V_{b_1}-V_{b_2}$ is continuously differentiable, there must exist $w'=\max\{w\mid V_{b_1}'(w)=V_{b_2}'(w), w\in[0,\bar{w}-\beta)\}$ and $V_{b_1}'(w)>V_{b_2}'(w)$ $\forall w>w'$. Equation (B.3) implies that $\mu(V_{b_1}(w'+\beta)-V_{b_2}(w'+\beta))=(\rho+\mu)(V_{b_1}(w')-V_{b_2}(w'))$. However, it contradicts with

$$\begin{aligned}0&>V_{b_1}(w'+\beta)-V_{b_2}(w'+\beta)\\&=V_{b_1}(w')-V_{b_2}(w')+\int_0^\beta[V_{b_1}'(w'+x)-V_{b_2}'(w'+x)]\,dx.\end{aligned}$$

Therefore, we must have $V_{b_1}'(w)-V_{b_2}'(w)>0$, $\forall w\in[0,\bar{w})$, and it implies that $V_{b_1}(w)-V_{b_2}(w)<0$, $\forall w\in[0,\bar{w})$. This implies that $V_b(0)$ is strictly increasing in b for $b<0$. Because $\underline{v}(0)=\bar{V}=(\mu R-c)/r$ and $\lim_{a\to-\infty}V_b(0)<V_b(\bar{w}-\beta)=-\infty$, there must exist a unique $b^*<0$ such that $V_{a^*}(0)=\underline{v}=\underline{\mu}R/r$. And V_{a^*} is strictly concave and increasing in $[0,\bar{w}]$. Q.E.D.

B.4. Proof of Proposition 1

Consider a process w_t according to (DW), in which the counting process N_t is generated from the effort level μ. Following Itô's Formula for jump processes (see, e.g., Bass 2011, Theorem 17.5),

$$\begin{aligned}dF(w_t)&=F'(w_t)r(w_t-\bar{w})\,dt\\&\quad+[F(w_t+\min\{\bar{w}-w_t,\beta\})-F(w_t)]\,dN_t.\end{aligned} \tag{B.6}$$

Therefore, for any $T\le\tau^*$, we have

$$\begin{aligned}e^{-rT}F(w_T)&=e^{0r}F(w_0)+\int_0^T F(w_t)\,de^{-rt}+\int_0^T e^{-rt}\,dF(w_t)\\&=F(w_0)+\int_0^T e^{-rt}\{[rw_t-\mu\beta+c]F'(w_t)-rF(w_t)\}\,dt\\&\quad+\int_0^T e^{-rt}(F(w_t+\min\{\bar{w}-w_t,\beta\})-F(w_t))\,dN_t.\end{aligned}$$

Applying Equation (12) to replace $F'(w_t)$, we have

$$\begin{aligned}e^{-rT}F(w_T)&=F(w_0)\\&\quad+\int_0^T e^{-rt}[F(\min\{w_t+\beta,\bar{w}\})-F(w_t)](dN_t-\mu\,dt)\\&\quad+\int_0^T e^{-rt}[(w+\beta-\bar{w})^+-R]\mu\,dt\\&=F(w_0)+\int_0^T e^{-rt}[F(\min\{w_t+\beta,\bar{w}\})-F(w_t)\\&\qquad+R-(w_t+\beta-\bar{w})^+](dN_t-\mu\,dt)\\&\quad-\int_0^T e^{-rt}(R\,dN_t-dL_t^*).\end{aligned}$$

Taking expectation on both sides, and noting that $F(w_{\tau^*})=F(0)=\underline{\mu}R/r$, we have

$$\begin{aligned}U(\Gamma^*)=F(w_0)+\mathbb{E}\bigg[\int_0^T e^{-rt}&[F(\min\{w_t+\beta,\bar{w}\})-F(w_t)\\&+R-(w_t+\beta-\bar{w})^+](dN_t-\mu\,dt)\bigg].\end{aligned} \tag{B.7}$$

Because

$$|F(\min\{w_t+\beta,\bar{w}\})-F(w_t)+R-(w_t+\beta-\bar{w})^+|<\infty,$$

the process $\{M_s\}_{s\ge0}$, defined as

$$\begin{aligned}M_s:=\int_0^s e^{-rt}&[F(\min\{w_t+\beta,\bar{w}\})-F(w_t)\\&+R-(w_t+\beta-\bar{w})^+](dN_t-\mu\,dt),\end{aligned}$$

is a martingale, which implies that the expectation term in (B.7) is zero following the optional stopping theorem, and hence the result. Q.E.D.

B.5. Proof of Proposition 2

First, we show the following technical lemma, similar to Lemma 1 in Biais et al. (2010).

Lemma 6. *For any contract Γ and effort process ν, the agent's continuation utility W_t (defined in (4)) satisfies the following differential equation.*

$$dW_t(\Gamma,\nu) = [rW_t(\Gamma,\nu) + c\,\mathbb{1}\{\nu_t=\mu\}]dt - H_t(\Gamma,\nu)[\nu_t dt - dN_t] - dL_t \quad t\in[0,\tau), \tag{B.8}$$

in which the counting process N is generated from the effort process ν and $\{H_t\}_{t\ge 0}$ is an $\mathcal{F}^N$-predictable process.

In particular, contract Γ satisfies (IC) if and only if $H_t(\Gamma,\bar\nu)\ge\beta$ for all $t\ge 0$.

Proof. For a generic contract Γ and effort process ν, we introduce the agent's total expected utility conditioned on the information available at time t as the following $\mathcal{F}_t^N$-adapted random variable,

$$u_t(\Gamma,\nu) = \mathbb{E}\left[\int_0^\tau e^{-rs}(dL_s - c\,\mathbb{1}\{\nu_s=\mu\}ds)\,\middle|\,\mathcal{F}_t^N\right] = \int_0^{t\wedge\tau} e^{-rs}(dL_s - c\,\mathbb{1}\{\nu_s=\mu\}ds) + e^{-rt}W_t(\Gamma,\nu). \tag{B.9}$$

Therefore, $u_0(\Gamma,\nu)=u(\Gamma,\nu)$.

Process $\{u_t\}_{t\ge0}$ is an $\mathcal{F}^N$-martingale. Define process

$$M_t^\nu = N_t - \int_0^t \nu_s\,ds, \tag{B.10}$$

which is also an $\mathcal{F}^N$-martingale. Following martingale representation theorem, there exists an $\mathcal{F}^N$-predictable process $H(\Gamma,\nu)=\{H_t(\Gamma,\nu)\}_{t\ge0}$ such that

$$u_t(\Gamma,\nu) = u_0(\Gamma,\nu) + \int_0^{t\wedge\tau} e^{-rs}H_s(\Gamma,\nu)\,dM_s^\nu, \quad \forall t\ge 0. \tag{B.11}$$

Differentiating (B.9) and (B.11) with respect to t yields

$$du_t = e^{-rt}H_t(\Gamma,\nu)\,dM_t^\nu = e^{-rt}(dL_t - c\,\mathbb{1}\{\nu_t=\mu\}dt) - re^{-rt}W_t(\Gamma,\nu)\,dt + e^{-rt}dW_t(\Gamma,\nu),$$

which implies (B.8).

Denote $\tilde u_t(\Gamma,\nu',\nu)$ to be a $\mathcal{F}_t^N$-measurable random variable, representing the agent's total payoff following an effort process ν' before time t and ν after t—that is,

$$\tilde u_t(\Gamma,\nu',\nu) = \int_0^{t\wedge\tau} e^{-rs}(dL_s - c\,\mathbb{1}\{\nu'=\mu\}ds) + e^{-rt}W_t(\Gamma,\nu). \tag{B.12}$$

Therefore,

$$\tilde u_0(\Gamma,\nu',\nu) = u_0(\Gamma,\nu) = u(\Gamma,\nu), \tag{B.13}$$

$$\mathbb{E}[\tilde u_\tau(\Gamma,\nu',\nu)\mid\mathcal{F}_0^N] = u(\Gamma,\nu'), \quad\text{and} \tag{B.14}$$

$$\mathbb{E}[\tilde u_t(\Gamma,\nu,\nu)\mid\mathcal{F}_0^N] = u(\Gamma,\nu), \quad \forall t\ge0. \tag{B.15}$$

For any given sample trajectory $\{N_s\}_{0\le s\le t}$ and effort processes ν and $\bar\nu$,

$$\begin{aligned}
\tilde u_t(\Gamma,\nu,\bar\nu) &= u_t(\Gamma,\bar\nu) + \int_0^{t\wedge\tau} e^{-rs}c(1-\mathbb{1}\{\nu_s=\mu\})\,ds \\
&= u_0(\Gamma,\bar\nu) + \int_0^{t\wedge\tau} e^{-rs}H_s(\Gamma,\bar\nu)\,dM_s^{\bar\nu} + \int_0^{t\wedge\tau} e^{-rs}c(1-\mathbb{1}\{\nu_s=\mu\})\,ds \\
&= u_0(\Gamma,\bar\nu) + \int_0^{t\wedge\tau} e^{-rs}H_s(\Gamma,\bar\nu)\,dM_s^{\nu} + \int_0^{t\wedge\tau} e^{-rs}H_s(\Gamma,\bar\nu)(\nu_s-\mu)\,ds + \int_0^{t\wedge\tau} e^{-rs}c(1-\mathbb{1}\{\nu_s=\mu\})\,ds \\
&= u_0(\Gamma,\bar\nu) + \int_0^{t\wedge\tau} e^{-rs}H_s(\Gamma,\bar\nu)\,dM_s^{\nu} + \int_0^{t\wedge\tau} e^{-rs}\Delta\mu(\mathbb{1}\{\nu_s=\mu\}-1)[-\beta+H_s(\Gamma,\bar\nu)]\,ds,
\end{aligned}$$

where the first equality follows from (B.9), the second equality follows (B.11), the third equality follows from (B.10), and the fourth equality follows from (1) and straightforward derivations.

Consider any two times $t'<t$,

$$\begin{aligned}
\mathbb{E}[\tilde u_t(\Gamma,\nu,\bar\nu)\mid\mathcal{F}_{t'}^N] &= u_0(\Gamma,\bar\nu) + \int_0^{t'\wedge\tau} e^{-rs}H_s(\Gamma,\bar\nu)\,dM_s^\nu + \int_0^{t'\wedge\tau} e^{-rs}\Delta\mu(\mathbb{1}\{\nu_s=\mu\}-1)[-\beta+H_s(\Gamma,\bar\nu)]\,ds \\
&\quad + \mathbb{E}\left[\int_{t'\wedge\tau}^{t\wedge\tau} e^{-rs}\Delta\mu(\mathbb{1}\{\nu_s=\mu\}-1)[-\beta+H_s(\Gamma,\bar\nu)]\,ds\,\middle|\,\mathcal{F}_{t'}^N\right] \\
&= \tilde u_{t'}(\Gamma,\nu,\bar\nu) + \mathbb{E}\left[\int_{t'\wedge\tau}^{t\wedge\tau} e^{-rs}\Delta\mu(\mathbb{1}\{\nu_s=\mu\}-1)[-\beta+H_s(\Gamma,\bar\nu)]\,ds\,\middle|\,\mathcal{F}_{t'}^N\right].
\end{aligned} \tag{B.16}$$

If $H_s(\Gamma,\bar\nu)\ge\beta$ for all $s\ge0$, then (B.16) implies that $\mathbb{E}[\tilde u_t(\Gamma,\nu,\bar\nu)\mid\mathcal{F}_{t'}^N]\le\tilde u_{t'}(\Gamma,\nu,\bar\nu)$. Therefore, $\{\tilde u_t\}_{t\ge0}$ is a supermartingale. Take $t'=0$, we have

$$u(\Gamma,\bar\nu) = \tilde u_0(\Gamma,\nu,\bar\nu) \ge \mathbb{E}[\tilde u_\tau(\Gamma,\nu,\bar\nu)\mid\mathcal{F}_0^N] = u(\Gamma,\nu),$$

in which the first equality follows from (B.13) and the last equality from (B.14), while the inequality follows from the Doob's optional stopping theorem. Therefore, the agent prefers the effort process $\bar\nu$ to any other effort process ν, which implies that Γ satisfies (IC) if $H_s(\Gamma,\bar\nu)\ge\beta$ for all $s\ge0$.

If, on the other hand, $H_s(\Gamma,\bar\nu)<\beta$ for s belonging to a positive measure set $\Omega\subset[0,t]$, define effort process ν to be such that

$$\nu_s = \begin{cases}\mu, & H_s(\Gamma,\bar\nu)\ge\beta\\ \underline{\mu}, & H_s(\Gamma,\bar\nu)<\beta\end{cases} \quad \text{for } s\in[0,t], \quad\text{and } \nu_s=\mu \quad \text{for } s>t.$$

Therefore, $\tilde u_t(\Gamma,\nu,\bar\nu)=\tilde u_t(\Gamma,\nu,\nu)$, and $\mathbb{E}[\int_0^{t\wedge\tau} e^{-rs}\Delta\mu(\mathbb{1}\{\nu_s=\mu\}-1)[-\beta+H_s(\Gamma,\bar\nu)]\,ds\mid\mathcal{F}_0^N]>0$. Equation (B.16) then implies that $\mathbb{E}[\tilde u_t(\Gamma,\nu,\bar\nu)\mid\mathcal{F}_0^N]>\tilde u_0(\Gamma,\nu,\bar\nu)$ and, therefore,

$$u(\Gamma,\bar\nu) = \tilde u_0(\Gamma,\nu,\bar\nu) < \mathbb{E}[\tilde u_t(\Gamma,\nu,\bar\nu)\mid\mathcal{F}_0^N] = \mathbb{E}[\tilde u_t(\Gamma,\nu,\nu)\mid\mathcal{F}_0^N] = u(\Gamma,\nu),$$

in which the last equality follows from (B.15). Therefore, the agent prefers effort process ν' over $\bar\nu$, which implies that Γ does not satisfy (IC) if $H_s(\Gamma,\bar\nu)<\beta$ for some $s\in[0,t]$. Q.E.D.

Now, we prove Proposition 2. In this proof, we suppress W_t and H_t's dependence on $(\Gamma,\bar\nu)$ in the expressions, in which contract $\Gamma = (L, \tau)$ satisfies (IC). Further, express a general payment process as

$$dL_t = dS_t^C + dS_t^D + (W_t + \beta - \bar{w})^+ dN_t, \tag{B.17}$$

in which S^C is a continuous process and S^D captures the jumps.

Now, consider the decomposition of $e^{-rT}F(W_T)$ for any $T \in [0, \tau]$. Following the Itô's Formula for jump processes (see, e.g., Bass 2011, Theorem 17.5) and (B.8), we obtain

$$\begin{aligned} &e^{-rT}F(W_T)\\ &= e^{-r0}F(W_0) + \int_0^T e^{-rt}\{[rW_t - \mu H_t + c]F'(W_t) - rF(W_t)\}\,dt\\ &\quad - \int_0^T e^{-rt}F'(W_t)\,dS_t{}^C\\ &\quad + \int_0^T e^{-rs}\,\{F[W_s + (H_s - (W_s + \beta - \bar{w})^+)dN_s - dS^D] - F(W_t)\}\\ &= F(W_0) + \int_0^T e^{-rt}[F(W_t + (H_t - (W_t + \beta - \bar{w})^+))\\ &\qquad - F(W_t)](dN_t - \mu dt) + A_1 + A_2, \end{aligned} \tag{B.18}$$

where A_1 is a standard integral with respect to time,

$$\begin{aligned} A_1 = \int_0^T &e^{-rt}\{[rW_t - \mu H_t + c]F'(W_t) - rF(W_t)\\ &- \mu[F(W_t) - F(W_t + H_t - (W_t + \beta - \bar{w})^+)]\}\,dt, \end{aligned} \tag{B.19}$$

A_2 accounts for changes in cumulative transfers,

$$\begin{aligned} A_2 = &-\int_0^T e^{-rt}F'(W_t)dS_t{}^C\\ &+ \int_0^T e^{-rt}\,[F(W_t + (H_t - (W_t + \beta - \bar{w})^+)dN_t - dS_t^D)\\ &\qquad - F(W_t + (H_t - (W_t + \beta - \bar{w})^+)dN_t)]. \end{aligned}$$

Further consider first A_1.

$$\begin{aligned} A_1 &= \int_0^T e^{-rt}\{[rW_t - \mu H_t + c]V'(W_t) - c - rV(W_t)\\ &\quad - \mu[V(W_t) - V(W_t + H_t - (W_t + \beta - \bar{w})^+)\\ &\quad - (W_t + \beta - \bar{w})^+]\}\,dt\\ &\le \int_0^T e^{-rt}\{[rW_t - \mu\beta + c]V'(W_t) - c - rV(W_t)\\ &\quad - \mu[V(W_t) - V(W_t + \beta - (W_t + \beta - \bar{w})^+)\\ &\quad - (W_t + \beta - \bar{w})^+]\}\,dt\\ &= \mu\int_0^T e^{-rt}[(W_t + \beta - \bar{w})^+ - R]\,dt, \end{aligned} \tag{B.20}$$

in which the inequality follows from function V being increasing concave and $V'(w) = 0$ for $w \ge \bar{w}$, and $H_t \ge \beta$ for incentive-compatible Γ from Lemma 6; the last equality follows from (13).

Now, consider A_2. Because F is concave, we have

$$\begin{aligned} &\int_0^T e^{-rt}\,[F(W_t + (H_t - (W_t + \beta - \bar{w})^+)\,dN_t - dS_t^D)\\ &\qquad - F(W_t + (H_t - (W_t + \beta - \bar{w})^+)\,dN_t)]\\ &\le \int_0^T -e^{-rt}F'(W_{t-} + (H_t - (W_t + \beta - \bar{w})^+)\,dN_t)\,dS_t^D\\ &\le \int_0^T e^{-rt}\,dS_t^D, \end{aligned}$$

in which the last inequality follows from $F' \ge -1$ (following Lemma 3). Apply $F' \ge -1$ again, we establish that

$$A_2 \le \int_0^T e^{-rt}\,dS_t{}^C + \int_0^T e^{-rt}\,dS_t^D. \tag{B.21}$$

Substituting the upper bounds (B.20) and (B.21) for A_1 and A_2, respectively, into (B.18), and applying (B.17), we have

$$F(W_0) \ge e^{-rT}F(W_T) + \int_0^T e^{-rt}(R\,dN_t - dL_t) + \Psi_T, \tag{B.22}$$

in which

$$\begin{aligned} \Psi_s = \int_0^s &e^{-rt}[F(W_t) - F(W_t + H_t - (W_t + \beta - \bar{w})^+)\\ &- R + (W_t + \beta - \bar{w})^+]\,(dN_t - \mu\,dt) \end{aligned} \tag{B.23}$$

is a martingale, because for each $t \ge 0$,

$$\begin{aligned} &\mathbb{E}\left[\int_0^{t\wedge\tau} |e^{-rs}[F(W_s) - F(W_s + H_s - (W_s + \beta - \bar{w})^+)\right.\\ &\qquad \left. - R + (W_s + \beta - \bar{w})^+]|\,ds\right]\\ &\le \mathbb{E}\left[\int_0^{t\wedge\tau} (e^{-rs}|R + 2\bar{V}| + W_s)\,ds\right] < \infty. \end{aligned}$$

Following the optional stopping theorem, $\mathbb{E}[\Psi_T] = 0$.

Take expectation in (B.22) conditional on $\mathcal{F}_0^N$, we have

$$\begin{aligned} F(W_0) &\ge \mathbb{E}\left[e^{-r\tau}F(W_\tau) + \int_0^\tau e^{-rt}\,(R\,dN_t - dL_t)\right]\\ &= \mathbb{E}\left[e^{-r\tau}F(0) + \int_0^\tau e^{-rt}\,(R\,dN_t - dL_t)\right] = U(\Gamma). \end{aligned}$$

Note that $W_0(\Gamma, \bar{v}) = u(\Gamma, \bar{v})$, which completes the proof. Q.E.D.

Appendix C. Proofs and Supplementary Materials for Section 4.1

C.1. Proof of Proposition 3

For simplicity of exposition, we suppress the dependence on J in this proof. Optimization of $(\mathrm{T}_\delta J)(w)$ is a concave maximization over a set of linear constraints. Therefore, the KKT condition is necessary and sufficient for optimality. It is clear that $(I_\delta^*, l_\delta^*, w_{A\delta}^*, w_{N\delta}^*) \in \Pi_w$ with the constraint (IC_δ) binding. Because $w_{A\delta}^* \le \hat{w}_\delta$ and $w_{N\delta}^* \le \hat{w}_\delta$, concavity of J implies that $\gamma J(w_{A\delta}^*)' - (\gamma - \varrho) \ge 0$ and $\gamma J(w_{A\delta}^*)' - (\gamma - \varrho) \ge 0$. Therefore, it remains to verify that there exist $y \ge 0$ and z such that

$$\begin{aligned} &\varrho(\mu\delta z - y) = \mu\delta[\gamma J(w_{A\delta}^*)' - (\gamma - \varrho)], \quad \text{and}\\ &\varrho[(1 - \mu\delta)z + y] = (1 - \mu\delta)[\gamma J(w_{N\delta}^*)' - (\gamma - \varrho)], \end{aligned}$$

which is satisfied with

$$\begin{aligned} y &= \frac{\gamma}{\varrho}\mu\delta(1 - \mu\delta)[J(w_{N\delta}^*)' - J(w_{A\delta}^*)'],\\ z &= \frac{\gamma}{\varrho}[\mu\delta J(w_{A\delta}^*)' + (1 - \mu\delta)J(w_{N\delta}^*)'] - \frac{\gamma - \varrho}{\varrho}. \end{aligned}$$

Because $w_{A\delta}^*$ and $w_{N\delta}^*$ remain the same for $w \ge \bar{w}_\delta(J)$, the nondecreasing function $(\mathrm{T}_\delta J)(w)$ reaches its maximum at $\bar{w}_\delta(J)$. Q.E.D.

C.2. Proof of Lemma 4

Consider the optimal decision (I, l, w_A, w_N) for the optimization $(\mathrm{T}_\delta J)(w)$ for $w \geq \underline{w}_\delta$. For any $w' > w$, it is clear that $(I + w' - w, l + (w' - w), w_A, w_N) \in \Pi_{w'}$ and the objective function value remains the same, which implies that $(\mathrm{T}_\delta J)(w)$ is nondecreasing on $[\underline{w}_\delta, \infty)$. Furthermore, $(\beta, 0, 0, 0) \in \Pi_{\underline{w}}$ and the corresponding objective function value is higher than $\underline{v}_\delta$, which implies monotonicity on $[0, \infty)$.

For any w^1 and w^2 in $[\underline{w}_\delta, \infty)$ and $\lambda \in [0,1]$, denote (I^i, l^i, w_A^i, w_N^i) to represent the optimal solution to the optimization problem $(\mathrm{T}_\delta J)(w^i)$ for $i = 1, 2$. Further define

$$w^\lambda = \lambda w^1 + (1-\lambda) w^2, \quad I^\lambda = \lambda I^1 + (1-\lambda) I^2,$$
$$l^\lambda = \lambda l^1 + (1-\lambda) l^2, \quad w_A^\lambda = \lambda w_A^1 + (1-\lambda) w_A^2,$$
$$\text{and} \quad w_N^\lambda = \lambda w_N^1 + (1-\lambda) w_N^2.$$

It is clear that $(I^\lambda, l^\lambda, w_A^\lambda, w_N^\lambda) \in \Pi_{w^\lambda}$. Furthermore,

$$\begin{aligned}
&\lambda(\mathrm{T}_\delta J)(w^1) + (1-\lambda)(\mathrm{T}_\delta J)(w^2) \\
&\quad = \mu\delta[R - I^\lambda + \gamma(\lambda J(w_A^1) + (1-\lambda) J(w_A^2))] \\
&\qquad + (1-\mu\delta)[-l^\lambda + \gamma(\lambda J(w_N^1) + (1-\lambda) J(w_N^2))] \\
&\qquad - (\gamma/\varrho)c\delta - (\gamma/\varrho - 1)(w^\lambda - \mu\delta I^\lambda - (1-\mu\delta) l^\lambda) \\
&\quad \leq \mu\delta[R - I^\lambda + \gamma J(w_A^\lambda)] + (1-\mu\delta)[-l^\lambda + \gamma J(w_N^\lambda)] \\
&\qquad - (\gamma/\varrho)c\delta - (\gamma/\varrho - 1)(w^\lambda - \mu\delta I^\lambda - (1-\mu\delta) l^\lambda) \\
&\quad \leq (\mathrm{T}_\delta J)(w^\lambda),
\end{aligned}$$

where the first inequality follows from the concavity of function J, and the second inequality follows from the feasibility of w^λ. Q.E.D.

C.3. A Lemma on the Convergence of Operator Υ_δ

For any positive integer k, define $\Upsilon_\delta^k J = \Upsilon_\delta(\Upsilon_\delta^{k-1} J)$, starting with $\Upsilon_\delta^1 = \Upsilon_\delta$.

Lemma 7. *The following limit exists and is the same for any bounded function J:*

$$J_\delta = \lim_{k\to\infty} \Upsilon_\delta^k J.$$

Furthermore, function J_δ has the following properties:

(1) *J_δ is the unique solution to the recursive equation $J = \Upsilon_\delta J$ among bounded functions;*

(2) *J_δ is increasing and concave, and $J_\delta(0) = \underline{v}_\delta$.*

Proof. Because both operators T_δ and Λ are monotone, the operator Υ_δ is also monotone. Furthermore, it is not hard to verify that $\Upsilon_\delta J \leq \Upsilon_\delta(J + \xi\mathbf{e}) \leq \Upsilon_\delta J + \gamma\xi\mathbf{e}$, for any positive constant ξ and the constant function $\mathbf{e}$ that takes value one. Therefore, Υ_δ is a contraction mapping, which implies the existence and uniqueness of the limit, and the solution to the fixed point equation.

Lemma 4 implies that function J_δ is increasing and concave, and $J_\delta(0) = \underline{v}_\delta$. Q.E.D.

C.4. More on Concavification

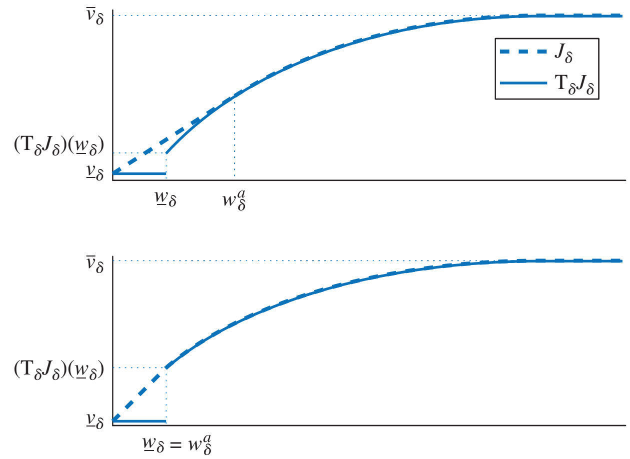
Figure C.1 depicts the concavification procedure in the limit. In particular, consider function $J = \mathrm{T}_\delta J_\delta$. There are two possibilities for the relationship between functions J (the solid curves) and $J_\delta = \Lambda J$ (the dashed curves). The top subfigure depicts the first possibility, where the ratio $[J_\delta(\underline{w}_\delta) - \underline{v}_\delta]/\underline{w}_\delta$ is less than the slope of the J function at $\underline{w}_\delta$. Consequently, the concavified function J_δ is strictly higher than J on the interval $[\underline{w}_\delta, w_\delta^a)$, in which

$$w_\delta^a := \min\{w\colon J_\delta(w) = (\mathrm{T}_\delta J_\delta)(w), w \geq \underline{w}_\delta\}. \tag{C.1}$$

And we have $w_\delta^a > \underline{w}_\delta$. The bottom subfigure of Figure C.1, on the other hand, depicts another possibility, where the ratio $[J_\delta(\underline{w}_\delta) - \underline{v}_\delta]/\underline{w}_\delta$ is larger than the slope of the J function at $\underline{w}_\delta$. In this case, with the same definition (C.1) for w_δ^a, we have $w_\delta^a = \underline{w}_\delta$.

Later, we show that as δ approaches zero, so does w_δ^a.

Figure C.1. (Color online) An Illustration of the Concavification Procedure



Appendix D. Proof and Supplementary Materials in Section 4.2

D.1. Proof of Proposition 4

We show the proposition in the following two steps.

1. For any such that $\tilde{w} < \bar{w}$, there exists a unique continuously differentiable function $V_{\tilde{w}}$ that satisfies (22) with $\tilde{w}$ in place of $\hat{w}$, and the boundary condition at $\tilde{w}$ while ignoring the boundary condition at zero.

2. Function $V_{\tilde{w}}(0)$ is strictly decreasing in $\tilde{w}$, which implies that there exists a unique $\hat{w} \in (0, \bar{w})$ such that $V_{\hat{w}}$ satisfies (22) with the boundary condition $V_{\hat{w}}(0) = \underline{v}$.

Step 1. For any $\tilde{w} < \bar{w}$, consider the following differential equation

$$\begin{aligned}
&\rho(\tilde{w} - w)V_{\tilde{w}}'(w) + (r+\mu)V_{\tilde{w}}(w) \\
&\quad = r\bar{V} + \mu V_{\tilde{w}}(\min\{w+\beta, \tilde{w}\}) - (\rho - r)w, \\
&\text{with the boundary condition } V_{\tilde{w}}(\tilde{w}) = \bar{V} + \frac{r-\rho}{r}\tilde{w}.
\end{aligned} \tag{D.1}$$

Case 1: $\tilde{w} \leq \beta$ ($\mu \leq r$). Rearrange Equation (D.1) as

$$(r+\mu)V_{\tilde{w}}(w) - \rho V_{\tilde{w}}'(w)(w - \tilde{w}) - r\bar{V} + (\rho - r)w = \mu V_{\tilde{w}}(\tilde{w}). \tag{D.2}$$

The above function is a linear ordinary differential equation with boundary condition that has a unique solution.

Case 2: $\tilde{w} > \beta$. Rearrange Equation (D.1) as

$$\begin{aligned}
&(r+\mu)V_{\tilde{w}}(w) - \rho V_{\tilde{w}}'(w)(w - \tilde{w}) - r\bar{V} + (\rho - r)w \\
&\quad = \mu V_{\tilde{w}}(\tilde{w}) \quad \text{for } w \in [\tilde{w} - \beta, \tilde{w}],
\end{aligned} \tag{D.3}$$

$$\begin{aligned}
&(r+\mu)V_{\tilde{w}}(w) - \rho V_{\tilde{w}}'(w)(w - \tilde{w}) - r\bar{V} + (\rho - r)w \\
&\quad = \mu V_{\tilde{w}}(w+\beta) \quad \text{for } w \in (0, \tilde{w} - \beta).
\end{aligned} \tag{D.4}$$

The solution to the linear ordinary differential equation (D.3) must have the following form,

$$V_{\tilde{w}}(w) = \frac{\rho - r}{r+\mu-\rho}(\tilde{w}-w) + \frac{\mu V_{\tilde{w}}(\tilde{w}) + r\bar{V} + (r-\rho)\tilde{w}}{r+\mu} + b_{\tilde{w}}(\tilde{w}-w)^{(r+\mu)/\rho} \quad \text{if } w \in [\tilde{w}-\beta, \tilde{w}],$$

with $b_{\tilde{w}} = ((r-\rho)/(r+\mu-\rho))(\rho/(r+\mu))(\bar{w}-\tilde{w})^{(\rho-r-\mu)/\rho}$ from the boundary condition. And we have $V'_{\tilde{w}}(\tilde{w}-) = 0 = V'_{\tilde{w}'}(\tilde{w}+) = 0$. Therefore, Equation (D.4) is reduced to a sequence of initial value problems over the intervals $[\tilde{w} - (k+1)\beta, \tilde{w} - k\beta)$, $k \in \mathbb{N}\setminus\{0\}$ that satisfy the assumptions of the Cauchy–Lipschitz theorem and, therefore, bear unique continuously differentiable solutions.

Finally, the delayed differential equation (D.1) becomes an ordinary differential equation for $w > \tilde{w}$, which bears a unique solution. It is easy to verify that $V_{\tilde{w}}(w)$ maintaining at $V_{\tilde{w}}(\tilde{w})$ solves this differential equation and, therefore, is the solution to (D.1) for $w \geq \tilde{w}$.

Step 2. In this step, we show that $V_{\tilde{w}}(0)$ is strictly decreasing in $\tilde{w}$. It is sufficient to show that if $\tilde{w}_1 < \tilde{w}_2 \in (0, \bar{w})$, then $V_{\tilde{w}_1}(w) > V_{\tilde{w}_2}(w)$ for $w \in [0, \tilde{w}_1]$. An equivalent argument is: if $\tilde{w}_1 < \tilde{w}_2 \in (0, \bar{w})$ and $\tilde{w}_2 - \tilde{w}_1 \leq \beta/2$, then $V_{\tilde{w}_1}(w) > V_{\tilde{w}_2}(w)$ for $w \in [0, \tilde{w}_1]$.

Because for any $\tilde{w} \in (0, \bar{w})$, $V_{\tilde{w}}$ is continuously differentiable, $V_{\tilde{w}_1}(w) - V_{\tilde{w}_2}(w)$ must also be continuously differentiable. In the interval $[\tilde{w}_1 - \beta/2, \tilde{w}_1]$, $V_{\tilde{w}_1}(w) > V_{\tilde{w}_2}(w)$ and $V'_{\tilde{w}_1}(w) < V'_{\tilde{w}_2}(w)$ since $b_{\tilde{w}_1} > b_{\tilde{w}_2}$. In the interval $[\tilde{w}_1, \tilde{w}_1 + \beta/2]$, on the other hand, $V_{\tilde{w}_1}(w) > V_{\tilde{w}_2}(w)$ and $0 = V'_{\tilde{w}_1}(w) < V'_{\tilde{w}_2}(w)$ since $V_{\tilde{w}_1}(\tilde{w}_1) > V_{\tilde{w}_2}(\tilde{w}_2)$.

Now, we claim that $V'_{\tilde{w}_1}(w) < V'_{\tilde{w}_2}(w)\ \forall\, w \in [0, \tilde{w}_1]$. Otherwise, because $V_{\tilde{w}_1}(w) - V_{\tilde{w}_2}(w)$ is continuously differentiable there must exists $\tilde{w}' = \max\{w \mid V'_{\tilde{w}_1}(w) - V'_{\tilde{w}_2}(w) = 0\}$. Then, we obtain $\mu(V_{\tilde{w}_1}(\tilde{w}'+\beta) - V'_{\tilde{w}_2}(\tilde{w}'+\beta)) = (r+\mu)(V_{\tilde{w}_1}(\tilde{w}') - V_{\tilde{w}_2}(\tilde{w}'))$. However, it contradicts with

$$0 < V_{\tilde{w}_1}(\tilde{w}'+\beta) - V_{\tilde{w}_2}(\tilde{w}'+\beta) = V_{\tilde{w}_1}(\tilde{w}') - V_{\tilde{w}_2}(\tilde{w}') + \int_0^\beta V'_{\tilde{w}_1}(\tilde{w}'+x) - V'_{\tilde{w}_2}(\tilde{w}'+x)\,dx.$$

Then, we must have $V'_{\tilde{w}_1}(w) < V'_{\tilde{w}_2}(w), V'_{\tilde{w}_1}(w) > V'_{\tilde{w}_2}(w)\ \forall\, w \in [0, \tilde{w}_1]$. Hence, $V_{\tilde{w}}(0)$ is strictly decreasing with $\tilde{w}$. And if $\tilde{w} = 0$, then the boundary condition states that $V_{\tilde{w}}(0) = \bar{V} > \underline{v}$. If we let $\tilde{w} \to \bar{w}$, then $b_{\tilde{w}} \to -\infty$, $V_{\tilde{w}}(\tilde{w}-\beta) \to -\infty$. Continuity of $V_{\tilde{w}}$ implies $V_{\tilde{w}}(0) \to -\infty$.

Therefore, there must exist a unique $\hat{w} \in (0, \bar{w})$ that satisfies the boundary additional condition $V_{\hat{w}}(0) = \underline{v}$. Q.E.D.

D.2. Additional Results Before Showing Proposition 5

To prove Proposition 5, we first argue that the limit of ψ_δ, which is the value function of repeatedly using policy $(I^*_\delta, l^*_\delta, w_{A\delta}^*, w_{N\delta}^*)_{J_\delta}$, exists and uniformly converges to function V_d, the solution to the differential equation (22)–(23), a step that resembles Lemma 7 of Biais et al. (2007). Define norm $\|\cdot\|_\delta$ as the L_∞ norm for a function on domain $[\underline{w}_\delta, \infty)$.

Proposition 8. *We have*

$$\hat{w} = \lim_{\delta\downarrow 0} \hat{w}_\delta(V_d) = \lim_{\delta\downarrow 0} \bar{w}_\delta(V_d), \quad \text{and} \quad \lim_{\delta\downarrow 0} \frac{\|\Xi_\delta V_d - V_d\|_\delta}{\delta} = 0, \tag{D.5}$$

which further implies that

$$\lim_{\delta\downarrow 0} \|V_d - \psi_\delta\|_0 = 0. \tag{D.6}$$

Proof. First, it is worth mentioning that because V_d is bounded and nondecreasing, and remains a constant on $[\hat{w}, \bar{w}]$, (22); and its derivative is nonnegative and bounded from above by $V'_d(\tilde{w})$ for some $\tilde{w} \in [0, \hat{w}]$.

Following the definition of $\hat{w}_\delta(J)$ and $w_\delta(J)$, it is easy to establish that $\hat{w} = \lim_{\delta\downarrow 0} \hat{w}_\delta(V_d) = \lim_{\delta\downarrow 0} \bar{w}_\delta(V_d)$. To prove the second part of (D.5), define $\Delta^\delta(w) = \Xi_\delta V_d(w) - V_d(w)$. Therefore, for $w \in [\underline{w}_\delta, \bar{w}_\delta(V_d)]$, we have

$$\begin{aligned}
\Delta^\delta(w) &= (\mu R - c)\delta + \mu\delta\left[\gamma V_d\left(\min\left\{\hat{w}_\delta(V_d), \frac{w - \underline{w}_\delta + \beta}{\varrho}\right\}\right)\right.\\
&\quad\left. - (\gamma-\varrho)\min\left\{\hat{w}_\delta(V_d), \frac{w-\underline{w}_\delta+\beta}{\varrho}\right\}\right]\\
&\quad + (1-\mu\delta)\left[\gamma V_d\left(\frac{w-\underline{w}_\delta}{\varrho}\right) - (\gamma-\varrho)\frac{w-\underline{w}_\delta}{\varrho}\right] - V_d(w)\\
&= \delta\left[(\mu R - c) + \mu V_d\left(\min\left\{\hat{w}_\delta(V_d), \frac{w-\underline{w}_\delta+\beta}{\varrho}\right\}\right)\right]\\
&\quad + (1-\mu\delta)\gamma\left[V_d\left(\frac{w-\underline{w}_\delta}{\varrho}\right) - V_d(w)\right]\\
&\quad + [(1-\mu\delta)\gamma - 1]V_d(w) - (1-\mu\delta)(\gamma-\varrho)\frac{w-\underline{w}_\delta}{\varrho}\\
&\quad - (\gamma-\varrho)\mu\delta\min\left\{\hat{w}_\delta(V_d), \frac{w-\underline{w}_\delta+\beta}{\varrho}\right\}.
\end{aligned}$$

Replacing $\mu R - c$ using (22), noting that $\underline{w}_\delta = \delta\rho\bar{w}$, and using the mean value theorem and the upper bound $V'_d(\tilde{w})$, we have

$$\begin{aligned}
&\frac{|\Delta^\delta(w)|}{\delta}\\
&\leq \mu\left|V_d\left(\min\left\{\hat{w}_\delta(V_d), \frac{w-\delta\rho\bar{w}+\beta}{\varrho}\right\}\right) - V_d(\min\{\hat{w}, w+\beta\})\right|\\
&\quad + V'_d(\tilde{w})\left|\left[1 - \frac{(1-\mu\delta)\gamma}{\varrho}\right]\rho\bar{w} + \left[(1-\mu\delta)\gamma\frac{1-\varrho}{\delta\varrho} - \rho\right]w\right|\\
&\quad + \left|r + \mu(1-\gamma) - \frac{1-\gamma}{\delta}\right|V_d(w) + (\gamma-\varrho)\frac{1-\mu\delta}{\varrho}\rho\bar{w}\\
&\quad + \frac{1-\mu\delta}{\varrho}\left|\frac{\gamma-\varrho}{\delta} - (\rho - r)\right|w\\
&\quad - (\gamma-\varrho)\mu\min\left\{\hat{w}_\delta(V_d), \frac{w-\underline{w}_\delta+\beta}{\varrho}\right\}\\
&\leq \mu V'_d(\tilde{w})\max\left\{|\hat{w}_\delta(V_d) - \bar{w}|,\right.\\
&\qquad\left.\left|\frac{\bar{w}_\delta(V_d) - \delta\rho\bar{w}+\beta}{\varrho} - [\bar{w}_\delta(V_d)+\beta]\right|, \left|\frac{\beta - \delta\rho\bar{w}}{\varrho} - \beta\right|\right\}\\
&\quad + V'_d(\tilde{w})\left[\left|1 - \frac{(1-\mu\delta)\gamma}{\varrho}\right|\rho\bar{w} + \left|(1-\mu\delta)\gamma\frac{1-\varrho}{\delta\varrho} - \rho\right|\bar{w}_\delta(V_d)\right]\\
&\quad + \left|r + \mu(1-\gamma) - \frac{1-\gamma}{\delta}\right|\bar{V}_d + (\gamma-\varrho)\frac{1-\mu\delta}{\varrho}\rho\bar{w}\\
&\quad + \frac{1-\mu\delta}{\varrho}\left|\frac{\gamma-\varrho}{\delta} - (\rho-r)\right|\bar{w}_\delta(V_d)\\
&\quad + (\gamma-\varrho)\mu\min\left\{\hat{w}_\delta(V_d), \frac{\bar{w}_\delta(V_d) - \underline{w}_\delta + \beta}{\varrho}\right\}.
\end{aligned} \tag{D.7}$$

Note that the right-hand side of inequality (D.7) does not depend on w and approaches zero with δ.

For $w \in [\bar{w}_\delta(V_d), \infty)$, we have

$$\Delta^\delta(w) = (\mu R - c)\delta + \gamma V_d(\hat{w}_\delta(V_d)) - (\gamma-\varrho)\hat{w}_\delta(V_d) - V_d(w).$$

Therefore, following (23), and again using the mean value theorem, for some w' between $\hat{w}$ and $\hat{w}_\delta(V_d)$, we have

$$\begin{aligned}\frac{\Delta^\delta(w)}{\delta} &\le (\mu R - c) + \frac{\gamma}{\delta}[V_d(\hat{w}) + V_d'(w')|\hat{w}_\delta(V_d) - \hat{w}|] \\ &\quad - \frac{(\gamma-\varrho)}{\delta}\hat{w}_\delta(V_d) - \frac{V_d(\hat{w})}{\delta} \\ &= (\mu R - c) + \frac{\gamma - 1}{\delta}\left(\bar{V} + \frac{r-\rho}{r}\hat{w}\right) - \frac{\gamma-\varrho}{\delta}\hat{w}_\delta(V_d) \\ &\quad + \frac{\gamma}{\delta}|\hat{w} - \hat{w}_\delta(V_d)|V_d'(\hat{w}_\delta(V_d)). \end{aligned} \tag{D.8}$$

The right-hand side of (D.8) does not depend on w, and converges to zero with δ, because

$$\lim_{\delta\downarrow 0}(\mu R - c) + \frac{\gamma-1}{\delta}\left(\bar{V} + \frac{r-\rho}{r}\hat{w}\right) - \frac{\gamma-\varrho}{\delta}\hat{w}_\delta(V_d) = 0, \quad \text{and}$$
$$\lim_{\delta\downarrow 0}\frac{\gamma}{\delta}|\hat{w} - \hat{w}_\delta(V_d)|V_d'(\hat{w}_\delta(V_d)) = \lim_{\delta\downarrow 0}|\hat{w} - \hat{w}_\delta(V_d)|V_d''(\hat{w}_\delta(V_d)) = 0.$$

Following similar logic,

$$\begin{aligned}\frac{\Delta^\delta(w)}{\delta} &\ge (\mu R - c) + \frac{\gamma}{\delta}[V_d(\hat{w}) - V_d'(w'')|\hat{w}_\delta(V_d) - \hat{w}|] \\ &\quad - \frac{\gamma-\varrho}{\delta}\hat{w}_\delta(V_d) - \frac{1}{\delta}V_d(\bar{w}_\delta(V_d)) \\ &\ge (\mu R - c) + \frac{\gamma-1}{\delta}\left(\bar{V} + \frac{r-\rho}{r}\hat{w}\right) - \frac{\gamma-\varrho}{\delta}\hat{w}_\delta(V_d) \\ &\quad - \frac{\gamma}{\delta}V_d'(\hat{w}_\delta(V_d))|\hat{w}_\delta(V_d) - \hat{w}| \\ &\quad - \frac{1}{\delta}|\hat{w} - \bar{w}_\delta(V_d)|V_d'(\bar{w}_\delta(V_d)). \end{aligned} \tag{D.9}$$

Similarly, the right-hand side of (D.8) does not depend on w and converges to zero with δ.

Together with (D.7)–(D.9), we establish the relation $\lim_{\delta\to 0}\|\Xi_\delta V_d - V_d\|_\delta/\delta = 0$.

Finally, we prove (D.6). Following the triangle inequality and contraction property of the DP operator, we have

$$\begin{aligned}\|\psi_\delta - V_d\|_\delta &\le \|\Xi_\delta\psi_\delta - \Xi_\delta V_d\|_\delta + \|\Xi_\delta V_d - V_d\|_\delta \\ &\le \gamma\|\psi_\delta - V_d\|_\delta + \|\Xi_\delta V_d - V_d\|_\delta, \end{aligned} \tag{D.10}$$

which implies that

$$\|\psi_\delta - V_d\|_\delta \le \frac{\delta}{1-\gamma}\frac{\|\Xi_\delta V_d - V_d\|_\delta}{\delta},$$

the right-hand side of which converges to zero with δ, following (D.5). If

$$\|\psi_\delta - V_d\|_0 = \max_{w\in[\underline{w}_\delta,\infty)}\{|\psi_\delta(w) - V_d(w)|\} = \|\psi_\delta - V_d\|_\delta,$$

Equation (D.6) follows directly. If, on the other hand,

$$\|\psi_\delta - V_d\|_0 = \max_{w\in[0,\underline{w}_\delta)}\{|\psi_\delta(w) - V_d(w)|\}$$

along a subsequence of δ's that goes to zero, then $\|\psi_\delta - V_d\|_0 \le |\underline{v}_\delta - V_d(\underline{w}_\delta)|$, which also converges to zero. Therefore, ψ_δ converges to V_d uniformly along this subsequence. This completes the proof. Q.E.D.

Because the function ψ_δ is the value function of a particular policy, it is a lower bound of the optimal value function. Next, we establish that in the limit, ψ_δ and the concave upper envelope of the optimal value function, J_δ, converge uniformly to each other. This establishes that V_d is the limit of the optimal value function and is concave.

To this end, we first show the following lemma, which implies that the effect of concavification diminishes as δ approaches zero.

Lemma 8. *We have*

$$\lim_{\delta\downarrow 0}(\mathrm{T}_\delta J_\delta)(\underline{w}_\delta) = \underline{v}, \quad \lim_{\delta\downarrow 0} w_\delta^a = 0, \quad \lim_{\delta\downarrow 0} J_\delta(w_\delta^a) = \underline{v}, \tag{D.11}$$

and

$$\limsup_{\delta\downarrow 0}\frac{J_\delta(w_\delta^a) - \underline{v}_\delta}{w_\delta^a} \le \frac{r+\mu}{r}\frac{\Delta\mu R - c}{\mu\beta - c}. \tag{D.12}$$

Proof. Define

$$\begin{aligned}\bar{v}_\delta &= \frac{\delta[(\mu R - c) - (\gamma - \varrho)(\mu\beta - c)]}{1-\gamma}, \quad \text{and} \\ J(w) &= \begin{cases}(\bar{v}_\delta - \underline{v}_\delta)(w/\underline{w}_\delta) + \underline{v}_\delta, & 0 \le w < \underline{w}_\delta, \\ \bar{v}_\delta, & w \ge \underline{w}_\delta.\end{cases}\end{aligned} \tag{D.13}$$

It is clear that $\Upsilon_\delta J \le J$. Therefore, $J_\delta = \lim_{k\to\infty}\Upsilon_\delta^k J \le J$. Following (20) and the definition of $\underline{v}_\delta$,

$$\begin{aligned}(\mathrm{T}_\delta J_\delta)(\underline{w}_\delta) - \underline{v}_\delta &\le (\mathrm{T}_\delta J)(\underline{w}_\delta) - \underline{v}_\delta \\ &= \delta[(\mu R - c) + \mu(\gamma J(\underline{w}_\delta) - (\gamma-\varrho)\underline{w}_\delta)] \\ &\quad + (1-\mu\delta)\gamma J(0) - \underline{v}_\delta \le \bar{a}_\delta\delta,\end{aligned}$$

in which

$$\bar{a}_\delta = (\mu R - c) + \mu[\gamma\bar{v}_\delta - (\gamma-\varrho)\underline{w}_\delta] + \frac{(1-\mu\delta)\gamma - 1}{1-\gamma}\underline{\mu}R.$$

Note that

$$\lim_{\delta\downarrow 0}\bar{a}_\delta = \frac{r+\mu}{r}(\Delta\mu R - c).$$

Therefore, the continuity of $\bar{a}_\delta$ in δ implies that $\bar{a}_\delta$ is bounded from above for δ close enough to zero.

Due to the concavity of $\mathrm{T}_\delta J_\delta$, the above result, which also implies that $\lim_{\delta\downarrow 0}(\mathrm{T}_\delta J_\delta)(\underline{w}_\delta) = \underline{v}$, further implies that $\lim_{\delta\downarrow 0} w_\delta^a = 0$.

Finally, we show that there exists $\hat{a} \ge 0$ such that $J_\delta(w_\delta^a) \le \hat{a}w_\delta^a$ for any δ small enough. If $w_\delta^a = \underline{w}_\delta$, then the right derivative

$$\begin{aligned}\lim_{\delta\downarrow 0} J_\delta(0)' &= \lim_{\delta\downarrow 0}\frac{(\mathrm{T}_\delta J_\delta)(\underline{w}_\delta) - \underline{v}_\delta}{\underline{w}_\delta} \le \lim_{\delta\downarrow 0}\frac{\bar{a}_\delta}{\mu\beta - c} \\ &= \frac{(r+\mu)(\Delta\mu R - c)}{r(\mu\beta - c)}.\end{aligned} \tag{D.14}$$

Now, suppose $w_\delta^a > \underline{w}_\delta$. We therefore have $J_\delta(0)' = J_\delta(w_\delta^a)' = (\mathrm{T}_\delta J_\delta)(w_\delta^a)'$ as well as $J_\delta(w_\delta^a) = (\mathrm{T}_\delta J_\delta)(w_\delta^a)$. Define $a_\delta = J_\delta(0)'$. We have

$$\begin{aligned}\underline{v}_\delta + a_\delta w_\delta^a &= J_\delta(w_a) = (\mathrm{T}_\delta J_\delta)(w_\delta^a) \\ &\le \delta(\mu R - c) + \delta\mu\left[\gamma\bar{v}_\delta - (\gamma-\varrho)\frac{w_\delta^a - \underline{w}_\delta + \beta}{\varrho}\right] \\ &\quad + (1-\delta\mu)\left[\gamma\left(\underline{v}_\delta + a_\delta\frac{w_\delta^a - \underline{w}_\delta}{\varrho}\right) - (\gamma-\varrho)\frac{w_\delta^a - \underline{w}_\delta}{\varrho}\right],\end{aligned}$$

in which we use $\bar{v}_\delta$ as an upper bound for $J_\delta(w_A)$ and apply (20) for the optimal w_N and w_A. This further implies that

$$\frac{1}{\delta}\left(w_\delta^a-(1-\mu\delta)\gamma\frac{w_\delta^a-\underline{w}_\delta}{\varrho}\right)a_\delta \le (\mu R-c)+\gamma\mu\bar{v}_\delta-\frac{1-\gamma(1-\mu\delta)}{1-\gamma}\underline{\mu}R-\frac{\gamma-\varrho}{\delta}\frac{w_\delta^a-\underline{w}_\delta+\delta\mu\beta}{\varrho}.$$

As δ approaches zero, the inequality above implies that

$$\lim_{\delta\downarrow 0} a_\delta \le \frac{(r+\mu)(\Delta\mu R-c)}{r(\mu\beta-c)}. \tag{D.15}$$

Continuity of a_δ in δ implies that it is upper bounded by some $\hat{a}$ for all δ small enough.

Note that (D.14) and (D.15) imply (D.12). Q.E.D.

Now, we are ready to show that the upper and lower bounds J_δ and ψ_δ uniformly converges to be identical as δ approaches zero, which implies that the effect of concavification diminishes as δ approaches zero.

D.3. Proof of Proposition 5

Consider the following function,

$$\underline{J}_\delta(w)=\underline{v}_\delta+\max\{0, J_\delta(w)-J_\delta(w_\delta^a)\}.$$

Following Proposition 3, for any point $w\ge w_\delta^a$, we have

$$(I_\delta^*, l_\delta^*, w_{A\delta}^*, w_{N\delta}^*)_{J_\delta}(w)=(I_\delta^*, l_\delta^*, w_{A\delta}^*, w_{N\delta}^*)_{\underline{J}_\delta}(w).$$

Further define function

$$\phi_\delta(w)=(\Xi_\delta\phi_\delta)(w)=\lim_{k\to\infty}(\Xi_\delta^k\underline{J}_\delta)(w),$$

for $w\ge w_\delta^a$, while $\phi_\delta(w)=\underline{v}_\delta$ for $w\in[0,w_\delta^a)$. Function ϕ_δ is the value function of repeatedly applying the policy specified in Proposition 3 when $w\ge w_\delta^a$, while function ψ_δ is the value function of repeatedly using the same policy when $w\ge \underline{w}_\delta$. Therefore, ϕ_δ is a lower bound of ψ_δ.

Next, we show that $\underline{J}_\delta$ is also a lower bound of ψ_δ. Recall that $(\Xi_\delta J_\delta)(w)=J_\delta(w)$ for $w\ge w_\delta^a$. Therefore,

$$\begin{aligned}(\Xi_\delta\underline{J}_\delta)(w) &= -c\delta+\mu\delta[R+\gamma\underline{J}_\delta(w_{A\delta}^*)-(\gamma-\varrho)w_{A\delta}^*]\\ &\quad+(1-\mu\delta)[\gamma\underline{J}_\delta(w_{N\delta}^*)-(\gamma-\varrho)w_{N\delta}^*]\\ &= -c\delta+\mu\delta\{R+\gamma[J_\delta(w_{A\delta}^*)-J_\delta(w_\delta^a)]-(\gamma-\varrho)w_{A\delta}^*\}\\ &\quad+(1-\mu\delta)\{\gamma[J_\delta(w_{N\delta}^*)-J_\delta(w_\delta^a)]-(\gamma-\varrho)w_{N\delta}^*\}+\gamma\underline{v}_\delta\\ &=(\Xi_\delta J_\delta)(w)-\gamma(J_\delta(w_\delta^a)-\underline{v}_\delta)>J_\delta(w)-J_\delta(w_\delta^a)+v_\delta=\underline{J}_\delta(w).\end{aligned}$$

Therefore, we have $\Xi_\delta\underline{J}_\delta\ge\underline{J}_\delta$, which further implies that

$$\psi_\delta\ge\phi_\delta=\lim_{k\to\infty}\Xi_\delta^k\underline{J}_\delta\ge\underline{J}_\delta.$$

Finally, optimality implies that $V_\delta\ge\psi_\delta$. Together with $J_\delta\ge V_\delta$, we have

$$J_\delta\ge V_\delta\ge\psi_\delta\ge\phi_\delta\ge\underline{J}_\delta,$$

which implies that

$$\|J_\delta-\psi_\delta\|_0\le\|J_\delta-\underline{J}_\delta\|_0.$$

Because $J_\delta(w_\delta^a)\le a w_\delta^a$ following Lemma 8 and w_δ^a approaches zero following (D.11), which implies that $\lim_{\delta\to 0}\|J_\delta-\underline{J}_\delta\|_0=0$. Together with (D.6), we have the result. Q.E.D.

Appendix E. Proof in Section 4.4

E.1. Proof of Proposition 6

Item 1. The proof of Proposition 6.1 is parallel to that of Proposition 1, with $F(\cdot)$ replaced with $F_d(\cdot)$, and most $\bar{w}$ replaced with $\hat{w}$ in all occasions except in (B.6), which, in this proof, should be

$$\begin{aligned}dF_d(w_t)&=F_d'(w_t)\rho(w_t-\bar{w})\,dt\\&\quad+[F_d(w_t+\min\{\hat{w}-w_t,\beta\})-F_d(w_t)\,dN_t.\end{aligned}$$

The remainder of the proof logic is essentially identical to that of Proposition 1 and is omitted here.

Item 2. We first note that the technical Lemma 6 is about the agent's utility and, therefore, only involves the agent's discount rate. Therefore, it still holds here, with ρ in place of r in Equation (B.8).

The proof of Proposition 6, based on Lemma 6, is parallel to the proof of Proposition 2. In fact, the proof logic is identical if we replace $\bar{w}$, $F(\cdot)$, $\bar{V}$, and Γ with $\hat{w}$, $F_d(\cdot)$, $\bar{V}_d$, and Γ_d here, respectively. Therefore, again, we omit the detailed replication here. Q.E.D.

Appendix F. Supplementary Materials for Section 5

F.1. Proof of Proposition 7 in Section 5.1

According to the technical lemma 6 in Appendix B.5, under any incentive-compatible contract, the agent's continuation utility satisfies Equation (B.8) with $H_t(\Gamma,\bar{\nu})\ge\beta$. Rearranging Equation (B.8) and replacing ν with $\bar{\nu}$, we obtain

$$dW_t(\Gamma,\nu)=[rW_t(\Gamma,\nu)+c-\mu H_t(\Gamma,\bar{\nu})]dt+H_t(\Gamma,\bar{\nu})\,dN_t-dL_t\qquad t\in[0,\tau).$$

For any contract starting with agent's utility $W_t\le\bar{w}$, we have $rW_t(\Gamma,\nu)+c-\mu H_t(\Gamma,\bar{\nu})\le 0$. This implies that without an arrival, utility W_t keeps decreasing. Therefore, starting from any continuation utility below $\bar{w}$, there is a positive probability that the promised utility decreases to zero before an arrival, which contradicts the requirement of $\tau=\infty$.

Furthermore, Propositions 4 and 6 imply that $F_d(w)$ is decreasing for $w>\bar{w}$ and is the optimal principal's value function starting from the agent's initial utility w. Therefore, the initial w for the required optimal contract should be $\bar{w}$. The corresponding optimal contract is $\bar{\Gamma}$. Q.E.D.

F.2. Proofs and Supplementary Materials for Section 5.2

Proposition 9. *Differential equation* (13) *with boundary condition* $V(0)=\max\{V(w^*)-w^*-k,\,\underline{v}\}$, *in which* w^* *is a maximizer of* $V(w)-w$, *has a unique solution* $V(w)$ *on* $[0,\bar{w}]$, *which is increasing and strictly concave. Furthermore, we have*

$$V(w)=\bar{V}:=\frac{\mu R-c}{r},\quad\forall\, w\ge\bar{w}, \tag{F.1}$$

and w^* *is increasing in* k.

Proof. The proof of this lemma follows the proof of Lemma 3. If the solution to (13) with boundary condition $V(0)=\underline{v}$ satisfies $\underline{v}\ge V(w^*)-w^*-k$, then the result has been established. The rest of the proof focuses on the case $\underline{v}<V(w^*)-w^*-k$, in which the boundary condition is $k=V(w^*)-w^*-V(0)=F(w^*)-F(0)$, in which $F(w)=V(w)-w$.

Up to Step 4 of the proof of Lemma 3, we have established that for $b_1 < b_2 < 0$, $V'_{b_1} > V'_{b_2}$ for $w \in [0, \bar{w}]$. Then $F'_{b_1} > F'_{b_2}$ for $w \in [0, \bar{w}]$. Hence,

$$F_{b_2}(w^*_{b_2}) - F_{b_2}(0) = \int_0^{w^*_{b_2}} F'_{b_2}(w)\,dw < \int_0^{w^*_{b_2}} F'_{b_1}(w)\,dw = F_{b_1}(w^*_{b_2}) - F_{b_1}(0),$$

where $w^*_b = \arg\max_w \{F_b(w)\}$. Further, we have $w^*_{b_2} < w^*_{b_1}$ and $F'_{b_1}(w) > 0$ for $w \in [w^*_{b_2}, w^*_{b_1}]$. Therefore, $F_{b_1}(w^*_{b_2}) - F_{b_1}(0) < F_{b_1}(w^*_{b_1}) - F_{b_1}(0)$. Hence, $F_b(w^*_b) - F_b(0)$ decreases in b.

Finally, if we let b diverge to $-\infty$, $F_b(w^*_b) - F_b(0) \geq F_b(\bar{w}) - F_b(0) \to \infty$; on the other hand, as $b \to 0$, $F_b(w^*) - F_b(0) \to 0$. Therefore, there must be a unique value $b(k)$ such that $F_{b(k)}(w^*_{b(k)}) - F(0) = k$.

Finally, the above argument also implies that if $k_1 > k_2$, then $b(k_1) \leq b(k_2) < 0$, and $w^*_{b(k_1)} \geq w^*_{b(k_1)}$. Q.E.D.

For the different discount factor case, we have the following result.

Proposition 10. *If $r < \rho$, there exists a unique value $\hat{w} \in [0, \bar{w}]$ and a unique function V_d that satisfy the following differential equation on $[0, \hat{w}]$:*

$$0 = (r+\mu)V_d(w) - \mu V_d(\min\{w+\beta, \hat{w}\}) + \rho(\bar{w} - w)V'_d(w) + (c - \mu R) + (\rho - r)w, \tag{F.2}$$

with boundary conditions

$$V_d(0) = \max\{\underline{v}, V_d(w^*_d) - w^*_d - k\}, \quad \text{and } V_d(\hat{w}) = \bar{V}_d := \bar{V} + \frac{r-\rho}{r}\hat{w}. \tag{F.3}$$

Furthermore, the derivative $V'_d(\hat{w}) = 0$, and both $\hat{w}$ and w^* are increasing in k, in which w^* is the unique maximizer of $V_d(w) - w$.

Proof. The existence and uniqueness of the solution V_d is established following similar lines of arguments as in the proofs of Proposition 4 along with Proposition 9.

Further, following the proof of Proposition 4, we have if $\tilde{w}_1 < \tilde{w}_2$, then $V_{\tilde{w}_1} > V_{\tilde{w}_2}$, $V'_{\tilde{w}_1} < V_{\tilde{w}_2}$, and $F'_{\tilde{w}_1} < F'_{\tilde{w}_2}$. And $w^*_2 = \arg\max_w \{F_{\tilde{w}_2}(w)\} > w^*_1 = \arg\max_w \{F_{\tilde{w}_1}(w)\}$.

Hence, we have $F_{\tilde{w}_2}(w^*_2) - F_{\tilde{w}_2}(0) > F_{\tilde{w}_2}(w^*_1) - F_{\tilde{w}_2}(0) = \int_0^{w^*_1} F_{\tilde{w}_2}(w)\,dw > \int_0^{w^*_1} F_{\tilde{w}_1}(w)\,dw = F_{\tilde{w}_1}(w^*_1) - F_{\tilde{w}_1}(0)$.

Therefore, if $k_1 < k_2$, then $\hat{w}_1 < \hat{w}_2$ and $w^*_1 < w^*_2$. Q.E.D.

Appendix G. Computation

In this appendix, we document the procedures to compute the optimal value functions from the HJB equation.

G.1. Equal Time Discount

Divide the interval $[0, \bar{w}]$ into N intervals, such that $\delta = \bar{w}/N$ and $w_{(i)} = i\delta$, for $i = 0, \ldots, N$. Further define $M = \lceil \beta/\delta \rceil$, and $V_i = V(w_{(i)})$. Values $w_{(i)}$ with $i = N-M, \ldots, N-1$ correspond to $w \in [\bar{w} - \beta, \bar{w}]$ in the continuous time model. In this interval, the delayed differential equation (13) is simplified to one with a boundary condition $V(\bar{w})$ and without delay. Consequently, there exists a general solution,

$$V(w) = \bar{V} + b(\bar{w} - w)^{(r+\mu)/r}, \quad \text{for } w \in [\bar{w} - \beta, \bar{w}],$$

parameterized with a parameter b.

Taking advantage of this, we have the following system of linear equations with N equations and variables ($V_1, \ldots, V_{N-1}$ and b):

$$\begin{aligned} &(\mu + r)V_i + r(N-i)(V_{i+1} - V_i) - \mu V_{i+M} = \mu R - c, \\ &\qquad\qquad i = 0, \ldots, N-M-1, \\ &V_i = \bar{V} + b(\bar{w} - w_{(i)})^{(r+\mu)/r}, \quad i = N-M, \ldots, N-1 \\ &V_0 = \underline{v}, \quad V_N = \bar{V}. \end{aligned} \tag{G.1}$$

Following similar computational procedures, we can calculate the probability as well as the expected time for the promised utility to reach either the absorbing state $\bar{w}$ or zero, starting from an initial value w.

Denote t_i to represent the time it takes for the performance score to decrease from state $w_{(i)}$ to $w_{(i-1)}$, without an arrival in between. The differential equation (DW) with $dN_t = 0$ implies that $e^{-rt_i} = (\bar{w} - w_{(i)})/(\bar{w} - w_{(i-1)})$. Therefore, we have

$$\pi_i := e^{-\mu t_i} = \left(\frac{N-i}{N-i+1}\right)^{\mu/r}, \quad \text{and} \quad t_i = [\ln(N-i+1) - \ln(N-i)]/r, \tag{G.2}$$

in which π_i is the probability that an arrival occurs after time t_i. Denote P_i to represent the probability of stopping with contract termination, starting from an initial score $w_{(i)}$. (Therefore, $1 - P_i$ is the probability of reaching $\bar{w}$.) The sequence of P_i is calculated from the following system of linear equations.

$$\begin{aligned} &P_0 = 1, \quad P_N = 0, \\ &P_i = (1-\pi_i)P_{i+M} + \pi_i P_{i-1}, \quad i = 1, 2, \ldots, N-M-1, \\ &P_i = \pi_i P_{i-1}, \quad i = N-M,\, N-M+1, \ldots, N-1 \end{aligned} \tag{G.3}$$

We can also compute T_i, the expected time it takes for the promised utility to reach either absorbing state, starting from $w_{(i)}$, according to the following linear equation system:

$$\begin{aligned} &T_0 = 0, \quad T_N = 0, \\ &T_i = (1-\pi_i)\left(\frac{1}{\mu} + T_{i+M}\right) + \pi_i T_{i-1}, \\ &\qquad\qquad i = 1, 2, \ldots, N-M-1, \\ &T_i = (1-\pi_i)\frac{1}{\mu} + \pi_i T_{i-1}, \quad i = N-M, \ldots, N-1. \end{aligned} \tag{G.4}$$

G.2. Different Time Discount

In this case, because we do not know the exact value of $\hat{w}$, we have to search for it iteratively. Each iterations involves solving a linear system of equation, similar to the one presented in Section G.1. Specifically, the function $V_d(w)$ on the interval $[\hat{w} - \beta, \hat{w}]$ bears the following solution given boundary condition $V_d(\hat{w}) = \bar{V}_d$:

$$V_d(w) = \bar{V} + \frac{\rho - r}{r + \mu - \rho}(\bar{w} - w) + \frac{r - \rho}{r + \mu}\left(\frac{\mu}{r}\hat{w} + \bar{w}\right) + b(\hat{w})(\bar{w} - w)^{(r+\mu)/\rho},$$

in which the coefficient $b(\hat{w})$ is uniquely specified for a given $\hat{w}$ as

$$b(\hat{w}) = \frac{(r-\rho)\rho}{(r+\mu-\rho)(r+\mu)}(\bar{w} - \hat{w})^{(\rho-r-\mu)/\rho}.$$

Define N, M, and $w_{(i)}$, $i = 0, \ldots, N$ the same as in Section G.1. For a given $\hat{N} \in \{0, \ldots, N\}$, we solve the following system of linear equations with $\hat{N} - M$ variables $(V_0, \ldots, V_{\hat{N}-M-1})$ and constraints.

$$(\mu + r)V_i + \rho(N - i)(V_{i+1} - V_i) - \mu V_{i+M} = (\mu R - c) - (\rho - r)w_{(i)}, \quad i = 0, \ldots, \hat{N} - M - 1, \tag{G.5}$$

where $V_{\hat{N}} = \bar{V} + \frac{r-\rho}{r} w_{(\hat{N})}$, and

$$V_i = \bar{V} + \frac{\rho - r}{r + \mu - \rho}(\bar{w} - w_{(i)}) - \frac{(\rho - r)\mu}{(r+\mu)r} w_{(\hat{N})} + b(w_{(\hat{N})})(\bar{w} - w_{(i)})^{(r+\mu)/r}, \quad i = \hat{N} - M, \ldots, \hat{N} - 1.$$

Following Proposition 4, in an outer loop, we conduct a binary search for $\hat{N}$ such that V_0 is as close to $\underline{v}$ as possible.

Similar to the equal time discount case, we can also compute the expected time, T_i, to reach the only absorbing state, $w = 0$, from state $w_{(i)}$. This involves computing the following system of linear equations:

$$\begin{aligned} T_0 &= 0, \\ T_i &= (1 - \pi_i)(1/\mu + T_{i+M}) + \pi_i T_{i-1}, \quad i = 1, 2, \ldots, \hat{N} - M - 1, \\ T_i &= (1 - \pi_i)(1/\mu + T_{\hat{N}}) + \pi_i T_{i-1}, \quad i = \hat{N} - M, \ldots, \hat{N}. \end{aligned} \tag{G.6}$$

Endnotes

[1] Standard results, for example, Theorem 6.2.9 of Aplebaum (2009), ensures that the stochastic differential equation (DW) has a unique solution w_t for any given starting point $w_0 \in [0, \bar{w}]$. Furthermore, $w_t \in [0, \bar{w}]$ almost surely.

[2] The stochastic differential equation (DWd) has a unique solution w_t for any given starting point $w_0 \in [0, \hat{w}]$, again, following Theorem 6.2.9 of Aplebaum (2009), such that $w_t \in [0, \hat{w}]$ almost surely.

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