Dynamic Moral Hazard with Adverse Selection – probation, sign-on bonus and delayed payment

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We study dynamic contracts that incentivize an agent to exert effort to increase the arrival rate of a Poisson process, where both the effort cost and the effort level at any time are the agent's private information. The principal needs to offer a menu of contracts, such that each type of agent (with a different effort cost) chooses the corresponding item on the menu. When there are two agent types, the essential idea of our design is to offer the good/low-cost agent a probation contract under which the bad/high-cost agent does not work. Therefore the first arrival during the probation period becomes a screening device. Our dynamic setting requires the contracts to also include a potential sign-on bonus and delayed payments. This idea allows us to design both an upper bound of the principal's utility and a lower bound from an incentive compatible menu of contracts. When the model parameters satisfy certain conditions, the two bounds meet, which implies that our approach yields an optimal menu of dynamic contracts. When these conditions for optimal solution do not hold, numerical studies show that our design outperforms a naïve menu constructed from optimal dynamic contracts based on known-costs. Comparison between the upper and lower bounds further demonstrates that our contract design often yields an optimal solution, and performs very well in general even when not optimal. We further provide a generalization of our design for the case with multiple agent types.

Key words: dynamic, moral hazard, optimal control, jump process, adverse selection

1. Introduction

Business operations often need long term contracts to manage incentives over time. For example, Plambeck and Zenios (2000) and Tian et al. (2021) study optimal maintenance contracts that motivate an agent to work on reducing the proportion of time a machine breaks down. Sun and Tian (2018) considers a company motivating its sales team or R&D branch to increase the arrival rate of customers or breakthroughs. All the aforementioned studies focus on dynamic moral hazard issues in agency problems. These models assume that the agent's capability is common knowledge among the principal and the agent, while the only asymmetric information is the hidden action that the agent takes over time. In practice, however, the principal often does not know the agent's capability. For example, an employer (principal) may not know whether it is easy or hard for a sale representative (agent) to increase the arrival rate of customers. Similarly, an investor (principal) may not know the cost structure of an entrepreneur (agent). In academia, funding agencies often do not know exactly how much it takes a research project to bear fruit. In all these settings, the principal needs to motivate effort from the agent while not knowing the exact cost of effort. In this paper, we study such a problem that involves both dynamic moral hazard and adverse selection.

In particular, we consider a setting in which a risk-neutral principal hires a risk-neutral agent in order to increase the arrival rate of a Poisson process, similar to Sun and Tian (2018). It may be instructive to think of the principal as a company, and the agent as a sales representative, who can increase the Poisson arrival rate of customers. Each arrival brings a positive revenue to the principal. The instantaneous arrival rate is a constant when the agent exerts full effort and bears its cost. Below full effort, the arrival rate is proportional to the agent's effort cost. Therefore, the principal and the agent's incentives are misaligned without a contract. A distinct feature of our model, compared with the previous literature, is that the agent's effort cost rate is private information, which represents heterogeneity in a potential agent's capability.

Following standard results in mechanism design (Laffont and Martimort 2009), the principal should provide a menu of contracts, such that an agent with a specific cost chooses a particular contract from this menu. Following the revelation principle (Myerson 1981), it is without loss of generality for us to consider direct mechanisms. In traditional adverse selection models, such mechanisms only involve allocation and payment decisions that depend on the agent's type. In our setting with dynamic moral hazard, however, payments and allocation (contract termination) need to be not only type-dependent, but also history-dependent. When designing the menu of dynamic contracts, the principal needs to consider effort responses and utilities of both agent types to ensure each type chooses the corresponding item in the menu. Consequently, the optimal design problem can no longer be formulated as a classic dynamic program/optimal control problem. The main idea of this paper is to use the first arrival as a screening device.

In particular, consider a setting in which the agent's cost can be either high or low. The low-cost agent is offered a probation contract, such that the agent is terminated if no arrival occurs during the probation period. Our design guarantees that the high-cost agent has no incentive to exert any effort facing this contract. Consequently, the first arrival during the probation period reveals that the agent must be of low-cost, and hence is a screening device. During the probation period, if an arrival does occur, how to set the agent's promised utility (Spear and Srivastava 1987) over time is determined by a convex optimization formulation based on a deterministic optimal control model. Delaying payments in the contract further reduces the high-cost agent's incentive to work if mimicking the low-cost type. The high-cost agent's contract, on the other hand, offers a sign-on-bonus up front. Under certain conditions, this approach yields the optimal menu of contracts. When the conditions do not hold, on the other hand, our (potentially sub-optimal) design based on using the first arrival as a screening device performs very well. It is worth noting that neither sign-on-bonus nor delayed payments are used in the optimal dynamic contract without adverse selection.

We generalize this approach to model multiple agent types. According to our contract design, the best type is offered a probation contract with delayed payments. The inefficient types are offered a sign-on-bonus up front and asked to leave immediately. Contracts for the middle types involve randomization among three forms.

Readers familiar with dynamic contracting may think of directly utilizing the optimal contracts with known costs to construct a menu that addresses the adverse selection issue. We present the corresponding naïve design, and show numerically that it performs much worse than our proposed menu of contracts.

An important step in our analysis is to establish an upper bound optimization. We use the upper bound to formally show optimality when certain conditions hold, and to demonstrate that our design performs very well even when not optimal.

The rest of the paper is organized as follows. After discussing the literature in Section 2, we introduce the model in Section 3. Section 4 presents the result when the agent's type is known by the principal, which serves as building blocks and benchmarks for our proposed design. We then present an upper bound optimization formulation in Section 5. The upper bound optimization yields a menu of contracts that relies on using the first arrival as a screening device. This design is indeed feasible (incentive compatible) and hence optimal when the model parameters satisfy certain conditions. When these conditions do not hold, we propose a feasible (incentive compatible) menu of contracts in Section 6. Numerical studies in Section 7 reveal that our design often achieves the upper bound and performs well overall even if not optimal. Finally, in Section 8 we generalize the model to include multiple types. For the reader's reference, we provide a table in Appendix A to summarize all notations introduced in the paper.

2. Literature Review

There have been previous attempts on the problem with both adverse selection and dynamic moral hazard. Ma (1991) focuses on renegotiation and actions with long term effects, whereas we give the principal full commitment power on contracting and hence the issue of renegotiation does not exist. Gershkov and Perry (2012) studies moral hazard with both persistent and repeated adverse selection in a discrete and finite horizon. In every period, the agent receives a task and different types of agent differ in the probability of success, while in our model types differ in the cost of exerting effort. They only consider two agent types while our design could be generalized to multiple types of agents. Furthermore, our payment contract has an arguably cleaner and simpler structure thanks to the continuous-time infinite horizon setting.

Our adverse selection and moral hazard setting is related to the one studied in Cvitanić and Zhang (2007). Their underlying stochastic process is Brownian motion over a finite time horizon. In comparison, our infinite horizon setting does not allow payment to be calculated as a lump-sum in the end of a finite time horizon, as in their paper. Furthermore, they suggest a relaxation-based procedure to obtain contracts that "are not necessarily optimal." In comparison, our Poisson uncertainty allows us to use the first arrival as a screening device to obtain contracts that are optimal under certain parameter settings. Cvitanić et al. (2013) considers a similar two-agent-type setting with Brownian motion uncertainties over an infinite horizon, which is the closest paper to ours. They propose an approach to characterize the set of promised utility pairs in order to numerically obtain certain types of contracts. For the two-type problem, the pairs of promised utilities can be described on a two-dimensional plane. In comparison, our approach, based on using the first arrival as a screening device in a Poisson setting, not only provides easily implementable contracts, but can also be scaled to more than two types, as we show in Section 8.

Another related strand of literature combines dynamic moral hazard with learning, such as Bhaskar (2012) and Kwon (2011). Unlike our private information setting, the agent also does not know the uncertainty, and hence the contract has to update and adjust the belief over time accordingly.

The dynamic moral hazard dimension of our model is based on Sun and Tian (2018), in which all model parameters (including the operating cost) are known. Similar dynamic moral hazard models based on Poisson arrivals include Chen et al. (2020), Tian et al. (2021), Cao et al. (2022, 2023). The key difference between our paper and these papers is the addition of the adverse selection component into the dynamic moral hazard model. The adverse-selection extension brings this line of work much closer to reality, because the agent's capability is often not transparent in real-world settings. The analysis and results are also strikingly different. First, due to adverse selection, our design involves a *menu* of contracts rather than a single contract as in each of the other papers. Second, the individual contract received by each type of agent needs to possess much richer structures, including a sign-on bonus or delayed payment, which do not appear in the aforementioned papers.

The promised utility formulation of continuous time moral hazard problem originates from Sannikov (2008), which provides a martingale representation of incentive compatibility constraint with an underlying Brownian motion uncertainty. This framework has been further applied to Poisson settings by Biais et al. (2010), in which the agent is hired to decrease the arrival rate of "bad news." Increasing the arrival rate of "good news," as in our model, has been studied in a stream of recent papers, (see, for example, Green and Taylor 2016, Shan 2017, Sun and Tian 2018, etc.), although without an adverse selection component.

3. Model

A principal contracts an agent to increase the arrival rate of a Poisson process over an infinite time horizon. At any point of time t, the agent can privately choose an effort level $\nu_t \in [0, \mu]$, which incurs a flow cost, and generates Poisson arrivals with an instantaneous rate ν_t. Each arrival yields a revenue R to the principal, and is observable to both the principal and the agent. Further denote a right-continuous counting process $N = \{N_t\}_{t\geq 0}$ to represent the total number of arrivals up to time t, which generates a filtration $\mathcal{F}^N = \{\mathcal{F}_t^N\}_{t\geq 0}$. Therefore, the instantaneous arrival rate of the counting process at time t is ν_t, and the left-continuous effort process $\nu = \{\nu_t\}_{t\geq 0}$ is $\mathcal{F}^N$-predictable.

The agent's capability, reflected in the operating cost, is linear in the effort level. We denote c to represent the operating cost when the agent chooses the highest possible effort level μ. Therefore, the per arrival rate per unit time unit operating cost is

$$\beta_c := c/\mu.$$

Hence, if the agent chooses $\nu_t \leq \mu$, then the flow operating cost is $\beta_c \cdot \nu_t$. That is, a more capable agent can generate arrivals with a lower operating cost. In this paper, we use "capability" and "cost" interchangeably. The operating cost is the agent's private information, and stays the same throughout the time horizon. The common prior distribution of the operating cost has a support $\mathcal{C}$. In this paper we mainly consider a binary set $\mathcal{C} = \{g, b\}$ with $g < b$, and extend the set to contain multiple values in Section 8. The operating cost c is also referred to as the agent's *type*.

We assume that the principal needs to reimburse the flow operating cost in real time, because the agent has limited liability and is cash constrained, a standard assumption in the dynamic contracting literature. In particular, at any point in time, the agent's effort choice is constrained by the flow reimbursement provided by the principal. Denote the flow reimbursement as ℓ_t, then for a type c agent who exerts effort at level ν_t, we require $\nu_t \cdot \beta_c \leq \ell_t$. (This situation is fairly common in contexts such as R&D and lobbying, where the principal has to provide a continuous flow of payments for the agent to operate. It may take the form of retainers in the case of lobbyists or repetitive payments in the case of R&D contracts.)[1]

Therefore, if the agent's type is b but pretends to be of a better (lower-cost) type g, and the principal only pays operating cost $\ell_t = g$, then this agent's effort choice can be no higher than $\mu g/b$. Generally speaking, if the agent chooses an effort level strictly less than $\mu \ell_t / b$, and therefore does not use up the operating payment, the part not being used can be diverted as a shirking benefit to the agent.[2]

Because the agent knows the operating cost at the beginning, following the *Revelation Principle*, it is without loss of generality to consider direct mechanisms (see, for example, Myerson 1986, Pavan et al. 2014). In our context, the principal designs a menu of contracts $\Gamma_{\mathcal{C}} = \{\gamma^c\}_{c\in\mathcal{C}}$, such that type c agent chooses contract γ^c. Any contract $\gamma^c = (L^c, \eta^c)$ includes an $\mathcal{F}$-predictable payment process L^c, and a $\mathcal{F}$-random time η^c representing contract termination. When it is not necessary in the context to stress the operating cost c, we also use the notation $\gamma = (L, \eta)$ without superscripts to represent a generic contract. As for the contract termination time η, if $\eta = \infty$, the contract continues throughout the infinite time horizon.

Specifically, for a payment process $L = \{L_t\}_{t\geq 0}$, at each time epoch $t \geq 0$, L_t represents the cumulative payment from the principal to the agent up to time t. For simplicity of expressions, in the rest of the paper we consider $\mathrm{d}L_t = \ell_t \mathrm{d}t + I_t$, in which ℓ_t represents the flow reimbursement mentioned before, and I_t the instantaneous payment at time t. The agent's limited liability and being cash constrained imply that payment is from the principal to the agent but not the other way around. Furthermore, the flow payment needs to cover the effort cost, which can be summarized in the following *limited liability* (LL) constraint for all contract $\gamma^c = (L^c, \eta^c) \in \Gamma_{\mathcal{C}}$,

$$I_t^c \geq 0, \quad \ell_t^c \geq \nu_t \cdot \beta_c, \quad \forall t \in [0, \eta] \text{ and } c \in \mathcal{C}. \tag{LL}$$

Both the principal and the agent discount future costs and payments with a discount rate r. Without loss of generality, and for simplicity of expressions, we normalize time unit such that

$$\mu + r = 1. \tag{1}$$

In order to formally define direct mechanisms, we start with expressing the agent's utility.

Agent utility Given a dynamic contract $\gamma = (L, \eta)$ and an effort process ν, the expected discounted utility of the agent with an operating cost c is

$$u(\gamma, \nu; c) = \mathbb{E}^{\nu}\left[\int_0^{\eta} e^{-rt}(\mathrm{d}L_t - \nu_t \cdot \beta_c \mathrm{d}t)\right], \tag{2}$$

in which the expectation $\mathbb{E}^{\nu}$ is taken with respect to probabilities generated from the effort process ν.

For a type c agent facing a contract γ, denote $\mathcal{N}(\gamma, c)$ to be the set of all $\mathcal{F}^N$-predictable effort processes ν that satisfy condition $\nu_t \cdot \beta_c \leq \ell_t, \forall t$ following (LL). Further denote $\mathfrak{N}(\gamma, c) \subseteq \mathcal{N}(\gamma, c)$ to represent the set of *best-response* effort processes, that is,

$$u(\gamma, \nu; c) \geq u(\gamma, \nu'; c), \ \forall \nu \in \mathfrak{N}(\gamma, c) \text{ and } \nu' \in \mathcal{N}. \tag{3}$$

We denote $\mathcal{F}_t^N$-predictable effort process $\bar{\nu} = \{\bar{\nu}_t\}_{t\geq 0}$ to be the *full-effort process* such that $\bar{\nu}_t = \mu$ almost surely for all t before contract termination. In this paper, we focus on the contract space where the agent of any type $c \in \mathcal{C}$ is willing to exert full effort under contract γ^c, which is commonly assumed in the dynamic contracting literature (see, for example, Biais et al. 2010) . That is, we assume the following *Full-Effort* (FE) constraint for the contracts,

$$\bar{\nu} \in \mathfrak{N}(\gamma^c, c). \tag{FE}$$

In the next section, we will explain in detail why we focus on this contract space.

A simple contract that induces the agent to always exert full effort is to pay the agent a constant β_c for each arrival besides reimbursing the operating cost rate c, such that the agent always exerts effort and receives a total discounted utility $\bar{w}_c$, where

$$\bar{w}_c = \frac{\mu\beta_c}{r} = \frac{c}{r}. \tag{4}$$

Although this simple contract is not optimal, the quantity $\bar{w}_c$ is useful in describing the optimal contracts.

Furthermore, the revelation principle implies that we can focus on direct mechanisms. Therefore, we need the following *Truth-Telling* (TT) constraint on the menu $\Gamma_{\mathcal{C}}$, which ensures that an agent with operating cost c indeed chooses contract γ^c from the menu, that is,

$$u(\gamma^c, \bar{\nu}; c) \geq u\left(\gamma^{c'}, \nu; c\right), \ \forall c, c' \in \mathcal{C}, \ \nu \in \mathcal{N}(\gamma^{c'}, c). \tag{TT}$$

It is standard by now to consider the agent's continuation utility (also called the *promised utility*, see, for example, Biais et al. 2010) at time t, defined as,[3]

$$W_t(\gamma, \nu; c) = \mathbb{E}^{\nu}\left[\int_{t+}^{\eta} e^{-r(s-t)}(\mathrm{d}L_s - \nu_s \cdot \beta_c \mathrm{d}s)\middle| \mathcal{F}_t^N\right]\mathbb{1}_{t<\eta}. \tag{5}$$

Following standard assumptions in the dynamic contracting literature, the principal has the commitment power to issue a long term contract, while the agent does not need to commit to staying in the contract. That is, we need the following *Individual Rationality* (IR) constraint to guarantee participation before contract termination,

$$W_t(\gamma, \nu; c) \geq 0, \ \forall t \in [0, \eta], \ c \in \mathcal{C}. \tag{IR}$$

The following result, which is parallel to Lemma 6 in Sun and Tian (2018), depicts the dynamic of the process W_t, and provides an equivalent condition to a best-response effort process.

LEMMA 1. *For any contract γ, effort process ν, and operating cost c, there exists an $\mathcal{F}^N$-adaptive process H_t such that*

$$\mathrm{d}W_t(\gamma,\nu;c) = \{[rW_{t-}(\gamma,\nu;c) - \nu_t H_t + \nu_t\beta_c]\mathrm{d}t + H_t\mathrm{d}N_t - \mathrm{d}L_t\}\,\mathbb{1}_{0\le t<\eta}. \tag{PK}$$

Furthermore, the following defined effort process is a best response to contract γ, or, $\{\nu_t\}_{t\in[0,\eta]} \in \mathfrak{N}(\gamma,c)$, in which

$$\nu_t = \begin{cases} \min\{\mu, \ell_t \cdot \mu/c\}, & \text{if } H_t \ge \beta_c, \\ 0, & o.w. \end{cases} \tag{IC}$$

It is worth explaining the condition (PK) for this paper to be self-contained. Overall, the promised utility is essentially total future payments, which explains that any payment at time t reduces the promised utility, and hence the $-\mathrm{d}L_t$ term in (PK). The first term $rW_{t-}(\gamma,\nu;c)\mathrm{d}t$ represents the interest accumulated over time. The term $H_t\mathrm{d}N_t$ in (PK) is an upward jump in the promised utility whenever there is an arrival ($\mathrm{d}N_t = 1$). Because this upward jump occurs at rate ν_t, the term $-\nu_t H_t \mathrm{d}t$ reflects a gradual decrease in the promised utility over time to balance the upward jump upon an arrival. The term $\nu_t\beta_c\mathrm{d}t$ reflects the effort cost to be reimbursed. Because the payment $\mathrm{d}L_t$ contains the reimbursement of effort cost according to (LL), the term $\nu_t\beta_c\mathrm{d}t$ cancels the corresponding term in $\mathrm{d}L_t$ in the promised utility.

Note that the quantity H_t can be perceived as a "reward" to the agent for each arrival, which is to be paid off later. Condition (IC) in Lemma 1 implies that the principal can motivate a type c agent to exert effort if and only if each arrival yields a reward (upward jump H_t in the agent's promised utility) of at least β_c. Later in the paper we show that in the optimal contract, the *Incentive Compatibility* (IC) constraint may not always be binding. That is, for certain operating cost c and time t, we need $H_t > \beta_c$, more than necessary to induce full effort.

Principal utility. Denote $U(\gamma,\nu)$ to represent the principal's total expected discounted utility from a contract γ while the agent's type is c and uses an effort process $\nu \in \mathcal{N}(\gamma,c)$. That is,

$$U(\gamma,\nu) := \mathbb{E}^{\nu}\left[\int_0^{\eta} e^{-rt}\left(R\mathrm{d}N_t - \mathrm{d}L_t\right)\right]. \tag{6}$$

Now we define $\mathcal{U}(\Gamma_{\mathcal{C}}) := \mathbb{E}\left[U(\gamma^c,\bar{\nu})\right]$ to represent the principal's total expected discounted utility from the *menu* of contracts $\Gamma_{\mathcal{C}}$. The principal's contract design problem is

$$\begin{aligned} \mathcal{Z}(\mathcal{C}) := \sup_{\Gamma_{\mathcal{C}}} \quad & \mathcal{U}(\Gamma_{\mathcal{C}}) \\ \text{s.t.} \quad & \text{(LL), (PK), (IC), (IR), (FE), and (TT).} \end{aligned} \tag{7}$$

Note that the expectation in the objective function is taken with respect to the operating cost c, while constraints (LL), (PK), (IC), (IR) and (FE) are for each $c \in \mathcal{C}$. In contrast, the constraint (TT) is for both pairs of operating costs $(c,c') = (b,g)$ and $(c,c') = (g,b)$. This implies that the maximization problem (7) cannot be decoupled in c. Finally, the objective function value $\mathcal{Z}(\mathcal{C})$ is the principal's optimal expected utility.

4. First Exploration of the Contract Design Problem

In a static setting, designing the optimal menu of contracts for a two-type case is fairly easy. One just needs to work out the information rent to prevent each type from mimicking the other type. In our dynamic setting, however, the optimal design is much more challenging. In particular, we cannot directly collect the optimal dynamic contracts with *known* operating costs to form the menu of contracts. This is because even for a given dynamic contract, each type of agent's best-response effort process may be non-trivial to compute. This further complicates the computation of the information rent for the truth-telling constraints. Cvitanić et al. (2013) faces similar challenges for their two type case, and is focused on characterizing the set of promised-utility pairs.

In order to circumvent this challenging issue, we limit the space of contracts to be ones that induce full effort for the corresponding agent type. We argue that this restriction makes the optimal contract design much more practical, because one may argue that it is unrealistic to assume that the agent has such a level of sophistication to compute general dynamic best-response effort processes. Solving the optimization under this restriction is still non-trivial, as we explain in this and the next section.

In this section, we first summarize essentially existing results on optimal dynamic contracts with known operating cost in Section 4.1, which will be some of the building blocks for our later contract design. Section 4.2 further provides a naïve approach to design a menu of contracts, which serves as a benchmark.

4.1. Known Operating Cost

In this subsection, we consider the operation cost c to be fixed and known to both the principal and agent. That is, the set $\mathcal{C}$ of operating costs is reduced to a singleton $\{c\}$. The corresponding contract design problem (7) becomes

$$U^c := \max_{\gamma^c} \quad U(\gamma^c, \bar{\nu}) \qquad (8)$$
$$\text{s.t. } (\text{LL}), (\text{PK}), (\text{IC}), \text{ and } (\text{IR}).$$

It is clear that constraint (TT) is no longer relevant here. So is constraints (TT), because when the principal and agent share the same discount rate, the optimal contract should induce full effort from the agent (Cao et al. 2023, Appendix A3).

This benchmark setting is similar to, although slightly more general than, the model in Sun and Tian (2018).[4] We first summarize the results following analysis in Sun and Tian (2018), and then comment on two key distinctions towards the end of this subsection. Since most technical results in this subsection can be adapted from Sun and Tian (2018), we try to be terse. Besides helping our paper to be self-contained, the content here, especially its connection and comparison with the general adverse selection problem, reveals important insights.

First, we formulate the optimal design problem (8) as an optimal control model, in which the promised utility is the state variable. The corresponding societal value function (total utility between the principal and the agent), $V_c(w)$, as a function of the promised utility w, satisfies the following delay differential equation (DDE).

$$r(w-\bar{w}_c)V_c'(w) = (c-\mu R) + V_c(w) - \mu V_c\left(w+\beta_c\right), \ \forall w \in [0,\bar{w}_c], \tag{9}$$

$$\text{with boundary conditions } \ V_c(0)=0 \ \text{ and } \ V_c(w) = \frac{\mu R - c}{r} \text{ for } w \geq \bar{w}_c. \tag{10}$$

LEMMA 2. *If $R > \beta_c$, the DDE (9)-(10) has a unique solution, $V_c(w)$, which is strictly concave and increasing on $[0,\bar{w}_c]$, and satisfies $V_c'(\bar{w}_c)=0$.*

Next, we define the following class of contracts, and establish that the optimal contract for the known operating cost case belongs to this class.

DEFINITION 1. For any $w \in [0,\bar{w}_c]$, define an *IC-binding contract* $\hat{\gamma}^c(w) = (L^c, \hat{\eta}^c)$, which generates a promised utility process $\{W_t^c\}$ following $W_0^c = w$, as well as a payment process $\{L_t^c\}$ and termination time $\hat{\eta}^c$, such that

$$\mathrm{d}W_t^c = \left[r(W_{t-}^c - \bar{w}_c)\mathrm{d}t + \min\left\{\bar{w}_c - W_{t-}^c, \beta_c\right\}\mathrm{d}N_t\right]\mathbb{1}_{W_{t-}^c \geq 0}, \tag{11}$$

$$\mathrm{d}L_t^c = \left[c\mathrm{d}t + (W_{t-}^c + \beta_c - \bar{w}_c)^+\mathrm{d}N_t\right]\mathbb{1}_{W_{t-}^c \geq 0}, \text{ and} \tag{12}$$

$$\hat{\eta}^c = \min\{t : W_{t-}^c = 0\}. \tag{13}$$

□

According to contract $\hat{\gamma}^c(w)$, the promised utility W_t^c starts from $W_{0-}^c = w$, and the dynamics (11) is consistent with (PK) with $H_t = \beta_c$. That is, the promised utility takes an upward jump of β_c upon each arrival, and gradually decreases at rate $r(\bar{w}_c - W_{t-}^c)$ as long as $W_{t-}^c < \bar{w}_c$. The contract terminates when W_t^c decreases to 0. The first instantaneous payment occurs when the promised utility W_t^c jumps above $\bar{w}_c$. After that, the promised utility stays at $\bar{w}_c$. The principal delivers this promised utility by paying the agent β_c for each future arrival, in addition to the flow payment $c\mathrm{d}t$ to reimburse the operating cost. In this case the termination time η^c is infinity. Therefore, contract $\hat{\gamma}^c(w)$ motivates the agent to always exert effort before contract termination ($\bar{\nu} \in \mathfrak{N}(\hat{\gamma}^c(w), c)$) by setting $H_t^c = \beta_c$ at all times, which satisfies the (IC) constraint.

The following proposition summarizes the results in Sun and Tian (2018). To this end, for any operating cost c and $w \geq 0$ we define $\Delta_c(w)$ to be the set of contracts $\gamma^c = (L^c, \eta^c)$ that satisfies (LL), (PK) and (IR) with a process $\{W_t^c\}_{t\geq 0}$ starting from $W_{0-}^c = w$. That is, any contract $\gamma^c \in \Delta_c(w)$ delivers a utility w to the agent.

Proposition 1. *Let $V_c(w)$ be the unique solution to (9)-(10) if $R > \beta_c$, and define $V_c(w) := 0$ if $R \le \beta_c$. Further define the principal's value function*

$$F_c(w) := V_c(w) - w. \tag{14}$$

For any $w \in [0, \bar{w}_c]$, we have

$$F_c(w) \ge U\big(\gamma^c, \nu^c\big),\ \forall \gamma^c \in \Delta_c(w),\ \nu^c \in \mathfrak{N}(\gamma^c, c). \tag{15}$$

Furthermore, if $R > \beta_c$, we have

$$u(\hat{\gamma}^c(w), \bar{\nu}; c) = w, \tag{16}$$

$$F_c(w) = U(\hat{\gamma}^c(w), \bar{\nu}). \tag{17}$$

Therefore, contract $\hat{\gamma}^c(w_*^c)$ is an optimal solution to (8), in which w_*^c is the unique maximizer of the strictly concave function $F_c(w)$ on $[0, \bar{w}_c]$. If $R \le \beta_c$, it is optimal for the principal not to hire this agent from the beginning.

In Proposition 1, condition (16) states that the contract $\hat{\gamma}^c(w)$ delivers the agent a utility w; (17) implies that function F_c is indeed the principal's value function for contract $\hat{\gamma}^c(w)$; and, finally, (15) claims that $F_c(w)$ is also an upper bound of the principal's utility under any contract and the corresponding best response effort process that delivers the agent a utility w. These conditions imply that an IC-binding contract is indeed optimal for this benchmark case.

It is worth noting that in the proof of Proposition 1, we formally establish the optimal contract for both $R > \beta_c$ and $R \le \beta_c$ cases. In particular, when $R \le \beta_c$, it is optimal for the principal to not hire the agent at all, or, equivalently, to set $\hat{\eta}_c = 0$. Note that in the general adverse selection case, which is the focus of this paper, we consider $R > \beta_g$ but β_b may be higher or lower than R. In later sections we demonstrate how the principal efficiently screens the high-cost agent, especially if the high type is also efficient ($R \ge \beta_b$).

For this purpose, we define a *delay-payment contract* that will be useful in later sections. According to Definition 1, the agent is paid $\bar{w}_c - W_{t-}^c$ at the time the promised utility reaches $\bar{w}_c$, and β_c for every future arrival. The *delay-payment contract*, on the other hand, delays the first number of payments to a later time while paying the corresponding interest. Intuitively, the good agent does not mind such a delay, because the interest is paid eventually. The delay, however, prevents the bad agent from trying to mimic the good one while not working. — As we will see in the probation contract later, with no effort and therefore no arrival, the bad agent would be terminated before the time to deliver the delayed payment.

Define the s-th arrival time after a time epoch t_0 as

$$\tau_{t_0}^{s} := \min\{t | N_t - N_{t_0} = s\}. \tag{18}$$

Definition 2. For any $w \geq \bar{w}_g$, $S \in \mathbb{N}$, and $t_0 \geq 0$, define a *delay-payment contract* $\gamma_{\mathtt{D}}^{g}(w, S, t_0) = (L^g, \eta_D^g)$, which generates a promised utility process according to

$$\mathrm{d}W_t = \left[(rW_{t-}/\mu)\mathrm{d}N_t \mathbb{1}_{t \leq \tau_{t_0}^{S}} + (\bar{w}_g - W_{t-})\mathrm{d}N_t \mathbb{1}_{t = \tau_{t_0}^{S+1}} \right] \mathbb{1}_{W_{t-} \geq 0}, \tag{19}$$

for $t \geq t_0$ following $W_{t_0} = w$. Furthermore, the payment process L_t^g follows

$$\mathrm{d}L_t^g = (W_{t-}/\mu - \bar{w}_g)\mathbb{1}_{t = \tau_{t_0}^{S+1}} + \beta_g \mathrm{d}N_t \mathbb{1}_{t > \tau_{t_0}^{S+1}}, \tag{20}$$

and the termination time $\eta_D^g = \infty$. □

Because the principal holds off the payment to the agent until the S-th arrival, Eq. (20) characterizes the corresponding payment process. It is worth connecting (19) with (PK). In particular, we set $H_t = rW_{t-}/\mu$ (which is higher than β_g for $W_{t-} \geq \bar{w}_g$), so that W_t remains a constant between arrivals before $\tau_{t_0}^{S+1}$. The payment at time $\tau_{t_0}^{S+1}$, $W_{\tau_{t_0}^{S+1}-}/\mu - \bar{w}_g = W_{\tau_{t_0}^{S+1}-} + H_{\tau_{t_0}^{S+1}} - \bar{w}_g$, includes all the interest that should be paid to the agent by then. Later in the paper we show how to utilize delay payment contracts with the appropriately chosen S and t_0 to construct incentive compatible contracts.

4.2. Unknown Operating cost with Two Types

Given the result for the known effort-cost case, it is tempting to think that one can use a pair of *IC-binding contracts* to solve the original contract design problem. For this purpose, we introduce a *sign-on-bonus contract* which slightly generalizes the *IC-binding contract*. As we will explain later in this section, the principal can use a sign-on-bonus up front to screen the bad type. We fully characterize the best sign-on-bonus contracts in this section. However, later in the paper we demonstrate that the best sign-on-bonus contracts are, in fact, not optimal for the original problem. In fact, the optimal contracts are not necessarily IC-binding, which highlights the difficulty of this contract design problem.

Definition 3. For an initial promised utility $w \geq 0$ and a sign-on-bonus $B \geq 0$, define a *sign-on-bonus contract* $\gamma_{\mathtt{B}}^{c}(w, B)$, which pays the agent $I_0 = B + \max\{w - \bar{w}_c, 0\}$ at time 0, and then follows the dynamics of an IC-binding contract $\hat{\gamma}^c(\min\{w, \bar{w}_c\})$ as defined in Definition 1. □

The principal can offer a sign-on-bonus contract $\gamma^g_{\mathtt{B}}(w^g, I^g)$ to the good agent and another sign-on-bonus contract $\gamma^b_{\mathtt{B}}(w^b, I^b)$ to the bad agent, where w^c represents the start of type-c agent's promised utility and I^c the sign-on bonus. As shown in the following lemma, if the good agent mimics the bad agent, then the good agent will always exert full effort; on the other hand, if the bad agent mimics the good agent, then the bad agent never works.

LEMMA 3. *We have $\nu^0 \in \mathfrak{N}(\gamma^g_{\mathtt{B}}(w^g, I^g), b)$ for any $w^g, I^g \ge 0$, and $\bar{\nu} \in \mathfrak{N}(\gamma^b_{\mathtt{B}}(w^b, I^b), g)$ for any $w^b, I^b \ge 0$, in which ν^0 represents a* never working *effort process, that is, $\nu^0_t = 0$ for all $t \ge 0$.*

In order to prevent the bad agent from mimicking the good agent and not working while collecting the flow payment, the principal can use a sign-on bonus to induce the bad agent to reveal its type. For this purpose, the next lemma shows both types' utilities when mimicking the other type under sign-on-bonus contracts. The result helps identify each type's rent.

LEMMA 4. *We have*

$$u\left(\gamma^b_{\mathtt{B}}(w^b, I^b), \bar{\nu}; g\right) = w^b + I^b + (b-g)\frac{V_b(w^b)}{\mu R - b}, \ \forall w^b, I^b \ge 0 \text{ and}$$
$$u\left(\gamma^g_{\mathtt{B}}(w^g, I^g), \nu^0; b\right) = u(\gamma^g_{\mathtt{B}}(w^g, I^g), \bar{\nu}; g) = w^g + I^g, \ \forall w^g, I^g \ge 0.$$

Following Lemma 4, when mimicking the bad type, the good agent "steals" the flow payment, whose present value is $(b-g)\frac{V_b(w^b)}{\mu R - b}$. The bad agent's mimicking utility turns out to be equal to the good agent's under the same contract.

If we restrict the class of contracts under consideration in the original contract design problem (7) to be sign-on-bonus contracts, then the optimization becomes

$$\begin{aligned} \mathcal{Y}_B := \max_{w^g, w^b, I^g, I^b} \quad & pF_g(w^g) + (1-p)F_b(w^b) - pI^g - (1-p)I^b \\ s.t. \quad & w^g + I^g \ge u(\gamma^b_{\mathtt{B}}(w^b, I^b), \bar{\nu}; g) = w^b + I^b + (b-g)\frac{V_b(w^b)}{\mu R - b}, \\ & w^b + I^b \ge u(\gamma^g_{\mathtt{B}}(w^g, I^g), \nu^0; b) = w^g + I^g. \end{aligned} \tag{21}$$

The following result characterizes its optimal solution.

LEMMA 5. *The optimal solution $(w^g_*, w^b_*, I^g_*, I^b_*)$ to (21) satisfies $w^g_* = I^b_* = \hat{B}$, $w^b_* = 0$, and $I^g_* = 0$, in which*

$$\hat{B} := \begin{cases} \min\{w | V_g'(w) = 1/p\}, & V_g'(0) > 1/p \\ 0, & V_g'(0) \le 1/p \end{cases}. \tag{22}$$

Overall, the following proposition summarizes the optimal sign-on-bonus contracts, which follows immediately from the lemmas in this section.

PROPOSITION 2. *The optimal menu of sign-on-bonus contracts is to issue an IC-binding contract $\hat{\gamma}^g(\hat{B})$ to the good agent, while paying the bad agent a bonus $\hat{B}$ for the agent to leave. The corresponding principal's utility is $\mathcal{Y}_B = pF_g(\hat{B}) - (1-p)\hat{B}$.*

Note that the pair of contracts described in Proposition 2 is equivalent to the case when we give both types of agents the good agent's IC-binding contract. While Sun and Tian (2018) showed that IC-binding contracts are optimal when the agent's type is public information, such a result does not extend to our model because of adverse selection issues. In Appendix E.6, Proposition 14 shows that this simple heuristic is always dominated by other easy-to-implement contracts, which may not be IC-binding, and can perform very poorly.

In the next section, we provide an upper bound to the optimal contract design problem. Under certain conditions, the upper bound optimization motivates a feasible menu of contracts that achieves the upper bound and hence is optimal. When the upper bound is not achievable, it helps us evaluate additional menus of dynamic contracts that we will propose later in this paper.

5. Upper bound optimization and corresponding contracts

In this section we present an optimization problem, which provides an upper bound to the original contract design problem (7). The intuition is the following. The upper bound optimization simply assumes, while ignoring necessary constraints to guarantee, that the bad agent does not work facing the good agent's contract. As a result, under the good agent's contract, only the good agent would be able to generate arrivals. Consequently, we can use the first arrival as a screening device. – If an arrival occurs, the principal knows that the agent must be good, and hence follow up with a dynamic contract $\hat{\gamma}^g$. If an arrival does not occur during a probation period of time, on the other hand, the principal has to treat the agent as bad and terminate the contract. Although a good agent may also fail to generate an arrival during the probation period, the principal with commitment power has to forfeit some efficiency due to information asymmetry.

We first present the upper bound optimization problem in Section 5.1. Then in Section 5.2 we present a menu of contracts, and conditions under which it is feasible and achieves the upper bound.

5.1. Upper bound optimization

In the following result, we use functions F_g and F_b as defined in (14) for $c \in \{g, b\}$.

PROPOSITION 3. *The following optimization problem yields an upper bound to the optimal value of the contract design problem (7). That is, $\mathcal{Y} \geq \mathcal{Z}(\{g, b\})$, where*

$$\mathcal{Y} := \max_{w_g, w_b, \tau, \xi} \; p \cdot G(w_g, \tau) + (1-p)\xi, \tag{23}$$

$$s.t. \; w_g \geq w_b \geq g(1 - e^{-r\tau})/r, \tag{24}$$

$$\tau \geq 0, \tag{25}$$

$$\xi \leq F_b(w_b), \tag{26}$$

$$\xi \leq \frac{w_g - w_b}{b-g}(\mu R - b)^+ - w_b, \tag{27}$$

in which we define the operator $(x)^+ := \max\{x, 0\}$*, and the function* $G(w,\tau)$ *through the following optimal control problem. If* $\tau < \infty$*,*

$$\begin{aligned} G(w,\tau) := \max_{W_t, H_t} \int_0^\tau \mu e^{-t}[R + F_g(W_t + H_t)]\mathrm{d}t - g(1-e^{-\tau}), \\ s.t.\ \frac{\mathrm{d}W_t}{\mathrm{d}t} = rW_{t-} - \mu H_t,\ \textit{for } t \in [0,\tau];\ W_0 = w,\ W_\tau = 0, \\ H_t \geq \beta_g,\ \forall t \in [0,\tau]; \end{aligned} \tag{28}$$

if $\tau = \infty$*, with a slight abuse of notation, we define*

$$\begin{aligned} G(w,\infty) := \max_{W_t, H_t} \int_0^\infty \mu e^{-t}[R + F_g(W_t + H_t)]\mathrm{d}t - g, \\ s.t.\ \frac{\mathrm{d}W_t}{\mathrm{d}t} = rW_{t-} - \mu H_t,\ \textit{for } t \geq 0;\ W_0 = w, \\ W_t \geq 0, H_t \geq \beta_g,\ \forall t \geq 0. \end{aligned} \tag{29}$$

It is instructive to explain the terms in the optimization problem (23)-(27). First, the decision variables w_g and w_b represent the utilities of type g and b agent under their respective contracts. The decision variable ξ and the term $G(w,\tau)$ represent the principal's expected utilities facing a type b and g agent, respectively, in which the decision variable τ is the duration of a "probation period" for the type g agent.

The constraint (24) states that the good agent's utility, w_g, needs to be as good as or better than the bad agent's w_b. Furthermore, the last inequality in (24) states that w_b needs to be no less than the total discounted expected operating cost that the agent would receive by pretending to be a good agent while exerting no effort. This is because receiving the operating cost g without working yields a utility $\int_0^\tau g e^{-rt}\mathrm{d}t = g(1-e^{-r\tau})/r$.

Constraint (24) reveals why the optimal value $\mathcal{Y}$ is only an upper bound. This is because the constraint itself (or any other constraint in this optimization) does not provide sufficient incentive for the bad agent not to work facing the good agent's contract. Restricting the bad agent's response (to zero effort) allows the principal to obtain a higher utility than reality, in which the agent may be able to obtain a utility higher than the right-hand-side of (24) with some effort.

Next, we articulate the meaning of $G(w_g,\tau)$ in the following remark.

Remark 1. First, $G(w_g,\tau)$ calculates the principal's expected utility where the good agent always exerts full effort. According to the optimal control problems (28) and (29), the principal designs a contract with an initial promised utility w and a (probation) time period τ. The control includes the promised utility process W_t, and the upward jump H_t associated with a potential arrival if

$t \le \tau$. During this probation period, if an arrival does occur, then the principal receives a revenue R, and the promised utility jumps to $W_{t-} + H_t$, at which point the principal follows the IC-binding contract, $\hat{\gamma}^g_{\mathtt{B}}(W_{t-} + H_t, 0)$, and earns a future utility $F_g(W_{t-} + H_t)$, following Proposition 1. Recall $\mu + r = 1$, which explains the term $e^{-t} = e^{-(\mu+r)t}$. The constraints in (28) and (29) capture the (PK), (IC) and (IR) constraints. Finally, the second term of the objective function in (28), $g(1 - e^{-\tau})$, is the total discounted operating cost that the principal needs to pay before the first arrival or at the end of the probation period, whichever comes first. Here, again, we use $\mu + r = 1$ as the effective discount rate. □

Finally, we focus on constraints (26) and (27). First, constraint (26) states that the principal's utility ξ is upper bounded by $F_b(w_b)$ when offering the type b agent a promised utility w_b, consistent with Proposition 2. Finally, constraint (27) ensures a type g agent does not pretend to be of type b, which is elaborated in the following remark.

Remark 2. Should the type g agent receive the type b contract, the agent is able to exert effort, and receive the same trajectory of payments as a type b agent. In addition to receiving the w_b reward, the type g agent also collects the extra operating cost $b - g$ for the duration of the contract. This (discounted) duration can be calculated as the (discounted) societal utility, $\xi + w_b$, divided by the societal utility rate, $\mu R - b$, if $\mu R > b$. This implies the following inequality,

$$w_g \ge w_b + (b - g)\frac{\xi + w_b}{\mu R - b}, \text{ or, equivalently, } \xi + w_b \le \frac{w_g - w_b}{b - g}(\mu R - b). \tag{30}$$

If $\mu R \le b$, on the other hand, the societal value of hiring the agent is negative, and, therefore, $\xi + w_b \le 0$. Constraint (27) captures both cases of $\mu R > b$ and $\mu R \le b$. □

So far we have provided intuitive interpretations of various components of the optimization problem (23)-(27). This optimization plays a central role in our contract design problem. In the next subsection we will convert the non-convex optimization problem (23)-(27) into an equivalent convex optimization problem, and obtain a menu of contracts based on its optimal solution. We further show that if the derived menu of contracts satisfies the truth-telling constraints (under a certain condition), then the performance of such a menu of contracts indeed achieves the upper bound $\mathcal{Y}$ of $\mathcal{Z}(\{g, b\})$.

In order to solve the optimization (23)-(27), we first provide a closed form solution to the deterministic optimal control problem (28). The solution approach is based on the Pontryagin minimum principle, as illustrated in the proofs of the following Lemmas, presented in the Appendix.

Lemma 6. *For any $\tau \in [0, \infty)$, define thresholds*

$$\breve{\omega}(\tau) := \frac{1 - e^{-r\tau}}{r} g, \quad \textit{and} \quad \hat{\omega}(\tau) := \frac{1 - e^{-\tau}}{r + \mu e^{-\tau}} g. \tag{31}$$

(i) If $w \in [\check{\omega}(\tau), \hat{\omega}(\tau))$*, then there exists a unique value* $\mathrm{z} \in [0, \hat{\omega}(\tau))$ *such that*[5]

$$w = \bar{w}_g - (\bar{w}_g - \mathrm{z})e^{r(\tau_{\mathrm{z}} - \tau)}, \tag{32}$$

where $\bar{w}_g$ *defined in* (4)*, and*

$$\tau_{\mathrm{z}} := \ln \frac{\mu(\mathrm{z} + \beta_g)}{r(\bar{w}_g - \mathrm{z})}. \tag{33}$$

Furthermore, the following W_t *and* H_t *solves the optimization* $G(w, \tau)$ *in* (28)*,*

$$W_t = \begin{cases} \bar{w}_g - (\bar{w}_g - \mathrm{z})e^{r(t + \tau_{\mathrm{z}} - \tau)}, & \text{for } t \in [0, \tau - \tau_{\mathrm{z}}], \\ \mu(\mathrm{z} + \beta_g)(1 - e^{t - \tau}), & \text{for } t \in [\tau - \tau_{\mathrm{z}}, \tau], \end{cases} \tag{34}$$

and

$$H_t = \begin{cases} \beta_g, & \text{for } t \in [0, \tau - \tau_{\mathrm{z}}], \\ \mathrm{z} + \beta_g - W_t, & \text{for } t \in [\tau - \tau_{\mathrm{z}}, \tau]. \end{cases} \tag{35}$$

(ii) If $w \geq \hat{\omega}(\tau)$*, then define*

$$\mathrm{z} := \frac{w}{\mu(1 - e^{-\tau})} - \beta_g. \tag{36}$$

For any $t \in [0, \tau]$*, the following* H_t *and* W_t *solves the optimization* $G(w, \tau)$ *in* (28)*,*

$$W_t = \mu(\mathrm{z} + \beta_g)(1 - e^{t - \tau}), \quad \text{and} \quad H_t = \mathrm{z} + \beta_g - W_t. \tag{37}$$

(iii) If $w < \check{\omega}(\tau)$*, the optimization problem* (28) *is infeasible, or, by convention,* $G(w, \tau) = -\infty$*.*

Remark 3. We use a figure to better illustrate the dynamics of W_t defined in Lemma 6. Figure 1 gives an illustrative example of the dynamics of W_t for the case that $\tau > \tau_{\mathrm{z}}$. The agent's promised utility trajectory W_t^g starts from W_0^g. Over time, if no arrival has occurred, the agent's promised utility drifts down, following the solid curve. If the first arrival occurs before τ, the promised utility jumps up to the dotted curve $\max\{W_t^g + \beta_g, \mathrm{z} + \beta_g\}$. Conceptually, the difference $H_t^g = \max\{\beta_g, \mathrm{z} + \beta_g - W_t^g\}$ represents the scale of upward jump in the agent's promised utility upon the first arrival. It is fixed and equal to β_g before time $\tau - \tau_{\mathrm{z}}$. After $\tau - \tau_{\mathrm{z}}$, however, the jump $H_t^g = \mathrm{z} + \beta_g - W_t^g > \beta_g$, and the (IC) constraint is not binding. After this first arrival, H_t^g is set to β_g. Finally, the dashed curve, which overlaps with the solid curve when $t < \tau - \tau_{\mathrm{z}}$, characterizes the movement of the promised utility following the dynamic (11) of the regular contract when H_t^g is kept at β_g. The figure implies that allowing the upward jump H_t^g to be higher than β_g effectively shortens the probation period. □

Similarly, we solve optimization problem (29) in the following Lemma.

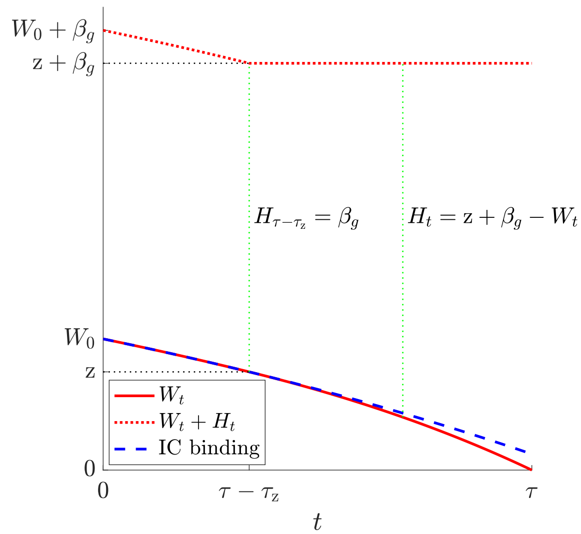

Figure 1 Good agent's promised utility dynamic before the first arrival for the case $\mu = 0.5$, $g = 1$, $\tau = 1$ and $W_0 = 0.85$. In this case $z = 0.63$, and $\tau - \tau_z = 0.34$.

Lemma 7. *If $\tau = \infty$, then define*

$$z := \frac{w}{\mu} - \beta_g, \tag{38}$$

(i) If $w \geq \frac{g}{r}$, then the following W_t and H_t solves the optimization (29),

$$W_t = w, \text{ and } H_t = \frac{w}{\mu} - w. \tag{39}$$

(ii) If $w < \frac{g}{r}$, the optimization problem (29) is infeasible, or, by convention, $G(w, \tau) = -\infty$.

We define the *discounted length* of the probation period to be

$$\bar{\tau} := \frac{1 - e^{-r\tau}}{r}. \tag{40}$$

Further define function

$$J(w, \bar{\tau}) := G\left(w, -\frac{\log(1 - r\bar{\tau})}{r}\right), \text{ for } \bar{\tau} \in \left[0, \frac{1}{r}\right], \ w \geq g\bar{\tau}. \tag{41}$$

Based on Lemma 6, we have the following result.

Proposition 4. *Function $J(w, \bar{\tau})$ is jointly concave in w and $\bar{\tau}$, and increasing in $\bar{\tau}$. Furthermore, we have*

$$\mathcal{Y} = \max_{w_g, w_b, \bar{\tau}} p \cdot J(w_g, \bar{\tau}) + (1-p) \min\left\{F_b(w_b), \ \frac{w_g - w_b}{b - g}(\mu R - b)^+ - w_b\right\} \tag{42}$$

$$s.t. \ w_g \geq w_b \geq g \cdot \bar{\tau} \tag{43}$$

$$0 \leq \bar{\tau} \leq \frac{1}{r}. \tag{44}$$

Because the minimum of two concave functions is concave, the objective function in (42) is concave. Therefore, Proposition 4 implies that we can convert the non-convex optimization problem (23)-(27) into a convex optimization problem with linear constraints, which can be solved efficiently.

5.2. Constructing contracts from the upper bound optimization problem

Now we define a menu of contracts based on this optimization problem. Denote $(w_g^*, w_b^*, \bar{\tau}^*)$ to represent an optimal solution of the convex optimization (42)-(44). Further define

$$\tau^* := -\frac{1}{r}\log(1 - r\bar{\tau}^*), \text{ and } z^* := \mathrm{z}(w_g^*, \tau^*), \tag{45}$$

in which the function $\mathrm{z}(w,\tau)$ is defined as z according to Lemmas 6 and 7. First, we construct the contract for the bad agent by using the *sign-on-bonus* contract of Definition 3. For this purpose, we need the following technical lemma.

LEMMA 8. *We have $w_b^* \le \bar{w}_b$. Furthermore, if $\mu R > b$, there exists a quantity $w \in [0, w_b^*]$ such that*

$$F_b(w) \le \frac{\left(w_g^* - w_b^*\right)(\mu R - b)}{b - g} - w. \tag{46}$$

Lemma 8 implies that the following threshold is well-defined, given (w_g^*, w_b^*),

$$w_B(w_g^*, w_b^*) := \begin{cases} \max\left\{ w \in [0, w_b^*] \;\middle|\; F_b(w) \le \frac{\left(w_g^* - w_b^*\right)(\mu R - b)}{b-g} - w \right\}, & \text{if } \mu R > b, \\ 0, & \text{if } \mu R \le b. \end{cases} \tag{47}$$

We claim, and will later show, that the principal should give the bad agent a sign-on-bonus contract $\gamma_{\mathtt{B}}^b\left(w_B(w_g^*, w_b^*), w_b^* - w_B(w_g^*, w_b^*)\right)$ of Definition 3.

Next, we construct a so-called *probation contract* for the good agent, whose structure is more intricate. Therefore it helps to understand why this structure makes intuitive sense. First, recall that it is necessary for the principal to pay a flow rate of at least c in order to induce full effort from a type c agent. In particular, the contract intended for the good agent offers a flow reimbursement g. The right-hand side of constraint (24) corresponds to the bad agent's utility of always shirking while collecting this flow reimbursement g. *If* the bad agent has no incentive to work, an arrival during this period would reveal that the agent's true type is indeed g. Hence, the principal can set a finite *probation* period, and terminate the agent at the end of probation if there is no arrival during this period. In this case, the adverse selection issue is resolved, and the principal can follow the contract structure $\hat{\gamma}^g(w)$ of Definition 1 after the first arrival. In order to ensure effort during probation for the good agent, the promised utility needs to take an upward jump of at least β_g, and possibly higher, at the first arrival. Later, we show that if this jump is not too high, then the bad agent indeed does not have any incentive to work.

The probation contract relies on the time of the first arrival, formally defined as

$$\tau_1^N := \min\{t \mid \mathrm{d}N_t = 1\}, \tag{48}$$

and the promised utility dynamics following Remark 3.

Definition 4. For any probation time period $\tau \geq 0$ and threshold $w \geq \check{w}(\tau)$, we define a *probation contract* $\gamma_{\mathtt{P}}^g(w,\tau) = (L^g, \eta^g)$, which pays $\mathrm{d}L_t^g = g\mathrm{d}t$ and generates a promised utility process W_t^g that evolves according to the following rules for $t \in [0,\tau)$, and $\eta^g = \tau$ if $\tau_1^N > \tau$.

- If $w \in [\check{w}(\tau), \hat{w}(\tau))$, then W_t^g follows (34),
- If $w \geq \hat{w}(\tau)$, then W_t^g follows (37),

in which $\check{w}(\tau)$ and $\hat{w}(\tau)$ are defined in (31).

If $\tau_1^N \leq \tau$, then the aforementioned dynamics lasts until τ_1^N, and define $\mathrm{z(w,\tau)}$ according to Lemmas 6 and 7. After the first arrival, there are two possibilities.

(1) If $\max\left\{W_{\tau_1^N}^g, \mathrm{z}(w,\tau)\right\} + \beta_g \leq \bar{w}_g$, the contract continues with $\hat{\gamma}^g\left(\max\left\{W_{\tau_1^N}^g, \mathrm{z}(w,\tau)\right\} + \beta_g\right)$ by resetting time τ_1^N to 0.

(2) If $\max\left\{W_{\tau_1^N}^g, \mathrm{z}(w,\tau)\right\} + \beta_g > \bar{w}_g$, it continues with $\gamma_{\mathtt{D}}^g\left(\max\left\{W_{\tau_1^N}^g, \mathrm{z}(w,\tau)\right\} + \beta_g, S(w,\tau), \tau_1^N\right)$, in which

$$S(w,\tau) := \left\lceil \frac{\ln(r(\max\{w, \mathrm{z}(w,\tau)\} + \beta_g)/g)}{\ln((br+\mu g)/g)} \right\rceil. \tag{49}$$

□

Remark 4. It is worth mentioning that the probation contract is built upon the delay-payment contract of Definition 2. Now we are ready to explain why delaying the first payment is helpful. If the good agent's promised utility after the first arrival exceeds $\bar{w}_g$, and if the principal immediately pays the agent while bringing the promised utility back to $\bar{w}_g$ as in the original IC-binding contract, then the benefit of the first arrival may be so high such that it induces the bad agent to mimic the good type while exerting partial effort (at level g/b). This would nullify the first arrival as a screening device and complicates contract design. In order to resolve this issue, the principal can delay the first payment to a later arrival, while paying the corresponding interest. This makes no difference to the good agent, who continues to work. The bad agent who mimics the good type, however, would have no incentive to exert even partial effort if the delay is sufficiently long, as guaranteed by the expression (49). □

We define the following menu of contracts and establish that it is optimal if z^*, as defined in (45), is not too large.

Definition 5. Given the optimal solution $(w_g^*, w_b^*, \bar{\tau}^*)$ to the convex optimization (42)-(44), define a menu of contracts $\Gamma_{\{g,b\}}^* := \left\{\gamma_{\mathtt{P}}^g(w_g^*, \tau^*),\ \gamma_{\mathtt{B}}^b\left(w_B(w_g^*, w_b^*), w_b^* - w_B(w_g^*, w_b^*)\right)\right\}$, in which τ^* is defined in (45), and w_B in (47). □

When the context is clear, we omit w_B's dependence on (w_g^*, w_b^*) in the expressions for the simplicity of exposition. The following lemma shows that it is still possible that we have $w_B = 0$ even if $b < \mu R$.

LEMMA 9. *There exists $\bar{b} \in [g, \mu R]$, such that $w_B = 0$ for $b \geq \bar{b}$ and $w_B > 0$ for $b < \bar{b}$.*

In summary, the good agent is always given a probation contract. The bad agent's contract depends on how high his operating cost b is. If $b \geq \bar{b}$, then the operating cost of the bad agent is too high to be worth hiring, or, $w_B = 0$. And the bad agent is paid an amount w_b^* up front to leave, which corresponds to a *pay-to-leave* contract $\gamma_{\mathsf{B}}^b(0, w_b^*)$. If $b < \bar{b}$, on the other hand, it is socially efficient to let the bad agent exert full effort. The corresponding sign-on-bonus contract $\gamma_{\mathsf{B}}^b(w_B, w_b^* - w_B)$ allows the bad agent to work from an initial promised utility w_B.

We are now ready to present the main result of this section.

THEOREM 1. *If*

$$\min\{z^* + \beta_g, \bar{w}_g\} < \beta_b, \tag{50}$$

in which z^* is defined in (45), then the menu of contracts $\Gamma^*_{\{g,b\}}$ satisfies (LL), (PK), (IC), (IR), (FE) and (TT) with $\mathcal{C} = \{g, b\}$. Furthermore, we have $\mathcal{U}(\Gamma^*_{\{g,b\}}) = \mathcal{Y}$, in which $\mathcal{Y}$ is defined in (23)-(27). Therefore, we have $\mathcal{U}(\Gamma^*_{\{g,b\}}) = \mathcal{Z}(\{g, b\})$, or, the menu of contract $\Gamma^*_{\{g,b\}}$ solves the optimal contract design problem (7) with two types.

Although the condition (50) is not based on primitive model parameters, whether it holds can be easily verified after solving the convex optimization (42)-(44). Following Remark 3 and Figure 1, the highest upward jump in the promised utility occurs at the end of the probation period, with a magnitude that is captured by the left-hand-side of (50). Condition (50) states that such an upward jump is lower than β_b, and hence the bad agent does not have any incentive to exert any effort throughout the probation period ((IC) in Lemma 1). Consequently, the upper bound optimization problem is tight, and the menu of contracts derived from the upper bound optimization problem is optimal.

Condition (50) immediately implies the following result.

COROLLARY 1. *The menu of contract $\Gamma^*_{\{g,b\}}$ solves the two-type optimal contract design problem when $g/r < b/\mu$, which is satisifed in particular when $\mu < r$.*

We present in Table 1 a summary of the menu of optimal contracts in the two-type case. Although the good agent's contract is always a probation contract regardless of b, the values w_g^* and τ^* are still functions of b.

	$b \geq \bar{b}$	$b < \bar{b}$
Good agent	Probation contract $\gamma_{\mathtt{P}}^{g}(w_g^*, \tau^*)$	Probation contract $\gamma_{\mathtt{P}}^{g}(w_g^*, \tau^*)$
Bad agent	Pay-to-leave contract $\gamma_{\mathtt{B}}^{b}(0, w_b^*)$	Sign-on-bonus contract $\gamma_{\mathtt{B}}^{b}(w_B, w_b^* - w_B)$

Table 1 Optimal menu of contracts under condition (50)

It is worth explaining incentives around the optimal menu of contracts $\Gamma^*_{\{g,b\}}$. In the case of $b \geq \bar{b}$, the initial bonus w_b^* to the bad agent equals the discounted total operating cost g that the agent can collect by mimicking the good type while shirking throughout the probation period. This initial bonus mitigates the bad agent's incentive to lie about the high cost. It is also worth noting that the (TT) constraint for the good agent is not binding. That is, the good agent's promised utility w_g^* under the probation contract is strictly higher than the bonus w_b^*.

If $b < \bar{b}$, however, the principal may allow the bad agent to work following contract $\gamma_{\mathtt{B}}^{b}(w_B, w_b^* - w_B)$, and provides sufficient information rent, w_b^*, for the bad agent to tell the truth. In order to discourage the good agent from mimicking the bad type and exerting effort while collecting a higher operating cost reimbursement, b, the principal needs to lower the bad agent's contract's initial promised utility w_B. If this initial promised utility w_B is lower than the information rent w_b^*, however, the principal needs to pay the difference as an initial sign-on-bonus to the bad agent.

Before closing this section, we highlight the following properties of the optimal menu of contracts $\Gamma^*_{\{g,b\}}$.

Property 1:

$$\ell_t^g = g, \text{ and } \ell_t^b = b.$$

The optimal contracts never over compensates the operating costs. That is, before contract termination, the good agent receives a flow payment g, and the bad agent receives a flow payment b.

Property 2:

$$\bar{\nu} \in \mathfrak{N}\Big(\gamma_{\mathtt{B}}^{b}\left(w_B(w_g^*, w_b^*), w_b^* - w_B(w_g^*, w_b^*)\right),\ g\Big), \text{ and } \nu^0 \in \mathfrak{N}\Big(\gamma_{\mathtt{P}}^{g}(w_g^*, \tau^*),\ b\Big).$$

The first part of the property states that a good agent who mimics the bad type would exert full effort. This result follows from Lemma 3. The second part of the property states that this contract induces the bad agent not to work. This is because: (1) according to Property 1, the good agent's contract only compensates the operating cost at rate g, and therefore the bad agent could only exert at most partial effort; (2) following condition (50), the first jump in the probation contract is small enough; and (3) the first payment is delayed to far enough into the future. As a result, under the probation contract $\gamma_{\mathtt{P}}^{g}(w_g^*, \tau^*)$, the good agent validates his type at the first arrival, resolving the adverse selection issue. This property plays an important role in showing that $\Gamma^*_{\{g,b\}}$ satisfies the (TT) constraint.

Property 3:

$$H_t^g = \beta_g,\ \forall t > \tau_1^N,\ \text{and}\ H_t^b = \beta_b,\ \forall t \geq 0.$$

This property indicates that under the menu $\Gamma^*_{\{g,b\}}$, the (IC) constraint is binding in the good agent's contract after the first arrival, and in the bad agent's contract the entire time. As mentioned earlier, the first arrival under the good agent's contract resolves adverse selection. Therefore the principal follows the most efficient contract, by setting the (IC) constraint binding. The bad agent's contract cannot screen agent's types using the first arrival, because as long as the contract allows the bad agent to work, the good agent is able to mimic the bad type and still generate arrivals. Therefore, the principal always offers a dynamically efficient contract (with binding (IC)) to the bad agent and adjusts other parameters in the menu to achieve optimality.

In the next section, we propose a feasible menu of contracts if condition (50) does not hold.

6. Lower Bound

In the upper-bound optimization, the bad agent is not allowed to work. If condition (50) does not hold, then the pair of sign-on bonus and probation contracts constructed in Section 5.2 do not satisfy the truth-telling constraints, and the upper bound cannot be achieved. In order to construct a menu of contracts that do satisfy truth-telling (although they may be suboptimal), we may restrict the contract space to induce the bad agent to always shirk facing the good agent's contract.

The reasons why we focus on such a class of lower bounds are twofold. First, by implementing the IC-binding contract for the good agent after the first arrival, we guarantee efficiency in the good agent's contract in the long run. This helps our lower bound to be near-optimal. Second, this class of contracts allows us to still use the first arrival to screen the two types, which helps us track the two types' promised utilities, and hence compute the corresponding information rent.

Proposition 5. *We have $\check{\mathcal{Y}}$, defined below, is lower than or equal to $\mathcal{Y}$.*

$$\check{\mathcal{Y}} := \max\{\check{\mathcal{Y}}_1, \check{\mathcal{Y}}_2\} \tag{51}$$

in which

$$\begin{aligned}\check{\mathcal{Y}}_1 := \max_{w_g, w_b}\ & p \cdot G(w_g, \infty) + (1-p)\min\left\{F_b(w_b),\ \frac{w_g - w_b}{b-g}(\mu R - b)^+ - w_b\right\} \\ \text{s.t. }\ & w_g \geq w_b \geq g/r,\end{aligned} \tag{52}$$

where the function $G(w,\infty)$ is defined in (29), and

$$\check{\mathcal{Y}}_2 := \max_{w_g, \bar{\tau}}\ p \cdot \check{G}\left(w_g, -\frac{1}{r}\ln(1 - r\bar{\tau})\right) + (1-p)\min\left\{F_b\left(\bar{\tau} g\right),\ \frac{w_g - \bar{\tau} g}{b-g}(\mu R - b)^+ - \bar{\tau} g\right\} \tag{53}$$

$$s.t.\ w_g \geq \bar{\tau} g \ \ and \ \bar{\tau} \in [0,\ 1/r],$$

where we define

$$\check{G}(w,\tau) := \max_{W_t, H_t} \int_0^{\tau} \mu e^{-t}[R + F_g(W_t + H_t)]\mathrm{d}t - g(1-e^{-\tau}), \tag{54}$$

$$s.t.\ \frac{\mathrm{d}W_t}{\mathrm{d}t} = rW_{t-} - \mu H_t,\ \ for\ t \in [0,\tau];\ \ W_0 = w,\ \ W_\tau = 0,$$

$$H_t \geq \beta_g,\ \ \forall t \in [0,\tau],$$

$$B_t = \frac{g}{r}\left(1 - e^{r(t-\tau)}\right),\ \ \forall t \in [0,\tau], \tag{55}$$

$$\min\{W_t + H_t, \bar{w}_g\} - B_t \leq \beta_b\ \ \forall t \in [0,\tau]. \tag{56}$$

In Proposition 5, we first separate the two cases between the infinite probation (52) and a finite probation (53). For the finite probation case, we further set the bad agent's utility w_b to be the total discounted value of mimicking the good agent while shirking through the probation period. Function $\check{G}$ in (53) replaces $G(w,\tau)$ in the upper bound optimization (23)-(29). The new process B_t represents the bad agent's total future utility while mimicking the good agent and shirking, which follows the expression (55). Constraint (56), which states that the upward jump in the promised utility upon an arrival is low enough, guarantees that the bad agent does not work facing the good agent's contract. Lemma 15 in the Appendix shows that this constraint, together with a delayed-payment, mitigates any incentive for the bad agent to work. Therefore, the good agent's contract indeed screens the types using the first arrival.

Next, we provide a process to solve the optimal control problem (54), in order to obtain the $\check{G}(w,\tau)$ function value for any given w and τ.

6.1. Solving $\check{G}$

First, it is worth noting that if $\bar{w}_g \leq \beta_b$, then constraint (56) is automatically satisfied. In this case, the calculation of lower bound $\check{\mathcal{Y}}$ coincides with the upper bound $\mathcal{Y}$, and Theorem 1 gives the corresponding optimal menu of contracts. Therefore, we focus on $\bar{w}_g > \beta_b$.

Before presenting the mathematical expressions and complete results, we first provide some insights on the optimal solution, before presenting the details in Proposition 6 later. Figure 2 demonstrates that the (w,τ) space is divided into five regions by thresholds $\check{\omega}(\tau)$, $\bar{\omega}(\tau)$, $\tilde{\omega}(\tau)$, and $\ln\rho$, to be defined later. In regions (1) and (5), there does not exist an admissible policy to solve (54), and hence $\check{G}(w,\tau) = -\infty$. In region (2), the optimal (W_t, H_t) that solves (54) follows the dynamic described in Lemma 6, which is further explained in Remark 3. In region (3), the optimal solution changes, as described in the following remark.

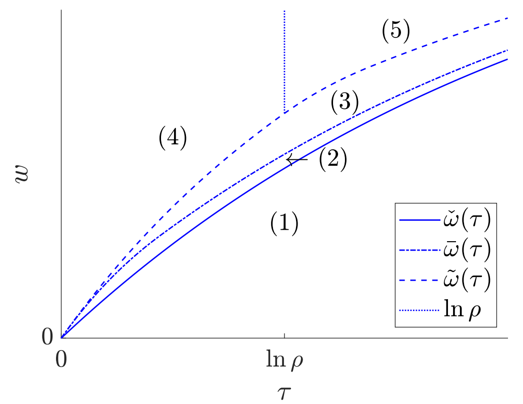

Figure 2 Different regions of (w, τ) for $\check{G}(w,\tau)$. In this case, $\mu = 0.7$, $g = 1$, $b = 1.5$.

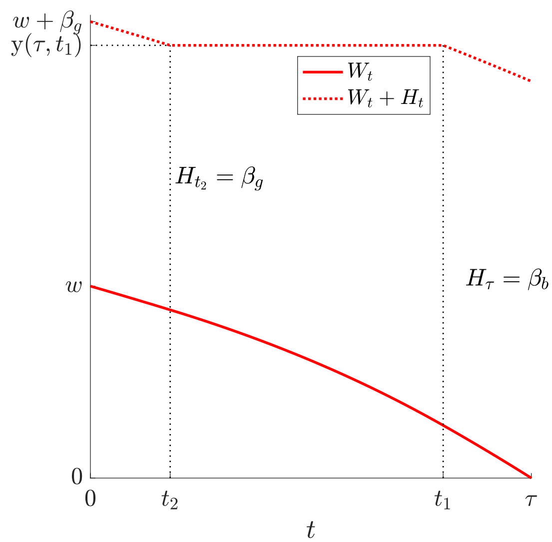

Figure 3 Good gent's promised utility dynamic before the first arrival for the case $\mu = 0.7$, $g = 1$, $b = 1.5$, $\tau = 1$, and $w = 1.04$. In this case $\mathrm{y}(\tau, \mathrm{t}_1) = 2.33$, $t_2 = 0.18$ and $t_1 = 0.8$.

REMARK 5. In this case, the probation period is divided into three phases by two thresholds, t_2 and t_1, as shown in Figure 3. In the first phase from 0 to t_2, the (IC) constraint is binding for the good agent. That is, an arrival during this period triggers the promised utility to take an upward jump of β_g, to the dotted curve. If the first arrival occurs during the second phase, between t_2 and t_1, the good agent's promised utility would jump to a fixed level ($\mathrm{y}(\tau, t_1)$, to be defined later). If the first arrival time t occurs after t_1 and before the end of probation period τ, the promised utility would jump to $\mathrm{y}(\tau, t)$, which is a decreasing function of t to be defined later. In the third phase, the upward jump is no higher than β_b, which dissuades the bad agent from working. □

Moving to region (4), Figure 3 changes to Figure 4 as a representation of the optimal solution.

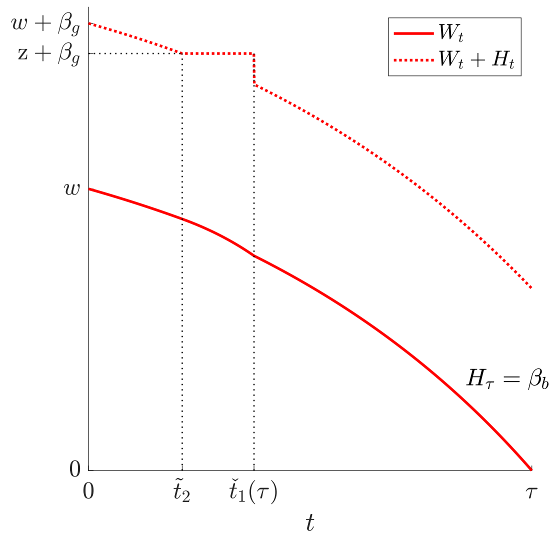

Figure 4 Good gent's promised utility dynamic before the first arrival for the case $\mu=0.7$, $g=1$, $b=1.1$, $\tau=4$ and $w=2.43$. In this case $\mathrm{z}=2.17$, $\tilde{t}_2=0.85$ and $\check{t}_1(\tau)=1.49$.

REMARK 6. Once again, there are three phases, marked by thresholds $\tilde{t}_2$ and $\check{t}_1(\tau)$, to be defined later. The promised utility takes an upward jump of β_g if an arrival occurs in the first phase, before $\tilde{t}_2$. At time $\tilde{t}_2$, the promised utility reaches a level z. From that time to $\check{t}_1(\tau)$, the first arrival triggers the promised utility to jump to a constant level, $\mathrm{z}+\beta_g$. In the third phase after time $\check{t}_1(\tau)$, the promised utility follows the same trajectory as the third phase described in Remark 6. Although the promised utility changes continuously, the level that it jumps to upon an arrival may involve discontinuity between phases 2 and 3 at time $\check{t}_1(\tau)$, as illustrated in Figure 4. □

The aforementioned dynamic involves many quantities to be defined next. We present these technical definitions in the remainder of this section in order to provide the complete result.

First define the following quantities,

$$\rho := (\bar{w}_g/\beta_b)^{1/r} > 1, \tag{57}$$

$$\check{t}_1(\tau) := \tau - \ln\rho, \text{ for } \tau \geq \ln\rho, \text{ and} \tag{58}$$

$$\check{t}_2(\tau) := \check{t}_1(\tau) + \ln\frac{rg}{rb+\mu(b-g)/\rho} \in [0, \check{t}_1(\tau)), \tag{59}$$

and functions,

$$\mathrm{W}(\tau,t) := \mu\left(\bar{w}_g+\beta_b\right)\left(1-e^{t-\tau}\right) + \bar{w}_g\left(e^{t-\tau}-e^{r(t-\tau)}\right), \text{ and} \tag{60}$$

$$\mathrm{y}(\tau,t) := \bar{w}_g\left(1-e^{r(t-\tau)}\right)+\beta_b. \tag{61}$$

In particular, the value $\mathrm{W}(\tau,\check{t}_1(\tau))$ no longer depends on τ. Define it as

$$\check{\mathrm{W}} := \mathrm{W}(\tau,\check{t}_1(\tau)) = \mu\left(\bar{w}_g+\beta_b\right)\left(1-\rho^{-1}\right)+\bar{w}_g\left(\rho^{-1}-\rho^{-r}\right). \tag{62}$$

We further define

$$\mathcal{W}(\tau) := \mu\bar{w}_g\left(1-e^{-\check{t}_1(\tau)}\right)+\check{W}e^{-\check{t}_1(\tau)},\ \text{for } \tau\geq\ln\rho, \tag{63}$$

$$\bar{\omega}(\tau) := \begin{cases} b(1-e^{-\tau}), & \text{if } \tau < -\ln\left(\frac{g-rb}{\mu b}\right), \\ \bar{w}_g+(\beta_b-\beta_g-\bar{w}_g)e^{-r\left(\tau+\ln\left(\frac{g-rb}{\mu b}\right)\right)}, & \text{if } \tau\geq -\ln\left(\frac{g-rb}{\mu b}\right), \end{cases} \quad \text{and} \tag{64}$$

$$\tilde{\omega}(\tau) := \begin{cases} \mathrm{W}(\tau,0), & \text{if } \tau<\ln\rho, \\ \mathcal{W}(\tau), & \text{if } \tau\geq\ln\rho \text{ and } \mathcal{W}(\tau)<\bar{w}_g-\beta_g, \\ \bar{w}_g(1-e^{-r\check{t}_2(\tau)})+e^{r\check{t}_2(\tau)}(\bar{w}_g-\beta_g), & \text{if } \tau\geq\ln\rho \text{ and } \mathcal{W}(\tau)\geq\bar{w}_g-\beta_g, \end{cases} \tag{65}$$

These notations are useful in the following result, corresponding to the five regions of Figure 2.

PROPOSITION 6. *Suppose $\bar{w}_g>\beta_b$. We have $\check{\omega}(\tau)\leq\bar{\omega}(\tau)\leq\tilde{\omega}(\tau)$ for any $\tau\geq 0$, where $\check{\omega}$ is defined in (31). Furthermore, for any pair (w,τ), the optimization problem $\check{G}(w,\tau)$ of (54) has the following five possibilities.*

(1) If $w<\check{\omega}(\tau)$, then (54) is infeasible, and hence, by convention, $\check{G}(w,\tau)=-\infty$.

(2) If $w\in[\check{\omega}(\tau),\bar{\omega}(\tau))$, then (W_t,H_t) defined in Lemma 6 is the optimal solution.

(3) If $w\in[\bar{\omega}(\tau),\tilde{\omega}(\tau)]$ there exists a unique pair of time epochs t_1 and t_2 with $0\leq t_2\leq t_1<\tau$, such that the following defined (W_t,H_t) is optimal,

$$W_t := \begin{cases} \bar{w}_g\left(1-e^{r(t-t_2)}\right)+e^{r(t-t_2)}\left[\mathrm{y}(\tau,t_1)\mu\left(1-e^{t-t_1}\right)+e^{t-t_1}\mathrm{W}(\tau,t_1)\right], & t\in[0,t_2), \\ \mathrm{y}(\tau,t_1)\mu\left(1-e^{t-t_1}\right)+e^{t-t_1}\mathrm{W}(\tau,t_1), & t\in[t_2,t_1), \\ \mathrm{W}(\tau,t), & t\in[t_1,\tau], \end{cases} \tag{66}$$

and

$$H_t = \begin{cases} \beta_g, & t\in[0,t_2), \\ \mathrm{y}(\tau,t_1)-W_t, & t\in[t_2,t_1), \\ \mathrm{y}(\tau,t)-W_t, & t\in[t_1,\tau]. \end{cases} \tag{67}$$

(4) If $w>\tilde{\omega}(\tau)$ and $\tau\geq\ln\rho$, there exists a unique time epoch $\tilde{t}_2\in[0,\check{t}_1(\tau)]$ and value z such that the following defined (W_t,H_t) is optimal,

$$W_t = \begin{cases} \bar{w}_g\left(1-e^{r(t-\tilde{t}_2)}\right)+e^{r(t-\tilde{t}_2)}\left[\mu(\mathrm{z}+\beta_g)\left(1-e^{\tilde{t}_2-\check{t}_1(\tau)}\right)+e^{\tilde{t}_2-\check{t}_1(\tau)}\check{\mathrm{W}}\right], & t\in[0,\tilde{t}_2), \\ \mu(\mathrm{z}+\beta_g)\left(1-e^{t-\check{t}_1(\tau)}\right)+e^{t-\check{t}_1(\tau)}\check{\mathrm{W}}, & t\in[\tilde{t}_2,\check{t}_1(\tau)), \\ \mathrm{W}(\tau,t), & t\in[\check{t}_1(\tau),\tau], \end{cases} \tag{68}$$

and

$$H_t = \begin{cases} \beta_g, & t\in[0,\tilde{t}_2), \\ \mathrm{z}+\beta_g-W_t, & t\in[\tilde{t}_2,\check{t}_1(\tau)), \\ \mathrm{y}(\tau,t)-W_t, & t\in[\check{t}_1(\tau),\tau]. \end{cases} \tag{69}$$

(5) If $w>\tilde{\omega}(\tau)$ and $\tau<\ln\rho$, then (54) is infeasible, and hence $\check{G}(w,\tau)=-\infty$.

Next, we present how to obtain quantities t_1, t_2, $\tilde{t}_2$ and z in Proposition 6 to complete the description of the optimal solution of (54).

6.1.1. Finding t_1 and t_2 Figure 3 illustrates (W_t, H_t) following (66) and (67). Given any $t_1 \in [0,\tau]$, we define $t_2(t_1) \le t_1$ to be the solution t that solves the equation

$$\mathrm{y}(\tau,t_1)\mu\left(1-e^{t-t_1}\right)+e^{t-t_1}\mathrm{W}(\tau,t_1)=\mathrm{y}(\tau,t_1)-\beta_g. \tag{70}$$

If no such $t\in[0,t_1]$ solves this equation (that is, $\mathrm{y}(\tau,t_1)\mu(1-e^{t_1})+e^{t_1}\mathrm{W}(\tau,t_1)<\mathrm{y}(\tau,t_1)-\beta_g$), on the other hand, define $t_2(t_1)=0$ and the first phase does not exist. Equipped with $t_2(t_1)$, we further search t_1 such that $W_0=w$ where W_t follows (66).

6.1.2. Finding $\tilde{t}_2$ and z Figure 4 illustrates (W_t, H_t) following (68) and (69), in which $\tilde{t}_2$ and z are defined as follows, depending on w and τ,

$$\text{If } w \ge \frac{g\left(1-e^{-\check{t}_1(\tau)}\right)+\check{\mathrm{W}}e^{-\check{t}_1(\tau)}}{r+\mu e^{-\check{t}_1(\tau)}},\ \text{then } \tilde{t}_2=0 \text{ and } \mathrm{z}=\frac{w-\check{\mathrm{W}}e^{-\check{t}_1(\tau)}}{\mu(1-e^{-\check{t}_1(\tau)})}-\beta_g; \tag{71}$$

$$\text{If } w < \frac{g\left(1-e^{-\check{t}_1(\tau)}\right)+\check{\mathrm{W}}e^{-\check{t}_1(\tau)}}{r+\mu e^{-\check{t}_1(\tau)}},\ \text{then } \tilde{t}_2=\check{t}_1(\tau)-\tau_{\mathrm{z}} \text{ and z is the unique solution of}$$

$$\bar{w}_g-(\bar{w}_g-\mathrm{z})\cdot e^{r(\tau_{\mathrm{z}}-\check{t}_1(\tau))}=w, \tag{72}$$

where

$$\tau_{\mathrm{z}}:=\ln\left(\frac{\mu(\mathrm{z}+\beta_g)-\check{\mathrm{W}}}{r(\bar{w}_g-\mathrm{z})}\right). \tag{73}$$

As a sanity check, when $w=\dfrac{g\left(1-e^{-\check{t}_1(\tau)}\right)+\check{\mathrm{W}}e^{-\check{t}_1(\tau)}}{r+\mu e^{-\check{t}_1(\tau)}}$, we have $\mathrm{z}=w$.

6.2. Contract Implementation

We are now ready to define a menu of contracts that achieves the lower bound $\check{\mathcal{Y}}$.

First, define the following probation contracts.[6]

Definition 6. For any probation period $\tau\ge 0$, and intial promised utility w such that $w\in[\breve{\omega}(\tau),\tilde{\omega}(\tau)]$, or $w>\tilde{\omega}(\tau)$ and $\tau\ge\ln\rho$, define a probation contract $\gamma^g_{\mathrm{P}'}(w,\tau,b)=(L^g,\eta^g)$, which pays $\mathrm{d}L^g_t=g\mathrm{d}t$ and generates a promised utility process W_t as follows.

1. If $w\in[\breve{\omega}(\tau),\bar{\omega}(\tau))$, $\gamma^g_{\mathrm{P}'}(w,\tau)=\gamma^g_{\mathrm{P}}(w,\tau)$ as in Definition 4.
2. If $w\in[\bar{\omega}(\tau),\tilde{\omega}(\tau)]$, W_t follows (66) for $t\in[0,\tau)$, and $\eta^g=\tau$ if $\tau_1^N>\tau$, in which t_1 and t_2 are obtained according to Section 6.1.1. If $\tau_1^N\le\tau$, on the other hand, the aforementioned dynamics lasts until τ_1^N, after which there are two possibilities:
   (1) if $W_{\tau_1^N}+H_{\tau_1^N}\le\bar{w}_g$, the contract continues with $\hat{\gamma}^g\left(W_{\tau_1^N}+H_{\tau_1^N}\right)$ of Definition 1 by resetting time τ_1^N to 0, where $H_{\tau_1^N}$ follows (69);
   (2) if $W_{\tau_1^N}+H_{\tau_1^N}>\bar{w}_g$, it continues with $\gamma^g_{\mathrm{D}}\left(W_{\tau_1^N}+H_{\tau_1^N},S'(w,\tau),\tau_1^N\right)$ of Definition 2, in which

$$S'(w,\tau)=\left\lceil\frac{\ln(g/r(\max\{w+\beta_g,\mathrm{y}(\tau,t_1),\bar{w}_g+\beta_b\}))}{\ln(g/(br+\mu g))}\right\rceil.$$

where $H_{\tau_1^N}$ follows (67).

3. If $w > \tilde{\omega}(\tau)$ and $\tau \geq \ln \rho$, W_t follows (68) for $t \in [0, \tau)$, and $\eta^g = \tau$ if $\tau_1^N > \tau$, in which $\tilde{t}_2$ and z are obtained according to Section 6.1.2. If $\tau_1^N \leq \tau$, on the other hand, the aforementioned dynamics lasts until τ_1^N, after which there are two possibilies:
   (1) if $W_{\tau_1^N} + H_{\tau_1^N} \leq \bar{w}_g$, the contract continues with $\hat{\gamma}^g\left(W_{\tau_1^N} + H_{\tau_1^N}\right)$ by resetting time τ_1^N to 0;
   (2) if $W_{\tau_1^N} + H_{\tau_1^N} > \bar{w}_g$, it continues with $\gamma^g_{\mathtt{D}}\left(W_{\tau_1^N} + H_{\tau_1^N}, S'(w,\tau), \tau_1^N\right)$, in which

$$S'(w,\tau) = \left\lceil \frac{\ln(g/r(\max\{w+\beta_g, \mathrm{z}+\beta_g, \bar{w}_g+\beta_b\}))}{\ln(g/(br+\mu g))} \right\rceil.$$

□

The following result demonstrates how to achieve the lower bound $\check{\mathcal{Y}}$ defined in Proposition 5. Denote $(\check{w}_g^1, \check{w}_b^1)$ to represent an optimal solution to (52), and $(\check{w}_g^2, \bar{\tau}_2)$ an optimal solution to (53). Define

$$\check{w}_g := \begin{cases} \check{w}_g^1, & \text{if } \check{\mathcal{Y}}_1 \geq \check{\mathcal{Y}}_2, \\ \check{w}_g^2, & \text{if } \check{\mathcal{Y}}_1 < \check{\mathcal{Y}}_2, \end{cases}$$
$$\check{w}_b := \begin{cases} \check{w}_b^1, & \text{if } \check{\mathcal{Y}}_1 \geq \check{\mathcal{Y}}_2, \\ g\bar{\tau}_2, & \text{if } \check{\mathcal{Y}}_1 < \check{\mathcal{Y}}_2, \end{cases} \quad \text{and}$$
$$\check{\tau} := \begin{cases} \infty, & \text{if } \check{\mathcal{Y}}_1 \geq \check{\mathcal{Y}}_2, \\ -\frac{1}{r}\ln\left(1 - r\bar{\tau}_2\right), & \text{if } \check{\mathcal{Y}}_1 < \check{\mathcal{Y}}_2. \end{cases}$$

PROPOSITION 7. *With $(\check{w}_g, \check{w}_b, \check{\tau})$, we define a menu of contracts*

$$\check{\Gamma}_{\{g,b\}} := \left\{\gamma^g_{\mathtt{P'}}\left(\check{w}_g, \check{\tau}, b\right), \gamma^b_{\mathtt{B}}\left(w_B(\check{w}_g, \check{w}_b), \check{w}_b - w_B(\check{w}_g, \check{w}_b)\right)\right\},$$

in which function w_B is defined in (47), contract $\gamma^b_{\mathtt{B}}$ in Definition 3, and contract $\gamma^g_{\mathtt{P'}}$ in Definition 6. We have $\check{\mathcal{Y}} = \mathcal{U}\left(\check{\Gamma}_{\{g,b\}}\right)$.

7. Numerical experiments

In this section we numerically test the performance of the menu of IC-binding contracts of Section 4 and the lower bound contracts of Section 6.

Following Corollary 1, we only consider model parameters such that $\mu > r$. We normalize $g = 1$, and take the following combinations of model parameters:

$$r \in \{0.1, 0.2, 0.3, 0.4\},$$
$$p \in \{0.1, 0.5, 0.9\},$$
$$b \in \{1.1g, 1.5g, 2g, 5g\},$$
$$R \in \{0.9\beta_g + 0.1\beta_b, 0.5(\beta_g + \beta_b), 0.1\beta_g + 0.9\beta_b, 1.1\beta_b, 1.2\beta_b, 1.3\beta_b, 2\beta_b, 5\beta_b\}.$$

Overall there are a total of $4 \times 3 \times 4 \times 8 = 384$ cases. Among all the cases, 144 of them yield a zero upper bound. That is, the principal should not hire the agent in these cases. For each of the remaining 240 "non-trivial" cases, we compute the ratio between the performances of the best IC-binding contracts $\mathcal{Y}_B$ (Proposition 2) and the upper bound $\hat{\mathcal{Y}}$. We also compute the ratio between our lower bound $\check{\mathcal{Y}}$ (Proposition 5) and the upper bound $\hat{\mathcal{Y}}$.

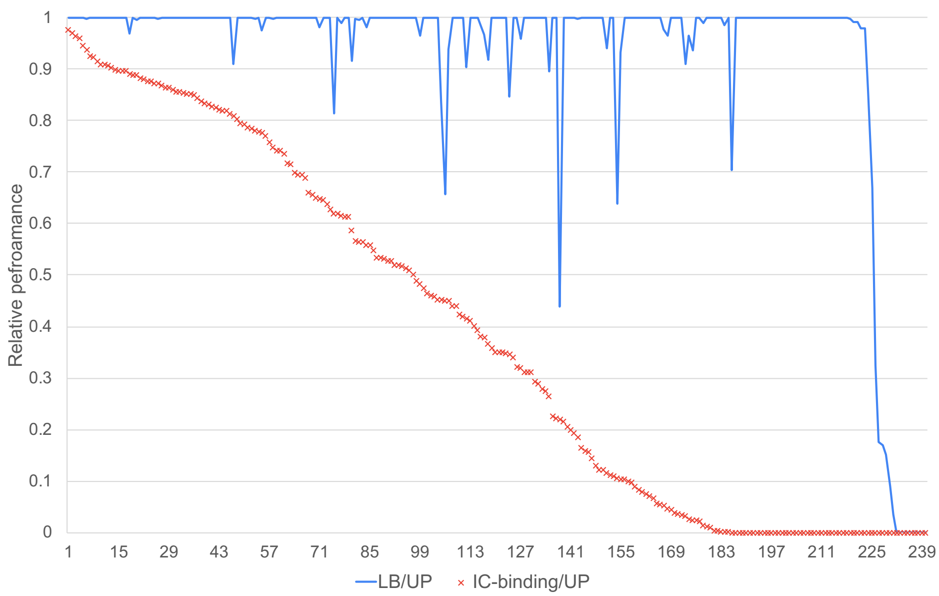

Figure 5 Comparing

In Figure 5, we arrange the cases in descending order of $\mathcal{Y}_B/\hat{\mathcal{Y}}$ and plot them in crosses ($\times$). We also plot the corresponding $\check{\mathcal{Y}}/\hat{\mathcal{Y}}$ as a solid line. It is clear that our lower bound $\check{\mathcal{Y}}$ always strictly outperforms the best IC-binding contracts, as long as $\check{\mathcal{Y}} > 0$. In each of the 8 cases such that our lower bound $\check{\mathcal{Y}} = 0$, the IC-binding contracts also yield $\mathcal{Y}_B = 0$. Among the 232 cases such that $\check{\mathcal{Y}} > 0$, the average of the ratio $\mathcal{Y}_B/\check{\mathcal{Y}}$, between the IC-binding and our lower bounds, is only 40.5%. Therefore, our constructed lower bound menu of contracts is much better than naívely relying on the known-cost optimal contracts.

Another observation from Figure 5 is that our lower bound $\check{\mathcal{Y}}$ often achieves the upper bound $\hat{\mathcal{Y}}$. In fact, among the 240 non-trivial cases, 76 of them satisfy the condition $g/r < b/\mu$. Following Corollary 1, it is not surprising that the lower bound meets the upper bound for these cases. What is more interesting is that 81 additional cases demonstrate $\check{\mathcal{Y}} = \hat{\mathcal{Y}}$ even though $g/r > b/\mu$. What about the remaining $240 - 76 - 81 = 83$ cases?

For these cases, one can also construct yet another "almost trivial" lower bound, by pooling both types together and offering both types the bad agent's optimal contract $\hat{\gamma}^b(w_*^b)$. Both types of agent will respond the same way facing this contract. The corresponding principal's utility is $\check{\mathcal{Y}}' := F_b(w_*^b)$, following Proposition 1. Among the 83 cases such that $\check{\mathcal{Y}} < \hat{\mathcal{Y}}$, the pooling contract outperforms the lower bound menu of contracts in 28 of them. A closer look at the 28 cases show that when b is close to g and r is relatively small, the pooling contract tends to perform better than the menu of lower bound contracts.

We then compute the relative performance of the better between these two lower bounds, $\max\left(\check{\mathcal{Y}}, \check{\mathcal{Y}}'\right)/\hat{\mathcal{Y}}$. Overall, the average relative performance among all 240 cases is 95.5%. This average relative performance becomes 86.9% among the 83 cases such that $\check{\mathcal{Y}} < \hat{\mathcal{Y}}$. Figure 6 depicts this relative performance for the 83 cases in a descending order. Overall, it appears that our contract design performs very well in most cases.

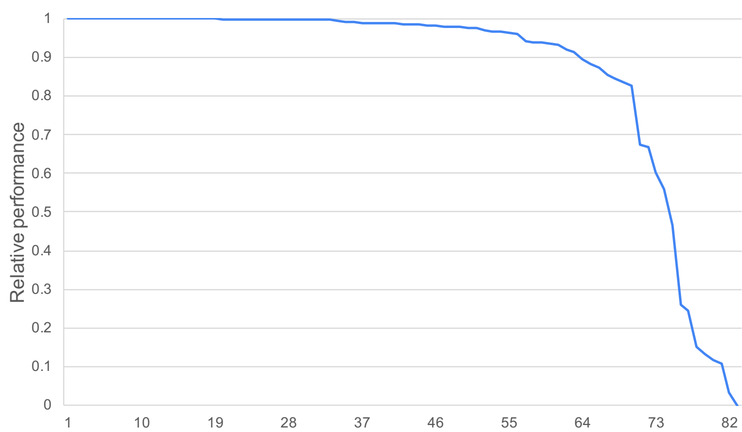

Figure 6 Relative performance $\max\left(\check{\mathcal{Y}}, \check{\mathcal{Y}}'\right)/\hat{\mathcal{Y}}$ of the 83 cases in which $\check{\mathcal{Y}} < \hat{\mathcal{Y}}$.

There are a total of only 20 cases such that the relative performance, $\max\left(\check{\mathcal{Y}}, \check{\mathcal{Y}}'\right)/\hat{\mathcal{Y}}$, is less than 90%. Table 2 lists all of them. These cases demonstrate the following characteristics: (1) the players are rather patient (r small), and (2) the agent is more likely to be bad (p no more than 0.5). Recall that our lower bound $\check{\mathcal{Y}}$ is based on restricting the contract space such that the bad agent has no incentive to exert effort. Alternatively, a patient principal could potentially design contracts that allow the bad agent to work for a while and screen the agent types through multiple arrivals. Such a construction is likely very complex and we leave it to future research.

8. Multiple Types

In this section we generalize the two-type case to a situation where the agents' operating cost may take its value from a set $\mathcal{C} := \{c_1, c_2, \ldots c_N\}$, such that the common prior probability of c_i is P_i, and $c_1 \leq \ldots \leq c_M < \mu R \leq c_{M+1} \leq \ldots \leq c_N$ for some positive integer $M \leq N$. The condition $c_1 < \mu R$ guarantees that the agent is efficient with a positive probability, which excludes the trivial case

Table 2 Parameters of the Cases with Highest Relative Losses

r	b	R	p	$\hat{\mathcal{Y}}$	$\max\left(\check{\mathcal{Y}}, \check{\mathcal{Y}}'\right)$	$\max\left(\check{\mathcal{Y}}, \check{\mathcal{Y}}'\right)/\hat{\mathcal{Y}}$
0.1	1.5	$1.2\beta_b = 2$	0.5	1.371	1.227	89.49%
0.1	1.1	$1.3\beta_b = 1.59$	0.1	0.512	0.453	88.41%
0.1	1.5	$2\beta_b = 3.33$	0.5	10.514	9.169	87.21%
0.3	2	$1.3\beta_b = 3.71$	0.1	0.031	0.027	85.53%
0.1	1.5	$1.3\beta_b = 2.17$	0.5	2.917	1.859	84.63%
0.1	5	$1.2\beta_b = 6.67$	0.1	3.519	2.975	84.54%
0.1	2	$2\beta_b = 4.44$	0.5	14.799	12.328	83.3%
0.2	1.1	$1.3\beta_b = 1.79$	0.1	0.03	0.021	67.39%
0.1	1.5	$1.3\beta_b = 2.17$	0.1	0.926	0.618	66.72%
0.1	1.1	$1.3\beta_b = 1.59$	0.5	0.75	0.453	60.39%
0.1	2	$1.3\beta_b = 2.89$	0.1	1.476	0.824	55.78%
0.1	1.1	$1.2\beta_b = 1.47$	0.1	0.041	0.019	46.51%
0.2	2	$1.3\beta_b = 3.25$	0.1	0.281	0.102	36.65%
0.2	1.1	$1.3\beta_b = 1.79$	0.5	0.079	0.021	26.1%
0.2	1.5	$1.3\beta_b = 2.44$	0.1	0.114	0.028	24.49%
0.1	2	$1.2\beta_b = 2.67$	0.1	0.59	0.089	15.06%
0.1	1.1	$1.2\beta_b = 1.47$	0.5	0.165	0.019	11.61%
0.1	1.5	$1.2\beta_b = 2$	0.1	0.243	0.026	10.73%
0.1	2	$1.1\beta_b = 2.44$	0.1	0.086	0.003	3.42%
0.2	2	$1.2\beta_b = 3$	0.1	0.027	0	0 %

where the agent is always inefficient and therefore should not be hired at all. Note that we make no assumption on the worst cost c_N. If $c_N > \mu R$, an agent with a cost higher than μR is not efficient, or, not worth hiring.

We still try to solve the contract design problem (7) with this newly defined set $\mathcal{C}$. In the contract design problem (7), the objective function becomes

$$\mathcal{U}(\Gamma_{\mathcal{C}}) = \sum_{i=1}^{N} P_i \cdot U(\gamma^{c_i}, \bar{\nu}) \tag{74}$$

in which the menu $\Gamma_{\mathcal{C}}$ offered by the principal contains a menu of contracts for each γ^{c_i} for $i \in \{1, 2, ..., N\}$. Similar to the two-type case, it is hard to characterize the optimal solution. Therefore, in this section, we focus on good approximations.

In Section 8.1, we first construct an optimization formulation similar to Section 5. Unfortunately, this upper bound is hard to compute. Therefore, in Section 8.1.1, we provide a further relaxation that is easy to compute, using a dynamic programming approach. This upper bound calculation also motivates the design of a menu of contracts. Therefore, in Section 8.2, we specify this menu of contracts, and compare its performance (a lower bound) with the upper bound. Our numerical study shows that the lower bound is fairly close to the upper bound, which implies that our design of contracts performs very well.

8.1. Upper Bound

Similar to Section 5, we present a new optimization problem, which provides an upper bound to the contract design problem (7). First, we expand the definition of the function J in (41) to include the cost variable as the following

$$\mathcal{J}(w,\bar{\tau},c) := \mathcal{G}\left(w, -\frac{1}{r}\log(1-r\bar{\tau}), c\right), \text{ for } \bar{\tau}\in\left[0,\frac{1}{r}\right],\ c\in\{c_1,\dots,c_M\}, \text{ and } w\geq c\bar{\tau}, \tag{75}$$

in which the function $\mathcal{G}$ is defined as

$$\begin{aligned}\mathcal{G}(w,\tau,c) := \max_{W_t,H_t} & \int_0^\tau \mu e^{-t}[R+F_c(W_t+H_t)]\mathrm{d}t - c(1-e^{-\tau}), \qquad (76)\\ \text{s.t. } & \frac{\mathrm{d}W_t}{\mathrm{d}t} = rW_{t-}-\mu H_t, \text{ for } t\in[0,\tau];\ W_0=w,\ W_\tau=0,\\ & H_t\geq\beta_c,\ \forall t\in[0,\tau],\end{aligned}$$

similar to the function G in Proposition 3. Therefore, the function $\mathcal{J}(w,\bar{\tau},c)$ represents the principal's optimal utility when offering a type c agent a probation period with a discounted length $\bar{\tau}$, and an initial promised utility level w. The relationship between the discounted length $\bar{\tau}$ and the real length τ of the probation period is defined in (40). Note that the function $\mathcal{J}$ is well-defined only for $\bar{\tau}\in[0,1/r]$, $c<\mu R$, and $w\geq c\bar{\tau}$, when the corresponding optimal control problem (76) is admissible.

Based on the definition of the value function $\mathcal{J}$ we are ready to present the following upper bound optimization problem,

THEOREM 2. *We have $\mathcal{Y}^N\geq\mathcal{Z}(\mathcal{C})$, where $\mathcal{Z}(\mathcal{C})$ is defined according to (7) and (74), and*

$$\mathcal{Y}^N := \max_{\{w_i,\xi_i\}_{i=1,\dots,M},w_{M+1}} \sum_{i=1}^{M}\xi_i P_i - w_{M+1}\sum_{i=M+1}^{N} P_i \tag{77}$$

$$s.t.\ w_i\geq w_{i+1}\geq 0,\ \forall i\in\{1,\dots,M\}, \tag{78}$$

$$\xi_1 = \mathcal{J}\left(w_1,\ \min\left\{\frac{w_{M+1}}{c_1},\ \frac{1}{r}\right\},\ c_1\right), \tag{79}$$

$$\xi_i \leq \mathcal{J}\left(w_i,\ \min\left\{\frac{w_{M+1}}{c_i},\ \frac{1}{r}\right\},\ c_i\right),\ \forall i\in\{2,\dots,M\}, \tag{80}$$

$$\xi_i \leq \min_{j<i}\left(\frac{w_j-w_i}{c_i-c_j}\right)\cdot(\mu R-c_i)-w_i,\ \forall i\in\{2,\dots,M\}. \tag{81}$$

The optimization problem (77)-(81) is a generalization of the one defined in Proposition 3 for the two type case. First, the decision variable w_i represents the initial promised utility assigned to the type c_i agent. If the agent is efficient ($c_i\leq\mu R$), the decision variable ξ_i represents the principal's utility facing a type c_j agent, similar to the variable ξ in Proposition 3. If the agent is inefficient ($c_i>\mu R$), on the other hand, the objective function (77) implies that the principal's utility is $-w_{M+1}$, that is, the agent should be paid off and terminated immediately.

The constraint (78) states that the principal needs to offer a higher promised utility to the agent with a better type (lower cost) than to one with a worse type (higher cost). This monotonicity constraint partly mitigates the agent's incentive to mimic a *worse* type. Constraint (80) indicates that the principal's utility from type c_i agent is upper bounded by the function $\mathcal{J}$, which calculates the principal's maximum expected utility when the type c_i agent always exerts effort under a probation contract. (Functions $\mathcal{J}$ and $\mathcal{G}$ in this section correspond to functions J and G in Section 5, respectively. See Remark 1 for a detailed explanation of G.) Constraint (81) ensures that the type c_i agent does not benefit from mimicking a worse type c_j. (This is similar to the constraint (26), as discussed in Remark 2.)

To understand how to mitigate the agent's incentive to mimic a *better* type, we need to look at the term $\mathcal{J}$ inside (80). If an agent with a higher cost c_k mimics the lower cost c_i and shirks through the probation period, the total discounted utility would be $c_i\bar{\tau}$, in which $\bar{\tau}$ represents the discounted probation period offered to type c_i. A constraint $c_i\bar{\tau} \le w_k$, or, equivalently, $\bar{\tau} \le w_k/c_i$, mitigates such an incentive. Monotonicity following (78) implies that we only need to require $\bar{\tau} \le w_N/c_i$. Since function $\mathcal{J}(w, \bar{\tau}, c)$ is increasing in $\bar{\tau}$, it helps to set $\bar{\tau}$ to the upper bound w_N/c_i in order to maximize $\mathcal{J}$. However, by definition, the discounted probation period $\bar{\tau}$ cannot be longer than $1/r$. This explains the second argument in function $\mathcal{J}$.

Finally, given constraint (78), if $\mu R \le c_N$, it is clear that the optimal w_i value for any $c_i \ge \mu R$ should be a constant, w_N.

8.1.1. Computing the upper bound In general, the optimization problem (77)-(78) is hard to solve. Therefore, we provide an efficient algorithm to compute an upper bound of $\mathcal{Y}^N$.

PROPOSITION 8. *Define*

$$\begin{aligned}
\hat{\mathcal{Y}}^N := \max_{w_i} \ & \sum_{i=1}^{M} P_i \cdot \xi_i - w_{M+1} \sum_{i=M+1}^{N} P_i \\
s.t.\ & w_i \ge w_{i+1}, \forall i \in \{1, \ldots, M\}, \\
& \xi_1 = \mathcal{J}\left(w_i, \min\left\{\frac{w_{M+1}}{c_i}, \frac{1}{r}\right\},\ c_i\right), \\
& \xi_i = \min\left\{\mathcal{J}\left(w_i, \min\left\{\frac{w_{M+1}}{c_i}, \frac{1}{r}\right\},\ c_i\right),\ \frac{w_{i-1}-w_i}{c_i - c_{i-1}}(\mu R - c_i) - w_i\right\}, \forall i \in \{2, \ldots, M\}.
\end{aligned} \tag{82}$$

We have

$$\mathcal{Y}^N \le \hat{\mathcal{Y}}^N. \tag{83}$$

It is worth noting the key difference between the upper bound optimization (82) and the original (77)-(81). For a type c_i and the corresponding w_i, the term $\min_{j<i}\left[\frac{w_j - w_i}{c_i - c_j}\right]$ in the original formulation is replaced with a higher value $\frac{w_{i-1}-w_i}{c_i - c_{i-1}}$, which yields a looser upper bound. The benefit of

this change is computational efficiency. In fact, the optimization problem (82) can be solved using a dynamic programming approach together with a single-dimensional line search.

Given a value $w_{M+1} \geq 0$, define the following boundary value function,

$$\mathfrak{J}_1(w|w_{M+1}) := P_1 \cdot \mathcal{J}\left(w, \min\left\{\frac{w_{M+1}}{c_1}, \frac{1}{r}\right\}, c_1\right), \quad \forall w \geq w_{M+1}. \tag{84}$$

For any $i \in \{1, \ldots, M\}$, define a deterministic dynamic programming recursion,

$$\begin{aligned}\mathfrak{J}_i(w|w_{M+1}) := \max_{w' \in [w,\infty)} \; & P_i \cdot \min\left\{\frac{w'-w}{c_i - c_{i-1}}(\mu R - c_i) - w,\ \mathcal{J}\left(w, \min\left\{\frac{w_{M+1}}{c_i}, \frac{1}{r}\right\}, c_i\right)\right\} \\ & + \mathfrak{J}_{i-1}(w'|w_{M+1}), \quad \forall w \geq w_{M+1},\end{aligned} \tag{85}$$

It is clear that

$$\hat{\mathcal{Y}}^N = \max_{w, w_{M+1}: 0 \leq w_{M+1} \leq w} \mathfrak{J}_M(w|w_{M+1}) - w_{M+1} \sum_{i=M+1}^{N} P_i, \tag{86}$$

which implies that we can obtain the upper bound approximation $\hat{\mathcal{Y}}^N$ by solving a sequence of the dynamic programming formulation (85) together with a one-dimensional search for the w_{M+1} value.

Proposition 15 of Appendix E.10 provides a closed-form solution to the maximization problem in (85). Furthermore, Proposition 16 of Appendix E.10 provides an upper bound $\bar{\mathrm{w}}$ for the optimal w_{M+1}. Therefore, we can focus on the search for the optimal w_{M+1} in the interval $[0, \bar{\mathrm{w}}]$. It is worth noting here that if $c_N < \mu R$, we have $\hat{\mathcal{Y}}^N = \max_{w_M \in [0, \mathrm{w})} \mathfrak{J}_M(w_M | w_M)$.

8.2. Lower bound and Contract design

Motivated by the upper bound optimization problem (82), we provide a lower bound optimization here, the solution of which yields a menu of implementable contracts to the original contract design problem. Similar to the definition of functions $\mathcal{J}$ and $\mathcal{G}$ in (75)-(76), we define the following functions,

$$\begin{aligned}\check{\mathcal{G}}(w, \tau, c, c') := \max_{W_t, H_t} & \int_0^\tau \mu e^{-t}[R + F_c(W_t + H_t)]\mathrm{d}t - c(1 - e^{-\tau}), \\ \text{s.t. } & \frac{\mathrm{d}W_t}{\mathrm{d}t} = rW_{t-} - \mu H_t, \text{ for } t \in [0, \tau];\ W_0 = w,\ W_\tau = 0, \\ & H_t \geq \beta_c,\ \forall t \in [0, \tau]. \\ & B_t = \frac{c}{r}\left(1 - e^{r(t-\tau)}\right), \forall t \in [0, \tau]. \\ & \min\{W_t + H_t, \bar{w}_g\} - B_t \leq c'/\mu, \forall t \in [0, \tau],\end{aligned} \tag{87}$$

and

$$\check{\mathcal{J}}(w, \bar{\tau}, c, c') := \check{\mathcal{G}}\left(w, -\frac{1}{r}\log(1 - r\bar{\tau}), c, c'\right), \text{ for } \bar{\tau} \in \left[0, \frac{1}{r}\right], c \in \{c_1, \ldots, c_M\}, c' > c, \text{ and } w \geq c\bar{\tau}. \tag{88}$$

Therefore, function $\check{\mathcal{J}}(w,\bar{\tau},c,c')$ represents the principal's optimal utility when offering a type c agent a probation period with discounted length $\bar{\tau}$ and an initial promised utility level w, which further guarantees that a "worse type c'" is not working while taking this contract. – For any efficient type $c=c_i$, we take $c'=c_{i+1}$. Since $\min\{W_t+H_t,\bar{w}_g\}-B_t\leq c_{i+1}/\mu\leq c_k/\mu$ for all $k>i$, we know that any type c_k agent with $k>i$ does not have the incentive to work while taking type c_i's contract.

Next, we present a dynamic programming approach which yields a menu of contracts implementable for the multiple-type problem. At the end of this section, a numerical study shows that the performance of the dynamic programming formulation is fairly good.

Given a value $w_{M+1}\geq 0$, define the following deterministic dynamic programming recursion, starting from the boundary condition,

$$\check{\mathfrak{J}}_1(w|w_{M+1})=P_i\cdot\check{\mathcal{J}}\left(w,\min\left\{\frac{w_{M+1}}{c_1},\frac{1}{r}\right\},\ c_1,c_2\right),\quad\forall w\geq w_{M+1},\tag{89}$$

and

$$u_1(w)=\infty,\quad\forall w\geq w_{M+1}.\tag{90}$$

For any $i\in\{1,\ldots,M\}$, define

$$\check{\mathfrak{J}}_i(w|w_{M+1})=\max_{u\in[0,u_{i-1}(w+u(c_i-c_{i-1}))]}\mathcal{Q}_i(w,u|w_{M+1}),\ \text{in which}\tag{91}$$

$$\begin{aligned}\mathcal{Q}_i(w,u|w_{M+1})=P_i\cdot\min\left\{u(\mu R-c_i)-w,\ \check{\mathcal{J}}\left(w,\min\left\{\frac{w_{M+1}}{c_i},\frac{1}{r}\right\},\ c_i,c_{i+1}\right)\right\}\\+\check{\mathfrak{J}}_{i-1}\big(w+u\cdot(c_i-c_{i-1})|w_{M+1}\big),\quad\forall w\geq w_{M+1}.\end{aligned}\tag{92}$$

Further define the optimal decision $u_i(w)$ such that

$$\check{\mathfrak{J}}_i(w|w_{M+1})=\mathcal{Q}_i(w,u_i(w)|w_{M+1}).\tag{93}$$

Finally, we define

$$\check{\mathcal{Y}}^N:=\max_{(w,w_{M+1})\geq 0:\ 0\leq w-w_{M+1}\leq u_M(w)(c_{M+1}-c_M)}\check{\mathfrak{J}}_M(w|w_{M+1})-w_{M+1}\sum_{i=M+1}^{N}P_i.\tag{94}$$

Denote an optimal solution of (94) as (w_M^*,w_{M+1}^*). For $i=2,\ldots,M$ define

$$w_{i-1}^*:=w_i^*+u_i(w_i^*)(c_i-c_{i-1}).\tag{95}$$

Following the constraint for u in (91), we know that $0\leq u_i(w_i^*)\leq u_{i-1}(w_{i-1}^*)$. Therefore, the sequence of w_i^* satisfies

$$\frac{w_{i-1}^*-w_i^*}{c_i-c_{i-1}}\geq\frac{w_i^*-w_{i+1}^*}{c_{i+1}-c_i}\ \text{and}\ w_i^*\geq w_{i+1}^*,\ \forall i\in\{2,\ldots,M\},\tag{96}$$

which is a key feature to construct an incentive compatible menu of contracts.

We are now ready to describe a menu of contracts based on the sequence $\{w_i^*\}_{i=1,...,N}$. In particular, for inefficient types $c \in \{c_{M+1},...,c_N\}$, the agent is paid w_{M+1}^* and terminated immediately, which corresponds to the sign-on-bonus contract $\gamma_{\mathtt{B}}^c(0, w_{M+1}^*)$ of Definition 3. If $c = c_1$, the agent is given a probation contract $\gamma_{\mathtt{P}'}^c(w_1^*, \tau_1)$, where the probation period is

$$\tau_1 := -\frac{1}{r}\log\left(1 - r\bar{\tau}_1\right), \text{ in which } \bar{\tau}_1 := \min\left\{\frac{w_{M+1}^*}{c_1}, \frac{1}{r}\right\}. \tag{97}$$

Next, we define two values that will be useful in designing contracts for type $c \in \{c_2,...,c_M\}$,

$$\mathcal{V}_i^* := \frac{w_{i-1}^* - w_i^*}{c_i - c_{i-1}}(\mu R - c_i), \text{ and } \check{\mathcal{V}}_i := \check{\mathcal{J}}\left(w_i^*, \min\left\{\frac{w_{M+1}^*}{c_i}, \frac{1}{r}\right\}, c_i, c_{i+1}\right) + w_i^*. \tag{98}$$

The first $\mathcal{V}_i^*$ is the maximum societal value such that the better agent does not mimic; the second $\check{\mathcal{V}}_i$ is the maximum societal value that is achievable using our baseline probation contract. Notice that our multi-type contracts will later hinge on these values because we are looking for an implementable contract that makes both the better type and worse type agents unwilling to mimic. If $c \in \{c_2,...,c_M\}$, the agent is given a probation contract with potential randomization at time 0, defined as the following.

Definition 7. For any probation period $\tau \geq 0$, initial promised utility w, and two probabilities p_0 and $p_{\hat{w}}$, define a *probation contract with randomization* $\gamma_{\mathtt{r}}^c(w, \tau, p_0, p_{\hat{w}}, c')$ for a type $c \in \mathcal{C}$ agent against another type $c' \in \mathcal{C}$. At time 0,

(1) with probability p_0, the contract is terminated;

(2) with probability $p_{\hat{w}}$, the contract continues with $\gamma_{\mathtt{D}}^c(\hat{w}, S'(\hat{w}), 0)$ of Definition 2, in which

$$\hat{w} := \frac{p_0 + p_{\hat{w}}}{p_{\hat{w}}} w, \text{ and } S'(\hat{w}) = \left\lceil \frac{\ln(c/r \cdot \hat{w}/\mu)}{\ln(c/(c'r + \mu c))} \right\rceil. \tag{99}$$

(3) With probability $1 - p_0 - p_{\hat{w}}$, the contract continues with $\gamma_{\mathtt{P}'}^c(w, \tau, c')$ following definition 6.

The contract defined above involves randomization among the following three special forms of contracts: (1) immediate termination at time 0; (2) zero probation with delayed payments; (3) a probation contract. The following remark examines the purpose of each contract.

Remark 7. All three cases of contracts guarantee that an agent with a worse type does not want to work when mimicking the focal type. In comparison, a better-type agent has the incentive to work in cases (2) and (3), and the corresponding mimicking utilities are easy to calculate following our previous analysis. For case (1), because the agent is immediately terminated, the corresponding utility is zero anyway. The right mixing probability guarantees that the expected utility obtained from mimicking is no higher than telling the truth. We need all three cases in order to have sufficient degrees of freedom to satisfy all truth-telling constraints.

In our construction, the principal offers a type c_i agent the contract $\gamma_{\mathtt{r}}^{c_i}(w_i^*, \tau_i, p_0^i, p_{\hat{w}}^i, c_{i+1})$, in which the parameters are defined in the following. For any $i \in \{2, \ldots, M\}$, the actual probation period length is

$$\tau_i := -\frac{1}{r} \ln(1 - r\bar{\tau}_i). \text{ in which } \bar{\tau}_i := \min\left\{\frac{w_{M+1}^*}{c_i}, \frac{1}{r}\right\}. \tag{100}$$

The probabilities p_0^i and $p_{\hat{\mathrm{W}}}^i$ are defined as

$$p_0^i := \begin{cases} 0, & \text{if } \mathcal{V}_i^* \geq \check{\mathcal{V}}_i, \\ p^i q^i, & \text{if } \mathcal{V}_i^* < \check{\mathcal{V}}_i, \end{cases} \text{ and} \tag{101}$$

$$p_{\hat{w}}^i := \begin{cases} 0, & \text{if } \mathcal{V}_i^* \geq \check{\mathcal{V}}_i, \\ (1-p^i)q^i, & \text{if } \mathcal{V}_i^* < \check{\mathcal{V}}_i, \end{cases} \tag{102}$$

where

$$p^i := \max\left\{1 - \frac{r\mathcal{V}_i^*}{2(\mu R - c_i)}, 1 - r\bar{\tau}_i, 1 - \frac{w_i^*}{\max(w_i^*, c_i/r)}\right\}, q^i := \frac{\check{\mathcal{V}}_i - \mathcal{V}_i^*}{\check{\mathcal{V}}_i - (1-p^i)(\mu R - c_i)/r},$$

with $\mathcal{V}_i^*$ and $\check{\mathcal{V}}_i$ defined in (98).

Lemma 10. *Define a menu of contracts $\hat{\Gamma}_{\mathcal{C}} := \{\gamma^c\}_{c \in \mathcal{C}}$, in which*

$$\gamma^c := \begin{cases} \gamma_{\mathtt{r}}^{c_i}(w_i^*, \tau_i, p_0^i, p_{\hat{w}}^i, c_{i+1}) \text{ of Definition 7}, & \text{if } i \in \{2, ..., M\}, \\ \gamma_{\mathtt{B}}^{c_i}(0, w_{M+1}^*) \text{ of Definition 3}, & \text{if } i \in \{M+1, ..., N\}, \end{cases}$$

and

$$\gamma^{c_1} := \gamma_{\mathtt{p}'}^{c_1}(w_1^*, \tau_1, c_2) \text{ of Definition 6, in which } \tau_1 \text{ is defined in (97).}$$

The menu of contracts $\hat{\Gamma}_{\mathcal{C}}$ satisfies (LL), (PK), (IC), (IR), (FE) and (TT).

Finally, we verify in the following proposition that the performance of the menu of contracts $\hat{\Gamma}_{\mathcal{C}}$ is indeed $\check{\mathcal{Y}}^N$ as defined in (94), which can be computed through the aforementioned dynamic program together with a line search.

Proposition 9. *We have*

$$\check{\mathcal{Y}}^N = \mathcal{U}\left(\hat{\Gamma}_{\mathcal{C}}\right). \tag{103}$$

We numerically study the performance of $\hat{\Gamma}_{\mathcal{C}}$, taking $\mu \in \{0.1, 0.2, ..., 0.9\}$, $R \in \{3, 4, ..., 20\}$, and $r = 1 - \mu$. We consider a case with three types, $c \in \{1, 3, 5\}$, and another case with five types, $c \in \{1, 3, 5, 7, 9\}$, and compute the performance loss as the relative sub-optimality $1 - \check{\mathcal{Y}}^N / \hat{\mathcal{Y}}^N$. For the three-type case, out of all the $9 \times 18 = 162$ instances, 112 are such that the upper bound $\hat{\mathcal{Y}}^N > 0$. The average performance loss of the 112 cases is only 1.46%. For the five-type case, among all the 162 instances, 94 are such that $\hat{\mathcal{Y}}^N > 0$. The average performance loss of the 94 cases is 3.40%. Therefore, the menu of contracts $\hat{\Gamma}_{\mathcal{C}}$ appears to be a good solution to the multiple types contract design problem.

9. Conclusion

We study an optimal incentive design problem in continuous time over an infinite horizon with both moral hazard and adverse selection. Specifically, the principal hires an agent to exert effort to increase the arrival rate of a Poisson process where the agent's efforts are unobservable by the principal and the agent's capabilities, measured by the operating cost, are unknown to the principal. This problem is generally very hard to solve. We rely on using the first arrival as a screening device to distinguish between different agent types to obtain incentive compatible menu of contracts and performance upper bounds. The comparison between the upper and lower bounds demonstrates that our design, which involves using a probation period before the first arrival and potential sign-on bonuses, yields solutions that are very close to optimal.

Endnotes

1. The charging of retainers by lobbyists is common, see for example, https://lobbyit.com/pricing/, https://arnoldpublicaffairs.com/faq/ and https://lobbying101.wordpress.com/about-lobbyists/how-much-do-they-charge/. Furthermore, it is common that R&D projects are funded for long durations of time and may not bring any results in the end.
2. Shirking and misuse of research funds are surprisingly common in R&D settings, see, for example, https://www.chron.com/news/houston-texas/article/Prof-accused-of-spending-NASA-grants-on-cars-1722521.php, https://www.nbcnews.com/news/us-news/philadelphia-professor-accused-spending-185-000-grant-funds-strip-clubs-n1118571, https://www.newsweek.com/fund-meant-vaccine-research-misused-least-145m-unrelated-expenses-almost-decade-1564954, and https://www.theguardian.com/higher-education-network/2015/mar/27/research-grant-money-spent.
3. It is worth noting that the integral is from $t+$. Therefore, any instantaneous payment at time t (for example, potential sign-on bonus at time 0) is not included in the promised utility. We use notation $W_{t-} := \lim_{s\uparrow t} W_t$, which includes the potential upwdard jump at time t.
4. First, in Sun and Tian (2018) the principal does not have to reimburse the operating cost rate c in real time. That is, the constraint (LL) reduces to L_t^c monotonically non-decreasing in time t. Second, in our paper, the agent can choose any effort level in $[0, \mu]$, while in Sun and Tian (2018), the agent can only choose between μ or $\underline{\mu}$.
5. We also express z as $\mathrm{z}(\mathrm{w}, \tau)$ if its dependence on (w, τ) is important to highlight.
6. We include the cost parameter b in the definition of $\gamma^g_{\mathrm{P}'}(w, \tau, b)$ for ease of exposition when introducing Definition 7 later in the paper.

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Appendix

A. Summary of Notations

Model parameters

R: revenue to the principal for each arrival.

μ: the highest possible effort.

$\mathcal{C}$: support of the distribution of the operating cost. $\mathcal{C} = \{g, b\}$ from Section 4-7 and $\mathcal{C} = \{c_1, c_2, \dots c_N\}$ in Section 8.

c: generic operating cost.

ν, ν^0 and $\bar{\nu}$: generic, always zero effort and always highest effort process, respectively.

r: principal and agent's discount rates.

Contracts and utilities

I and ℓ: instantaneous and flow payments, respectively.

L: payment process $dL_t = I_t + \ell_t dt$.

η: stopping time.

γ: generic contract, $\gamma = (L, \eta)$.

$\Gamma_{\mathcal{C}}$: generic menu of contracts.

$\Gamma^*_{\{g,b\}}$: optimal menu of contracts in two-type case under condition (50) (Section 5).

$\check{\Gamma}_{\{g,b\}}$: a menu of contracts defined in Proposition 7 (Section 6.2).

$\hat{\Gamma}_{\mathcal{C}}$: a menu of contracts defined in Lemma 10.

$\hat{\gamma}^c(w)$: *IC-binding contract* in Definition 1.

$\gamma^g_{\mathtt{D}}(w, S, t_0)$: *delay-payment contract* in Definition 2.

$\gamma^c_{\mathtt{B}}(w, B)$: *sign-on-bonus contract* in Definition 3.

$\gamma^c_{\mathtt{P}}(w, \tau)$: *probation contract* in Definition 4.

$\gamma^c_{\mathtt{P}'}(w, \tau, b)$: *probation contract* in Definition 6.

$\gamma^c_{\mathtt{r}}(w, \tau, p_0, p_{\hat{w}}, c')$: *probation contract with randomization* in Definition 7.

u and U: agent's and principal's utilities, respectively.

W_t: agent's continuation utility.

$\mathcal{U}$ and $\mathcal{Z}$: principal's total expected discounted utility and principal's optimal expected utility, respectively.

Derived quantities

β_c: per arrival rate per unit time unit operating cost.

$\bar{w}_c$: defined in (4).

$\check{\omega}(\tau)$ and $\hat{\omega}(\tau)$: defined in (31).

τ_{z}: defined in (33).

$\mathrm{z}(w,\tau)$: a unique value solves (32) if $w \in [\check{\omega}(\tau), \hat{\omega}(\tau))$; defined in (36) if $w \geq \hat{\omega}(\tau)$.

$\bar{\tau}$: defined in (40).

$(w_g^*, w_b^*, \bar{\tau}^*)$: an optimal solution of the optimization (42)-(44).

$w_B(w_g^*, w_b^*)$: defined in (47).

$\bar{b}$: defined in Lemma 9.

$(\check{w}_g, \check{w}_b, \check{\tau})$: parameters used in define $\check{\Gamma}_{\{g,b\}}$.

$\bar{\tau}_i$, p_0^i, and $p_{\hat{w}}^i$: defined in (100) - (102).

$\{w_i^*\}_{i=1,\dots,M+1}$: defined in (94) and (95).

Value functions and optimization problems

V_c: unique solution to differential equation (9) with boundary condition (10).

G: a function defined in (28) and (29).

$\check{G}$: a function defined in (54).

J: a function defined in (41).

$\mathcal{J}$ and $\mathcal{G}$: functions defined in (75) and (76), respectively.

$\check{\mathcal{J}}$ and $\check{\mathcal{G}}$: functions defined in (88) and (87), respectively.

$\mathcal{Y}$: the optimal value of the upper bound optimization problem (23)-(27).

$\mathcal{Y}^N$: the optimal value of the upper bound optimization problem (77)-(81).

$\hat{\mathcal{Y}}^N$: the optimal value of the optimization problem (82).

$\check{\mathcal{Y}}^N$: the optimal value of the optimization problem (94).

B. Proof in Section 3

B.1. Proof of Lemma 1

To characterize how the agent's continuation utility evolves over time, it is useful to consider her lifetime expected utility, evaluated conditionally upon the information available at time t

$$u_t(\gamma,\nu;c) = \mathbb{E}^\nu\left[\int_0^\eta e^{-rs}\left(\mathrm{d}L_s - \beta_c\cdot\nu_s\mathrm{d}s\right)\middle|\mathcal{F}_t^N\right]$$
$$= \int_0^{t\wedge\eta-} e^{-rs}(\mathrm{d}L_s - \beta_c\cdot\nu_s\mathrm{d}s) + e^{-rt}W_t(\gamma,\nu;c) \tag{104}$$

Since $u_t(\gamma,\nu;c)$ is the expectation of a given random variable conditional on $\mathcal{F}_t^N$, the process $\mathbf{u}(\gamma,\nu;c) = \{u_t(\gamma,\nu;c)\}_{t\geq 0}$ is an martingale under the probability measure $\mathbf{P}^\nu$. Relying on this martingale property, we now offer an alternative representation of $\mathbf{u}(\gamma,\nu;c)$. Consider the process $M^\nu = \{M_t^\nu\}_{t=\geq 0}$ defined by

$$M_t^\nu = N_t - \int_0^t \nu_s\mathrm{d}s \tag{105}$$

for all $t\geq 0$. The martingale representation theorem for point processes implies that the martingale $\mathbf{u}(\gamma,\nu;c)$ satisfies

$$u_t(\gamma,\nu;c) = u_0(\gamma,\nu;c) + \int_0^{t\wedge\eta} e^{-rs}H_s(\gamma,\nu;c)\mathrm{d}M_s^\nu \tag{106}$$

for all $t \geq 0$, $\mathbf{P}^{\nu}$-almost surely, for some $\mathcal{F}^N$-predictable process $H(\gamma,\nu;c) = \{H_t(\gamma,\nu;c)\}_{t\geq 0}$. Then, (104) and (106) imply (PK). Next, we show that $\{\nu_t\}_{t\in[0,\eta]}$ defined in (IC) is a best response to contract γ.

Let u_t' denote the agent's lifetime expected payoff, given the information available at date t, when he acts according to $\nu' = \{\nu_t'\}_{t\geq 0}$ until date t and then reverts to $\nu = \{\nu_t\}_{t\geq 0}$:

$$u_t' = \int_0^{t\wedge\eta^-} e^{-rs}\left(\mathrm{d}L_s - \nu_s'\cdot\beta_c \mathrm{d}s\right) + e^{-rt}W_t(\gamma,\nu;c) \tag{107}$$

Following Sannikov (2008) (Proposition 2), the proof now proceeds as follows. First, we show that if $u' = \{u_t'\}_{t\geq 0}$ is an $\mathcal{F}^N$-submartingale under $\mathbf{P}^{\nu}$ that is not a martingale, then ν is suboptimal for the agent. Indeed, in that case, there exists some $t > 0$ such that

$$u_{0-}(\gamma,\nu;c) = u_{0-}' < \mathbb{E}^{\nu'}[u_t'] \tag{108}$$

where $u_{0-}(\gamma,\nu;c)$ and u_{0-}' correspond to unconditional expected payoffs at date 0. By (107), the agent is then strictly better off acting according to ν' until date t and then reverting to ν. The claim follows. Next, we show that if u' is a $\mathcal{F}^N$-supermartingale under $\mathbf{P}^{\nu'}$, then ν is at least as good as ν' for the agent. From (104) and (107),

$$u_t' = u_t(\gamma,\nu;c) + \int_0^{t\wedge\eta} e^{-rs}(\nu_s - \nu_s')\cdot\beta_c \mathrm{d}s \tag{109}$$

for all $t \geq 0$. Hence, since $u_t(\gamma,\nu;c)$ is right-continuous with left-hand limit, so is u'. Moreover, since u' is non-negative, it has a last element. Hence, by the optional sampling theorem (Dellacherie and Meyer (2011), Chapter VI, Theorem 10)),

$$u_0' \geq \mathbb{E}^{\nu'}[u_\eta'] = u_0(\gamma,\nu';c) \tag{110}$$

where again $u_{0-}(\gamma,\nu')$ is an unconditional expected payoff at date 0. Since $u_0' = u_0(\gamma,\nu)$ by (107), the claim follows. Now, for each $t \geq 0$,

$$\begin{aligned}
u_t' &= u_t(\gamma,\nu;c) + \int_0^{t\wedge\eta} e^{-rs}(\nu_s - \nu_s')\beta_c \mathrm{d}s \\
&= u_0(\gamma,\nu;c) + \int_0^{t\wedge\eta} e^{-rs}H_s(\gamma,\nu;c)\mathrm{d}M_s^{\nu'} + \int_0^{t\wedge\eta} e^{-rs}H_s(\gamma,\nu;c)(\nu_s' - \nu_s)\mathrm{d}s + \int_0^{t\wedge\eta} e^{-rs}(\nu_s - \nu_s')\beta_c \mathrm{d}s \\
&= u_0(\gamma,\nu;c) + \int_0^{t\wedge\eta} e^{-rs}H_s(\gamma,\nu;c)\mathrm{d}M_s^{\nu'} + \int_0^{t\wedge\eta} e^{-rs}(\nu_s - \nu_s')\left[\beta_c - H_s(\gamma,\nu;c)\right]\mathrm{d}s
\end{aligned} \tag{111}$$

Since $H(\gamma,\nu;c)$ is $\mathcal{F}^N$-predictable and $M^{\nu'}$ is an $\mathcal{F}^N$-martingale under $P^{\nu'}$, the drift of u' has the same sign as

$$(\nu_s - \nu_s')\left[\beta_c - H_s(\gamma,\nu;c)\right]$$

for all $t \in [0,\eta)$. If (IC) holds, then this drift remains non-positive for all $t \in [0,\eta)$ and all choices of ν'. This implies that for any effort process ν', u' is an $\mathcal{F}^N$-supermartingale under $P^{\nu'}$ and, thus, that ν is at least as good as ν' for the agent. If (IC) does not hold for the effort process ν, then choose ν' such that for each $t \in [0,\eta)$, $\nu_t' = \mu$ if $H_t \geq \beta_c$ and $\nu_t' = 0$ if $H_t < \beta_c$. The drift of u' is then everywhere non-negative and strictly positive over a set of $P^{\nu'}$-strictly positive measure. As a result of this, u' is an $\mathcal{F}^N$-submartingale under $P^{\nu'}$ that is not a martingale and, thus, ν is suboptimal for the agent. This concludes the proof.

C. Proofs in Section 4

C.1. Proof of Lemma 2

Case 1. $\bar{w}_c \leq \beta_c$. Rearrange equation (9) as

$$V_c(w) - rV_c'(w)(w - \bar{w}_c) + c - \mu R = \mu V_c(\bar{w}_c)\ . \tag{112}$$

Consider the above equation in $[0,\bar{w}_c)$, it is a linear ordinary differential equation with boundary condition. The solution is

$$V_c(w) = V_c(\bar{w}_c) + c_1(\bar{w}_c - w)^{\frac{1}{r}} \quad \text{for} \quad w \in [0,\bar{w}_c]\ .$$

with $c_1 = -(\mu R - c)/r\bar{w}_c^{-\frac{1}{r}} < 0$. (Our initial assumption of $R > \beta$ is equivalent to $0 < \mu R - c$.)

Then with $V_c'(w) = -c_1(\bar{w}_c - w)^{\frac{\mu}{r}}/r > 0, V_c''(w) = c_1\mu(\bar{w}_c - w)^{\frac{\mu - r}{r}}/r^2 < 0$ for $w \in [0, \bar{w}_c]$. Hence V_c is increasing and strictly concave on $[0, \bar{w}_c]$. Furthermore, it can be verified that $V(\bar{w}_c) = (\mu R - c)/r$, $V_c'(\bar{w}_c) = 0$, and $V_c(w) = (\mu R - c)/r$ for $w \in [\bar{w}_c, \infty)$ solves (9).

Case 2. $\bar{w}_c > \beta_c$. Rearrange equation (9) as

$$V_c(w) - rV_c'(w)(w - \bar{w}_c) + c - \mu R = \mu V_c(\bar{w}_c) \quad \text{for} \quad w \in [\bar{w}_c - \beta_c, \infty) \ , \tag{113}$$

$$V_c(w) - rV_c'(w)(w - \bar{w}_c) + c - \mu R = \mu V_c(w + \beta_c) \quad \text{for} \quad w \in (0, \bar{w}_c - \beta_c) \ . \tag{114}$$

We then show the result according to the following steps.

1. Demonstrate the solution of (113) as a parametric function V_b, with parameter b.
2. Show that the solution of (114) is unique and twice continuously differentiable for any b, also called V_b.
3. Show that the V_b is convex and decreasing for $b > 0$ and concave and increasing for $b < 0$.
4. Show that $V_b(0)$ is increasing in b for $b < 0$, which implies that the boundary condition $V_b(0) = 0$ uniquely determines b, and therefore the solution of the original differential equation.

Step 1. The solution to the linear ordinary differential equation (113) on $[\bar{w}_c - \beta_c, \bar{w}_c)$ must have the following form, for any scalar b.

$$V_b(w) = (\mu R - c)/r + b(\bar{w}_c - w)^{\frac{1}{r}} \quad \text{for } w \in [\bar{w}_c - \beta_c, \bar{w}_c) \ . \tag{115}$$

Also define $V_b(w) = (\mu R - c)/r$ for $w \in [\bar{w}_c, \infty)$, which satisfies (113), so that V_b is continuously differentiable on $[\bar{w}_c - \beta_c, \infty)$.

Step 2. Using (115) as the boundary condition, we show that differential equation (114) has a unique solution (also called $V_b(w)$, on $(0, \bar{w}_c - \beta_c)$), which is continuously differentiable. In fact, differential equation (114) is equivalent to a sequence of initial value problems over the intervals $[\bar{w}_c - (k+1)\beta_c, \bar{w}_c - k\beta_c), k = 1, 2, ...$ This sequence of initial value problems satisfies the Cauchy-Lipschitz Theorem and, therefore, bears unique solutions. Also, computing $V_b'(\bar{w}_c - \beta_c)$ from (115), and comparing it with (114), we see that V_b is continuously differentiable at $\bar{w}_c - \beta_c$, and therefore on $[0, \infty)$.

Further, deriving the expressions for $V_b''(w)$ following (114) and (115), respectively, confirms that V_b is twice continuously differentiable on $([0, \infty))$. In particular, (114) implies that

$$V_b''(w) = \frac{\mu[V_b'(w + \beta_c) - V_b'(w)]}{r(\bar{w}_c - w)} \ . \tag{116}$$

Step 3. Next, we argue that in order to satisfy the boundary condition $V_b(0) = 0$, we must have $b < 0$. Equivalently, we show that if $b > 0$, V_b must be convex and decreasing, which violates $V_b(0) = 0 < (\mu R - c)/r = V_b(\bar{w}_c)$. In fact, if $b > 0$, (115) implies that V_b is decreasing and convex on $[\bar{w}_c - \beta_c, \bar{w}_c)$, and therefore $V_b''(w) > 0$ on this interval. Assume there exists $\breve{w} \in [0, \bar{w}_c - \beta_c)$, such that $V_b''(\breve{w}) \leq 0$, then V_b twice continuously differentiable implies that there must $\tilde{w} = \max\{w \in [0, \bar{w} - \beta) | V_b''(w) = 0\}$, and $V_b''(w) > 0, \forall w > \tilde{w}$. Equation (116) implies that $V_b'(\tilde{w} + \beta_c) = V_b'(\tilde{w})$. However, it contradicts with

$$V_b'(\tilde{w} + \beta_c) = V_b'(w) + \int_0^{\beta_c} V_b''(\tilde{w} + x)dx > V_b'(\tilde{w}) \ .$$

Therefore, we must have $V_b''(w) > 0$ and V_b is decreasing on $[0, \bar{w}_c)$ if $b > 0$. In case $b = 0$, $V_b(w)$ is a constant following (114) and (115), which also contradicts the boundary condition. Therefore we must have $b < 0$.

The same logic implies that for $b < 0$, V_b must best be increasing and strictly concave on $[0, \bar{w}_c)$.

Step 4. Finally, we show that $V_b(0)$ is strictly increasing in b for $b < 0$, which allows us to uniquely determine b that satisfies $V_b(0) = 0$. For any $b_1 < b_2 < 0$, it can be verified that $V_{b_1}(w) < V_{b_2}(w)$, $V_{b_1}'(w) > V_{b_2}'(w)$, for $w \in [\bar{w}_c - \beta_c, \bar{w}_c)$ from (115). We claim that $V_{b_1}' > V_{b_2}'$ $\forall w \in [0, \bar{w}_c]$. Otherwise, because $V_{b_1} - V_{b_2}$ is continuously differentiable, there must exists $w' = \max\left\{w | V_{b_1}'(w) = V_{b_2}'(w), w \in [0, \bar{w} - \beta)\right\}$ and $V_{b_1}'(w) > V_{b_2}'(w) \forall w > w'$. Equation (114) implies that $\mu(V_{b_1}(w' + \beta_c) - V_{b_2}(w' + \beta_c)) = (V_{b_1}(w') - V_{b_2}(w'))$. However, it contradicts with

$$0 > V_{b_1}(w' + \beta) - V_{b_2}(w' + \beta) = V_{b_1}(w') - V_{b_2}(w') + \int_0^{\beta} V_{b_1}'(w' + x) - V_{b_2}'(w' + x)dx \ .$$

Therefore, we must have $V_{b_1}'(w) - V_{b_2}'(w) > 0$, $\forall w \in [0, \bar{w}_c)$ and it implies that $V_{b_1}(w) - V_{b_2}(w) < 0$, $\forall w \in [0, \bar{w}_c)$. This implies that $V_b(0)$ is strictly increasing in b for $b < 0$. Because $V_0(0) = (\mu R - c)/r$ and $\lim_{b \to -\infty} V_b(0) < V_b(\bar{w}_c - \beta_c) = -\infty$, there must exists a unique $b^* < 0$ such that $V_{b^*}(0) = 0$. And V_{b^*} is strictly concave and increasing in $[0, \bar{w}_c]$. Finally, we denote $V_{b^*}(w)$ as $V_c(w)$. Q.E.D.

C.2. An optimality condition to prove Proposition 1

In this section, we present an optimality condition, which will help us prove the optimality of contracts in the later sections.

LEMMA 11. *Suppose $F(w)$ is a differentiable, concave and upper-bounded function, with $F(0)=0$ and $F'(w)\geq -1$. Consider any $\gamma^c\in\Delta_c(w)$ with $\nu^c\in\mathfrak{N}(\gamma^c,c)$, followed by the promised utility process $\{W_t\}_{t\geq 0}$ according to (PK). Define a stochastic process $\{\Psi_t\}_{t\geq 0}$, where*

$$\Psi_t := F'(W_{t-})rW_{t-} - rF(W_{t-}) + \nu_t^c\left[R - F'(W_{t-})H_t + F(W_{t-}+H_t) - F(W_{t-})\right] - \nu_t^c\beta_c. \tag{117}$$

If the process $\{\Psi_t\}_{t\geq 0}$ is non-positive almost surely, then we have $F(w)\geq U(\gamma^c,\nu^c)$.

Proof. Following Ito's Formula for jump processes (see, for example, Theorem 17.5 of Bass (2011)), and considering (PK), we have

$$\begin{aligned} e^{-r(T\wedge\eta)}F(W_{T\wedge\eta}) &= F(W_{0-}) + \int_0^{T\wedge\eta}\left[e^{-rt}dF(W_{t-}) - re^{-rt}F(W_{t-})dt\right] \\ &= F(W_{0-}) + \int_0^{T\wedge\eta} e^{-rt}(-RdN_t + dL_t) + \int_0^{T\wedge\eta} e^{-rt}\mathcal{A}_t \end{aligned} \tag{118}$$

where

$$\begin{aligned} \mathcal{A}_t :=& dF(W_{t-}) - rF(W_{t-})dt + RdN_t - dL_t \\ =& F'(W_{t-})\left[rW_{t-} - \nu_t^c H_t + \nu_t^c\beta_c - \ell_t\right]dt + F(W_{t-}+H_tdN_t - I_t) - F(W_{t-}) - rF(W_{t-})dt + RdN_t - dL_t. \end{aligned}$$

Further, define

$$\mathcal{B}_t := [F(W_{t-}+H_t) - F(W_{t-})](dN_t - \nu_t^c dt) + R(dN_t - \nu_t^c dt).$$

Because function $F(w)$ is concave, $F'(w)\geq -1$ and ν_t^c follows (IC), we have

$$\begin{aligned} \mathcal{A}_t \leq& F'(W_{t-})\left(rW_{t-} - \nu_t^c H_t\right)dt + F(W_{t-}+H_tdN_t) - F'(W_{t-})(\ell_t - \nu_t^c\beta_c)dt \\ &- F'(W_{t-}+H_tdN_t)I_t - F(W_{t-}) - rF(W_{t-})dt + RdN_t - dL_t \\ \leq& F'(W_{t-})\left(rW_{t-} - \nu_t^c H_t\right)dt - rF(W_{t-})dt - \nu_t^c\beta_c dt + RdN_t + F(W_{t-}+H_tdN_t) - F(W_{t-}) \\ =& F'(W_{t-})\left(rW_{t-} - \nu_t^c H_t\right)dt - rF(W_{t-})dt - \nu_t^c\beta_c dt + [F(W_{t-}+H_t) - F(W_{t-})]dN_t + \nu_t^c Rdt + \mathcal{B}_t \\ =& \mathcal{B}_t + \Psi_t dt. \end{aligned} \tag{119}$$

Taking the expectation on both sides of (118) and letting $T\to\infty$, we have

$$\begin{aligned} F(W_{0-}) &\geq \mathbb{E}\left[e^{-r\eta}F(W_\eta) + \int_0^\eta e^{-rt}(RdN_t - dL_t) - \int_0^\eta e^{-rt}\mathcal{B}_t - \int_0^\eta e^{-rt}\Psi_t dt\right] \\ &\geq \mathbb{E}\left[e^{-r\eta}F(W_\eta) + \int_0^\eta e^{-rt}(RdN_t - dL_t)\right] = U(\gamma^c,\nu^c), \end{aligned} \tag{120}$$

where the first inequality follows from (119), the second inequality follows from $\Psi_t\leq 0$ and $\mathbb{E}\left[\int_0^\eta e^{-rt}\mathcal{B}_t\right]=0$, and the last equality follows from $F(W_\eta)=F(0)=0$. Q.E.D.

A Simplification of Ψ_t. Following (117), if we define $V(w):=F(w)+w$, then

$$\begin{aligned} \Psi_t &= V'(W_{t-})rW_{t-} - rW_{t-} - rV(W_{t-}) + rW_{t-} + \nu_t^c\left[R - V'(W_{t-})H_t + V(W_{t-}+H_t) - V(W_{t-})\right] - \nu_t^c\beta_c \\ &\leq V'(W_{t-})rW_{t-} - rV(W_{t-}) + \nu_t^c\left[R - V'(W_{t-})\beta_c + V(W_{t-}+\beta_c) - V(W_{t-})\right] - \nu_t^c\beta_c, \end{aligned} \tag{121}$$

where the inequality follows from $\beta_c = \arg\max_{H_t\geq\beta_c}\{-V'(W_{t-})H_t + V(W_{t-}+H_t)\}$ and ν_t^c follows (IC).

C.3. Proof of Proposition 1

First, we show (14). If $R > \beta_c$, based on Lemma 11, to prove $F_c(w) \geq U(\gamma^c, \nu^c)$, we only need to show that $\{\Psi_t\}_{t\geq 0}$ is non-positive almost surely when we let $F(w) = F_c(w)$. Following (121), we have

$$\begin{aligned}\Psi_t &\leq V_c'(W_{t-})rW_{t-} - rV_c(W_{t-}) + \nu_t^c\left[R - V_c'(W_{t-})\beta_c + V_c(W_{t-}+\beta_c) - V_c(W_{t-})\right] - \nu_t^c\beta_c \\ &= V_c'(W_{t-})rW_{t-} - rV_c(W_{t-}) + \nu_t^c\left[R - \beta_c + V_c'(W_{t-})\beta_c + V_c(W_{t-}+\beta_c) - V_c(W_{t-})\right] := f(\nu_t^c).\end{aligned} \tag{122}$$

First,

$$f(0) = V_c'(W_{t-})rW_{t-} - rV_c(W_{t-}) = rW_{t-}\left[V_c'(W_{t-}) - \frac{V_c(W_{t-}) - V_c(0)}{W_{t-}}\right] \leq 0$$

where the inequality follows from the concavity of V_c. Second, following (9), we have $f(\mu) = 0$. Further since $f(\nu_t^c)$ is linear in ν_t^c, $f(\nu_t^c) \leq 0$ for any $\nu_t^c \in [0,\mu]$. Therefore, $\{\Psi_t\}_{t\geq 0}$ is non-positive almost surely.

If $R \leq \beta_c$, based on Lemma 11, to prove $-w \geq U(\gamma^c, \nu^c)$, we only need to show that $\{\Psi_t\}_{t\geq 0}$ is non-positive almost surely when we let $F(w) = -w$. Following (121), we have

$$\Psi_t \leq V_c'(W_{t-})rW_{t-} - rV_c(W_{t-}) + \nu_t^c\left[R - V_c'(W_{t-})\beta_c + V_c(W_{t-}+\beta_c) - V_c(W_{t-})\right] - \nu_t^c\beta_c = \nu_t^c\left[R - \beta_c\right] \leq 0.$$

Second, we show (16) and (17) if $R > \beta_c$. In the following, we show that

$$W_t(\hat{\gamma}^c, \bar{\nu}; c) = W_t^c, \text{for } t \in [0, \hat{\eta}^c], \tag{123}$$

where the left-hand side is defined in (5) and the right-hand side follows (11). For any $t \in [0, \hat{\eta}^c]$ we have

$$\begin{aligned}e^{-rt}W_t^c &= W_0^c e^{r0} + \int_{0+}^t d(e^{-rs}W_s^c) \\ &= W_0^c + \int_0^t W_s^c de^{-rs} + \int_{0+}^t e^{-rs}dW_s^c \\ &= W_0^c + \int_0^t W_s^c(-r)e^{-rs}ds + \int_{0+}^t e^{-rs}\left[r(W_s^c - \bar{w}_c)ds + \min\{\bar{w}_c - W_s^c, \beta_c\}dN_s\right] \\ &= W_0^c + \int_{0+}^t e^{-rs}\left[cds + \beta_c(dN_s - \mu ds) - dL_s^c\right] ,\end{aligned}$$

in which the third equality follows from (PK), and the fourth equality from the definition of L^c.

Because W_t^c is bounded in $[0, \bar{w}_c]$, and $W_{\hat{\eta}^c}^c = 0$ if $\hat{\eta}^c < \infty$, we have $e^{-r\hat{\eta}^c}W_{\hat{\eta}^c}^c = 0$, and

$$e^{-r\hat{\eta}^c}W_{\hat{\eta}^c}^c = W_0^c + \int_{0+}^{\hat{\eta}^c} e^{-rs}\left[cds + \beta_c(dN_s - \mu ds) - dL_s^c\right].$$

Therefore

$$e^{-rt}W_t^c = \int_{t+}^{\hat{\eta}^c} e^{-rs}\left[dL_s^c + cds - \beta_c(dN_s - \mu ds)\right].$$

Taking conditional expectation on both side, and noting that W_t is $\mathcal{F}_t^N$-adapted, we obtain

$$\begin{aligned}e^{-rt}W_t^c &= \mathbb{E}\left[\int_t^{\hat{\eta}^c} e^{-rs}\left[dL_s^c - cds - \beta_c(dN_s - \mu ds)\right]\middle|\mathcal{F}_t^N\right] \\ &= \mathbb{E}\left[\int_t^{\hat{\eta}^c} e^{-rs}\left[dL_s^c - \mu\beta_c ds\right]\middle|\mathcal{F}_t^N\right] = e^{-rt}W_t(\hat{\gamma}^c, \bar{\nu}; c) ,\end{aligned}$$

in which the second equality follows from the (potentially non-homogeneous) Poisson process, which implies that

$$\mathbb{E}[N_{\hat{\eta}^c} - N_t|\mathcal{F}_t^N] = \mathbb{E}\left[\int_t^{\hat{\eta}^c}\mu ds\middle|\mathcal{F}_t^N\right].$$

Therefore, (123) implies (16) if we let $t = 0$.

Finally, we verify (17). Consider the process W_t^c according to (11). Following Itô's Formula for jump processes (see, for example, Bass 2011, Theorem 17.5),

$$dF(W_t^c) = F'(W_t^c)r(W_t^c - \bar{w}_c)dt + \left[F(W_t^c + \min\{\bar{w}_c - W_t^c, \beta_c\}) - F(W_t^c)\right]dN_t , \tag{124}$$

Therefore, for any $T \le \hat{\eta}^c$, we have

$$\begin{aligned} e^{-rT}F_c(W_T^c) &= e^{0r}F_c(W_0^c) + \int_0^T F_c(W_t^c)de^{-rt} + \int_0^T e^{-rt}dF_c(W_t^c) \\ &= F_c(W_0^c) + \int_0^T e^{-rt}\{[rW_t^c - \mu\beta_c]F_c'(W_t^c) - rF(W_t^c)\}dt \\ &\quad + \int_0^T e^{-rt}\big(F_c(W_t^c + \min\{\bar{w}_c - W_t^c, \beta_c\}) - F_c(W_t^c)\big)dN_t \end{aligned}$$

Applying Equation (9) to replace $F_c'(W_t^c)$, we have

$$\begin{aligned} e^{-rT}F_c(W_T^c) &= F(W_0^c) + \int_0^T e^{-rt}[F_c(\min\{W_t^c + \beta_c, \bar{w}_c\}) - F_c(W_t^c)](dN_t - \mu dt) \\ &\quad + \int_0^T e^{-rt}[(W_t^c + \beta_c - \bar{w}_c)^+ - R]\mu dt \\ &= F_c(W_0^c) + \int_0^T e^{-rt}[F_c(\min\{W_t^c + \beta_c, \bar{w}_c\}) - F_c(W_t^c) + R - (W_t^c + \beta_c - \bar{w}_c)^+](dN_t - \mu dt) \\ &\quad - \int_0^T e^{-rt}(RdN_t - dL_t^c). \end{aligned}$$

Taking expectation on both sides, and noting that $F_c(W_{\hat{\eta}^c}) = F_c(0) = 0$, we have

$$U(\hat{\gamma}^c(w), \bar{\nu}) = F_c(W_0^c) + \mathbb{E}\left[\int_0^T e^{-rt}[F_c(\min\{W_t^c + \beta_c, \bar{w}_c\}) - F_c(W_t^c) + R - (W_t^c + \beta_c - \bar{w}_c)^+](dN_t - \mu dt)\right]. \tag{125}$$

Because

$$\left|F_c(\min\{W_t^c + \beta_c, \bar{w}_c\}) - F_c(W_t^c) + R - (W_t^c + \beta_c - \bar{w}_c)^+\right| < \infty,$$

the process $\{M_s\}_{s \ge 0}$, defined as

$$M_s := \int_0^s e^{-rt}[F_c(\min\{W_t^c + \beta_c, \bar{w}_c\}) - F_c(W_t^c) + R - (W_t^c + \beta_c - \bar{w}_c)^+](dN_t - \mu dt),$$

is a martingale, which implies that the expectation term in (125) is 0 following the Optional Stopping Theorem, and hence the result. Q.E.D.

C.4. Proof of Lemma 3

First, we show that $\bar{\nu} \in \mathfrak{N}(\gamma_{\mathtt{B}}^b(w^b, I^b), g)$ for any $w^b, I^b \ge 0$. By the definition of *IC-binding contracts* and *sign-on-bonus contracts*, we have $\bar{\nu} \in \mathfrak{N}(\gamma_{\mathtt{B}}^b(w^b, I^b), b)$. Define bad agent's (type b) lifetime expected utility, evaluated conditionally upon the information available at time t under contract $\gamma_{\mathtt{B}}^b(w^b, I^b)$ and effort process $\bar{\nu}$ as u_t^b, then

$$u_t^b = \mathbb{E}^{\bar{\nu}}\left[\int_0^{\hat{\eta}^b} e^{-rs}(\mathrm{d}L_s^b - b\mathrm{d}s)\middle|\mathcal{F}_t^N\right] = u_0^b + \int_0^t \beta_b dM_s^{\bar{\nu}}$$

where $M_t^{\bar{\nu}} = N_t - \mu t$. Define good agent's (type g) lifetime expected utility, evaluated conditionally upon the information available at time t under contract $\gamma_{\mathtt{B}}^b(w^b, I^b)$ and effort process $\bar{\nu}$ as u_t^g, then

$$\begin{aligned} u_t^g &= \mathbb{E}^{\bar{\nu}}\left[\int_0^{\hat{\eta}^b} e^{-rs}(\mathrm{d}L_s^b + (b-g)\mathrm{d}s)\middle|\mathcal{F}_t^N\right] = u_t^b + \mathbb{E}^{\bar{\nu}}\left[\int_0^{\hat{\eta}^b} e^{-rs}(b-g)\mathrm{d}s\middle|\mathcal{F}_t^N\right] \\ &= u_0^b + \int_0^t \beta_b dM_s^{\bar{\nu}} + \mathbb{E}^{\bar{\nu}}\left[\int_0^{\hat{\eta}^b} e^{-rs}(b-g)\mathrm{d}s\middle|\mathcal{F}_t^N\right]. \end{aligned}$$

Next, we denote $u_t^{g'}$ as the good agent's lifetime expected payoff, given the information available at time t, when he acts according to $\nu' = \{\nu_t'\}_{t \ge 0}$ until time t and then reverts to $\bar{\nu}$, then

$$u_t^{g'} = u_t^g + \int_0^{t \wedge \hat{\eta}^b -} e^{-rs}(g - \nu_s'\beta_g)ds$$

$$= u_0^b + \int_0^{t\wedge\hat{\eta}^b-} \beta_b dM_s^{\bar{\nu}} + \mathbb{E}^{\bar{\nu}}\left[\int_0^{\hat{\eta}^b} e^{-rs}(b-g)\mathrm{d}s \middle| \mathcal{F}_t^N\right] + \int_0^{t\wedge\hat{\eta}^b-} e^{-rs}(g-\nu_s'\beta_g)ds$$

$$= u_0^b + \int_0^{t\wedge\hat{\eta}^b-} \beta_b dM_s^{\nu'} + \mathbb{E}^{\bar{\nu}}\left[\int_0^{\hat{\eta}^b} e^{-rs}(b-g)\mathrm{d}s \middle| \mathcal{F}_t^N\right] + \int_0^{t\wedge\hat{\eta}^b-} e^{-rs}(\mu-\nu_s')(\beta_g-\beta_b)ds$$

Then, for any $t' > t$,

$$\mathbb{E}^{\nu'}[u_{t'}^{g'}|\mathcal{F}_t^N] = \mathbb{E}^{\nu'}\left[u_0^b + \int_0^{t'\wedge\hat{\eta}^b-} H_s^b dM_s^{\nu'} + \mathbb{E}^{\bar{\nu}}\left[\int_0^{\hat{\eta}^b} e^{-rs}(b-g)\mathrm{d}s \middle| \mathcal{F}_{t'}^N\right] + \int_0^{t'\wedge\hat{\eta}^b-} e^{-rs}(\mu-\nu_s')(\beta_g-\beta_b)ds \middle| \mathcal{F}_t^N\right]$$

$$= u_0^b + \int_0^{t\wedge\hat{\eta}^b-} H_s^b dM_s^{\nu'} + \mathbb{E}^{\bar{\nu}}\left[\int_0^{\hat{\eta}^b} e^{-rs}(b-g)\mathrm{d}s \middle| \mathcal{F}_t^N\right] + \mathbb{E}^{\nu'}\left[\int_0^{t'\wedge\hat{\eta}^b-} e^{-rs}(\mu-\nu_s')(\beta_g-\beta_b)ds \middle| \mathcal{F}_t^N\right]$$

$$\leq u_0^b + \int_0^{t\wedge\hat{\eta}^b-} H_s^b dM_s^{\nu'} + \mathbb{E}^{\bar{\nu}}\left[\int_0^{\hat{\eta}^b} e^{-rs}(b-g)\mathrm{d}s \middle| \mathcal{F}_t^N\right] + \int_0^{t\wedge\hat{\eta}^b-} e^{-rs}(\mu-\nu_s')(\beta_g-\beta_b)ds = u_t^{g'},$$

where the second equality follows from the law of iterated expectation and the first inequality follows from that $(\mu-\nu_s')(\beta_g-\beta_b) \leq 0, \forall t$. Hence, $u_t^{g'}$ is $\mathcal{F}^N$-supermartingale under $P^{\nu'}$. Therefore, by the optional sampling theorem (Dellacherie and Meyer (2011), Chapter VI, Theorem 10),

$$u(\gamma_{\mathtt{B}}^b(w^b, I^b), \bar{\nu}; g) = u_0^{g'} \geq \mathbb{E}^{\nu'}[u_\eta^{g'}] = u(\gamma_{\mathtt{B}}^b(w^b, I^b), \nu'; g).$$

which implies that $\bar{\nu}$ is at least as good as ν' for the good agent. Similarly, we are able to show that for the bad agent, ν^0 is at least as good as ν', where ν' deviates from ν^0 before time t. Hence, $\nu^0 \in \mathfrak{N}(\gamma_{\mathtt{B}}^g(w^g, I^g), b)$.

C.5. Useful Definitions

Shirking Duration:

$$\bar{\tau}(\gamma) := \inf\{t : W_t = 0, N_s = 0 \forall s \leq t\}, \tag{126}$$

which represents the time an agent stays in contract γ given no arrival.

Effective Cumulated Effort:

$$\bar{T}(\gamma, \nu) := \mathbb{E}^\nu\left[\int_0^\eta e^{-rt}\nu_t \mathrm{d}t\right], \tag{127}$$

which measures the agent's expected effective cumulated effort under contract γ when the agent chooses the effort process ν.

Societal Value:

$$S(\gamma, \nu; c) = \mathbb{E}^\nu\left[\int_0^\eta e^{-rt}(R\mathrm{d}N_t - \nu_t \cdot \beta_c \mathrm{d}t)\right], \tag{128}$$

which measures the expected total value net of cost produced with effort ν when the agent's cost is c.

LEMMA 12. *The societal value produced is proportional to the working duration, i.e.,* $S(\gamma,\nu;c) = (R - \beta_c)\bar{T}(\gamma,\nu)$.

Proof:

$$S(\gamma, \nu; c) = \mathbb{E}^\nu\left[\int_0^\eta e^{-rt}(R\mathrm{d}N_t - \nu_t \cdot \beta_c \mathrm{d}t)\right] = \mathbb{E}^\nu\left[\int_0^\eta e^{-rt}(R \cdot \nu_t \mathrm{d}t - \nu_t\beta_c \mathrm{d}t)\right] = (R-\beta_c)\bar{T}(\gamma, \nu). \tag{129}$$

Hence, for each moment the agent exerts effort ν_t, he produces an expected revenue of $(R-\beta_c)\nu_t$.

C.6. Proof of Lemma 4

First, we verify that $u\Big(\gamma^g_{\mathtt{B}}(w^g,I^g),\nu^0;b\Big)=w^g+I^g$. Since $\nu^0\in\mathfrak{N}(\gamma^g_{\mathtt{B}}(w^g,I^g),b)$ (followed from Lemma 3), following the definition of *IC-binding contracts*, we have

$$\hat{\tau}^g=-\frac{1}{r}\log\left(1-\frac{r\min\{w_g,\bar{w}_g\}}{g}\right)$$

and

$$\begin{aligned}u\Big(\gamma^g_{\mathtt{B}}(w^g,I^g),\nu^0;b\Big)&=\mathbb{E}^{\nu^0}\left[\int_{0-}^{\hat{\tau}^g}e^{-rs}(\mathrm{d}L_s-\beta_g\cdot 0\mathrm{d}s)\middle|\mathcal{F}^N_t\right]=I^g+\max\{w_g-\bar{w}_g,0\}+\int_{0+}^{\hat{\tau}^g}e^{-rs}gds\\&=I^g+\max\{w_g-\bar{w}_g,0\}+\frac{g}{r}(1-e^{-r\hat{\tau}^g})=I^g+\max\{w_g-\bar{w}_g,0\}+\min\{w_g,\bar{w}_g\}=I^g+w^g.\end{aligned}\tag{130}$$

Second, we verify that $u\left(\gamma^b_{\mathtt{B}}(w^b,I^b),\bar{\nu};g\right)=w^b+I^b+(b-g)\dfrac{V_b(w^b)}{\mu R-b}$. Since $\bar{\nu}\in\mathfrak{N}(\gamma^b_{\mathtt{B}}(w^b,I^b),g)$, we have

$$\begin{aligned}u\left(\gamma^b_{\mathtt{B}}(w^b,I^b),\bar{\nu};g\right)&=\mathbb{E}^{\bar{\nu}}\left[\int_{0-}^{\hat{\tau}^b}e^{-rs}(\mathrm{d}L^b_s-g\mathrm{d}s)\middle|\mathcal{F}^N_t\right]=I^b+\max\{w_b-\bar{w}_b,0\}+\mathbb{E}^{\bar{\nu}}\left[\int_{0-}^{\hat{\tau}^b}e^{-rs}(\mathrm{d}L^b_s-b+(b-g)\mathrm{d}s)\middle|\mathcal{F}^N_t\right]\\&=I^b+\max\{w_b-\bar{w}_b,0\}+\min\{w_b,\bar{w}_b\}+\mathbb{E}^{\bar{\nu}}\left[\int_{0-}^{\hat{\tau}^b}e^{-rs}(b-g)\mathrm{d}s\middle|\mathcal{F}^N_t\right]\\&=I^b+w^b+(b-g)/\mu\cdot\bar{T}(\gamma^b_{\mathtt{B}}(w^b,I^b),\bar{\nu})=I^b+w^b+(b-g)\frac{V_b(w_b)}{\mu R-b}.\end{aligned}\tag{131}$$

where the fourth equality follows from (127), and the fifth equality follows from Lemma 12 and Proposition 1.

C.7. Proof of Lemma 5

The second constraint of the optimization problem (21) further implies that

$$w^b+I^b\geq w^g+I^g\geq w^b+I^b+(b-g)\frac{V_b(w^b)}{\mu R-b}.$$

Since $V_b(w^b)\geq 0$ for any w^b, we should have $V_b(w^b)=0$. Hence, $w^b+I^b=w^g+I^g$. Therefore, the objective becomes

$$\begin{aligned}&\max_{w^g,w^b,I^g,I^b}\quad pF_g(w^g)+(1-p)F_b(w^b)-pI^g-(1-p)I^b\\=&\max_{w^g,w^b,I^g,I^b}\quad pF_g(w^g)+(1-p)-w^b-pI^g-(1-p)I^b\\=&\max_{w^g\geq 0,I^g\geq 0}\quad pF_g(w^g)-(1-p)w^g-I^g,\end{aligned}\tag{132}$$

without any further constraints. Since the objective is decreasing in I^g, the optimization further becomes

$$\max_{w^g}\quad pF_g(w^g)-(1-p)w^g=pV_g(w^g)-w^g.$$

Since V_g is strictly concave on w^g in $[0,\bar{w}_g]$, we have $w^g_*=\min\{w|V_g'(w)=1/p\}$ if $V_g'(0)>1/p$ and w^g_* if $V_g'(0)\leq 1/p$. Furthermore, $I^g_*=0$, $w^b_*=0$ and $I^b_*=w^g_*$.

D. Proof in Section 5

D.1. Proof of Proposition 3

For any contract pair of contracts (γ^g,γ^b) that satisfy (LL), (PK), (IC), (IR), (FE), and (TT), we create a vector (w_g,w_b,τ,ξ) such that, they satisfy the constraints (24) - (27), and $p\cdot G(w_g,\tau)+(1-p)\xi\geq p\cdot U(\gamma^g,\bar{\nu})+(1-p)U(\gamma^b,\bar{\nu})$. Then, we have $\mathcal{Y}\geq\mathcal{Z}(\{g,b\})$.

We let $w_g:=u(\gamma^g,\nu^g;g)$, $w_b:=u(\gamma^b,\nu^b;b)$, $\tau:=\bar{\tau}(\gamma^g)$ where $\bar{\tau}(.)$ is defined in (126) and $\xi:=\min\left\{F_b(w_b),\dfrac{w_g-w_b}{b-g}(\mu R-b)^+-w_b\right\}$.

Step 1: We check the constraints (24)-(27). First, (TT) implies that

$$w_g \geq \max_{\nu} u(\gamma^b, \nu; g) \geq u(\gamma^b, \nu^b; g) = w_b + (b-g)\bar{T}(\gamma^b, \nu^b) \geq w_b, \tag{133}$$

where $\nu^b \in \mathfrak{N}(\gamma^b, b)$ and $\bar{T}$ is defined in (127). Second,

$$w_b \geq \max_{\nu} u(\gamma^g, \nu; b) \geq u(\gamma^g, \nu^0; b) = g \int_0^{\tau} e^{-rt} \mathrm{d}t = g/r \cdot (1 - e^{-r\tau}). \tag{134}$$

Finally, by the definition of τ and ξ, (25)-(27) are automatically satisfied. Hence, constraints (24)-(27) are satisfied.

Step 2: We prove that $p \cdot G(w_g, \tau) + (1-p)\xi \geq p \cdot U(\gamma^g, \bar{\nu}) + (1-p)U(\gamma^b, \bar{\nu})$.

Step 2.1: If $R > \beta_b$, then following (133), we have

$$w_g \geq w_b + (b-g)\bar{T}(\gamma^b, \bar{\nu}) = w_b + \frac{(b-g)S(\gamma^b, \bar{\nu}; b)}{\mu R - b} = w_b + \frac{(b-g)(U(\gamma^b, \bar{\nu}) + w_b)}{\mu R - b} \tag{135}$$

where the first equality follows from Lemma 12 and the last equality follows from $S(\gamma^b, \nu^b; b) = U(\gamma^b, \bar{\nu}) + u(\gamma^b, \bar{\nu}; b)$. Rearrange (135), we have

$$U(\gamma^b, \bar{\nu}) \leq \frac{(w_g - w_b)(\mu R - b)}{b-g} - w_b, \text{ if } R > \beta_b. \tag{136}$$

Further, following Proposition 1, we have

$$U(\gamma^b, \bar{\nu}) \leq F_b(w_b). \tag{137}$$

On the other hand, if $R \leq \beta_b$, then

$$U(\gamma^b, \bar{\nu}) \leq F_b(w_b) = -w_b, \text{ if } R \leq \beta_b \tag{138}$$

Hence, following (136) - (138), we have

$$U(\gamma^b, \bar{\nu}) \leq \min\left\{ \frac{(w_g - w_b)}{b-g} \max\{\mu R - b, 0\} - w_b, F_b(w_b) \right\} = \xi. \tag{139}$$

Step 2.2: Denote $\hat{W}_t$ as the agent's continuation utility under contract γ^g and τ_1^g as the time of the first arrival. Then, following Lemma 1, we have

$$d\hat{W}_t = [r\hat{W}_t - \mu\hat{H}_t + g]dt - d\hat{L}_t, \hat{H}_t \geq \beta_g, \quad \text{for} \quad t < \min\{\tau_1^g, \tau\}. \tag{140}$$

Furthermore, denote $\hat{I}_{\tau_1^g}$ as the payment upon the first arrival. Thus,

$$\begin{aligned}
U(\gamma^g, \bar{\nu}) &= \mathbb{E}^{\bar{\nu}} \left[\int_0^{\eta} e^{-rt}(RdN_t - d\hat{L}_t) \right] \\
&\leq \left\{ \mathbb{E}_{\tau_1^g} \left[e^{-r\tau_1^g} \left(R - I_{\tau_1^g} + U_{\tau_1^g}(\hat{\gamma}^g, \bar{\nu}) \right) \mathbb{1}_{\tau_1^g < \tau} - \int_0^{\min\{\tau_1^g, \tau\}} e^{-rt} g \mathrm{d}t \right] \right\} \\
&= \left\{ \int_0^{\tau} \left[e^{-r\tau_1^g} \left(R - I_{\tau_1^g} + U_{\tau_1^g}(\hat{\gamma}^g, \bar{\nu}) \right) - \int_0^{\tau_1^g} e^{-rt} g \mathrm{d}t \right] \mu e^{-\mu\tau_1^g} d\tau_1^g - \int_{\tau}^{\infty} \int_0^{\tau} e^{-rt} g \mathrm{d}t \cdot \mu e^{-\mu\tau_1^g} d\tau_1^g \right\} \\
&= \left[\int_0^{\tau} \mu e^{-t}(R - I_t + U_t(\hat{\gamma}^g, \bar{\nu}))\mathrm{d}t - \int_0^{\tau} g e^{-t} \mathrm{d}t \right] \qquad (141)
\end{aligned}$$

where the first inequality follows from that $d\hat{L}_t \geq 0$ and $\hat{\ell}_t \geq g$ for $t < \tau$ and the inequality is binding if and only if $d\hat{L}_t = gdt$. Finally, following Proposition 1 (since W_t is the state variable of the optimal control problem, we can easily generalize (15) to it at time t), we have

$$-I_t + U_t(\hat{\gamma}^g, \bar{\nu}) \leq -I_t + F_g(\hat{W}_{t-} + \hat{H}_t - I_t) \leq F_g(\hat{W}_{t-} + \hat{H}_t), \tag{142}$$

where the first inequality follows from that the agent's continuation utility $\hat{W}_t = \hat{W}_{t-} + H_t - I_t$ and the second inequality follows from $F_g' \geq -1$. Therefore, (140) - (142) imply that

$$U(\gamma^g, \bar{\nu}) \leq G(w_g, \tau).$$

Therefore, inequality (139) and (142) together imply that $p \cdot G(w_g, \tau) + (1-p)\xi \geq p \cdot U(\gamma^g, \bar{\nu}) + (1-p)U(\gamma^b, \bar{\nu})$.

D.2. Proof of Lemma 6

First, we verify (iii). If $H_t = \beta_g$, $\forall t \in [0,\tau]$, then $W_0 = \check{w}(\tau)$. Further since $H_t \ge \beta_g$, $\forall t \in [0,\tau]$, we have $W_0 \ge \check{w}(\tau)$. Hence, if $w < \check{w}(\tau)$, then the optimization problem (28) is infeasible, or, by convention, $G(w,\tau) = -\infty$.

Next, we verify (i) and (ii) by solving the optimization problem (28). Since $g(1-e^{-\tau})$ is given when τ is given, we only need to maximize the integral $\int_0^\tau \mu e^{-t}[R + F_g(W_t + H_t)]dt$. To solve the optimization problem, we can write down the Hamiltonian:

$$\mathcal{H} = e^{-t}\{\mu[R + F_g(W_t + H_t)]\} + \lambda(t)(rW_t - \mu H_t) + \eta(t)(H_t - \beta_g) \tag{143}$$

The optimality conditions are

$$\frac{\partial \mathcal{H}}{\partial H} = \mu e^{-t} F_g'(W_t + H_t) - \lambda(t)\mu + \eta(t) = 0 \tag{144}$$
$$\eta(t)(H_t - \beta_g) = 0; \quad \eta(t) \ge 0 \tag{145}$$
$$\frac{\partial \mathcal{H}}{\partial W} = \mu e^{-t} F_g'(W_t + H_t) + \lambda(t) r = -\lambda'(t) \tag{146}$$

Since the objective of the optimal control problem is jointly concave in (W_t, H_t), it is sufficient to verify the above optimality conditions.

Next, we verify (ii). If $W_0 = w \ge \hat{w}(\tau)$, then $W_t + H_t = z + \beta_g, \forall t \in [0,\tau]$, where $z + \beta_g = w/(\mu(1-e^{-\tau}))$. We can easily verify the optimality conditions (144) - (146) by letting

$$\lambda(t) = F'(z+\beta_g)e^{-t} \quad \text{and} \quad \eta(t) = 0.$$

Furthermore, we can verify that

$$\begin{aligned} W_t &= \mu(z+\beta_g) - \mu(z+\beta_g)e^{t-\tau}, \\ H_t &= z + \beta_g - W_t \ge z + \beta_g - W_0 \ge w\left(\frac{1}{\mu(1-e^{-\tau})} - 1\right) \ge \beta_g. \end{aligned}$$

where the last inequality follows from $w = W_0 \ge \hat{w}(\tau)$.

Finally, we verify (i). If $w = W_0 \in [\check{w}(\tau), \hat{w}(\tau))$, we firstly prove the following technical lemma.

Lemma 13. *If $w = W_0 \in [\check{w}(\tau), \hat{w}(\tau))$, there exists a unique $z \in [0, g(1-e^{-\tau})/(r+\mu e^{-\tau}))$ such that*

$$\bar{w}_g - (\bar{w}_g - z)e^{r(\tau_z - \tau)} = w.$$

Proof. Define

$$h(z_1) := \bar{w}_g - (\bar{w}_g - z)e^{r(\tau_z - \tau)},$$

then we can easily obtain that

$$h(0) = \bar{w}_g - \bar{w}_g \cdot e^{-r\tau} = g/r \cdot (1 - e^{-r\tau})$$

where the first equality follows from $\tau_0 = 0$ and

$$\lim_{z_1 \to g(1-e^{-\tau})/(r+\mu e^{-\tau})} h(z_1) = \bar{w}_g - (\bar{w}_g - z) = g(1-e^{-\tau})/(r+\mu e^{-\tau})$$

where the first equality follows from that $\lim_{z \to g(1-e^{-\tau})/(r+\mu e^{-\tau})} \tau_z = \tau$. Furthermore, we have

$$\begin{aligned} h'(z_1) &= e^{-r\tau} e^{r\tau_z}\left(1 + (z - \bar{w}_g) r \cdot \frac{\partial \tau_z}{\partial z_1}\right) \\ &= e^{r(\tau_z - \tau)}\left(1 + (z - \bar{w}_g)\frac{r(\bar{w}_g + \beta_g)}{(z_1 + \beta_g)(\bar{w}_g - z_1)}\right) \\ &= e^{r(\tau_z - \tau)}\left(1 - \frac{r(\bar{w}_g + \beta_g)}{z + \beta_g}\right) = e^{r(\tau_z - \tau)}\frac{z}{z + \beta_g} > 0 \end{aligned}$$

Since $w \in [h(0), \lim_{z \to g(1-e^{-\tau})/(r+\mu e^{-\tau})})h(z))$ and h is continuous, we have there exists a unique $z \in [0, g(1-e^{-\tau})/(r+\mu e^{-\tau}))$ such that $w = h(z)$. Q.E.D.

It is easy to verify that $\tau_z < \tau$ since $z < g(1-e^{-\tau})/(r+\mu e^{-\tau})$. Hence, if we let H_t follows (35), then W_t follows (34).

We can easily verify the optimality conditions (144) - (146) by letting

$$\lambda(t)=\begin{cases} \left[\int_t^{\tau-\tau_z}\mu e^{-\mu\xi}F_g'(W_\xi+\beta)d\xi+F_g'(z+\beta_g)e^{-\mu(\tau-\tau_z)}\right]e^{-rt}, & t\in[0,\tau-\tau_z],\\ F_g'(z+\beta_g)e^{-t}, & t\in[\tau-\tau_{z_1},\tau],\end{cases} \tag{147}$$

and

$$\eta(t)=\begin{cases} \mu e^{-rt}\gamma(t), & t\in[0,\tau-\tau_z],\\ 0, & t\in[\tau-\tau_{z_1},\tau],\end{cases} \tag{148}$$

where

$$\gamma(t):=\left[\int_t^{\tau-\tau_{z_1}}\mu e^{-\mu\xi}F_g'(W_\xi+\beta_g)d\xi+F_g'(z_1+\beta_g)e^{-\mu s}-e^{-\mu t}F'(W_t+\beta_g)\right]\geq 0$$

and $\gamma(t)$ follows from that $\gamma(t)$ is decreasing in t and $\gamma(\tau-\tau_z)=0$. Furthermore, we have

$$H_t=\beta, t\in[0,\tau-\tau_z],$$
$$H_t=z+\beta_g-W_t\geq z+\beta_g-W_{\tau-\tau_z}=\beta, t\in[\tau-\tau_z,\tau],$$

For $w\in[0,t_1]$, $H_t=\beta_g$. For $w\in[t_1,t_2]$, $W_t+H_t=y_1^*$ and for $w\in[t_2,\tau]$, $W_t+H_t-B_t=\beta_b$.

$$\mathcal{H}=e^{-t}\{\mu[R+F_g(W_t+H_t)]\}+\lambda(t)(rW_t-\mu H_t)+\eta(t)(H_t-\beta_g)+\phi(t)(B_t+\beta_b-W_t-H_t) \tag{149}$$

The optimality conditions are

$$\frac{\partial\mathcal{H}}{\partial H}=\mu e^{-t}F_g'(W_t+H_t)-\lambda(t)\mu+\eta(t)-\phi(t)=0 \tag{150}$$
$$\eta(t)(H_t-\beta_g)=0;\quad \eta(t)\geq 0 \tag{151}$$
$$\frac{\partial\mathcal{H}}{\partial W}=\mu e^{-t}F_g'(W_t+H_t)+\lambda(t)r-\phi(t)=-\lambda'(t) \tag{152}$$
$$\phi(t)(B_t+\beta_b-W_t-H_t)=0;\phi(t)\geq 0 \tag{153}$$

For $w\in[t_1,t_2]$, we have $\eta(t)=\phi(t)=0$ and $\lambda(t)=F_g'(y_1^*)e^{-t}$. For $w\in[t_2,\tau]$, $\eta(t)=0$, $\lambda(t)=F_g'(y_1^*)e^{-t}$ and $\phi(t)=\mu e^{-t}[F_g'(B_t+\beta_b)-F_g'(y_1^*)]\geq 0$.

D.3. Proof of Lemma 7

For the optimization problem (29), it is sufficient to verify the optimality conditions (144) - (146). We can simply let

$$\lambda(t)=F'\left(\frac{w}{\mu}\right)e^{-t},\eta(t)=0$$

and

$$W_t=w>0, H_t=\frac{w}{\mu}-w\geq\frac{r}{\mu}\cdot\frac{g}{r}=\beta_g.$$

D.4. Proof of Proposition 4

First, we look at the function if $w\geq\hat{w}(\tau)$. According to the optimal solution in (37), we have

$$G(w,\tau)=\int_0^\tau\mu e^{-t}F_g(y^*)\mathrm{d}t+\int_0^\tau(\mu R-c)e^{-t}\mathrm{d}t \tag{154}$$

where $y^*=z+\beta_g=\dfrac{w}{\mu(1-e^{-\tau})}$. Hence,

$$\begin{aligned}\frac{\partial G(w,\tau)}{\partial\tau}&=\mu e^{-\tau}F_g(y^*)+\int_0^T\mu e^{-t}F'(y^*)\frac{\partial y^*}{\partial\tau}\mathrm{d}t+(\mu R-g)e^{-\tau}\\&=\mu e^{-\tau}F(y^*)+(\mu R-g)e^{-\tau}+F'(y^*)\frac{-we^{-\tau}}{\mu(1-e^{-\tau})^2}\int_0^T\mu e^{-t}\mathrm{d}t\\&=\mu e^{-\tau}(F(y^*)-y^*F'(y^*))+(\mu R-c)e^{-\tau}\end{aligned}$$

$$= \mu e^{-\tau} y^* \left(\frac{F(y^*) - F(0)}{y^*} - F'(y^*) \right) + (\mu R - c) e^{-\tau} > 0 \tag{155}$$

where the inequality follows from the concavity of F. As a result, we have

$$\frac{\partial J(w,\bar{\tau})}{\partial \bar{\tau}} = \frac{\partial \tau}{\partial \bar{\tau}} \frac{\partial G(w,\tau)}{\partial \tau} > 0, \tag{156}$$

where the inequality follows from (155) and $\frac{\partial \tau}{\partial \bar{\tau}} > 0$, and

$$\frac{\partial J(w,\bar{\tau})}{\partial w} = F_g'(y^*). \tag{157}$$

Hence, J is increasing in $\bar{\tau}$ when $w \geq \hat{w}(\tau)$. Next, we verify the concavity of J. Following (156) and (157), we have that the Hessian matrix of J is

$$\begin{bmatrix} \frac{\partial^2 J(w,\bar{\tau})}{\partial^2 w} & \frac{\partial^2 J(w,\bar{\tau})}{\partial w \partial \bar{\tau}} \\ \frac{\partial^2 J(w,\bar{T})}{\partial w \partial \bar{T}} & \frac{\partial^2 J(w,\bar{T})}{\partial^2 \bar{T}} \end{bmatrix} \tag{158}$$

where

$$\frac{\partial^2 J(w,\bar{\tau})}{\partial^2 w} = \frac{F''(y^*)}{\mu(1-e^{-\tau})} < 0, \tag{159}$$

where the inequality follows from the concavity of F,

$$\frac{\partial^2 J(w,\bar{\tau})}{\partial w \partial \bar{\tau}} = \frac{\partial y^*}{\partial \bar{\tau}} F''(y^*) = \frac{\partial y^*}{\partial \tau} \frac{\partial \tau}{\partial \bar{\tau}} F''(y^*) > 0, \tag{160}$$

where the inequality follows from $\frac{\partial y^*}{\partial \tau} < 0$, $\frac{\partial \tau}{\partial \bar{\tau}} > 0$, and the concavity of F, and

$$\frac{\partial^2 J(w,\bar{\tau})}{\partial^2 \bar{\tau}} = -\mu(1-r\bar{\tau})^{\frac{\mu}{r}} \cdot y^* \cdot F''(y^*) \cdot \frac{\partial y^*}{\partial \bar{\tau}} - \mu^2 (1-r\bar{\tau})^{\frac{\mu-r}{r}} [F(y^*) - y^* F'(y^*)] - (\mu R - g)\mu(1-r\bar{\tau})^{\frac{\mu-r}{r}} < 0. \tag{161}$$

where the inequality follows from $\frac{\partial y^*}{\partial \bar{\tau}} < 0$ and $F(y^*) - y^* F'(y^*) = y^*[(F(y^*) - F(0))/y^* - F'(y^*)] \geq 0$ (implied by the concavity of F).

Further, with (159) - (161), we can show that the Hessian matrix of J is negative definite which implies that J is jointly concave when $w \geq \hat{w}(\tau)$.

Second, we look at the function if $w \in [\check{w}(\tau), \hat{w}(\tau))$. According to (34) and (35), we have

$$W_{\tau_1} = z = \mu(z + \beta_g)(1 - e^{\tau_1 - \tau}) = \bar{w}_g + (w - \bar{w}_g) e^{r\tau_1}$$

by denoting $\tau_1(w,\tau) := \tau - \tau_z$ and simplifying $\tau_1(w,\tau)$ with τ_1. Further, we denote $y_1^* = z + \beta_g$. Hence,

$$y_1^* = \frac{g}{\mu(r + \mu e^{\tau_1 - \tau})},$$

and $\tau_1(w,\tau)$ is the solution of

$$\frac{g}{r + \mu e^{\tau_1 - \tau}} (1 - e^{\tau_1 - \tau}) = \bar{w}_g + (w - \bar{w}_g) e^{r\tau_1}, \tag{162}$$

where we again simplify $\tau_1(w,\tau)$ with τ_1. Therefore,

$$G(w,\tau) = \int_0^{\tau_1(w,\tau)} \mu e^{-t} F_g\left(\bar{w}_g + (w - \bar{w}_g) e^{rt} + \beta_g\right) \mathrm{d}t + \int_{\tau_1(w,\tau)}^{\tau} \mu e^{-t} F_g(y_1^*) \mathrm{d}t + \int_0^{\tau} (\mu R - g) e^{-t} \mathrm{d}t. \tag{163}$$

Then,

$$\begin{aligned} \frac{\partial G(w,\tau)}{\partial \tau} &= [\mu e^{-\tau_1} F_g\left(\bar{w}_g + (w - \bar{w}_g) e^{r\tau_1} + \beta_g\right) - \mu e^{-\tau_1} F(y_1^*)] \frac{\partial \tau_1(w,\tau)}{\partial \tau} + \int_{\tau_1(w,\tau)}^{\tau} \mu e^{-t} F'(y_1^*) \frac{\partial y_1^*}{\partial \tau} \mathrm{d}t + \mu e^{-\tau} F(y_1^*) \\ &= \int_{\tau_1(w,\tau)}^{\tau} \mu e^{-t} \mathrm{d}t \cdot F'(y_1^*) \frac{-g\mu e^{\tau_1(w,\tau) - \tau} \cdot (\frac{\partial \tau_1(w,\tau)}{\partial \tau} - 1)}{\mu(r + \mu e^{\tau_1(w,\tau) - \tau})^2} + \mu e^{-\tau} F(y_1^*) + (\mu R - g) e^{-\tau} \end{aligned} \tag{164}$$

Since (162) implies that

$$\frac{\partial \tau_1(w,\tau)}{\partial \tau} - 1 = \frac{r+\mu e^{\tau_1-\tau}}{\mu(1-e^{\tau_1-\tau})}, \tag{165}$$

we have

$$\frac{\partial G(w,\tau)}{\partial \tau} = \mu e^{-\tau}(F(y_1^*) - y_1^* F'(y_1^*)) + (\mu R - g)e^{-\tau} > 0, \tag{166}$$

where the inequality follows from $F(y_1^*) - y_1^* F'(y_1^*) = y_1^*[(F(y_1^*) - F(0))/y_1^* - F'(y_1^*)] \geq 0$. As a result, we have

$$\frac{\partial J(w,\bar{\tau})}{\partial \bar{\tau}} = \frac{\partial \tau}{\partial \bar{\tau}} \frac{\partial G(w,\tau)}{\partial \tau} > 0, \tag{167}$$

where the inequality follows from (166) and $\frac{\partial \tau}{\partial \bar{\tau}} > 0$, and

$$\frac{\partial J(w,\bar{\tau})}{\partial w} = \mu \int_0^{\tau_1} e^{-\mu t} F'(\bar{w}_g + (w-\bar{w}_g)e^{rt} + \beta_g)\mathrm{d}t + F'(y_1^*)\cdot e^{-\mu\tau_1}. \tag{168}$$

Hence, J is increasing in $\bar{\tau}$ when $w \in [\check{w}(\tau), \hat{w}(\tau))$. Next, we verify the concavity of J. Following (167) and (168), we have that the Hessian matrix of J is

$$\begin{bmatrix} \frac{\partial^2 J(w,\bar{T})}{\partial^2 w} & \frac{\partial^2 J(w,\bar{T})}{\partial w \partial \bar{T}} \\ \frac{\partial^2 J(w,\bar{T})}{\partial w \partial \bar{T}} & \frac{\partial^2 J(w,\bar{T})}{\partial^2 \bar{T}} \end{bmatrix} \tag{169}$$

where

$$\frac{\partial^2 J(w,\bar{\tau})}{\partial^2 w} = \mu \int_0^{\tau_1} e^{-\mu t} F''(\bar{w}_g + (w-\bar{w}_g)e^{rt} + \beta_g)e^{rt}\mathrm{d}t + F''(y_1^*)\cdot e^{-\mu\tau_1}\cdot \frac{\partial y_1^*}{\partial w}, \tag{170}$$

and

$$\frac{\partial^2 J(w,\bar{\tau})}{\partial^2 \bar{\tau}} = -\mu(1-r\bar{\tau})^{\frac{\mu}{r}}\cdot y_1^*\cdot F''(y_1^*)\cdot \frac{\partial y_1^*}{\partial \bar{\tau}} - \mu^2(1-r\bar{\tau})^{\frac{\mu-r}{r}}\cdot [F(y_1^*) - y_1^* F'(y_1^*)] - (\mu R - g)\cdot \mu \cdot (1-r\bar{\tau})^{\frac{\mu-r}{r}} < 0, \tag{171}$$

where the inequality follows from $\frac{\partial y^*}{\partial \bar{\tau}} < 0$ and $F(y^*) - y^* F'(y^*) = y^*[(F(y^*) - F(0))/y^* - F'(y^*)] \geq 0$ (implied by the concavity of F), and

$$\frac{\partial^2 J(w,\bar{\tau})}{\partial w \partial \bar{\tau}} = -\mu \cdot \frac{\partial \tau}{\partial \bar{\tau}} \cdot e^{-\tau}\cdot y_1^* \cdot F''(y_1^*)\cdot \frac{\partial y_1^*}{\partial w}. \tag{172}$$

Further, the concavity of F and

$$\frac{\partial y_1^*}{\partial w} = \frac{\partial y_1^*}{\partial \tau_1}\cdot \frac{\partial \tau_1}{\partial w} = -e^{r\tau_1}\frac{(r+\mu e^{-(\tau-\tau_1)})^2}{g e^{-(\tau-\tau_1)}\mu(1-e^{-(\tau-\tau_1)})}\cdot \frac{\partial y_1^*}{\partial \tau_1} > 0.$$

implies that

$$\frac{\partial^2 J(w,\bar{\tau})}{\partial^2 w} < 0, \frac{\partial^2 J(w,\bar{\tau})}{\partial w \partial \bar{\tau}} > 0. \tag{173}$$

Furthermore, following (170)-(172) we can show that the Hessian matrix of J is negative definite which implies that J is jointly concave when $w \in [\check{w}(\tau), \hat{w}(\tau))$. Finally, following (154) and (163), we can show that $G(w,\tau)$ is continuously differentiable when $w = \hat{w}(\tau)$. This concludes the proof.

D.5. Proof of Lemma 8

First, it is clear that at optimality, either (26) or (27) holds as equality. Otherwise, we can increase ξ^* to improve the objective value without violating any constraint, which contradicts optimality.

If (26) is binding at optimality, (46) holds with $w = w_b^*$, following (27). If (27) is binding, on the other hand, (26) implies $F_b(w_b^*) + w_b^* \geq \xi^* \geq 0$. Furthermore, (10) implies $F_b(0) + 0 = 0$. Finally, $F_b(w) + w$ is increasing following Lemma 2. Therefore, there exists a $w \in [0, w_b^*]$ such that (46) holds as an equality. Q.E.D.

D.6. Proof of Lemma 9

First, we present a technical lemma.

LEMMA 14. *For any $k \geq 0$, we have*

$$\frac{V_{b_1}(k \cdot b_1)}{\mu R - b_1} = \frac{V_{b_2}(k \cdot b_2)}{\mu R - b_2}, \forall b_1, b_2 < \mu R, \tag{174}$$

where V_b is defined in (9) and (10) in Section 4.1.

Proof. Following Proposition 2, we have

$$\begin{aligned} V_b(w) &= F_b(w) + w = U(\gamma_{\mathrm{B}}^b(w,0), \bar{\nu}) + w \\ &= S(\gamma_{\mathrm{B}}^b(w,0), \bar{\nu}; b) = (\mu R - b)\bar{T}(\gamma_{\mathrm{B}}^b(w,0), \bar{\nu}) \\ &= (\mu R - b)\mathbb{E}\left[\int_0^{\hat{\eta}^b} e^{-rt} dt\right], \end{aligned} \tag{175}$$

where the fourth equality follows from Lemma 9, and the fifth equality follows from (127). Following (11) and (13), we have for any b,

$$dW_t^b = [r(W_{t-}^b - b) + \min\{b/r - W_{t-}^b, b/\mu\} dN_t] \mathbb{1}_{W_{t-}^b \geq 0}, \hat{\eta}^b = \min\{t : W_{t-}^b = 0\}.$$

Define process $w_t := W_t^b / b$, then we have

$$dw_t = [r(w_t - 1) + \min\{1/r - w_{t-}, 1/\mu\} dN_t] \mathbb{1}_{w_{t-} \geq 0}, \hat{\eta}^b = \min\{t : w_{t-} = 0\},$$

Hence, for any b_1, b_2, $\hat{\eta}^{b_1}$ and $\hat{\eta}^{b_2}$ follows the same distribution if $W_0^{b_1}/b_1 = W_0^{b_2}/b_2$. Therefore,

$$\frac{V_{b_1}(k \cdot b_1)}{\mu R - b_1} = \mathbb{E}\left[\int_0^{\hat{\eta}^{b_1}} e^{-rt} dt\right] = \mathbb{E}\left[\int_0^{\hat{\eta}^{b_2}} e^{-rt} dt\right] = \frac{V_{b_2}(k \cdot b_2)}{\mu R - b_2},$$

which verifies (174). Q.E.D.

Lemma 14 implies that for any $w > 0$ and $b_1 < b_2$, we have

$$V_{b_1}(w) = V_{b_2}\left(w \frac{b_2}{b_1}\right) \frac{\mu R - b_1}{\mu R - b_2} > V_{b_2}(w) \frac{\mu R - b_1}{\mu R - b_2} > V_{b_2}(w) \tag{176}$$

Next, we prove the desired results. For any given b, we denote the solution of optimization problem (42)-(44) as $(w_g^*(b), w_b^*(b), \bar{\tau}^*(b))$ and w_B defined in (47) as $w_B(b)$. If there exists $\check{b} \in [b, \mu R]$ such that $w_B(\check{b}) = 0$, then we prove by contradiction that for any $\hat{b} > \check{b}$, we have $w_B(\hat{b}) = 0$. It implies that there exists $\bar{b} \in [g, \mu R]$ such that $w_B = 0$ if and only if $w_B > 0$.

If $w_B(\hat{b}) > 0$, following (47), we have $\mu R > b$ and $w_g^*(\hat{b}) > w_b^*(\hat{b})$. Following (43) and (44), $(w_g^*(\hat{b}), w_b^*(\hat{b}), \bar{\tau}^*(\hat{b}))$ and $(w_g^*(\check{b}), w_b^*(\check{b}), \bar{\tau}^*(\check{b}))$ are feasible for both $b = \hat{b}$ and $b = \check{b}$. Therefore,

$$\begin{aligned} &p \cdot J(w_g^*(\hat{b}), \bar{\tau}^*(\hat{b})) + (1-p)\min\left\{F_{\hat{b}}(w_b^*(\hat{b})), \frac{w_g^*(\hat{b}) - w_b^*(\hat{b})}{\hat{b} - g} - w_b^*(\hat{b})\right\} > p \cdot J(w_g^*(\check{b}), \bar{\tau}^*(\check{b})) + (1-p) w_b^*(\check{b}) \\ \geq &p \cdot J(w_g^*(\hat{b}), \bar{\tau}^*(\hat{b})) + (1-p)\min\left\{F_{\check{b}}(w_b^*(\hat{b})), \frac{w_g^*(\hat{b}) - w_b^*(\hat{b})}{\check{b} - g} - w_b^*(\hat{b})\right\} \end{aligned}$$

which implies that

$$\min\left\{F_{\hat{b}}(w_b^*(\hat{b})), \frac{w_g^*(\hat{b}) - w_b^*(\hat{b})}{\hat{b} - g} - w_b^*(\hat{b})\right\} > \min\left\{F_{\check{b}}(w_b^*(\hat{b})), \frac{w_g^*(\hat{b}) - w_b^*(\hat{b})}{\check{b} - g} - w_b^*(\hat{b})\right\}$$

which is equivalent to

$$\min\left\{V_{\hat{b}}(w_b^*(\hat{b})), \frac{w_g^*(\hat{b}) - w_b^*(\hat{b})}{\hat{b} - g}\right\} \geq \min\left\{V_{\check{b}}(w_b^*(\hat{b})), \frac{w_g^*(\hat{b}) - w_b^*(\hat{b})}{\check{b} - g}\right\}$$

which contradicts with

$$V_{\hat{b}}(w_b^*(\hat{b})) < V_{\check{b}}(w_b^*(\hat{b})), \text{and}, \frac{w_g^*(\hat{b}) - w_b^*(\hat{b})}{\hat{b} - g} < \frac{w_g^*(\hat{b}) - w_b^*(\hat{b})}{\check{b} - g}.$$

where the first inequality follows from (176).

D.7. Preparation to prove Theorem 1

LEMMA 15. *Following a delay-payment contract $\gamma^g_{\mathrm{D}}(w,S,t_0)=(L^g,\eta^g_D)$, which generates a promised utility process according to (19) for $t\geq t_0$ following $W_{t_0}=w$. Furthermore, the payment process L^g_t follows (20). Then, a bad agent with cost $b>g$ will never work after t_0 if*

$$w\left(\frac{g}{br+\mu g}\right)^{S+1}\leq\frac{g}{r}. \tag{177}$$

LEMMA 16. *For any w_g,τ such that $\min\{\mathrm{z}(w_g,\tau)+\beta_g,\bar{w}_g\}\leq\beta_b$, we have $\nu^0\in\mathfrak{N}(\gamma^g_{\mathrm{P}}(w_g,\tau,b)$.*

PROPOSITION 10. *(i) For any $w\geq 0$ and $B\geq 0$, we have $\bar{\nu}\in\mathfrak{N}(\gamma^c_{\mathrm{B}}(w,B),c)$, and*

$$u\Big(\gamma^c_{\mathrm{B}}(w,B),\bar{\nu};c\Big)=w+B.$$

(ii) For any $\tau\geq 0$ and $w\geq c/r(1-e^{-r\tau})$, we have $\bar{\nu}\in\mathfrak{N}(\gamma^c_{\mathrm{P}}(w,\tau),c)$, and

$$u\Big(\gamma^c_{\mathrm{P}}(w,\tau),\bar{\nu};c\Big)=w.$$

PROPOSITION 11. *For any $g<\mu R$, we have*

$$U(\gamma^g_{\mathrm{P}}(w,\tau),\bar{\nu})=G(w,\tau) \tag{178}$$

D.7.1. Proof of Lemma 15 After the i-th ($i\in\{0,1,...,S,S+1\}$) arrival after t_0, the good agent's continuation value is $\frac{w}{\mu^i}$. First, since after the $S+1$-th arrival, the contract $\gamma^g_{\mathrm{D}}(w,S,t_0)$ will reward $\beta_g<\beta_b$ for every arrival, the bad agent does not work after the $S+1$-th arrival. Assuming the bad agent works until the $S+1-$th arrival, his continuation value right after the $S+1$-th arrival will be $B^{S+1}_t=\frac{w}{\mu^{S+1}}$ (achieved by mimicking the good agent). After the S-th arrival, the bad agent's value follows

$$\begin{aligned}B^S_t&=\max_{\nu_t\in[0,g/b]}g\delta-b\nu_t\delta+\mu\nu_tB^{S+1}_t+(1-\mu\nu_t\delta)e^{-r\delta}B^S_{t+\delta}\\&=g\delta+e^{-r\delta}B^S_{t+\delta}+\delta\max_{\nu_t\in[0,g/b]}\nu_t\left[-b+\mu B^{S+1}_t-\mu B^S_{t+\delta}\right]\end{aligned} \tag{179}$$

Since the objective is linear in ν_t, the bad agent chooses either $\nu_t=g/b$ or $\nu_t=0$. Hence, the bad agent works between S-th arrival and $S+1$-th arrival if and only if

$$B^{S+1}_t-B^S_{t+\delta}\geq\beta_b \tag{180}$$

If the agent works, we have $B^S_t=\frac{\mu g}{br+\mu g}B^{S+1}_t$ and $B^S_t=g/r$ if the agent does not work (if the agent does not work, he never gets the $S+1$-arrival and he stays in the contract to steal the flow payment g). Hence, the agent works if and only if

$$B^{S+1}_t>\frac{g}{r}+\beta_b\Longleftrightarrow w>\mu^{S+1}\left(\frac{g}{r}+\beta_b\right) \tag{181}$$

Similarly, assuming the agent works after the S-th arrival, the bad agent works between the $S-1$-th arrival and S-th arrival if and only if

$$B^S_t>\frac{g}{r}+\beta_b\Longleftrightarrow w>\mu^{S+1}\left(\frac{g}{r}+\beta_b\right)\frac{br+\mu g}{\mu g} \tag{182}$$

If the agent works, then $B^{S-1}_t=\left(\frac{\mu g}{br+\mu g}\right)^2B^{S+1}_t$. Therefore, assuming the agent works after the i-th arrival, the agent also works between the $i-1$-th arrival and i-th arrival if and only if

$$B^i_t>\frac{g}{r}+\beta_b\Longleftrightarrow w>\mu^{S+1}\left(\frac{g}{r}+\beta_b\right)\left(\frac{br+\mu g}{\mu g}\right)^{S+1-i} \tag{183}$$

Finally, if $i=1$, then (183) is equivalent to

$$w>\mu^{S+1}\left(\frac{g}{r}+\beta_b\right)\left(\frac{br+\mu g}{\mu g}\right)^S=\frac{g}{r}\left(\frac{br+\mu g}{g}\right)^{S+1} \tag{184}$$

If condition (177) holds (equivalently, (184) does not hold), then before the first arrival, the agent will not work even if assuming he works after the first arrival. Therefore, by induction, the bad agent never works if (177) holds.

D.7.2. Proof of Lemma 16 Following definition 4, it is straightforward to verify that $S(w_g,\tau)$ satisfies (177) if we let $S=S(w_g,\tau)$ and $w=w_g$. Hence, if $\max\left\{W^g_{\tau_1^N},\mathrm{z}(w_g,\tau)\right\}+\beta_g>\bar{w}_g$, the contract after the first arrival would be $\gamma^g_{\mathtt{D}}\left(\max\left\{W^g_{\tau_1^N},\mathrm{z}(w_g,\tau)\right\}+\beta_g,S(w_g,\tau),\tau_1^N\right)$. Then, following Lemma 15, since $\max\left\{W^g_{\tau_1^N},\mathrm{z}(w_g,\tau)\right\}+\beta_g\le\max\{w_g,\mathrm{z}(w_g,\tau)\}+\beta_g$, after the first arrival, the bad agent with cost b will never work. As a result, the bad agent's continuation utility after the first arrival (if he tried to mimic the good agent) is $\bar{w}_g$ if $\max\left\{W^g_{\tau_1^N},\mathrm{z}(w_g,\tau)\right\}+\beta_g>\bar{w}_g$. On the other hand, if $\max\left\{W^g_{\tau_1^N},\mathrm{z}(w_g,\tau)\right\}+\beta_g\le\bar{w}_g$, the contract after the first arrival would be an *IC-Binding contract* $\hat{\gamma}^g$. Then, following Lemma 4, the bad agent will never work after the first arrival. As a result, the bad agent's continuation utility after the first arrival (if he tried to mimic the good agent) is $\max\left\{W^g_{\tau_1^N},\mathrm{z}(w_g,\tau)\right\}+\beta_g$ if $\max\left\{W^g_{\tau_1^N},\mathrm{z}(w_g,\tau)\right\}+\beta_g\le\bar{w}_g$. To summarize, agent's continuation utility after the first arrival (if he tried to mimic the good agent) is $\min\left\{\max\left\{W^g_{\tau_1^N},\mathrm{z}(w_g,\tau)\right\}+\beta_g,\bar{w}_g\right\}$.

If the bad agent never works in the good agent's contract, then for $t\le\tau$, we have the bad agent's continuation utility as $B_t=g/r\cdot(1-e^{r(t-\tau)})$. In the following, we use a one-shot deviation argument to show that if $\min\{\mathrm{z}(w_g,\tau)+\beta_g,\bar{w}_g\}\le\beta_b$, then the bad agent will always prefer no effort in the good agent's contract. For any $t\in[0,\tau]$, if the bad agent deviates from no effort to ν_t in the next δ time interval, then the bad agent's utility becomes

$$\begin{aligned}&g\delta-b\nu_t\delta+\mu\nu_t+\mu\nu_t\min\{\max\{W^g_t,\mathrm{z}(w_g,\tau)\}+\beta_g,\bar{w}_g\}+(1-\mu\nu_t\delta)e^{-r\delta}B_{t+\delta}\\=&g\delta+e^{-r\delta}B_{t+\delta}+\delta\nu_t[-b+\mu\min\{\max\{W^g_t,\mathrm{z}(w,\tau)\}+\beta_g,\bar{w}_g\}-\mu B_{t+\delta}]\end{aligned}$$

where W^g_t follows (34). Hence, as long as $\min\{\max\{W^g_t,\mathrm{z}(w_g,\tau)\}+\beta_g,\bar{w}_g\}-B_t\le\beta_b$, there is no benefit for the bad agent to deviate from no effort. Following Lemma 6, we can show that for any $t\in[0,\tau],\max\{W^g_t,\mathrm{z}(w_g,\tau)\}+\beta_g-B_t\le W^g_\tau+H^g_\tau-B_\tau=\mathrm{z}(w_g,\tau)+\beta_g$. Therefore, $\min\{\mathrm{z}(w_g,\tau)+\beta_g,\bar{w}_g\}\le\beta_b$ implies that the bad agent will not work in the good agent's contract, i.e., $\nu^0\in\mathfrak{N}(\gamma^g_{\mathtt{P}}(w_g,\tau),b)$. Q.E.D.

D.7.3. Proof of Proposition 10

(i) Following contract $\gamma^c_{\mathtt{B}}(w,B,0)$, since the promised utility process W^c_t follows (11) and the payment process follows (12), then $H^c_t=\beta_c$. Hence, $\bar{\nu}\in\mathfrak{N}(\gamma^c_{\mathtt{B}}(w,B,0),c)$. Meanwhile,

$$\begin{aligned}u\left(\gamma^c_{\mathtt{B}}(w,B,0),\bar{\nu};c\right)&=\mathbb{E}^{\bar{\nu}}\left[\int_{0-}^{\eta}e^{-rs}(\mathrm{d}L_s-\beta_c\mu\mathrm{d}s)\middle|\mathcal{F}^N_t\right]\\&=B+\mathbb{E}^{\bar{\nu}}\left[\int_{0+}^{\eta}e^{-rs}(\mathrm{d}L_s-\beta_c\mu\mathrm{d}s)\middle|\mathcal{F}^N_t\right]=B+w\end{aligned}\tag{185}$$

where the third equality follows from the definition of $\gamma^c_{\mathtt{B}}(w,B,0)$.

(ii) If $\max\left\{W^c_{\tau_1^N},\mathrm{z}(w_c,\tau)\right\}+\beta_c>\bar{w}_c$, the contract after the first arrival would be $\gamma^c_{\mathtt{D}}\left(\max\left\{W^c_{\tau_1^N},\mathrm{z}(w_c,\tau)\right\}+\beta_c,S(w_c,\tau),\tau_1^N\right)$. By definition 4, we have that the promised utility of $\gamma^c_{\mathtt{P}}(w,\tau)$ follows (PK), where $H^c_t=\max\{\beta_c,\mathrm{z}(w_c,\tau)+\beta_c-W^c_{t-}\}\mathbb{1}_{t\le\tau_1^N}+(rW^c_{t-}/\mu)\mathbb{1}_{t\in(\tau_1^N,\tau_{t_0}^{S+1}]}+\beta_c\mathbb{1}_{t>\tau_{t_0}^{S+1}}$, $dL^c_t=cdt+(W^c_{t-}+H^c_t-\bar{w}_c)^+\mathbb{1}_{t\ge\tau_{t_0}^{S+1}}dN_t$. On the other hand, if $\max\left\{W^c_{\tau_1^N},\mathrm{z}(w_c,\tau)\right\}+\beta_c\le\bar{w}_c$, since the contract after the first arrival would be an *IC-Binding contract* $\hat{\gamma}^c$. Similarly, by definition 4, we have that the promised utility of $\gamma^c_{\mathtt{P}}(w,\tau)$ follows (PK), where $H^c_t=\max\{\beta_c,\mathrm{z}(w_c,\tau)+\beta_c-W^c_{t-}\}\mathbb{1}_{t\le\tau_1^N}+\beta_c\mathbb{1}_{t>\tau_1^N}$, $dL^c_t=cdt+(W^c_{t-}+H^c_t-\bar{w}_c)^+dN_t$. Furthermore, $\tau^c=\min\{t:W^c_{t-}=0\}$. Hence, following (IC), we have $\bar{\nu}\in\mathfrak{N}(\gamma^c_{\mathtt{P}}(w,\tau),c)$. Further, by definition 4, we have

$$u\left(\gamma^c_{\mathtt{P}}(w,\tau),\bar{\nu};c\right)=W^c_0=w.$$

Q.E.D.

D.7.4. Proof of Proposition 11 Following definition 4, we have

$$\begin{aligned}U(\gamma^g_{\mathtt{P}}(w,\tau),\bar{\nu})&=\mathbb{E}^{\bar{\nu}}\left[\int_0^{\eta^g}e^{-rt}(RdN_t-dL^g_t)\right]\\&=\mathbb{E}_{\tau_1^N}\left[e^{-r\tau_1^N}\left(R+U_{\tau_1^N}(\hat{\gamma}^g,\bar{\nu})\mathbb{1}_{\max\left\{W^g_{\tau_1^N},\mathrm{z}(w,\tau)\right\}+\beta_g\le\bar{w}_g}\right.\right.\end{aligned}$$

$$+U_{\tau_1^N}(\gamma_{\mathtt{D}}^g,\bar{\nu})\mathbb{1}_{\max\left\{W^g_{\tau_1^N},\mathrm{z}(w,\tau)\right\}+\beta_g>\bar{w}_g}\Bigg)\mathbb{1}_{\tau_1^N<\tau^*}-\int_0^{\min\{\tau_1^N,\tau\}}e^{-rt}g\mathrm{d}t\Bigg]$$
$$=\int_0^{\tau^*}\Bigg[e^{-r\tau_1^N}\Bigg(R+U_{\tau_1^N}(\hat{\gamma}^g,\bar{\nu})\mathbb{1}_{\max\left\{W^g_{\tau_1^N},\mathrm{z}(w,\tau)\right\}+\beta_g\le\bar{w}_g}$$
$$+U_{\tau_1^N}(\gamma_{\mathtt{D}}^g,\bar{\nu})\mathbb{1}_{\max\left\{W^g_{\tau_1^N},\mathrm{z}(w,\tau)\right\}+\beta_g>\bar{w}_g}\Bigg)-\int_0^{\tau_1^N}e^{-rt}g\mathrm{d}t\Bigg]\mu e^{-\mu\tau_1^N}d\tau_1^N$$
$$-\int_{\tau^*}^{\infty}\int_0^{\tau^*}e^{-rt}g\mathrm{d}t\cdot\mu e^{-\mu\tau_1^N}d\tau_1^N$$
$$=\int_0^{\tau^*}\mu e^{-t}(R+U_t(\hat{\gamma}^g,\bar{\nu})\mathbb{1}_{\max\left\{W^g_t,\mathrm{z}(w,\tau)\right\}+\beta_g\le\bar{w}_g}+U_t(\gamma_{\mathtt{D}}^g,\bar{\nu})\mathbb{1}_{\max\left\{W^g_t,\mathrm{z}(w,\tau)\right\}+\beta_g>\bar{w}_g})\mathrm{d}t$$
$$-\int_0^{\tau^*}ge^{-t}\mathrm{d}t \tag{186}$$

where W_t^g follows (34) if $w\in[\check{\omega}(\tau),\hat{\omega}(\tau)]$, (37) if $w>\hat{\omega}(\tau)$, (39) if $\tau=\infty$. Finally, following Proposition 1 (since W_t is the state variable of the optimal control problem, we can easily generalize (15) to it at time t), we have

$$U_t(\hat{\gamma}^g\left(\max\left\{W_t^g,\mathrm{z}(w,\tau)\right\}+\beta_g\right),\bar{\nu})=F_g\left(\max\left\{W^g_{\tau_1^N},\mathrm{z}(w,\tau)\right\}+\beta_g\right), \tag{187}$$

if $\max\left\{W^g_{\tau_1^N},\mathrm{z}(w,\tau)\right\}+\beta_g\le\bar{w}_g$. Furthermore, following definition 4, the contract is never terminated after τ_1^N if $\max\left\{W^g_{\tau_1^N},\mathrm{z}(w,\tau)\right\}+\beta_g>\bar{w}_g$. Hence, the principal's future utility right after time t is $(\mu R-c)/r$ minus the agent's continuation utility,

$$U_t(\gamma_{\mathtt{D}}^g\left(\max\left\{W_t^g,\mathrm{z}(w_g,\tau)\right\}+\beta_g,S(w,\tau),\tau_1^N\right),\bar{\nu})=\bar{V}-\left(\max\left\{W^g_{\tau_1^N},\mathrm{z}(w,\tau)\right\}+\beta_g\right)=F_g\left(\max\left\{W^g_{\tau_1^N},\mathrm{z}(w,\tau)\right\}+\beta_g\right), \tag{188}$$

if $\max\left\{W^g_{\tau_1^N},\mathrm{z}(w,\tau)\right\}+\beta_g>\bar{w}_g$. Therefore, (186) - (188), definition of $G(w,\tau)$ in Proposition 3, and the solution in Lemma 6 and 7 together imply (178). Q.E.D.

D.8. Proof of Theorem 1

Following proposition 10, $(\gamma_{\mathtt{P}}^g(w_g^*,\tau^*),\ \gamma_{\mathtt{B}}^b\left(w_B(w_g^*,w_b^*),w_b^*-w_B(w_g^*,w_b^*)\right))$ satisfy constraints (LL), (PK), (IC), (IR), and (FE). In the following, we verify (TT).

$$\begin{aligned}u(\gamma_{\mathtt{B}}^b\left(w_B(w_g^*,w_b^*),w_b^*-w_B(w_g^*,w_b^*)\right),\bar{\nu};b)&=w_b^*\ge g/r\cdot(1-e^{-r\tau^*})=u(\gamma_{\mathtt{P}}^g(w_g^*,\tau^*),\nu^0;b)\\&=\max_{\nu}u(\gamma_{\mathtt{P}}^g(w_g^*,\tau^*),\nu;b),\end{aligned}$$

where the first equality follows from Proposition 10, the first inequality follows from constraint (43), and the last equality follows from $\nu^0\in\mathfrak{N}(\gamma_{\mathtt{P}}^g(w_g^*,\tau^*),b)$ (Lemma 16). If $\mu R\le b$, following Lemma 9, we have $w_B^*=0$, and

$$u(\gamma_{\mathtt{P}}^g(w_g^*,\tau^*),\bar{\nu};g)=w_g^*\ge w_b^*=\max_{\nu}u(\gamma_{\mathtt{B}}^b(0,w_b^*),\nu;g)$$

where the first equality follows from proposition 10 and the first inequality follows from constraint (43). On the other hand, if $\mu R>b$ and $F_b(w_b^*)\le\dfrac{w_g^*-w_b^*}{b-g}(\mu R-b)-w_b^*$, following (47), we have $w_B(w_g^*,w_b^*)=w_b^*$, and

$$\begin{aligned}u(\gamma_{\mathtt{P}}^g(w_g^*,\tau^*),\bar{\nu};g)=w_g^*&\ge w_b^*+(b-g)\frac{V_b(w_B(w_g^*,w_b^*))}{(\mu R-b)}=w_b^*+(b-g)\bar{T}(\gamma_{\mathtt{B}}^b(w_B,w_b^*-w_B),\bar{\nu})\\&=u(\gamma_{\mathtt{B}}^b(w_B,w_b^*-w_B),\bar{\nu};g)=\max_{\nu}u(\gamma_{\mathtt{B}}^b(w_B,w_b^*-w_B),\nu;g)\end{aligned}$$

where the first equality follows from proposition 10, the first inequality follows from constraint (27), the last equality follows from lemma 3.

If $\mu R > b$ and $F_b(w_b^*) \geq \frac{w_g^* - w_b^*}{b-g}(\mu R - b) - w_b^*$, following (47), we have $w_B(w_g^*, w_b^*)$ satisfies

$$F_b(w_B(w_g^*, w_b^*)) = \frac{w_g^* - w_b^*}{b-g}(\mu R - b) - w_B(w_g^*, w_b^*)$$

Hence,

$$\begin{aligned} u(\gamma_{\mathtt{P}}^g(w_g^*, \tau^*), \bar{\nu}; g) &= w_g^* = w_b^* + (b-g)\frac{V_b(w_B(w_g^*, w_b^*))}{(\mu R - b)} = w_b^* + (b-g)\bar{T}(\gamma_{\mathtt{B}}^b(w_B, w_b^* - w_B), \bar{\nu}) \\ &= u(\gamma_{\mathtt{B}}^b(w_B, w_b^* - w_B), \bar{\nu}; g) = \max_{\nu} u(\gamma_{\mathtt{B}}^b(w_B, w_b^* - w_B), \nu; g) \end{aligned}$$

where the first equality follows from proposition 10, the first inequality follows from constraint (27), the last equality follows from lemma 3.

Next, we verify that $\mathcal{U}(\Gamma_{\{g,b\}}^*) = \mathcal{Y}$. First, we verify that

$$U(\gamma_{\mathtt{B}}^b\left(w_B(w_g^*, w_b^*), w_b^* - w_B(w_g^*, w_b^*)\right), \bar{\nu}) = \min\left\{F_b(w_b^*),\ \frac{w_g^* - w_b^*}{b-g}(\mu R - b)^+ - w_b^*\right\} \tag{189}$$

If $R \leq \beta_b$, then $w_B(w_g^*, w_b^*) = 0$, hence,

$$U(\gamma_{\mathtt{B}}^b\left(w_B(w_g^*, w_b^*), w_b^* - w_B(w_g^*, w_b^*)\right), \bar{\nu}) = -w_b^* = F_b(w_b^*) = \min\left\{F_b(w_b^*),\ \frac{w_g^* - w_b^*}{b-g}(\mu R - b)^+ - w_b^*\right\}, \tag{190}$$

where the second equality follows from the definition of F_b, and the last equality follows from $w_g^* \geq w_b^*$. If $R > \beta_b$ and $F_b(w_b^*) \leq \frac{w_g^* - w_b^*}{b-g}(\mu R - b) - w_b^*$, following (47), we have $w_B(w_g^*, w_b^*) = w_b^*$, and

$$U(\gamma_{\mathtt{B}}^b\left(w_B(w_g^*, w_b^*), w_b^* - w_B(w_g^*, w_b^*)\right), \bar{\nu}) = U(\gamma_{\mathtt{B}}^b\left(w_b^*, 0\right), \bar{\nu}) = F_b(w_b^*) = \min\left\{F_b(w_b^*),\ \frac{w_g^* - w_b^*}{b-g}(\mu R - b)^+ - w_b^*\right\}$$

where the second equality follows from Proposition 1. If $R > \beta_b$ and $F_b(w_b^*) \geq \frac{w_g^* - w_b^*}{b-g}(\mu R - b) - w_b^*$, following (47), we have $w_B(w_g^*, w_b^*)$ satisfies

$$F_b(w_B(w_g^*, w_b^*)) = \frac{w_g^* - w_b^*}{b-g}(\mu R - b) - w_B(w_g^*, w_b^*),$$

we have

$$\begin{aligned} U(\gamma_{\mathtt{B}}^b\left(w_B(w_g^*, w_b^*), w_b^* - w_B(w_g^*, w_b^*)\right), \bar{\nu}) &= F_b(w_B(w_g^*, w_b^*)) - (w_b^* - w_B(w_g^*, w_b^*)) = \frac{w_g^* - w_b^*}{b-g}(\mu R - b) - w_B(w_g^*, w_b^*) \\ &= \min\left\{F_b(w_b^*),\ \frac{w_g^* - w_b^*}{b-g}(\mu R - b)^+ - w_b^*\right\} \end{aligned}$$

Following Proposition 11, we have

$$U(\gamma_{\mathtt{P}}^g(w_g^*, \tau^*), \bar{\nu}) = J(w_g^*, \bar{\tau}^*) \tag{191}$$

Finally, (189) and (191) imply

$$\begin{aligned} \mathcal{U}(\Gamma_{\{g,b\}}^*) &= p \cdot U(\gamma_{\mathtt{P}}^g(w_g^*, \tau^*), \bar{\nu}) + (1-p) U(\gamma_{\mathtt{B}}^b\left(w_B(w_g^*, w_b^*), w_b^* - w_B(w_g^*, w_b^*)\right), \bar{\nu}) \\ &= p \cdot J(w_g^*, \bar{\tau}^*) + (1-p)\min\left\{F_b(w_b^*),\ \frac{w_g^* - w_b^*}{b-g}(\mu R - b)^+ - w_b^*\right\} \end{aligned}$$

E. Proofs in Section 6

E.1. Proof of Proposition 5

For any (w_g, w_b) that satisfy the constraints of the optimization problem (52), if we let $\tau = -1/r \ln(1 - r w_b/g)$ and $\xi = \min\{F_b(w_b), (w_g - w_b)/(b-g)(\mu R - b)^+ - w_b\}$, then (w_g, w_b, τ, ξ) also satisfy the constraints of the optimization problem (23). Furthermore, $G(w_g, \tau) \leq \breve{G}(w_g, \tau)$ since $\breve{G}$ has one more constraint (i.e., (55)) than G. Therefore, $\breve{\mathcal{Y}}_1 \leq \mathcal{Y}$.

Similarly, for any $(w_g, \bar{\tau})$ that satisfy the constraints of the optimization problem (53), if we let $\tau = -1/r \ln(1 - r\bar{\tau})$, $w_b = g\bar{\tau}$ and $\xi = \min\{F_b(w_b), (w_g - w_b)/(b-g)(\mu R - b)^+ - w_b\}$ then (w_g, w_b, τ, ξ) also satify the constraints of the optimization problem (23). Furthermore, $G(w_g, \tau) \leq \breve{G}(w_g, \tau)$ since $\breve{G}$ has one more constraint (i.e., (55)) than G. Therefore, $\breve{\mathcal{Y}}_2 \leq \mathcal{Y}$.

Therefore, $\max\{\breve{\mathcal{Y}}_1, \breve{\mathcal{Y}}_2\} \leq \mathcal{Y}$.

E.2. Preparations to the proof of Proposition 6

LEMMA 17. *If $\bar{w}_g > \beta_b$ and $w \geq \breve{\omega}(\tau)$, then $\mathrm{z}(w,\tau)$ in Lemma 6 is well-defined. We have $\mathrm{z}(w,\tau)+\beta_g \leq \beta_b$ if and only if $w \in [\breve{\omega}(\tau), \bar{\omega}(\tau)]$.*

LEMMA 18. *If $\bar{w}_g > \beta_b$ and $w \in [\bar{\omega}(\tau), \tilde{\omega}(\tau)]$, then there exists unique pair of time epochs t_1, t_2 with $0 \leq t_2 \leq t_1 < \tau$, such that (W_t, H_t) defined in (66) and (67) satisfy*

$$W_0 = w, \tag{192}$$

and

$$\mathrm{y}(\tau, t_1) - \beta_g \geq W_0 \quad \textit{if} \quad t_2 = 0, \tag{193}$$

and the constraints in the optimization problem (54).

LEMMA 19. *If $w > \tilde{\omega}(\tau)$ and $\tau \geq \ln\rho$, there exists a unique time epoch $\tilde{t}_2 \in [0, \check{t}_1(\tau)]$ and value z such that (W_t, H_t) defined in (68) and (69)satisfy the constraints in the optimization problem (54).*

E.2.1. Proof of Lemma 17 First, we show that $\dfrac{\partial \mathrm{z}}{\partial w} > 0$. For $w \in [\breve{w}(\tau), \hat{\omega}(\tau)]$, z solves equation (36). We have

$$\frac{\partial \mathrm{z}}{\partial w} = e^{r(\tau_{\mathrm{z}}-\tau)}\left(-1 + \frac{r(\bar{w}_g+\beta_g)}{\mathrm{z}+\beta_g}\right) < 0.$$

For $w \geq \hat{\omega}(\tau)$, then $\mathrm{z}(w,\tau) = w/(\mu(1-e^{-\tau})) - \beta_g$. Clearly, $\dfrac{\partial \mathrm{z}}{\partial w} > 0$. If $\tau < -\ln\left(\frac{g-rb}{\mu b}\right)$, then $\bar{\omega}(\tau) > \hat{\omega}(\tau)$, and $\mathrm{z}(\bar{\omega}(\tau),\tau)+\beta_g = \beta_b$. Similarly, if $\tau \geq -\ln\left(\frac{g-rb}{\mu b}\right)$, then $\bar{\omega}(\tau) \leq \hat{w}(\tau)$, and

$$\mathrm{z}(\bar{\omega}(\tau),\tau)+\beta_g = \beta_b. \tag{194}$$

Hence, we have $\mathrm{z}(w,\tau)+\beta_g \leq \beta_b$ if and only if $w \in [\breve{\omega}(\tau), \bar{\omega}(\tau)]$. Q.E.D.

E.2.2. Proof of Lemma 18 Given $t_1 \in [0,\tau]$, $\mathrm{y}(\tau,t_1) = \bar{w}_g\left(1-e^{r(t_1-\tau)}\right)+\beta_b$ and $W_{t_1} = \mu(\bar{w}_g+\beta_b)(1-e^{t_1-\tau}) + \bar{w}_g \cdot e^{t_1-\tau} - \bar{w}_g \cdot e^{r(t_1-\tau)}$ are well-defined. Define $f(t_2;t_1) := \mu\mathrm{y}(\tau,t_1)(1-e^{t_2-t_1}) + W_{t_1}e^{t_2-t_1} + \beta_g - \mathrm{y}(\tau,t_1)$. It is straightforward to verify that $f(t_1;t_1) < 0$ for $t_1 \in [0,\tau]$ and $f'(t_2;t_1) < 0$ for $t_2 \in [0,t_1]$ and $t_1 \in [0,\tau]$. Hence, if $f(0;t_1) > 0$, then there exists unique $t_2 \in [0,t_1]$ such that $f(t_2;t_1) = 0$. Define $t_2(t_1)$ solves $f(t_2(t_1);t_1) = 0$ if $f(0;t_1) > 0$ and $t_2(t_1) = 0$ if $f(0;t_1) \leq 0$. By the definition of $f(t_2;t_1)$, we have $t_2(t_1)$ is continuous in t_1.

Given $t_2(t_1)$, we denote

$$W_0(t_1) = \bar{w}_g(1-e^{-rt_2(t_1)}) + e^{-rt_2(t_1)}\left(\mathrm{y}(\tau,t_1)\mu(1-e^{t_2-t_1}) + e^{t_2-t_1}\mathrm{W}(\tau,t_1)\right)$$

By definition of $t_2(t_1)$, we can verify that $W_0(t_1)$ strictly decreases in t_1 on $[0,\tau]$. Furthermore, $W_0(0) = \tilde{\omega}(\tau)$. If we let $t_1 \to \tau$, then $\lim_{t_1\to\tau}\mathrm{y}(\tau,t_1) = \beta_b$. Hence, following (194), we have $\lim_{t_1\to\tau} W_0(t_1) = \bar{\omega}(\tau)$. Therefore, for any $w \in [\bar{\omega}(\tau), \tilde{\omega}(\tau)]$, there exists unique t_1 such that $W_0(t_1) = w$. It is worth noting here that,

$$\mathrm{y}(\tau,t_1) = \bar{w}_g \,\mathrm{if}\, w = \tilde{w}(\tau). \tag{195}$$

Finally, we verify that W_t and H_t defined in (66) and (67), with t_1 and t_2 defined above satisfy the constraints in the optimization problem (54). It is straightforward to show that W_t and H_t defined in (66) and (67) satisfy $dW_t/dt = rW_t - \mu H_t$. Then, $H_t \geq \beta_g$ for $t \in [0,\tau]$ can be verified by: first, H_t is continuous in t on $[0,\tau]$. Second, $H_t = \beta_g$ for $t \in [0,t_2]$, $H_t = \mathrm{y}(\tau,t_1) - W_t$ increases in t for $t \in [t_1,t_2]$, and H_t increases in t for $t \in [t_2,\tau]$ ($\frac{\partial H_t}{\partial t} = (b-g)e^{t-\tau} > 0$ for $t \in [t_2,\tau]$). Next, $\min\{W_t+H_t, \bar{w}_g\} - B_t \leq \beta_b$ can be verified by: first, W_t+H_t is continuous in t, and $W_t+H_t-B_t = \beta_b$ for $t \in [t_1,\tau]$, Second, $W_t+H_t-B_t = \mathrm{y}(\tau,t_1) - B_t$ increases in t for $t \in [t_2,t_1]$, and $W_t+H_t-B_t = W_t+\beta_g-B_t = e^{r(t-t_2)}(W_{t_2}-B_{t_2})+\beta_g$ increases in t for $t \in [0,t_2]$. Q.E.D.

E.2.3. Proof of Lemma 19 If $\tau \geq \ln\rho$, we have $\check{t}_1(\tau) \in [0,\tau]$ is well-defined and following (68) and (69), we have

$$W_{\check{t}_1(\tau)} + H_{\check{t}_1(\tau)} = \bar{w}_g. \tag{196}$$

Case 1: $w \geq \dfrac{g\left(1-e^{-\check{t}_1(\tau)}\right) + W_{\check{t}_1(\tau)}e^{-\check{t}_1(\tau)}}{r+\mu e^{-\check{t}_1(\tau)}}$. Let $\mathrm{z} := \frac{w - W_{\check{t}_1}e^{-\check{t}_1}}{\mu(1-e^{-\check{t}_1})} - \beta_g$ and $\tilde{t}_2 = 0$. Clearly, $W_0 = w$. Further, W_t and H_t satisfy $dW_t/dt = rW_t - \mu H_t$. Then, we verify that $H_t \geq \beta_g$ for $t \in [0,\tau]$. First, $H_t = g(t,\tau) := \bar{w}_g + \beta_b - \mu(\bar{w}_g + \beta_b)(1-e^{t-\tau}) - \bar{w}_g e^{t-\tau}$ for $t \in [\check{t}_1(\tau),\tau]$. $\partial g(t,\tau)/\partial t = (b-g)e^{t-\tau} > 0$. Hence, $g(t,\tau) \geq g(0,\tau) = \bar{w}_g + \beta_b - \mu(\bar{w}_g + \beta_b)(1-e^{-\tau}) - \bar{w}_g e^{-\tau}$. $\partial g(0,\tau)/\partial\tau = [\bar{w}_g - \mu(\bar{w}_g + \beta_b)]e^{-\tau} = (g-b)e^{-\tau} < 0$, which implies that $g(0,\tau) > g(0,\infty) = r(\bar{w}_g + \beta_b) > r(\bar{w}_g + \beta_g) = \beta_g$. Therefore, $H_t \geq \beta_g$ for $t \in [\check{t}_1(\tau),\tau]$.

Second, $H_t = \mathrm{z} + \beta_g - W_t$ increases in t on $[0,\check{t}_1(\tau))$. Hence, $w \geq \dfrac{g\left(1-e^{-\check{t}_1(\tau)}\right) + W_{\check{t}_1(\tau)}e^{-\check{t}_1(\tau)}}{r+\mu e^{-\check{t}_1(\tau)}}$ implies that $H_t \geq H_0 = \mathrm{z} + \beta_g - w \geq \beta_g$. Finally, for $t \in [\check{t}_1(\tau),\tau]$, we have $W_t + H_t - B_t = \beta_b$ and following (196), we have $W_{\check{t}_1(\tau)} + H_{\check{t}_1(\tau)} - B_{\check{t}_1(\tau)} = \bar{w}_g - B_{\check{t}_1(\tau)} = \beta_b$. Hence, for $t \in [0,\check{t}_1(\tau)]$, $\bar{w}_g - B_t$ increases in t, and $\bar{w}_g - B_t \leq \beta_b$. Therefore, $\min\{W_t + H_t, \bar{w}_g\} - B_t \leq \beta_b$ for $t \in [0,\tau]$.

Case 2: $w < \frac{g\left(1-e^{-\check{t}_1(\tau)}\right) + W_{\check{t}_1(\tau)}e^{-\check{t}_1(\tau)}}{r+\mu e^{-\check{t}_1(\tau)}}$. Following the same logic of the proof of Lemma 13, we can show that for any $w > \tilde{\omega}(\tau)$ and $w < \frac{g\left(1-e^{-\check{t}_1}\right) + W_{\check{t}_1}e^{-\check{t}_1}}{r+\mu e^{-\check{t}_1}}$, there exists a unique $\mathrm{z} \in \left[W_{\check{t}_1}, \frac{g\left(1-e^{-\check{t}_1}\right) + W_{\check{t}_1}e^{-\check{t}_1}}{r+\mu e^{-\check{t}_1}}\right]$, which satisfy equations

$$\bar{w}_g - (\bar{w}_g - \mathrm{z}) \cdot e^{r(\tau_{\mathrm{z}} - \check{t}_1(\tau))} = w, \tag{197}$$

where

$$\tau_{\mathrm{z}} = \ln\left(\frac{\mu(\mathrm{z} + \beta_g) - W_{\check{t}_1(\tau)}}{r(\bar{w}_g - \mathrm{z})}\right). \tag{198}$$

Further, we let $\tilde{t}_2 = \check{t}_1(\tau) - \tau_{\mathrm{z}}$. It is straightforward to verify that W_t and H_t satisfy $dW_t/dt = rW_t - \mu H_t$. Then, $H_t = \beta_g$ for $t \in [0,\tilde{t}_2]$, and $H_t = \mathrm{z} + \beta_g - W_t \geq \mathrm{z} + \beta_g - W_{\tilde{t}_2} = \beta_g$. Furthermore, similar to Case 1, we can show that $\min\{W_t + H_t, \bar{w}_g\} - B_t \leq \beta_b$ for $t \in [0,\tau]$.

Finally, we present a property that will be useful in the optimality proof. If $w > \tilde{w}(\tau)$ and $\tau \geq \ln\rho$, (195) implies that

$$\mathrm{z} + \beta_g > \bar{w}_g. \tag{199}$$

Q.E.D.

E.3. Proof of Proposition 6

Scenario 1: We verify that if $w < \check{w}(\tau)$, then the optimization problem is infeasible. If $H_t = \beta_g$ for $t \in [0,\tau]$, then $W_0 = \check{w}(\tau)$. Since the constraint of the optimization problem requires that $H_t \geq \beta_g$ for $t \in [0,\tau]$, we have $W_0 \geq \check{w}(\tau)$.

Scenario 2: If $\bar{w}_g > \beta_b$, following Lemma 17, we have if $w \in [\breve{\omega}(\tau), \bar{\omega}(\tau)]$, then $\mathrm{z}(w,\tau)$ in Lemma 6 is well-defined and $\mathrm{z}(w,\tau) + \beta_g \leq \beta_b$. Furthermore, take W_t and H_t defined in Lemma 6, we have $W_t + H_t \leq \mathrm{z}(w,\tau) + \beta_g \leq \beta_b$. Hence, W_t and H_t defined in Lemma 6 are optimal for the optimization $\check{G}(w,\tau)$ in (54).

Scenario 3: Since $g(1-e^{-\tau})$ is fixed when τ is given, we only need to maximize the integral $\int_0^\tau \mu e^{-t}[R + F_g(W_t + H_t)]dt$. To solve the optimization problem, we can write down the Hamiltonian:

$$\mathcal{H} = e^{-t}\{\mu[R + F_g(W_t + H_t)]\} + \lambda(t)(rW_t - \mu H_t) + \eta(t)(H_t - \beta_g) + \phi(t)[B_t - \min(W_t + H_t, \bar{w}_g) + \beta_b]. \tag{200}$$

The optimality conditions are

$$\frac{\partial\mathcal{H}}{\partial H} = \mu e^{-t}F_g'(W_t + H_t) - \lambda(t)\mu + \eta(t) - \phi(t)\mathbb{1}_{W_t + H_t \leq \bar{w}_g} = 0, \tag{201}$$

$$\eta(t)(H_t - \beta_g) = 0, \tag{202}$$

$$\eta(t) \geq 0, \tag{203}$$

$$\phi(t)[B_t - \min(W_t + H_t, \bar{w}_g) + \beta_b] = 0, \tag{204}$$

$$\phi(t) \geq 0, \tag{205}$$

$$\frac{\partial \mathcal{H}}{\partial W} = \mu e^{-t} F_g'(W_t + H_t) + \lambda(t) r - \phi(t) \mathbb{1}_{W_t + H_t \le \bar{w}_g} = -\lambda'(t). \tag{206}$$

Since the objective of the optimal control problem is jointly concave in (W_t, H_t), to verify that W_t and H_t defined in (66) and (67) are optimal, it is sufficient to verify the above optimality conditions and the corresponding constraints. Following Lemma 18, W_t and H_t defined in (66) and (67) satisfy the constraints of the optimization problem. Next, we verify that they also satisfy the optimality conditions.

Since $\mathrm{y}(\tau, t_1) \le \bar{w}_g$, then $W_t + H_t \le \bar{w}_g$ for $t \ge t_2$ and we can verify the optimality conditions (201), (202), (204), and (206) by letting

$$\begin{aligned}
\phi(t) &= \left[\mu e^{-t} F_g'\left(g/r(1 - e^{t-\tau}) + \beta_b\right) - \mu e^{-t} F_g'\left(y_1^*\right)\right] \mathbb{1}_{t \ge t_1} \ge 0 \\
\eta(t) &= \mu e^{-rt} \gamma(t) \mathbb{1}_{t \le t_2} \\
\lambda(t) &= \begin{cases} \left[\int_t^{t_2} \mu e^{-\mu\xi} F_g'(W_\xi + \beta) d\xi + F_g'(y_1^*) e^{-\mu t_2}\right] e^{-rt}, & t \in [0, t_2], \\ F_g'(y_1^*) e^{-t}, & t \in [t_2, \tau], \end{cases}
\end{aligned}$$

where

$$\gamma(t) := \left[\int_t^{t_2} \mu e^{-\mu\xi} F_g'(W_\xi + \beta_g) d\xi + F_g'(y_1^*) e^{-\mu t} - e^{-\mu t} F_g'(W_t + \beta_g)\right] \ge 0,$$

for $t \in [0, t_2]$ and the first inequality follows from that F_g is concave and the second inequality follows from that $\gamma(t)$ is decreasing in t and $\gamma(t_2) = 0$. The inequalities further imply (203) and (205) and complete the proof.

Scenario 4: Following Lemma 19, W_t and H_t defined in (68) and (69) satisfy the constraints of the optimization problem. Next, similar to scenario (3), we verify that they also satisfy the optimality conditions (201) - (206). We can verify the optimality conditions

$$\begin{aligned}
\phi(t) &= \left[\mu e^{-t} F_g'\left(g/r(1 - e^{t-\tau}) + \beta_b\right) - \mu e^{-t} F_g'\left(z_1 + \beta_g\right)\right] \mathbb{1}_{t \ge \tilde{t}_1(\tau)} \ge 0 \\
\eta(t) &= \mu e^{-rt} \gamma(t) \mathbb{1}_{t \le \tilde{t}_2} \\
\lambda(t) &= \begin{cases} \left[\int_t^{\tilde{t}_2} \mu e^{-\mu\xi} F_g'(W_\xi + \beta) d\xi + F_g'(z_1 + \beta_g) e^{-\mu \tilde{t}_2}\right] e^{-rt}, & t \in [0, \tilde{t}_2], \\ F_g'(z_1 + \beta_g) e^{-t}, & t \in [\tilde{t}_2, \tau], \end{cases}
\end{aligned}$$

where

$$\gamma(t) := \int_t^{\tilde{t}_2} \mu e^{-\mu\xi} F_g'(W_\xi + \beta_g) d\xi + F_g'(z_1 + \beta_g) e^{-\mu t} - e^{-\mu t} F_g'(W_t + \beta_g) \ge 0,$$

for $t \in [0, \tilde{t}_2]$ and the first inequality follows from that F_g is concave and $g/r(1 - e^{t-\tau}) + \beta_b \le g/r(1 - e^{\tilde{t}_1(\tau) - \tau}) + \beta_b = \bar{w}_g < \mathrm{z} + \beta_g$, followed by (199). The second inequality follows from that $\gamma(t)$ is decreasing in t and $\gamma(\tilde{t}_2) = 0$. The inequalities further imply (203) and (205) and complete the proof.

Scenario 5: We verify that if $w > \tilde{w}(\tau)$ and $\tau < \ln \rho$, then the optimization problem is infeasible. Constraint (56) requires that

$$\min\{W_t + H_t, \bar{w}_g\} \le B_t + \beta_b = \bar{w}_g(1 - e^{r(t-\tau)}) + \beta_b < \bar{w}_g,$$

for $t \in [0, \tau)$, where the second inequality follows from that $\tau < \ln \rho$. Hence, $W_t + H_t < \bar{w}_g$ and $W_t + H_t \le B_t + \beta_b$. Hence, we have

$$\begin{aligned}
&\frac{dW_t}{dt} = rW_t - \mu H_t \\
&\frac{dW_t}{dt} - W_t = -\mu(W_t + H_t) \\
&(e^{-t} W_t)' = -\mu e^{-t}(W_t + H_t) \\
&W_0 = \int_0^\tau e^{-t} \mu(W_t + H_t) dt \le \int_0^\tau e^{-t} \mu(B_t + \beta_b) dt = \tilde{w}(\tau).
\end{aligned}$$

Therefore, if $w > \tilde{w}(\tau)$, $W_0 < w$ and the optimization problem is infeasible. Q.E.D.

E.4. Preparations to prove Proposition 7

LEMMA 20. *For any w,τ such that $\check{G}(w,\tau) > -\infty$, we have $\nu^0 \in \mathfrak{N}(\gamma^g_{\mathrm{P}'}(w,\tau,b),b)$.*

LEMMA 21. *If $\tau = \infty$, we have $\gamma^g_{\mathrm{P}'}(w,\tau,b) = \gamma^g_{\mathrm{P}}(w,\tau)$ and $\nu^0 \in \mathfrak{N}(\gamma^g_{\mathrm{P}'}(w,\tau,b),b)$.*

PROPOSITION 12. *For any $\tau \geq 0$, $w \in [\breve{\omega}(\tau), \tilde{\omega}(\tau)]$, or $w > \tilde{\omega}(\tau)$ and $\tau \geq \ln \rho$, we have $\bar{\nu} \in \mathfrak{N}(\gamma^g_{\mathrm{P}'}(w,\tau,b),g)$, and*

$$u\Big(\gamma^g_{\mathrm{P}'}(w,\tau,b), \bar{\nu}; g\Big) = w.$$

The proof of Proposition 12 can be adapted from the proof of Proposition 10 and is omitted here.

PROPOSITION 13. *For any $c < \mu R$, we have*

$$U(\gamma^c_{\mathrm{P}'}(w,\tau), \bar{\nu}) = \check{G}(w,\tau) \tag{207}$$

The proof of Proposition 13 can be adapted from the proof of Proposition 11 and is omitted here.

E.4.1. Proof of Lemmas 20 and 21

For Lemma 20, we only need to consider the following cases of (w_g,τ) since otherwise, by definition of $\check{G}$, $\check{G}(w_g,\tau) = -\infty$.

1. $w \in [\breve{\omega}(\tau), \bar{\omega}(\tau))$: $\gamma^g_{\mathrm{P}'}(w,\tau,b) = \gamma^g_{\mathrm{P}}(w,\tau)$. Furthermore, Lemma 17 implies that $\mathrm{z}(w,\tau) + \beta_g \leq \beta_b$. Then, Lemma 16 implies that $\nu^0 \in \mathfrak{N}(\gamma^g_{\mathrm{P}}(w,\tau),b)$.
2. $w \in [\bar{\omega}(\tau), \tilde{\omega}(\tau)]$: Following definition 6, it is straightforward to verify that $S'(w_g,\tau)$ satisfies (177) if we let $S = S'(w_g,\tau)$ and $w = w_g$. Hence, if $W_{\tau_1^N} + H_{\tau_1^N} > \bar{w}_g$, the contract after the first arrival would be $\gamma^g_{\mathrm{D}}\left(W_{\tau_1^N} + H_{\tau_1^N}, \tau_1^N, S'(w,\tau)\right)$. Then, following Lemma 15, after the first arrival, the bad agent with cost b will never work. As a result, the bad agent's continuation utility after the first arrival (if he tried to mimic the good agent) is $\bar{w}_g$ if $W_{\tau_1^N} + H_{\tau_1^N} > \bar{w}_g$. On the other hand, if $\bar{w}_g$ if $W_{\tau_1^N} + H_{\tau_1^N} \leq \bar{w}_g$, the contract after the first arrival would be an *IC-Binding contract* $\hat{\gamma}^g$. Then, following Lemma 4, the bad agent will never work after the first arrival. As a result, the bad agent's continuation utility after the first arrival (if he tried to mimic the good agent) is $W_{\tau_1^N} + H_{\tau_1^N}$ if $W_{\tau_1^N} + H_{\tau_1^N} \leq \bar{w}_g$. To summarize, the agent's continuation utility after the first arrival (if he tried to mimic the good agent) is $\min\left\{W_{\tau_1^N} + H_{\tau_1^N}, \bar{w}_g\right\}$.

   If the bad agent never works in the good agent's contract, then for $t \leq \tau$, we have the bad agent's continuation utility as $B_t = g/r \cdot (1 - e^{r(t-\tau)})$. In the following, we use a one-shot deviation argument to show that the bad agent will always prefer no effort in the good agent's contract. For any $t \in [0,\tau]$, if the bad agent deviates from no effort to ν_t in the next δ time interval, then the bad agent's utility becomes

   $$\begin{aligned}&g\delta - b\nu_t\delta + \mu\nu_t + \mu\nu_t \min\{W_t + H_t, \bar{w}_g\} + (1-\mu\nu_t\delta)e^{-r\delta}B_{t+\delta}\\=&g\delta + e^{-r\delta}B_{t+\delta} + \delta\nu_t\left[-b + \mu\min\{W_t + H_t, \bar{w}_g\} - \mu B_{t+\delta}\right]\end{aligned}$$

   where W_t follows (66) and H_t follows (67). Hence, as long as $\min\{W_t + H_t, \bar{w}_g\} - B_t \leq \beta_b$, there is no benefit for the bad agent to deviate from no effort. Following the optimization problem (54), if $\check{G}(w,\tau) > -\infty$, we have (56) is satisfied, i.e., $\min\{W_t + H_t, \bar{w}_g\} - B_t \leq \beta_b$ for any $t \in [0,\tau]$. Therefore, the bad agent will not work in the good agent's contract, i.e., $\nu^0 \in \mathfrak{N}(\gamma^g_{\mathrm{P}'}(w,\tau,b),b)$.
3. $w > \tilde{\omega}(\tau)$ and $\tau \geq \ln\rho$: The proof of this case is similar to that in Case 2 and is omitted here.

For Lemma 21, if $\tau = \infty$, then $w \geq g/r$. Following Proposition 6, we have $W_t = w$ and $H_t = w/\mu - w$. Hence, following definition 4, we have $\gamma^g_{\mathrm{P}'}(w,\tau,b) = \gamma^g_{\mathrm{P}}(w,\tau)$. Since $W_{\tau_1^N} + H_{\tau_1^N} > \bar{w}_g$, the contract after the first arrival would be $\gamma^g_{\mathrm{D}}\left(W_{\tau_1^N} + H_{\tau_1^N}, \tau_1^N, S'(w,\tau)\right)$. Then, following Lemma 15, after the first arrival, the bad agent with cost b will never work. As a result, the bad agent's continuation utility after the first arrival (if he tried to mimic the good agent) is $\bar{w}_g$. Since $\tau = \infty$, the bad agent's continuation utility stays at $\bar{w}_g$ even if he does not exert effort to create the arrival. Hence, the bad agent will never exert effort, i.e., $\nu^0 \in \mathfrak{N}(\gamma^g_{\mathrm{P}'}(w,\tau,b),b)$. Q.E.D.

E.5. Proof of Proposition 7

Following Propositions 10 and 12, $\{\gamma^{g}_{\mathrm{P}'}(\check{w}_g,\check{\tau},b),\gamma^{b}_{\mathrm{B}}(w_B(\check{w}_g,\check{w}_b),\check{w}_b-w_B(\check{w}_g,\check{w}_b))\}$ satisfy constraints (LL), (PK), (IC), (IR), and (FE). In the following, we verify (TT).

$$\begin{aligned}u(\gamma^{b}_{\mathrm{B}}(w_B(\check{w}_g,\check{w}_b),\check{w}_b-w_B(\check{w}_g,\check{w}_b)),\bar{\nu};b)&=\check{w}_b\geq g/r\cdot(1-e^{-r\check{\tau}})=u(\gamma^{g}_{\mathrm{P}'}(\check{w}_g,\check{\tau},b),\nu^0;b)\\&=\max_{\nu}u(\gamma^{g}_{\mathrm{P}'}(\check{w}_g,\check{\tau},b),\nu;b),\end{aligned}$$

where the first equality follows from Proposition 10, the first inequality follows from the constraints of (52), i.e., $\check{w}_b\geq g/r$ if $\check{\mathcal{Y}}=\check{\mathcal{Y}}_1$ and constraints of (53), i.e., $\check{w}_b=g/r(1-e^{-r\check{\tau}})$ if $\check{\mathcal{Y}}=\check{\mathcal{Y}}_2$, and the last equality follows from $\nu^0\in\mathfrak{N}(\gamma^{g}_{\mathrm{P}'}(\check{w}_g,\check{\tau},b),b)$ (Lemmas 20 and 21). If $\mu R\leq b$, following Lemma 9, we have $w_B=0$, and

$$u(\gamma^{g}_{\mathrm{P}'}(\check{w}_g,\check{\tau},b),\bar{\nu};g)=\check{w}_g\geq\check{w}_b=\max_{\nu}u(\gamma^{b}_{\mathrm{B}}(0,\check{w}_b),\nu;g)$$

where the first equality follows from proposition 12 and the first inequality follows from constraints of (52) if $\check{\mathcal{Y}}=\check{\mathcal{Y}}_1$ and constraints of (53) if $\check{\mathcal{Y}}=\check{\mathcal{Y}}_2$. On the other hand, if $\mu R>b$ and $F_b(\check{w}_b)\leq\dfrac{\check{w}_g-\check{w}_b}{b-g}(\mu R-b)-\check{w}_b$, following (47), we have $w_B(\check{w}_g,\check{w}_b)=\check{w}_b$, and

$$\begin{aligned}u(\gamma^{g}_{\mathrm{P}'}(\check{w}_g,\check{\tau},b),\bar{\nu};g)&=\check{w}_g\geq\check{w}_b+(b-g)\frac{V_b(w_B(\check{w}_g,\check{w}_b))}{(\mu R-b)}=\check{w}_b+(b-g)\bar{T}(\gamma^{b}_{\mathrm{B}}(w_B,\check{w}_b-w_B),\bar{\nu})\\&=u(\gamma^{b}_{\mathrm{B}}(w_B,\check{w}_b-w_B),\bar{\nu};g)=\max_{\nu}u(\gamma^{b}_{\mathrm{B}}(w_B,\check{w}_b-w_B),\nu;g)\end{aligned}$$

where the first equality follows from Proposition 12, the first inequality follows from the constraints of (52) if $\check{\mathcal{Y}}=\check{\mathcal{Y}}_1$ and constraints of (53) if $\check{\mathcal{Y}}=\check{\mathcal{Y}}_2$, the last equality follows from lemma 3.

If $\mu R>b$ and $F_b(\check{w}_b)\geq\dfrac{\check{w}_g-\check{w}_b}{b-g}(\mu R-b)-\check{w}_b$, following (47), we have $w_B(\check{w}_g,\check{w}_b)$ satisfies

$$F_b(w_B)=\frac{\check{w}_g-\check{w}_b}{b-g}(\mu R-b)-w_B$$

Hence,

$$\begin{aligned}u(\gamma^{g}_{\mathrm{P}'}(\check{w}_g,\check{\tau},b),\bar{\nu};g)&=\check{w}_g=\check{w}_b+(b-g)\frac{V_b(w_B)}{(\mu R-b)}=\check{w}_b+(b-g)\bar{T}(\gamma^{b}_{\mathrm{B}}(w_B,\check{w}_b-w_B),\bar{\nu})\\&=u(\gamma^{b}_{\mathrm{B}}(w_B,\check{w}_b-w_B),\bar{\nu};g)=\max_{\nu}u(\gamma^{b}_{\mathrm{B}}(w_B,\check{w}_b-w_B),\nu;g)\end{aligned}$$

where the first equality follows from proposition 10, the first inequality follows from the constraints of (52) if $\check{\mathcal{Y}}=\check{\mathcal{Y}}_1$ and constraints of (53) if $\check{\mathcal{Y}}=\check{\mathcal{Y}}_2$, the last equality follows from lemma 3.

Similar to (189), we can verify that

$$U(\gamma^{b}_{\mathrm{B}}(w_B(\check{w}_g,\check{w}_b),\check{w}_b-w_B(\check{w}_g,\check{w}_b)),\bar{\nu})=\min\left\{F_b(\check{w}_b),\ \frac{\check{w}_g-\check{w}_b}{b-g}(\mu R-b)^+-\check{w}_b\right\}.$$

Finally, following Proposition 13, we have

$$\begin{aligned}\mathcal{U}(\check{\Gamma}_{\{g,b\}})&=p\cdot U(\gamma^{g}_{\mathrm{P}'}(\check{w}_g,\check{\tau},b),\bar{\nu})+(1-p)U(\gamma^{b}_{\mathrm{B}}(w_B(\check{w}_g,\check{w}_b),\check{w}_b-w_B(\check{w}_g,\check{w}_b)),\bar{\nu})\\&=p\cdot\check{G}(\check{w}_g,\check{\tau})+(1-p)\min\left\{F_b(\check{w}_b),\ \frac{\check{w}_g-\check{w}_b}{b-g}(\mu R-b)^+-\check{w}_b\right\}\end{aligned}$$

i.e., $\check{\mathcal{Y}}=\mathcal{U}\left(\check{\Gamma}_{\{g,b\}}\right)$.

E.6. Comparison between two lower bounds

The following proposition verifies that the lower bound proposed in Proposition 5 is always greater than the lower bound in Proposition 2. Together with Proposition 7, we can show that the simple heuristic is always dominated by the easy-to-implement contracts $\check{\Gamma}_{\{g,b\}}$.

Proposition 14.

$$\check{\mathcal{Y}}\geq\mathcal{Y}_B \tag{208}$$

where $\check{\mathcal{Y}}$ is defined in Proposition 5 and $\mathcal{Y}_B$ is defined in Proposition 2.

Given $\hat{B}$ that is defined in (22), we have $\hat{B} < \bar{w}_g$. Hence, $\hat{B}/g < 1/r$. We let $w_g = \hat{B}$, $\tau = -\log(1 - r\hat{B}/g)$, and $\xi = -\hat{B}$. It is straightforward to verify that $(w_g, \bar{\tau})$ satisfies the constraints of optimization problem (53). Next, in the definition of $\check{G}$ ((54)), $H_t = \beta_g$ and $W_t = B_t$ for $t \in [0, \tau]$ satisfy the constraints. Further, if we let $H_t = \beta_g$, the value of $\int_0^\tau \mu e^{-t}[R + F_g(W_t + H_t)]dt - g(1 - e^{-\tau})$ is exactly principal's value $U(\hat{\gamma}^g, \bar{\nu})$, which is also $F_g(\hat{B})$. Hence, by definition of $\check{G}$, we have $\check{G}(w_g, \tau) \geq F_g(\hat{B})$. Finally, we have

$$p\check{G}(w_g, \tau) + (1-p)\xi \geq F_g(\hat{B}) - (1-p)\hat{B},$$

which completes the proof. Q.E.D.

E.7. Property of function $\mathcal{J}$

LEMMA 22. *For $\bar{\tau} \in \left[0, \frac{1}{r}\right]$, $c \in [\underline{c}, \min\{\bar{c}, \mu R\}]$, and $w \geq c\bar{\tau}$, we have the following properties for function $\mathcal{J}(w, \bar{\tau}, c)$,*

(i) $\mathcal{J}(w, \bar{\tau}, c)$ is jointly concave in w and $\bar{\tau}$;

(ii) $\mathcal{J}(w, \bar{\tau}, c)$ is increasing in $\bar{\tau}$ and $\mathcal{J}(w, 0, c) = -w$;

(iii) $\mathcal{J}(w, \bar{\tau}, c) + w$ is non-decreasing in w;

(iv) $0 \leq \mathcal{J}(w, \bar{\tau}, c) + w \leq \dfrac{\mu R - c}{r}$.

Proof.

(i) Following Proposition 4, $\mathcal{J}(w, \bar{\tau}, c)$ is jointly concave in $(w, \bar{\tau})$.

(ii) Following Proposition 4, $\mathcal{J}(w, \bar{\tau}, c)$ is is increasing in $\bar{\tau}$. If $\bar{\tau} = 0$, then $\tau = -\log(1 - r\bar{\tau})/r = 0$. Hence, $\hat{w}(0) = 0$. Hence, for any $w \geq 0$, following (154), we have

$$\begin{aligned}\mathcal{J}(w, 0, c) &= \lim_{\tau \to 0} \int_0^\tau \mu e^{-t} F_g\left(\frac{w}{\mu(1 - e^{-\tau})}\right) dt = \lim_{\tau \to 0} \int_0^\tau \mu e^{-t}\left[\frac{\mu R - c}{r} - \frac{w}{\mu(1 - e^{-\tau})}\right] dt \\ &= \lim_{\tau \to 0} -\mu(1 - e^{-\tau}) \frac{w}{\mu(1 - e^{-\tau})} = -w\end{aligned}$$

(iii) It is equivalent to show that $\mathcal{J}_1'(w, \bar{\tau}, c) = J_1'(w, \bar{\tau}) \geq -1$. First, J is concave in w. Hence, we only need to show $J_1'(w, \bar{\tau}) \geq -1$ when w is large enough. Denote $\tau := \dfrac{\log(1 - r\bar{\tau})}{r}$. Following (157), we have, for $w \geq \hat{w}(\tau)$, $J_1'(w, \bar{\tau}) = F_c'\left(\dfrac{w}{\mu(1 - e^{-\tau})}\right) \geq -1$.

(iv) Following (ii) and (iii), we have

$$\mathcal{J}(w, \bar{\tau}, c) + w \geq \mathcal{J}(w, 0, c) + w = 0$$

Again, following (ii) and definition of $\mathcal{J}$, we have

$$\mathcal{J}(w, \bar{\tau}, c) + w \leq \mathcal{J}\left(w, \frac{w}{c}, c\right) + w = U(\gamma_{\mathrm{P}}^c(\tau, 0), \bar{\nu}) = U(\hat{\gamma}^c, \bar{\nu}) + w = F_c(w) + w = V_c(w) \leq \frac{\mu R - c}{r},$$

where $\tau := \dfrac{\log(1 - r\bar{\tau})}{r}$ and $\bar{\tau} = \dfrac{w}{c}$, the last inequality follows from 2. Q.E.D.

E.8. Proof of Theorem 2

For any contract menu $\Gamma_{\mathcal{C}}, \mathcal{C} = \{c_1, ..., c_N\}$ that satisfy (LL), (PK), (IC), (IR), (FE), and (TT), we create a vector $\{w_i, \xi_i\}_{i=1,...,M}, w_{M+1}$ such that, they satisfy the constraints (78), and

$$\mathcal{U}(\Gamma_{\mathcal{C}}) \leq \sum_{i=1}^{M} \xi_i P_i - w_{M+1} \sum_{i=M+1}^{N} P_i.$$

First, we let ξ_1 follows (79), and for $i \in \{2, ..., M\}$, ξ_i is defined as the minimum of the right-hand side of (80) and (81). Then, we have $\mathcal{Y} \geq \mathcal{Z}(\mathcal{C})$. Define $\check{w}_i := u(\gamma^{c_i}, \bar{\nu}; c_i)$ for $i = 1, ..., N$. We further let $w_i = \check{w}_i$ for $i \in \{1, ..., M\}$ and $w_{M+1} = \check{w}_N$.

Step 1: We check constraint (78). For any $i \in \{1, ..., N-1\}$, we have

$$\check{w}_i = u(\gamma^{c_i}, \bar{\nu}; c_i) \geq \max_{\nu} u(\gamma^{c_{i+1}}, \nu; c_i) \geq u(\gamma^{c_{i+1}}, \bar{\nu}; c_i) = \check{w}_{i+1} + (c_i - c_{i+1})\bar{T}(\gamma^{c_{i+1}}, \bar{\nu}) \geq \check{w}_{i+1}, \tag{209}$$

Clearly, (209) implies constraint (78)

Step 2: If $R > \beta_{c_i}$, then for any $c_j < c_i$, (TT) implies that

$$w_j = u(\gamma^{c_j}, \bar{\nu}; c_j) \ge \max_{\nu} u(\gamma^{c_i}, \nu; c_j) \ge u(\gamma^{c_i}, \bar{\nu}; c_j) \ge w_i + (c_i - c_j)\bar{T}(\gamma^{c_i}, \bar{\nu}) = w_i + \frac{(c_i - c_j)S(\gamma^{c_i}, \bar{\nu}; c_i)}{\mu R - c_i}$$
$$= w_i + \frac{(c_i - c_j)(U(\gamma^{c_i}, \bar{\nu}) + w_i)}{\mu R - c_i} \tag{210}$$

where the first equality follows from Lemma 12 and the last equality follows from $S(\gamma^c, \bar{\nu}; c) = U(\gamma^c, \bar{\nu}) + u(\gamma^c, \bar{\nu}; c)$. Rearrange (214), we have, for any $c_j < c_i$,

$$U(\gamma^{c_i}, \bar{\nu}) \le \frac{(w_j - w_i)(\mu R - c_i)}{c_i - c_j} - w_i, \text{ if } R > \beta_c. \tag{211}$$

Hence, for any $c_i < \mu R$, we have

$$U(\gamma^{c_i}, \bar{\nu}) \le \min_{j<i} \left[\frac{w_j - w_i}{c_i - c_j}\right](\mu R - c_i) - w_i. \tag{212}$$

where the first inequality follows from Proposition 1, and the second inequality follows from (209). On the other hand, if $R \le \beta_{c_i}$ ($c_i \ge \mu R$), then

$$U(\gamma^{c_i}, \bar{\nu}) \le -\breve{w}_i \le -\breve{w}_N = -w_{M+1}, \text{ if } R \le \beta_{c_i}. \tag{213}$$

Furthermore, for any $c_i < c_N$, we have

$$w_{M+1} = \breve{w}_N \ge \max_{\nu} u(\gamma^{c_i}, \nu; c_N) \ge u(\gamma^{c_i}, \nu^0; c_N) = c_i \int_0^{\tau^0(\gamma^{c_i})} e^{-rt}\mathrm{d}t = c_i/r \cdot (1 - e^{-r\tau^0(\gamma^{c_i})}). \tag{214}$$

which implies that

$$1/r \cdot (1 - e^{-r\tau^0(\gamma^{c_i})}) \le \breve{w}_N / c_i \tag{215}$$

Finally, following Step 2.2 of the proof of Proposition 3 and definition of $\mathcal{J}$, we have for any $c_i < \mu R$,

$$U(\gamma^{c_i}, \bar{\nu}) \le \mathcal{J}(w_i, 1/r \cdot (1 - e^{-r\tau^0(\gamma^{c_i})}), c) \le \mathcal{J}\left(w_i, \min\left\{\frac{w_N}{c}, \frac{1}{r}\right\}, c\right) \tag{216}$$

where the second inequality follows from that the function $\mathcal{J}(w, \bar{\tau}, c)$ is increasing in $\bar{\tau}$ (Lemma 22(ii)) and (215). Therefore, (212) and (216) imply that for any $c < \mu R$,

$$U(\gamma^{c_i}, \bar{\nu}) \le \xi_i. \tag{217}$$

With (213), we established that

$$\mathcal{U}(\Gamma_{\mathcal{C}}) = \sum_{i=1}^{N} U(\gamma^{c_i}, \bar{\nu}) \cdot P_i \le \sum_{i=1}^{M} \xi_i P_i - w_{M+1} \sum_{i=M+1}^{N} P_i \tag{218}$$

E.9. Proof of Proposition 8

For any $\{w_i, \xi_i\}_{i=1,\dots,M}, w_{M+1}$ that satisfy (78). Clearly, $\{w_1, \dots, w_{M+1}\}$ satisfy the constraints of optimization problem (82). Furthermore, the objective of (77) is smaller than the objective of (82). Therefore, $\mathcal{Y}^N \le \hat{\mathcal{Y}}^N$.

E.10. Calculation of $\mathfrak{J}_i$

Proposition 15. *For any given $i = 1, \dots, M$ and $w_{M+1} \ge 0$, function $\mathfrak{J}_i(w|w_{M+1})$ is concave in w. Use $\mathfrak{J}'_{i-1}(w|w_{M+1})$ to represent the its left-derivative at w. Further fix a value $w_i \ge w_{M+1}$, and define*

$$\breve{w} := \sup\left\{w \;\middle|\; w \ge w_i \text{ and } \mathfrak{J}'_{i-1}(w|w_{M+1}) \ge 0\right\},$$
$$\hat{w} := \inf\left\{w \;\middle|\; w \ge w_i \text{ and } \mathfrak{J}'_{i-1}(w|w_{M+1}) \le -(\mu R - c_i)P_i\right\}, \text{ and}$$
$$\bar{u} := \begin{cases} \dfrac{\mathcal{J}\left(w_i, \min\left\{w_{M+1}/c_i, 1/r\right\}, c_i\right) + w_i}{\mu R - c_i}, & \text{if } \mu R - c_i > 0 \\ 0, & \text{if } \mu R - c_i = 0. \end{cases}$$

We have $\breve{w} \le \hat{w}$, and the following defined w^*_{i-1} solves the right-hand-side optimization problem in (85),

$$w^*_{i-1} := \begin{cases} \breve{w}, & \text{if } w_i \le \breve{w} - \bar{u}\delta, \\ w_i + \bar{u}\delta, & \text{if } w_i \in (\breve{w} - \bar{u}\delta,\ \hat{w} - \bar{u}\delta], \\ \hat{w}, & \text{if } w_i \in (\hat{w} - \bar{u}\delta,\ \hat{w}], \\ w_i, & \text{if } w_i > \hat{w}. \end{cases}$$

Concavity of $\mathfrak{J}_i(w|w_{M+1})$ follows from an induction proof showing that the objective of the maximization in (85) is jointly concave in w_i and w_{i-1}. This concavity property is crucial for us to obtain the closed-form optimal solution w^*.

Proof. First, we show that $\mathfrak{J}_i(w|w_{M+1})$ is concave in w by induction. $\mathfrak{J}_1(w|w_{M+1})=0$ is clearly concave in w (follows from the concavity of $\mathcal{J}$) . Next, if $\mathfrak{J}_{i-1}(w|w_N)$ is concave in w, we verify that $\mathfrak{J}_i(w|w_{M+1})$ is also concave in w.

Denote

$$f(w_{i-1},w_i):=P_i\min\left\{\frac{w_{i-1}-w_i}{c_i-c_{i-1}}(\mu R-c_i)-w_i,\ \mathcal{J}\left(w_i,\min\left\{\frac{w_{M+1}}{c_i},\frac{1}{r}\right\},c_i\right)\right\}+\mathfrak{J}_{i-1}(w_{i-1}|w_{M+1}).$$

Since $\dfrac{w_{i-1}-w_i}{c_i-c_{i-1}}(\mu R-c_i)-w_i$ is linear in (w_{i-1},w_i) and $\mathcal{J}\left(w_i,\min\left\{\dfrac{w_{M+1}}{c_i},\dfrac{1}{r}\right\},c_i\right)$ is concave in w_i (follows Lemma 22), then $f(w_{i-1},w_i)$ is jointly concave in (w_{i-1},w_i). Hence, $\mathfrak{J}_i(w_i|w_N)$ is concave in w_i.

Since $\mathfrak{J}_{i-1}(w_{i-1}|w_N)$ is concave in w_{i-1}, $\check{w}$ and $\hat{w}$ are well-defined and $\check{w}\le\hat{w}$. Next, we verify the optimal solution in the following 3 cases. Further, following Lemma 22 (iii) and $c_i<\mu R$, we have

Case 1. If $w_i\le\check{w}-\bar{u}\delta$, then we verify that $w^*_{i-1}=\check{w}$. If $w_{i-1}\ge\check{w}$, then $w_{i-1}\ge\check{w}\ge w_i+\bar{u}\delta$. Hence

$$f(w_{i-1},w_i)=P_i\mathcal{J}\left(w_i,\min\left\{\frac{w_{M+1}}{c_i},\frac{1}{r}\right\},c_i\right)+\mathfrak{J}_{i-1}(w_{i-1}|w_{M+1})$$

and

$$f_1'(w_{i-1},w_i)=\mathfrak{J}'_{i-1}(w_{i-1}|w_{M+1})\le 0$$

for $w_{i-1}\ge\check{w}$, where the last inequality follows from the definition of $\check{w}$. If $w_{i-1}<\check{w}$, then

$$f_1'(w_{i-1},w_i)=\begin{cases}P_i(\mu R-c_i)+\mathfrak{J}'_{i-1}(w_{i-1}|w_{M+1})>0, & \text{if } w_{i-1}\le w_i+\bar{u}\delta,\\ \mathfrak{J}'_{i-1}(w_{i-1}|w_{M+1})\ge 0, & \text{if } w_i\in(w_i+\bar{u}\delta,\ \check{w}].\end{cases}$$

where the second inequality follows from the definition of $\check{w}$. Hence, $f(w_{i-1},w_i)$ is increasing in w_{i-1} if $w_{i-1}<\check{w}$ and decreasing in w_{i-1} if $w_{i-1}\ge\check{w}$which imply that $w^*_{i-1}=\check{w}$.

Case 2. If $w_i\in(\check{w}-\bar{u}\delta,\hat{w}-\bar{u}\delta]$, then we verify that $w^*_{i-1}=w_i+\bar{u}\delta$. For $w_{i-1}<w_i+\bar{u}\delta$,

$$f_1'(w_{i-1},w_i)=P_i(\mu R-c_i)+\mathfrak{J}'_{i-1}(w_{i-1}|w_{M+1})\ge 0,$$

where the inequality follows from $w_{i-1}<w_i+\bar{u}\delta\le\hat{w}$. Further, for $w_{i-1}>\check{w}+\bar{u}\delta$,

$$f_1'(w_{i-1},w_i)=\mathfrak{J}'_{i-1}(w_{i-1}|w_{M+1})<0,$$

where the inequality follows from $w_{i-1}>w_i+\bar{u}\delta>\check{w}$.

Case 3. If $w_i\in(\hat{w}-\bar{u}\delta,\hat{w}]$, then we verify that $w^*_{i-1}=\hat{w}$. For $w_{i-1}<\hat{w}<w_i+\bar{u}\delta$, we have

$$f_1'(w_{i-1},w_i)=P_i(\mu R-c_i)+\mathfrak{J}'_{i-1}(w_{i-1}|w_N)\ge 0,$$

where the inequality follows from $w_{i-1}<\hat{w}$. And for $w_{i-1}>\hat{w}$, then

$$f_1'(w_{i-1},w_i)=\begin{cases}P_i(\mu R-c_i)+\mathfrak{J}'_{i-1}(w_{i-1}|w_N)\le 0, & \text{if } w_{i-1}\in(\hat{w},w_i+\bar{u}\delta],\\ \mathfrak{J}'_{i-1}(w_{i-1}|w_N)\le 0, & \text{if } w_{i-1}>w_i+\bar{u}\delta.\end{cases} \tag{219}$$

where the first inequality follows from $w_{i-1}>\hat{w}$. Hence, $f(w_{i-1},w_i)$ is increasing in w_{i-1} if $w_{i-1}<\hat{w}$ and decreasing in w_{i-1} if $w_{i-1}\ge\hat{w}$ which imply that $w^*_{i-1}=\hat{w}$.

Case 4. If $w_i>\hat{w}$, we verify that $w^*_{i-1}=w_i$. Following (219), we have $f(w_{i-1},w_i)$ is decreasing in w_{i-1} for $w_{i-1}\ge w_i$. Hence, $w^*_{i-1}=w_i$. Q.E.D.

Finally, we have the following result, which provides an upper bound for the optimal w_N.

PROPOSITION 16. *Define* $\bar{\mathrm{w}}:=\min\{\mu R-c_1,c_N\}/r$. *For any* $w_{M+1}\ge\bar{\mathrm{w}}, w_M\ge w_{M+1}$, *we have* $\mathfrak{J}_M(w_M|w_{M+1})\le\max_{w_M\ge w_{M+1},w_{M+1}\le\bar{\mathrm{w}}}\mathfrak{J}_M(w_M|w_{M+1})$.

Proposition 16 implies that we can focus the search for the optimal w_{M+1} that solves (86) in the interval $[0, \bar{\mathrm{w}}]$.

Proof. First, if $w_{M+1} \geq \dfrac{\mu R - c_1}{r}$, then

$$\begin{aligned}\mathfrak{J}_N(w_N|w_N) &= \sum_{i=1}^{N} P_i \min\left\{\frac{w_{i-1}-w_i}{c_i-c_{i-1}}(\mu R - c_i) - w_i,\ \mathcal{J}\left(w_i, \min\left\{\frac{w_{M+1}}{c_i}, \frac{1}{r}\right\}, c_i\right)\right\} \\ &\leq \sum_{i=1}^{N} P_i \mathcal{J}\left(w_i, \min\left\{\frac{w_{M+1}}{c_i}, \frac{1}{r}\right\}, c_i\right) \\ &\leq \sum_{i=1}^{N} P_i \left[\frac{\mu R - c_i}{r} - w_i\right] \leq 0 \leq \mathcal{J}_N(0|0)\end{aligned}$$

where the last inequality follows from $w_i \geq w_{M+1} \geq \dfrac{\mu R - c_1}{r}$.

On the other hand, for any $w_{M+1} > c_N/r$, then denote the corresponding optimal solution as $\{w_i^*\}_{i=1,\dots,M+1}$. Since $w_{M+1} > c_N/r$, we have $w_i^* > c_N/r \geq c_i/r$ for $i = 1, \dots, M+1$. Hence, $\min\{w_{M+1}/c_i, 1/r\} = 1/r$ for $i = 1, \dots, N$. Following (157), we have

$$\frac{\partial \mathcal{J}(w, 1/r, c)}{\partial w} = -1, \tag{220}$$

if $w \geq \hat{w}(\infty) = c/r$ and $w \geq \mu c/r$. Hence,

$$\mathcal{J}(w, 1/r, c_i) \leq \mathcal{J}(\bar{c}/r, 1/r, c) \tag{221}$$

for any $i \in \{1, \dots, M+1\}$ and $w > \bar{c}/r$. Define $\{\tilde{w}_i\}_{i=1,\dots,M+1}$ as

$$\tilde{w}_i = w_i^* - (w_{M+1}^* - \bar{c}/r).$$

Hence, if $w_N \geq \bar{c}/r$,

$$\begin{aligned}\mathfrak{J}_M(w_M|w_{M+1}) &= \sum_{i=1}^{N} P_i \min\left\{\frac{w_{i-1}^*-w_i^*}{c_i-c_{i-1}}(\mu R - c_i) - w_i^*,\ \mathcal{J}\left(w_i^*, \min\left\{\frac{w_{M+1}^*}{c_i}, \frac{1}{r}\right\}, c_i\right)\right\} \\ &\leq \sum_{i=1}^{N} [P(c_i) - P(c_{i-1})] \min\left\{\frac{\tilde{w}_{i-1}-\tilde{w}_i}{\delta}(\mu R - c_i) - \tilde{w}_i,\ \mathcal{J}\left(\tilde{w}_i, \min\left\{\frac{\tilde{w}_{M+1}}{c_i}, \frac{1}{r}\right\}, c_i\right)\right\} \\ &\leq \mathcal{J}_M(\tilde{w}_M|\tilde{w}_{M+1}) = \mathcal{J}_M(\tilde{w}_M|c_N/r)\end{aligned}$$

where the first inequality follows from (221). Therefore, we have

$$\mathfrak{J}_M(w_M|w_{M+1}) \leq \max_{w_M \geq w_{M+1}, w_{M+1} \leq \bar{\mathrm{w}}} \mathfrak{J}_M(w_M|w_{M+1}).$$

if $w_{M+1} \geq \bar{\mathrm{w}}, w_M \geq w_{M+1}$. Q.E.D.

E.11. Preparations to prove Lemma 10

Proposition 17. *For $i \in \{2\dots,M\}$, we have $\bar{\nu} \in \mathfrak{N}(\gamma_{\mathrm{r}}^{c_i}(w_i^*, \tau_i, p_0^i, p_{\hat{w}}^i, c_{i+1}), c_i)$, where τ_i, p_0^i, p_{W}^i are defined in (100) - (102), and*

$$u\Big(\gamma_{\mathrm{r}}^{c_i}(w_i^*, \tau_i, p_0^i, p_{\hat{w}}^i, c_{i+1}), \bar{\nu}; c\Big) = w_i^*,$$

$$U\Big(\gamma_{\mathrm{r}}^{c_i}(w_i^*, \tau_i, p_0^i, p_{\hat{w}}^i, c_{i+1}), \bar{\nu}\Big) + w_i^* = \min\{\mathcal{V}_i^*, \check{\mathcal{V}}_i\},$$

and $p_0^i, p_{\hat{w}}^i \in [0,1]$, and $\hat{w}$ in (99) is well-defined.

Since $c_j \geq c_{j+1}$ and $\beta_{c_j} \geq \beta_{c_{j+1}}$, we have the following results.

Lemma 23. *Given $\{w_i^*\}_{i=1,\dots,M+1}$ defined in (95), we have for any $i < j, j \in \{2, \dots, N\}$, $\nu^0 \in \mathfrak{N}(\gamma^{c_i}, c_j)$ and $\nu^0 \in \mathfrak{N}(\gamma^{c_1}, c_j)$, where γ^{c_i}s are defined in Lemma 10. Furthermore, for $i < j$ and $i \in \{1, \dots, M\}$, we have*

$$u(\gamma^{c_i}, \nu^0; c_j) \leq c_i \bar{\tau}_i \tag{222}$$

Lemma 24. *For any $b > g$, and γ^b such that $\bar{\nu} \in \mathfrak{N}(\gamma^b, b)$, we have $\bar{\nu} \in \mathfrak{N}(\gamma^b, g)$.*

E.11.1. Proof of Proposition 17 First, we verify $\bar{\nu} \in \mathfrak{N}(\gamma_{\mathtt{r}}^{c_i}(w_i^*, \tau_i, p_0^i, p_{\hat{w}}^i, c_{i+1}), c_i)$. With probability p_0^i, the contract is directly terminated. With probability $p_{\hat{w}}^i$, the contract continues with a delay payment contract $\gamma_{\mathtt{D}}^{c_i}$. Then, following the proof of Proposition 17 (ii), we can show that type c_i will always exert effort. With probability $1 - p_0^i - p_{\hat{w}}^i$, the contract continues with $\gamma_{\mathtt{P}'}^{c}(w_i^*, \tau_i, c')$. Following Proposition 12, we have type c_i that will always exert effort.

Second, we verify $u\Big(\gamma_{\mathtt{r}}^{c_i}(w_i^*, \tau_i, p_0^i, p_{\hat{w}}^i, c_{i+1}), \bar{\nu}; c\Big) = w_i^*$. Following Proposition 12, we have $u\Big(\gamma_{\mathtt{P}'}^{g}(w_i^*, \tau_i, c_{i+1}), \bar{\nu}; c_i\Big) = w_i^*$. Hence, if $\mathcal{V}_i^* \geq \check{\mathcal{V}}_i$, then $u\Big(\gamma_{\mathtt{r}}^{c_i}(w_i^*, \tau_i, p_0^i, p_{\hat{w}}^i, c_{i+1}), \bar{\nu}; c\Big) = u\Big(\gamma_{\mathtt{P}'}^{g}(w_i^*, \tau_i, c_{i+1}), \bar{\nu}; c_i\Big) = w_i^*$. On the other hand, if $\mathcal{V}_i^* < \check{\mathcal{V}}_i$, then

$$u\Big(\gamma_{\mathtt{r}}^{c_i}(w_i^*, \tau_i, p_0^i, p_{\hat{w}}^i, c_{i+1}), \bar{\nu}; c\Big) = p_0^i \cdot 0 + p_{\hat{w}}^i \cdot \hat{w} + (1 - p_0^i - p_{\hat{\mathrm{W}}}^i) u\Big(\gamma_{\mathtt{P}'}^{g}(w_i^*, \tau_i, c_{i+1}), \bar{\nu}; c_i\Big)$$
$$= (1-p^i) q^i \cdot \hat{w} + (1 - p_0^i - p_{\hat{w}}^i) w_i^* = (1-p^i) q^i \cdot \frac{q^i w_i^*}{(1-p^i) q^i} + (1-q^i) w_i^* = w_i^*$$

where the second equality follows from (101) and (102), the third equality follows from (99).

Third, we verify $U\Big(\gamma_{\mathtt{r}}^{c_i}(w_i^*, \tau_i, p_0^i, p_{\hat{w}}^i, c_{i+1}), \bar{\nu}\Big) + w_i^* = \min\{\mathcal{V}_i^*, \check{\mathcal{V}}_i\}$. Following Proposition 13, definition of $\check{\mathcal{J}}$ in (88), and (98), we have $U(\gamma_{\mathtt{P}'}^{c_i}(w_i^*, \tau_i, c_{i+1}), \bar{\nu}) = \check{\mathcal{V}}_i - w_i^*$. Furthermore, since $p^i \geq 1 - \dfrac{w_i^*}{\max(w_i^*, c_i/r)}$, then $\hat{w} = (p_0^i + p_{\hat{w}}^i) w_i^* / p_{\hat{w}}^i \geq c_i/r$. Hence, following Definition 2, the agent is never terminated under contract $\gamma_{\mathtt{D}}^{c_i}(\hat{\mathrm{W}}_i, S'(\hat{\mathrm{W}}_i), 0)$. Therefore, $U(\gamma_{\mathtt{D}}^{c_i}(\hat{w}, S'(\hat{w}), 0), \bar{\nu}) = (\mu R - c_i)/r - \hat{w}$. Hence, if $\check{\mathcal{V}}_i \leq \mathcal{V}_i^*$, then $p_0^i = p_{\hat{w}}^i = 0$, and

$$U\Big(\gamma_{\mathtt{r}}^{c_i}(w_i^*, \tau_i, p_0^i, p_{\hat{w}}^i, c_{i+1}), \bar{\nu}\Big) + w_i^* = U(\gamma_{\mathtt{P}'}^{c_i}(w_i^*, \tau_i, c_{i+1}), \bar{\nu}) + w_i^* = \check{\mathcal{V}}_i.$$

On the other hand, if $\check{\mathcal{V}}_i > \mathcal{V}_i^*$, then

$$U\Big(\gamma_{\mathtt{r}}^{c_i}(w_i^*, \tau_i, p_0^i, p_{\hat{w}}^i, c_{i+1}), \bar{\nu}\Big) + w_i^* = p_0^i \cdot 0 + p_{\hat{w}}^i \cdot (\mu R - c_i)/r + (1 - p_0^i - p_{\hat{w}}^i) \Big(U\Big(\gamma_{\mathtt{P}'}^{g}(w_i^*, \tau_i, c_{i+1}), \bar{\nu}\Big) + w_i^*\Big)$$
$$= (1-p^i) q^i \cdot (\mu R - c_i)/r + (1-q^i) \check{\mathcal{V}}_i$$
$$= (1-p^i) \frac{\check{\mathcal{V}}_i - \mathcal{V}_i^*}{\check{\mathcal{V}}_i - (1-p^i)(\mu R - c_i)/r} \cdot (\mu R - c_i)/r + \left(1 - \frac{\check{\mathcal{V}}_i - \mathcal{V}_i^*}{\check{\mathcal{V}}_i - (1-p^i)(\mu R - c_i)/r}\right) \check{\mathcal{V}}_i = \check{\mathcal{V}}_i - (\check{\mathcal{V}}_i - \mathcal{V}_i^*) = \mathcal{V}_i^*.$$

Fourth, we verify that $p_0^i \in [0,1]$, $p_{\hat{w}}^i \in [0,1]$, and $\hat{w}$ is well-defined. If $\check{\mathcal{V}}_i \leq \mathcal{V}_i^*$, then $p_0^i = p_{\hat{w}}^i = 0$. On the other hand, if $\check{\mathcal{V}}_i > \mathcal{V}_i^*$, clearly, the statements are followed from $p^i \in (0,1]$ and $q^i \in [0,1]$. Since $U(\gamma_{\mathtt{P}'}^{c_i}(w_i^*, \tau_i, c_{i+1}), \bar{\nu}) + w_i^* = \check{\mathcal{V}}_i$, we have $\mathcal{V}_i^* < \check{\mathcal{V}}_i \leq (\mu R - c_i)/r$. Hence,

$$1 - \frac{r\mathcal{V}_i^*}{2(\mu R - c_i)} \geq 1 - 1/2 > 0$$

Furthermore, since $w_{i-1}^* \geq w_i^*$, we have $1 - \frac{r\mathcal{V}_i^*}{2(\mu R - c_i)} \leq 1$. By definition of $\bar{\tau}_i$ in (100), we have $1 - r\bar{\tau}_i \in [0,1]$. Further, $1 - \dfrac{w_i^*}{\max(w_i^*, c_i/r)} \in [0,1]$. Hence, $p^i \in (0,1]$. Since $p^i \geq 1 - \frac{r\mathcal{V}_i^*}{2(\mu R - c_i)}$, we have $(1-p^i)(\mu R - c_i)/r \leq \mathcal{V}_i^*/2$ and

$$q^i = \frac{\check{\mathcal{V}}_i - \mathcal{V}_i^*}{\check{\mathcal{V}}_i - (1-p^i)(\mu R - c_i)/r} \in [0,1].$$

E.11.2. Proof of Lemma 23 First, following Lemma 20, we have $\nu^0 \in \mathfrak{N}(\gamma^{c_1}, c_2)$. Further, since $\beta_{c_j} \geq \beta_{c_2} (j \geq 2)$, following the same logic of the proof of Lemma 20, we can show that $\nu^0 \in \mathfrak{N}(\gamma^{c_1}, c_j)$. Following definition 6, for $j > 1$, we have

$$u(\gamma^{c_1}, \nu^0; c_j) = u(\gamma_{\mathtt{P}'}^{c_1}(w_1^*, \tau_1, c_2), \nu^0; c_j) = c_i \bar{\tau}_1$$

where $\bar{\tau}_1$ is defined in (97).

Second, given $i < j$ and $i \in \{1, ..., M\}$, we have $\nu^0 \in \mathfrak{N}(\gamma_{\mathtt{P}'}^{g}(w_i^*, \tau_i, c_{i+1}), c_j)$. Next, following Lemma 15, since $S'(\hat{w})$ satisfies inequality (177) with $g = c_i$, $b = c_j$, we have $\nu^0 \in \mathfrak{N}(\gamma_{\mathtt{D}}^{c_i}(\hat{w}, S'(\hat{w}), 0), c_j)$. Therefore, $\nu^0 \in \mathfrak{N}(\gamma^{c_i}, c_j)$. Following definition 6, for $j > i$, we have $u(\gamma_{\mathtt{P}'}^{c_i}(w_i^*, \tau_i, c_{i+1}), \nu^0; c_j) = c_i \bar{\tau}_1$. Following Definition 2, the agent is never terminated under contract $\gamma_{\mathtt{D}}^{c_i}(\hat{w}, S'(\hat{w}), 0)$. Therefore, $u(\gamma_{\mathtt{D}}^{c_i}(\hat{w}, S'(\hat{w}), 0), \nu^0; c_j) = c_i/r$. Hence, if $\check{\mathcal{V}}_i \leq \mathcal{V}_i^*$, then $p_0^i = p_{\hat{w}}^i = 0$, and

$$u(\gamma^{c_i}, \nu^0; c_j) = u(\gamma_{\mathtt{P}'}^{c_i}(w_i^*, \tau_i, c_{i+1}), \nu^0; c_j) = c_i \bar{\tau}_i$$

On the other hand, if $\check{\mathcal{V}}_i > \mathcal{V}_i^*$, then

$$u\Big(\gamma_{\mathtt{r}}^{c_i}(w_i^*, \tau_i, p_0^i, p_{\hat{w}}^i, c_{i+1}), \nu^0; c_j\Big) = p_0^i \cdot 0 + p_{\hat{w}}^i \cdot c_i/r + (1 - p_0^i - p_{\hat{w}}^i) u\Big(\gamma_{\mathtt{P}'}^{g}\left(w_i^*, \tau_i, c_{i+1}\right), \bar{\nu}; c_i\Big)$$
$$= (1-p^i) q^i \cdot c_i/r + (1-q^i) c_i \bar{\tau}_i \leq c_i \bar{\tau}_i,$$

where the last inequality follows from $p^i \geq 1 - r\bar{\tau}_i$.

Third, given $i < j$ and $i \in \{M+1, ..., N-1\}$, since $\gamma^{c_i} = \gamma_{\mathtt{B}}^{c_i}(0, w_{M+1}^*)$, i.e., the contract is immediately terminated, we have $\nu^0 \in \mathfrak{N}(\gamma^{c_i}, c_j)$. Q.E.D.

E.11.3. Proof of Lemma 24 Define the bad agent's lifetime expected utility, evaluated conditionally upon the information available at time t under contract $\gamma^b = (L^b, \eta^b)$ and effort process $\bar{\nu}$ as u_t^b, then

$$u_t^b = \mathbb{E}^{\bar{\nu}} \left[\int_0^{\eta^b} e^{-rs} (\mathrm{d}L_s^b - b\mathrm{d}s) \middle| \mathcal{F}_t^N \right] = u_0^b + \int_0^t H_s^b dM_s^{\bar{\nu}}$$

where $M_t^{\bar{\nu}} = N_t - \mu t$ and $H_s^b \geq \beta_b$ for any s. Define good agent's lifetime expected utility, evaluated conditionally upon the information available at time t under contract γ^b and effort process $\bar{\nu}$ as u_t^g, then

$$u_t^g = \mathbb{E}^{\bar{\nu}} \left[\int_0^{\eta^b} e^{-rs} (\mathrm{d}L_s^b + (b-g)\mathrm{d}s) \middle| \mathcal{F}_t^N \right] = u_t^b + \mathbb{E}^{\bar{\nu}} \left[\int_0^{\eta^b} e^{-rs} (b-g)\mathrm{d}s \middle| \mathcal{F}_t^N \right]$$
$$= u_0^b + \int_0^t H_s^b dM_s^{\bar{\nu}} + \mathbb{E}^{\bar{\nu}} \left[\int_0^{\eta^b} e^{-rs} (b-g)\mathrm{d}s \middle| \mathcal{F}_t^N \right].$$

Next, we denote $u_t^{g'}$ as the good agent's lifetime expected payoff, given the information available at time t, when he acts according to $\nu' = \{\nu_t'\}_{t\geq 0}$ until time t and then reverts to $\bar{\nu}$, then

$$u_t^{g'} = u_t^g + \int_0^{t\wedge\eta-} e^{-rs}(1 - \mathbb{1}_{\nu_s'=\mu}) g ds$$
$$= u_0^b + \int_0^{t\wedge\eta-} H_s^b dM_s^{\bar{\nu}} + \mathbb{E}^{\bar{\nu}} \left[\int_0^{\eta^b} e^{-rs} (b-g)\mathrm{d}s \middle| \mathcal{F}_t^N \right] + \int_0^{t\wedge\eta-} e^{-rs} (g - \nu_s' \beta_g) ds$$
$$= u_0^b + \int_0^{t\wedge\eta^b-} H_s^b dM_s^{\nu'} + \mathbb{E}^{\bar{\nu}} \left[\int_0^{\eta^b} e^{-rs} (b-g)\mathrm{d}s \middle| \mathcal{F}_t^N \right] + \int_0^{t\wedge\eta^b-} e^{-rs} (\mu - \nu_s')(\beta_g - H_s^b) ds$$

Then, for any $t' > t$,

$$\mathbb{E}^{\nu'}[u_{t'}^{g'} | \mathcal{F}_t^N] = \mathbb{E}^{\nu'} \left[u_0^b + \int_0^{t'\wedge\eta^b-} H_s^b dM_s^{\nu'} + \mathbb{E}^{\bar{\nu}} \left[\int_0^{\eta^b} e^{-rs}(b-g)\mathrm{d}s \middle| \mathcal{F}_{t'}^N \right] + \int_0^{t'\wedge\eta^b-} e^{-rs}(\mu - \nu_s')(\beta_g - H_s^b) ds \middle| \mathcal{F}_t^N \right]$$
$$= u_0^b + \int_0^{t\wedge\eta^b-} H_s^b dM_s^{\nu'} + \mathbb{E}^{\bar{\nu}} \left[\int_0^{\eta^b} e^{-rs}(b-g)\mathrm{d}s \middle| \mathcal{F}_t^N \right] + \mathbb{E}^{\nu'} \left[\int_0^{t'\wedge\eta^b-} e^{-rs}(\mu - \nu_s')(\beta_g - H_s^b) ds \middle| \mathcal{F}_t^N \right]$$
$$\leq u_0^b + \int_0^{t\wedge\eta^b-} H_s^b dM_s^{\nu'} + \mathbb{E}^{\bar{\nu}} \left[\int_0^{\eta^b} e^{-rs}(b-g)\mathrm{d}s \middle| \mathcal{F}_t^N \right] + \int_0^{t\wedge\eta^b-} e^{-rs}(\mu - \nu_s')(\beta_g - H_s^b) ds = u_t^{g'}$$

where the second equality follows from the law of iterated expectation and the first inequality follows from that $(\mu - \nu_s')(\beta_g - H_s^b) \leq 0, \forall t$. Hence, $u_t^{g'}$ is $\mathcal{F}^N$-supermartingale under $P^{\nu'}$. Therefore, by the optional sampling theorem (Dellacherie and Meyer (2011), Chapter VI, Theorem 10),

$$u(\gamma^b, \bar{\nu}; g) = u_0^{g'} \geq \mathbb{E}^{\nu'}[u_\eta^{g'}] = u(\gamma^b, \nu'; g).$$

which implies that $\bar{\nu}$ is at least as good as ν' for the agent.

E.12. Proof of Lemma 10

First, following Proposition 12, we have $\bar{\nu} \in \mathfrak{N}(\gamma^{c_1}, c_1)$ and

$$u\left(\gamma^{c_1}, \bar{\nu}; c_1\right) = w_1^*.$$

Second, following Proposition 17, we have $\bar{\nu} \in \mathfrak{N}(\gamma^{c_i}, c_i)$ and

$$u\left(\gamma^{c_i}, \bar{\nu}; c_i\right) = w_i^*.$$

for $i \in \{2, ..., M\}$. Next, following definition 3, for type c_i agent where $i \in \{M+1, ...N\}$, the agent is terminated at time 0. Hence, The menu of contracts $\hat{\Gamma}_{\mathcal{C}}$ satisfies (LL), (PK), (IC), (IR), and (FE). In the following, we verify (TT). Conditions (96) implies that for any $j < i, i \in \{2, ..., M\}$,

$$\frac{w_j^* - w_i^*}{c_i - c_j}(\mu R - c_i) - w_i \geq \frac{w_{i-1}^* - w_i^*}{c_i - c_{i-1}}(\mu R - c_i) - w_i^* \geq \check{\xi}_i. \tag{223}$$

where $\check{\xi}_i$ is defined as

$$\check{\xi}_i := \min\left\{\frac{w_{i-1}^* - w_i^*}{c_i - c_{i-1}}(\mu R - c_i) - w_i^*,\ \check{\mathcal{J}}\left(w_i^*, \min\left\{\frac{w_{M+1}^*}{c_i}, \frac{1}{r}\right\},\ c_i, c_{i+1}\right)\right\}. \tag{224}$$

Following Proposition 17, we have that the societal value

$$S(\gamma^{c_i}, \bar{\nu}; c_i) = \check{\xi}_i + w_i^*. \tag{225}$$

We first prove that type c_1 will not mimic any other types. For type c_1, and any type c_i, $i \in \{2, ..., M\}$,

$$\begin{aligned} u(\gamma^{c_1}, \bar{\nu}; c_1) = w_1^* &\geq w_i + (c_i - c_1)\frac{\check{\xi}_i + w_i^*}{\mu R - c_i} = w_i^* + (c_i - c_1)\bar{T}(\gamma^{c_i}, \bar{\nu}) \\ &= u(\gamma^{c_i}, \bar{\nu}; c_1) = \max_{\nu} u(\gamma^{c_i}, \nu; c_1) \end{aligned}$$

where the first inequality follows from (223) by letting $j = 1$, the third equality follows from (225) and the last equality follows from Lemma 24. For type c_1, and any type c_i, $i \in \{M+1, ..., N\}$,

$$u(\gamma^{c_1}, \bar{\nu}; c_1) = w_1^* \geq w_{M+1}^* = \gamma_{\mathtt{B}}^{c_i}(0, w_{M+1}^*), \bar{\nu}; c_1) = \max_{\nu} u(\gamma_{\mathtt{B}}^{c_i}(0, w_{M+1}^*), \nu; c_1)$$

where the inequality follows from (96). Second, we prove that type $c_i, i \in \{1, ..., M\}$ will not mimic any other types. For type c_i, and any type $c_j, j \in \{i+1, ..., M\}$

$$\begin{aligned} u(\gamma^{c_i}, \bar{\nu}; c_i) = w_i &\geq w_j^* + (c_j - c_i)\frac{\check{\xi}_j + w_j^*}{\mu R - c_j} = w_j^* + (c_j - c_i)\bar{T}(\gamma^{c_j}, c_{j+1}), \bar{\nu}) \\ &= u(\gamma^{c_j}, \bar{\nu}; c_i) = \max_{\nu} u(\gamma^{c_j}, \nu; c_i) \end{aligned}$$

where the first inequality follows from (223) by letting $j = i, i = j$, the third equality follows from (225) and the last equality follows from Lemma 24. For type c_i, and any type $c_j, j \in \{1, ..., i-1\}$,

$$\begin{aligned} u(\gamma^{c_i}, \bar{\nu}; c_i) = w_i^* \geq w_{M+1}^* &\geq c_j \min\left\{\frac{w_{M+1}^*}{c_j}, \frac{1}{r}\right\} \geq c_j \bar{\tau}_j \\ &\geq u(\gamma^{c_j}, \nu^0; c_i) = \max_{\nu} u(\gamma^{c_j}, \nu; c_i) \end{aligned}$$

where the first inequality follows from (96), the fourth inequality and the last equality follow from Lemma 23. For type c_i, and any type $c_j, j \in \{M+1, ..., N\}$, we have

$$u(\gamma^{c_i}, \bar{\nu}; c_o) = w_i^* \geq w_{M+1}^* = \gamma_{\mathtt{B}}^{c_i}(0, w_{M+1}^*), \bar{\nu}; c_i) = \max_{\nu} u(\gamma_{\mathtt{B}}^{c_i}(0, w_{M+1}^*), \nu; c_i)$$

where the inequality follows from (96). Finally, we prove that type $c_i, i \in \{M+1, ..., N\}$ will not mimic any other types. For type c_i, and any type $c_j, j \in \{1, ..., M\}$

$$\begin{aligned} u(\gamma_{\mathtt{B}}^{c_i}(0, w_{M+1}^*), \bar{\nu}; c_i) = w_{M+1}^* \geq c_j \min\left\{\frac{w_{M+1}^*}{c_i}, \frac{1}{r}\right\} \geq c_j \bar{\tau}_j \geq u(\gamma^{c_j}, \nu^0; c_i) \\ = \max_{\nu} u(\gamma^{c_j}, \nu; c_i) \end{aligned}$$

where the first inequality follows from (96), the fourth inequality and the last equality follow from Lemma 23. For type c_i and any other type $c_j, j \in \{M+1, ..., N\}$, we have

$$u(\gamma_{\mathtt{B}}^{c_i}(0, w_{M+1}^*), \bar{\nu}; c_i) = w_{M+1}^* = u(\gamma_{\mathtt{B}}^{c_j}(0, w_{M+1}^*), \bar{\nu}; c_i).$$

Therefore, we have verified (TT) and completed the proof. Q.E.D.

E.13. Proof of Proposition 9

First, following the dynamic programming formulation in (89) - (94) and optimal solutions $(w_1^*, ..., w_{M+1}^*)$ defined in (95), we have

$$\check{\mathcal{Y}}^N = P_1 \cdot \check{\mathcal{J}}\left(w_1^*, \min\left\{\frac{w_{M+1}^*}{c_1}, \frac{1}{r}\right\},\ c_1, c_2\right) + \sum_{i=1}^{M} P_i \cdot \check{\xi}_i - w_{M+1}^* \sum_{i=M+1}^{N} P_i,$$

where $\check{\xi}_i$ is defined in (224). Second, following Lemma 10, we have the menu of contracts $\hat{\Gamma}_{\mathcal{C}}$ satisfies (LL), (PK), (IC), (IR), (FE) and (TT). First, it is straightforward to see that for type $c_i, i \in \{M+1, ..., N\}$, the contracts are directly terminated, hence,

$$U(\gamma^{c_i}, \bar{\nu}) = -w_{M+1}^*. \tag{226}$$

Second, for type c_1, following Proposition 13, we have

$$U(\gamma^{c_1}, \bar{\nu}) = \check{\mathcal{J}}\left(w_1^*, \min\left\{\frac{w_{M+1}^*}{c_1}, \frac{1}{r}\right\},\ c_1, c_2\right). \tag{227}$$

Third, for type $c_i, i \in \{2, ..., M\}$, following Proposition 17, we have

$$U(\gamma^{c_i}, \bar{\nu}) = \min\left\{\frac{w_{i-1}^* - w_i^*}{c_i - c_{i-1}}(\mu R - c_i) - w_i^*, \check{\mathcal{J}}\left(w_i^*, \min\left\{\frac{w_{M+1}^*}{c_i}, \frac{1}{r}\right\}, c_i, c_{i+1}\right)\right\} = \check{\xi}_i \tag{228}$$

where $\check{\xi}_i$ is defined in (224). Therefore, following (226)-(228), we have

$$\mathcal{U}(\hat{\Gamma}_{\mathcal{C}}) = \sum_{i=1}^{N} P_i \cdot U(\gamma^{c_i}, \bar{\nu}) = P_1 \cdot \check{\mathcal{J}}\left(w_1^*, \min\left\{\frac{w_{M+1}^*}{c_1}, \frac{1}{r}\right\},\ c_1, c_2\right) + \sum_{i=1}^{M} P_i \cdot \check{\xi}_i - w_{M+1}^* \sum_{i=M+1}^{N} P_i,$$

i.e., $\check{\mathcal{Y}}^N = \mathcal{U}\left(\hat{\Gamma}_{\mathcal{C}}\right)$, which completes the proof. Q.E.D.