Wait Time Based Pricing for Queues with Customer-Chosen Service Times

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This paper studies a pricing problem for a single-server queue where customers arrive according to a Poisson process. For each arriving customer, the service provider announces a price rate and a system wait time, and the customer decides whether to join the queue, and, if so, the duration of the service time. The objective is to maximize either the long-run average revenue or social welfare. We formulate this problem as a continuous-time control model whose optimality conditions involve solving a set of delay differential equations. We develop an innovative method to obtain the optimal control policy, whose structure reveals interesting insights. The optimal dynamic price rate policy is not monotonic in wait time. In particular, in addition to the congestion effect often reported in the literature, i.e., the optimal price rate increases in the queue length (measured by the wait time in our setting), we find a compensation effect, meaning that the service provider should lower the price rate when the wait time is longer than a threshold. Compared with the prevalent flat pricing policy, our optimal dynamic pricing policy improves the objective value through admission control, which, in turn, increases the utilization of the server. We use a real data set obtained from a public charging station to calibrate our model with an objective of maximizing the average revenue. We find that our optimal pricing policy outperforms the best flat pricing policy, especially when the arrival rate is low and drivers are impatient. Interestingly, our revenue-maximizing pricing policy also improves social welfare over the flat pricing policy in most of the tested cases.

Key words: dynamic pricing, queueing, wait time, service system, electric vehicle, charging station

1. Introduction

We consider a single-server system in which customers arrive according to a Poisson process. The service provider (either a private firm or a public entity) announces the current wait time (the time between when a customer enters the system and when he starts receiving the service) and the corresponding price rate ($ per unit time in service) to each arriving customer. In response to the announced wait time and price rate, the customer first decides whether to join the queue, and, if so, the duration of service that maximizes his utility. The provider can learn the service time for each customer joining the queue, and, therefore, knows the total wait time to announce to the next arriving customer. We study congestion-dependent, linear pricing policies in this paper.

Specifically, the service provider sets a price rate for per unit of service time, which depends on the system wait time, to maximize either the average revenue or social welfare. We focus on linear pricing because of its simplicity. More sophisticated nonlinear pricing schemes certainly allows the service provider to extract more consumer surplus. Nevertheless, complex nonlinear pricing schemes may not be practical.

The proposed model can be applied to any service system whose customers are sensitive to price and wait time, and allowed to determine service times based on a price rate, e.g., beauty services, delivery services, etc. One focal application of this paper is public fast-charging stations for electric vehicles (EVs). In this context, customers are EV drivers and service time corresponds to each driver's charging time.

The development of fast-charging stations has drawn considerable attention from both the public and private sectors as it is an enabler to promote electric vehicle adoption. Although regular at-home charging remains the most common way of charging EVs, it does not fulfill all charging needs. It is often necessary to install fast-charging stations at work or in public to extend travel range and increase charging access to those drivers without home charging. Nevertheless, according to Engel et al. (2018), one barrier to the development of public fast-charging stations is financial viability. Installing fast-charging stations requires a considerable upfront investment. The prevalent flat-rate (static) pricing policy does not respond to demand variability, and therefore may not generate sufficient returns to justify the investment. A dynamic pricing policy, on the other hand, allows a charging station to change the price with regards to the congestion level. For example, when congestion starts to increase, raising the price rate induces each arriving EV to shorten the charging time, which may allow the station to accept more drivers and thereby collect more revenue. In view of public interests, it is also important to understand whether a pricing policy improves not only revenue, but also social welfare. We shall address this question in a numerical study.

We assume that the service provider knows the service time when each arriving customer decides to enter the queue. This can be implemented in traditional services by asking customers to choose from a menu of service times depending on the set prices. Mobile terminals make such communications even easier. In the context of fast-charging stations, we note that many mobile apps, such as ChargePoint, are able to show real-time prices. We expect that similar technology can be applied to allow the station to announce the current waiting time and the price rate, and receive the chosen service time from the driver upon entering the queue.

Under our pricing scheme, congestion is measured by wait time, instead of queue length. This appears to be a unique feature, compared with the existing queueing-pricing literature, in which dynamic pricing often depends on the queue length while the wait time is uncertain. In our setting, a customer knows the exact time to wait for the service, which may not be available in the traditional queue-length-based control models under uncertain service times. Knowing the exact wait time may bring additional benefits to customers in practice (e.g., reducing uncertainty and anxiety due to waiting), which are not incorporated in our analysis.

We formulate the problem as a continuous-time optimal control model where the control is a history-dependent price rate process. The optimality (Hamilton-Jacobi-Bellman, or HJB) equation is a set of delay differential equations (DDEs). We are not able to verify the existence of the optimal solution using standard approaches based on the Cauchy-Lipschitz theorem for ordinary differential equations (ODEs). Instead, we devise a shooting-method algorithm, which proves the existence of the optimal value function by construction, and helps characterize the optimal dynamic price rate policy from the optimality equations.

The optimal history-dependent price rate only depends on the wait time, and its structure consists of five sections as the wait time increases. The first section involves a low flat-rate when the system wait time is relatively short. This low rate induces the customer to choose the longest service time permitted under this policy. In the second section, the optimal price increases with the wait time. This suggests that the firm should raise the price rate to induce arriving customers to reduce their chosen service time when the system is getting more congested. This result is consistent with the congestion effect often obtained in the literature (e.g., Low 1974, Ata and Shneorson 2006). Interestingly, when the wait time continues to grow, the optimal price rate starts to decrease in the third section. This suggests that when the system becomes even more congested, the firm will induce a customer to join the queue by lowering the price so the increased service time can compensate the customer's substantial waiting cost. We refer to this new phenomenon as the "compensation effect." In the fourth section, the maximum possible service time is reached, and the firm can continue lowering the price to allow arriving customers to join the queue. Finally, when the wait time is excessively long, the firm should set a high price rate to block the arriving customer from entering the queue.

We compare our optimal dynamic pricing policy with the best flat pricing policy under the objective of maximizing the average revenue rate for an EV charging station based on a real data set. We find that our optimal dynamic pricing policy achieves a revenue rate that is 6% higher than the best flat pricing policy on average. The benefit of dynamic pricing over flat pricing is most significant when the arrival rate is low and drivers are impatient. A numerical study suggests that our optimal pricing policy increases the revenue and utilization due to offering a lower (higher, respectively) price, leading to a longer (shorter, respectively) service time when the wait time is short (long, respectively). Thus, our pricing policy helps improve the load shape and capacity utilization of the station. Interestingly, our optimal policy improves not only the station's revenue, but often also social welfare. Specifically, compared with the revenue-maximizing flat pricing policy, the proposed dynamic pricing policy increases social welfare when the arrival rate is low and drivers are impatient, or when the arrival rate is high and drivers are patient. Our findings have important implications for policy makers. In particular, governments should implement a dynamic pricing policy for a charging station when a service area has moderate traffic flows and drivers are impatient, if their goal is to improve both revenue and social welfare.

This paper contributes to the literature in several aspects. First, as mentioned earlier, we propose an innovative dynamic pricing model in a queueing system, in which the pricing decision is based on the wait time rather than the queue length. This pricing approach appears widely applicable to discretionary service systems (Hopp et al. 2007), where firms or consumers can decide their service duration. Second, the optimal pricing policy reveals new insights. In addition to the congestion effect often documented in the literature, we also identify a compensation effect, due to the considered linear pricing policy. The optimal dynamic policy is easy to compute and implement, and outperforms the flat pricing policy. Finally, from a technical perspective, our proof approach to show the existence of a solution to the optimality conditions appears to be new, and may be applicable to other similar optimal control models.

The rest of paper is organized as follows: Section 2 provides a literature review. Section 3 introduces the system and formulates the problem as a continuous-time control model. We then derive the optimality conditions and the corresponding DDEs. Section 4 provides the proof, by construction, that the optimality equations indeed yield the optimal value and an optimal control policy. Section 5 characterizes the structure of the optimal policy. Section 6 conducts a numerical study based on a real data set to compare the optimal pricing policy and the best flat-price policy in terms of revenue and social welfare. Section 7 concludes the paper with comments on our modeling choices, and discusses future research directions.

2. Literature Review

The pricing literature on service systems is extensive and spans a broad range of topics. Our paper studies dynamic pricing to maximize revenue or social welfare for a queueing system where customers can choose their service time and are sensitive to price and wait times. Thus, our review is focused on real-time, dynamic control of queueing systems with utility-maximizing customers. We refer readers to Kim and Randhawa (2018) for an excellent review and discussion on static pricing policies.

Dynamic control on queueing systems has a long history. Early papers focus on controlling either the arrival rate through dynamic pricing or the service rate at the expense of additional costs. Recent papers then combine these two features. Our review groups the literature into the aforementioned three categories.

Controlling arrival process. Papers in this category assume that customers are sensitive to price but not wait time. The firm changes prices dynamically to control the arrival rate because congestion is costly to the firm. Examples include Low (1974) for single-class customers and Paschalidis and Tsitsiklis (2000) for multi-class customers. One key result of Low (1974) is that optimal prices are non-decreasing with respect to the number of customers in the system, which is extended to a setting with multiple classes of customers in Paschalidis and Tsitsiklis (2000). Yoon and Lewis (2004) allow non-stationarity in arrival and service rates, and establish structural properties for the optimal policy. This stream of literature has recently been extended to include learning issues. Afèche and Ata (2013) study dynamic pricing for admission control of different customer classes with unknown proportions. They show that a standard Bayesian learning approach may lead to incomplete learning. We also use dynamic pricing for admission control, similar to this stream of research.

Controlling service process. Our pricing decisions affect not only admission, but also service time on each customer. This is related to queueing control models in which the decision maker can choose either service time or quality. Hopp et al. (2007) consider a queueing model with Poisson arrivals. The service provider can choose the (deterministic) service time for each arriving customer based on the queue length (congestion level). They call such services "discretionary services," and show that the optimal service time decreases with queue length. Our paper can be perceived as using pricing control for discretionary services based on the wait time. Adjusting service rate is another approach to control service processes, which has a longer history in the literature, see, e.g., Crabill et al. (1977), Stidham and Weber (1993), and George and Harrison (2001). These papers consider a speed-congestion trade-off in queuing models where firms can choose service rate to balance congestion and service costs. An insight from these papers is that service rate generally increases with queue length. Alizamir et al. (2013) consider a learning problem in which the service provider decides how many tests to perform, each taking a random time, before committing to a diagnosis. Their optimal policy structure reflects the trade-off between the accuracy of posterior belief on a customer's type and the congestion cost.

Controlling both arrival and service processes. This literature incorporates both admission control through pricing and service rate adjustments. One feature of these papers is that the congestion (or the queue length) affects customers' utility. When the queue length increases, each customer's utility decreases, which further decreases the arrival rate. The decision maker can use both pricing and service rate as two levers to maximize the system objective. Representative papers include Chen and Frank (2001) on revenue maximization, and Ata and Shneorson (2006) on social welfare maximization. Both papers study dynamic pricing for price- and delay-sensitive customers who can observe the queue length. Kim and Randhawa (2018) study a similar system using an asymptotic large system approach, which reveals interesting insights and characterizes the benefit of dynamic pricing. The main difference between our work and these papers is that customers in our model can choose the service time based on the system wait time, whereas in those papers, the decision maker chooses the service rate based on the queue length. In addition, our model reveals that under linear pricing, the optimal price rate may decrease as the system becomes more congested, which appears a new insight.

Our paper is also related to studies of pricing and lead-time quotation, make-to-order systems. Çelik and Maglaras (2008) consider a firm that sets a menu of prices and quoted lead-times to affect the arrival rates of multiple classes of customers. The service time of each class follows a general distribution. The firm controls order sequencing at the server in order to meet the quoted lead-time and prevent expediting costs. The objective is to maximize the average profit rate. Using an approximate diffusion control approach, Çelik and Maglaras obtain near-optimal dynamic pricing, lead-time quotation, order sequencing, and expediting policies. Afeche and Pavlin (2016) extends the model to consider customer choice and strategic delay. The modeling framework of Çelik and Maglaras (2008) is very general, encompassing various features of our model. More specifically, the different service times and the system wait time in our model correspond to the service times of different customer classes and the quoted lead times, respectively, in their model. In addition, they assume prices and quoted lead-times affect the arrival process in a general manner, while our model specifies a utility function to serve the same purpose. Because Çelik and Maglaras consider a very general model, they need to use an approximate solution approach. Our model, on the other hand, allows for an exact analysis, which yields a simple solution algorithm and reveals different insights.

Finally, in terms of the charging station application, there is a different way of managing these stations, by making day-ahead reservations. The day-ahead schedule usually is applied to level-1 or level-2 charging stations. We refer the reader to Wu et al. (2019) for the reservation model.

3. Model and Problem Formulation

We consider a single-server firm where customers arrive according to a Poisson process with an arrival rate λ. The firm announces the current wait time w and the corresponding price rate p ($ per unit time of service) to each arriving customer. The wait time is defined as the time interval from when the customer enters the system to when he starts receiving the service. In response to the announced wait time and price rate, the customer first decides whether to join the queue, and, if so, the duration of service that maximizes his utility. We assume no abandonment after a customer joins the queue, and that the utility function of the customer is public information.

We use t to represent the service time that a customer chooses, which generates a usage utility $U(t)$ to the customer. We assume that the maximum service time is C. In the case of charging stations, C represents the time it takes to fully charge a vehicle. Furthermore, waiting in the queue for w length of time incurs a waiting cost $c(w)$ to the customer. We assume that the usage utility function $U(t)$ is three times differentiable, and the wait time cost function $c(w)$ twice differentiable. These functions satisfy the following conditions.

Assumption 1.
(i) $U(t) \geq 0$, $U(0) = 0$, $0 \leq U'(t) \leq U'(0) < \infty$, *and* $U''(t) < 0$, $\forall t \in [0, C]$,
(ii) $\frac{d^2}{dt^2}[U'(t)t] < 0, \quad \forall t \in [0, C]$,
(iii) $c(0) = 0$, $c'(w) > 0$ *and* $c''(w) \geq 0$, $\forall w \geq 0$.

The first condition in Assumption 1 states that the usage utility is non-negative, concave and increasing in service time, and with finite derivative. In the second condition, the term $U'(t)t$ represents the total revenue (as will be explained later), which is concave in service time. These assumptions are standard in the queueing-pricing literature (see, e.g., Paschalidis and Tsitsiklis 2000, Kim and Randhawa 2018). The third condition states that the wait time cost is increasing and convex.

In the remainder of this section we introduce the model in three subsections. First, we detail customers' decisions in Section 3.1. Based on that, we define the long-run average dynamic optimal control model in Section 3.2. Finally, we present the optimality conditions for the optimal control model in Section 3.3.

3.1. Customers' Decisions

Customers' decisions are based on the following net utility function when an arriving customer faces the wait time w and price rate p.

$$\mathcal{U}(w, p) := \max\left\{\max_{0 \leq t \leq C}\{U(t) - pt - c(w)\}, 0\right\}. \tag{1}$$

Here, the outer maximization represents the customer's joining decision, assuming the customer's outside option has a zero value. The inner maximization determines the service time, given that the customer joins the queue.

One can infer the correspondence between the price rate p and the service time t from (1). For a given wait time w and price rate p, the firm knows that a customer would choose a service time t that maximizes his utility $U(t) - pt - c(w)$. Following the first order condition, the firm can set the price $p = U'(t)$ to induce the customer to choose a service time $t \leq C$ as long as the corresponding customer utility $U(t) - U'(t)t - c(w) \geq 0$. This explains the revenue term $U'(t)t$ in Assumption 1(ii). Furthermore, Assumption 1(i) and (iii) together imply that

$$U(t) - U'(t)t - c(0) > 0, \quad \forall t \in (0, C], \tag{2}$$

which indicates that an arriving customer has an incentive to enter an empty queue as long as the chosen service time t is positive. Finally, the firm can use $p = U'(0)$ to block a customer to enter the queue. Thus, it is sufficient to consider $[0, U'(0)]$ as the feasible set for the price.

If the service time is at the maximum level C and the wait time w is so large that $U(C) - U'(C)C - c(w) < 0$, the firm may still be able to attract a customer to join the queue by setting a price p lower than $U'(C)$, as long as $U(C) - pC - c(w) \geq 0$. The next proposition formally illustrates this scenario.

PROPOSITION 1. *(i) Under Assumption 1, there exists a unique positive threshold $\hat{w}$ such that*

$$U(C) - U'(C)C - c(\hat{w}) = 0. \tag{3}$$

(ii) For any $w \in [0, \hat{w}]$, there is a unique lower bound $L(w) \in [0, C]$, such that

$$U(t) - U'(t)t - c(w) \geq 0, \text{ if and only if } w \leq \hat{w} \text{ and } t \in [L(w), C]. \tag{4}$$

Proposition 1 states that for any wait time $w \leq \hat{w}$, there exists a service time $t \in [0, C]$, such that the firm can use a price $p = U'(t) \in [U'(C), U'(0)]$ to induce a customer to join the queue and choose the service time t. For longer wait time $w > \hat{w}$, however, the firm can no longer use the price $p \in [U'(C), U'(0)]$ to induce a customer to join the queue. Instead, the provider may still use a price $p < U'(C)$ and the maximum service time C to attract a customer to join the queue. This understanding is useful when we discuss the structure of optimal pricing decisions in Section 5, along with the definitions $\hat{w}$ and $L(w)$.

We now summarize the relationship between the service time and the price rate using the following expressions. Define the service time function for a given price p, which solves the inner maximization problem in (1), as

$$\mathtt{t}(p) := \min\left\{ (U')^{-1}(p),\ C \right\}, \text{ for } p \in [0, U'(0)], \tag{5}$$

in which $(U')^{-1}$ is the inverse function of the derivative U', well-defined following Assumption 1(i). It is worth noting that $\mathtt{t}(p)$ is independent of the wait time w. Considering the joining decision, represented in the outer maximization problem in (1), however, the customer's chosen service time depends on both price p and wait time w, which is[1]

$$\mathbf{t}(p, w) := \mathtt{t}(p) \cdot \mathbb{1}_{\{U(\mathtt{t}(p)) - p \cdot \mathtt{t}(p) - c(w) \geq 0\}}. \tag{6}$$

Here the indicator function takes value 1 if the customer joins the queue, and 0 otherwise.

[1] It is worth noting that the "≥" sign in the indicator function implies that an arriving customer who is indifferent between joining the queue or not joining would choose to join. This is to guarantee that in our optimization models the feasible region is compact and, hence, the optimal solution can be attained.

Clearly, the wait time w affects not only the joining decision, but also the service time and the corresponding price rate. For example, if the wait time w is excessively long, a customer would not join the queue even if offered the maximum service time C at price $p=0$. Therefore, the system wait time never exceeds an upper bound $\bar{w}$, and, for any wait time $w \in [0, \bar{w}]$, the price cannot exceed an upper bound $\Gamma(w)$ in order for a customer to joint the queue. Proposition 2 formalizes this argument.

Proposition 2. *(i) Under Assumption 1, there exist a unique positive threshold $\bar{w}$ such that*

$$U(C) = c(\bar{w}). \tag{7}$$

(ii) For any $w \in [0, \bar{w}]$, there exists an upper bound $\Gamma(w) \le U'(0)$, such that $\mathtt{t}(p,w) = 0$ for $p > \Gamma(w)$ and $\mathtt{t}(p,w) = \mathtt{t}(p)$, or, equivalently,

$$U(\mathtt{t}(p)) - p \cdot \mathtt{t}(p) - c(w) \ge 0, \text{ if and only if } p \in [0, \Gamma(w)]. \tag{8}$$

(iii) $\Gamma(0) = U'(0)$ and $\Gamma(w)$ is decreasing in w.

The definition of $\bar{w}$ in (7) specifies the wait time at which a customer is indifferent between joining the queue for the maximum service time C with zero price and not joining, i.e., $U(C) - c(\bar{w}) = 0$. Thus, if the wait time is higher than $\bar{w}$, no price can induce the customer to join the queue. On the other hand, if the wait time is less than $\bar{w}$, there exists a price p to induce a customer to join the queue. Clearly, the maximum acceptable price to induce customers to join the queue depends on the wait time, which is defined as $\Gamma(w)$ in (8). Proposition 2(iii) specifies the intuition that the higher the wait time, the lower a price can the firm charge to attract an incoming customer to join the queue. The upper bound $\Gamma(w)$ is important for us to define the optimal control problem in the following sections. The monotonicity of $\Gamma(w)$ plays an important role when we study the structure of the optimal control policy in Section 5.

3.2. Optimal Control

We first define the dynamics of the wait time process. We use $s \in [0, \infty)$ to represent time epoch. Denote a counting process $\{N_s\}_{s \ge 0}$ to represent the total number of arrivals up to but not including time s, which further generates a filtration $\{\mathcal{F}_s\}_{s \ge 0}$, with $\mathcal{F}_s$ capturing the arriving time epochs up to s. For simplicity, in the sequel we follow conventions in the literature and use the short-hand notation $\mathcal{F}_s$ to represent $\{\mathcal{F}_s\}_{s \ge 0}$. Furthermore, we write $\{\mathcal{X}_s\}$ to represent a generic process $\{\mathcal{X}_s\}_{s \ge 0}$.

Denote an $\mathcal{F}_s$-adapted process $\{w_s\}$ for the wait time, and an $\mathcal{F}_s$-predictable process $\{p_s\}$ for the price rate control. The dynamic of wait time process $\{w_s\}$ under control $\{p_s\}$ is defined as,

$$dw_s = -\mathbb{1}_{\{w_{s-} > 0\}} ds + \mathtt{t}(p_s, w_{s-}) dN_s, \tag{9}$$

where we set the initial wait time $w_{0-} = 0$, and define $w_{s-} := \lim_{\tau \uparrow s} w_\tau$ for $s > 0$. The first term in the right-hand side of (9) indicates that the wait time reduces due to time passing if it has been positive. The second term captures that if a customer joins the queue at this time epoch, the wait time takes an instantaneous upward jump, the magnitude of which equals the service time.

Now we are ready to define the objective of our model. We aim to maximize the long run average rate of either the firm's revenue or the social welfare. The social welfare includes both the firm's revenue and consumer surplus. If we consider the firm's revenue, the expression $p_s \mathbf{t}(p_s, w_{s-})$ represents the total payment generated from a customer arriving at time s, facing wait time w_{s-} and price p_s. If considering the social welfare, on the other hand, payments from customers to the firm are internal transfers, and each arriving costumer generates a social welfare of $[U(\mathbf{t}(p_s, w_{s-})) - c(w_{s-})]\mathbb{1}_{\{\mathbf{t}(p_s,w_{s-})>0\}}$ under price control p_s. The indicator function implies that the social welfare is zero if the price yields a zero service time. Therefore, the optimization formulations to maximize the long run average revenue and social welfare are

$$\sup_{\{p_s\}\in\Pi} \liminf_{T\to\infty} \frac{1}{T}\mathbb{E}\left[\int_0^T p_s \mathbf{t}(p_s, w_{s-}) dN_s\right], \text{ and}$$
$$\sup_{\{p_s\}\in\Pi} \liminf_{T\to\infty} \frac{1}{T}\mathbb{E}\left[\int_0^T [U(\mathbf{t}(p_s, w_{s-})) - c(w_{s-})]\mathbb{1}_{\{\mathbf{t}(p_s,w_{s-})>0\}} dN_s\right], \text{ respectively.}$$

Here the set of admissible policies Π contains all pricing control $\{p_s\}$ such that $p_s \in [0, U'(0)]$ for all $s \geq 0$, and the system wait time w_s follows dynamics (9). Because the service is assumed to be always available, we ignore the cost of providing/maintaining the service in our model.

More generally, we can unify these two cases by defining a function $\mathbf{v}(p)$ and a constant α such that in both cases of revenue and social welfare maximization, the objective for each customer entering the queue at time s is $\mathbf{v}(p_s) - \alpha c(w_{s-})$. This is guaranteed by the following lemma.

LEMMA 1. *For any $w \in [0, \bar{w}]$ and $p \in [0, \Gamma(w)]$,*
(i) if $\mathbf{v}(p) = p\mathtt{t}(p)$ and $\alpha = 0$, we have $p\mathbf{t}(p, w) = \mathbf{v}(p) - \alpha c(w)$;
(ii) if $\mathbf{v}(p) = U\big(\mathtt{t}(p)\big)$ and $\alpha = 1$, then $[U(\mathbf{t}(p, w)) - c(w)]\mathbb{1}_{\{\mathbf{t}(p,w)>0\}} = \mathbf{v}(p) - \alpha c(w)$.

Therefore, in either case, as long as the price rate $p_s \leq \Gamma(w_{s-})$, the expression $\mathbf{v}(p_s) - \alpha c(w_{s-})$ captures the revenue/social welfare that a customer generates by entering the queue. If the price $p_s > \Gamma(w_{s-})$, however, the customer would not enter the queue, resulting in a zero revenue/social welfare. Overall, we express the general objective of maximizing the long-run average rate that include both cases as

$$g^* := \sup_{\{p_s\}\in\Pi} \mathcal{G}(\{p_s\}_{s\geq 0}), \text{ where } \mathcal{G}(\{p_s\}) := \liminf_{T\to\infty} \frac{1}{T}\mathbb{E}\left[\int_0^T \Big(\mathbf{v}(p_s) - \alpha c(w_{s-})\Big)\mathbb{1}_{\{p_s\leq\Gamma(w_{s-})\}} dN_s\right]. \tag{10}$$

Here the indicator function reflects that a customer generates a value $\mathbf{v}(p_s) - \alpha c(w_{s-})$ only if the price p_s is within the upper bound $\Gamma(w_{s-})$.

Finally, we present the following result, which summarizes all properties of function $\mathbf{v}$ and constant α that we need to establish the optimality condition.

PROPOSITION 3. *Following Assumption 1, the function* $\mathbf{v}$ *defined as either of the two cases in Lemma 1 has the following properties,*

$$\mathbf{v}(p) \leq U'(0)\mathtt{t}(p), \ \forall p \geq 0, \ \textit{and} \tag{V1}$$

$$\mathbf{v}(p) = 0 \ \forall p = U'(0), \textit{and } \exists p \in [0, U'(0)) \textit{ such that } \mathbf{v}(p) > 0. \tag{V2}$$

Proposition 3 summarizes all the properties that we need for the function $\mathbf{v}$ to establish our optimality conditions. In particular, the property (V1), which indicates that the value generated from a customer is upper bounded, is useful in showing the existence of the solution to the optimality condition in Section 4.1. The first part of (V2) implies that setting the price p at the upper bound $U'(0)$ generates zero revenue/welfare when the wait time is zero, because no customers joins the system at this price; in contrast, the second part implies that it is always beneficial to let the next customer join an empty system (when $w = 0$). Consequently, it is optimal not to shut down the operation when the system is empty. This property is useful in the optimality proof of Section 4.2. Equipped with this proposition, we do not need to consider specific expressions of the function $\mathbf{v}$ for either the revenue or social welfare any more, until Section 5, when we study the structure of the optimal policies. That is, as long as the function $\mathbf{v}$ satisfy conditions (V1) and (V2), and for any value $\alpha \geq 0$ as well as a function $c(w)$ that satisfies Assumption 1(iii), we are able to establish the optimality condition in the next subsection, and in Section 4 show its connection with the original optimization model (10).

3.3. Optimality Equations

We provide a heuristic derivation towards the optimality equation, which shows that the optimal control policy is stationary, and a function of the wait time. For this purpose, we first restrict the pricing control to be a function of w_{s-}, i.e., $p_s = \mathcal{P}(w_{s-})$ for some function $\mathcal{P}$. The corresponding service time is also a function of w_{s-}, i.e.,

$$\mathcal{T}(w_{s-}) := \mathbf{t}(\mathcal{P}(w_{s-}), w_{s-}).$$

The wait time dynamics (9) becomes

$$dw_s = -\mathbb{1}_{\{w_{s-}>0\}} ds + \mathcal{T}(w_{s-}) dN_s,$$

which implies that w_s is a Markov process under this restricted class of stationary pricing policy. (Later in Section 4 we will show that the performance of the optimal policy under this restriction is no worse than any $\mathcal{F}_s$-predictive control policies.) The corresponding optimal control problem (10) becomes

$$\sup_{\mathcal{P}(\cdot)\in\Pi_{\mathcal{P}}} \liminf_{T\to\infty} \frac{1}{T}\mathbb{E}\left[\int_0^T \left[\mathbf{v}\left(\mathcal{P}(w_{s-})\right) - \alpha c\left(w_{s-}\right)\right] dN_s\right], \tag{11}$$

in which $\Pi_{\mathcal{P}}$ represents the set of control policies that depend on wait time w_{s-}.

Now we are ready to present the optimality condition (HJB equation) for the optimization problem defined in (11). To provide some intuition, we follow a heuristic derivation based on a discrete time approximation that yields the optimality condition. Consider a discrete time counterpart of the problem, in which each time period corresponds to a short time interval with length Δ, and a corresponding Markov decision process on a state space $[0,\bar{w}]$ of wait time. The optimal long-run average revenue, denoted as $\mathtt{g}\Delta$, and the relative value function $V(w)$ for any wait time $w\in(\Delta,\bar{w}]$, satisfy the following Bellman equation (see, for example, Bertsekas 2000),

$$\mathtt{g}\Delta + V(w) = (1-\lambda\Delta)V(w-\Delta) + \lambda\Delta \max\left\{\max_{p\geq 0}\left[\mathbf{v}(p) - \alpha c(w) + V\left(w+\mathbf{t}(p,w)-\Delta\right)\right], V(w-\Delta)\right\},$$

where $\lambda\Delta$ approximates the probability that a customer arrives during this time interval. Following (8) of Proposition 2, we convert the feasible region of the inner maximization to be $p\in[0,\Gamma(w)]$, and, correspondingly, replace $\mathbf{t}(p,w)$ with $\mathtt{t}(p)$ as defined in (5). Subtracting $V(w)$ from both sides of the equation, then dividing Δ on both sides, and finally letting Δ approach 0, we obtain the following delay differential equation (DDE):

$$\mathtt{g} = -V'(w) + \lambda \max\left\{\max_{p\in[0,\Gamma(w)]}\left[\mathbf{v}(p) - \alpha c(w) + V(w+\mathtt{t}(p)) - V(w)\right],\ 0\right\}, \quad \forall w\in(0,\bar{w}], \tag{12}$$

The discrete time Bellman equation at state $w=0$ is

$$\mathtt{g}\Delta + V(0) = (1-\lambda\Delta)V(0) + \lambda\Delta \max\left\{\max_{p\in[0,\Gamma(0)]}\left[\mathbf{v}(p) - \alpha c(0) + V(\mathtt{t}(p)-\Delta)\right],\ V(0)\right\}.$$

Following the same procedure and note $c(0)=0$ from Assumption 1(iii), we obtain

$$\mathtt{g} = \lambda \max\left\{\max_{p\in[0,\Gamma(0)]}\left[\mathbf{v}(p) + V\big(\mathtt{t}(p)\big) - V(0)\right],\ 0\right\}. \tag{13}$$

Finally, if the wait time $w>\bar{w}$, an arriving customer would balk the system. Thus, the corresponding discrete time Bellman equation is $\mathtt{g}\Delta + V(w) = V(w-\Delta)$, yielding

$$\mathtt{g} = -V'(w), \quad \forall w > \bar{w}. \tag{14}$$

Therefore, we claim that the optimal long-run average revenue rate $\mathtt{g}$, together with a differentiable function $V(w)$, must satisfy the DDE (12), with boundary conditions (13) and (14). It is clear that if any value g and function $V(w)$ solve (12)-(14), then the value g and function $V(w)+x$ for any constant x also satisfy (12)-(14). Therefore, in order to specify a particular function $V(w)$, we also require

$$V(\bar{w}) = 0. \tag{15}$$

Starting from the next section, we construct a value g and a function $V(w)$ that solve Equations (12)-(15), and establish that the value g indeed equals to the maximum revenue rate g^* defined in (10). Therefore, we refer to (12)-(15) as the optimality conditions.

4. Optimality Analysis

The optimality conditions proposed in the previous section involve a system of DDEs. Unlike discrete-time Markov decision processes, or standard stochastic optimal control problems with Brownian motion uncertainties, we are not aware of existing results that readily connects these optimality conditions with the original optimal control model (10). Therefore, in this section we establish this connection. In Section 4.1, we first show, by construction, the existence of a function $V(w)$ and value g that solve the DDE (12) with the boundary conditions (13)-(15). In Section 4.2, we show that the value g that satisfies these equations is indeed the optimal long-run average rate g^*, and a stationary control policy obtained from these equations achieves g^*.

4.1. Existence of Solution to Optimality Conditions

Because the optimality equation (12) is a DDE, rather than an ODE, we cannot use the usual approach based on Cauchy-Lipschitz theorem to verify the existence of a solution. In fact, we are not aware of any result directly applicable to show that a solution exists. Therefore, we develop a shooting method based algorithm to construct a piece-wise linear approximation, and show that when the number of pieces approaches infinity, our algorithm yields a solution to (12)-(15). We first present the following Algorithm 1, inspired by the explicit Euler method in numerical analysis.

Algorithm 1 takes a non-negative value g and an integer value N, and returns a piece-wise linear function $V_N^g(w)$ defined on the interval $[0, \bar{w}+C]$. The upper bound of the domain $\bar{w}+C$ comes from the potential control of admitting a customer and offering the highest service time C when the wait time is at the upper bound $\bar{w}$. The algorithm is based on a uniform grid $x_0, \ldots, x_N$ on the interval $[0, \bar{w}]$, where $x_0 = 0$ and $x_N = \bar{w}$. The initial step in (16) guarantees that $V_N^g(\bar{w}) = 0$ and the derivative of $V_N^g(w)$ for $w > \bar{w}$ is $-g$, motivated from boundary conditions (14) and (15). We construct the piece-wise linear function by obtaining a linear piece for each interval $[x_{N-i}, x_{N-i+1})$ from the right end point x_{N-i+1} with a slope of $-g + K_N^g(x_{N-i+1})$, as shown in (21). One can view the term $K_N^g(x_{N-i+1})$ as an adjustment to the slope in each interval from the initial slope $-g$ at $\bar{w}$.

Algorithm 1 Constructing $V_N^g(w)$ for a given value g and an integer N

set

$$V_N^g(w) = -(w-\bar{w})g, \quad \forall w \in [\bar{w}, \bar{w}+C], \tag{16}$$

$$K_N^g(w) = 0, \quad \forall w \in (\bar{w}, \bar{w}+C], \tag{17}$$

$$\epsilon_N = \frac{\bar{w}}{N},$$

$$x_0 = 0, \text{ and } x_{i+1} = x_i + \epsilon_N \text{ for } i \in \{0,1,2,...N-1\}. \tag{18}$$

for $i = 1,2,...N$ **do**

$$K_N^g(x_{N-i+1}) = \lambda \max\left\{ \max_{p\in[0,\Gamma(x_{N-i+1})]} \mathbf{v}(p) - \alpha c(x_{N-i+1}) + V_N^g\big(x_{N-i+1} + \mathtt{t}(p)\big) - V_N^g(x_{N-i+1}),\ 0 \right\}, \tag{19}$$

$$K_N^g(w) = K_N^g(x_{N-i+1}), \quad \forall w \in (x_{N-i}, x_{N-i+1}), \tag{20}$$

$$V_N^g(w) = V_N^g(x_{N-i+1}) + (-g + K_N^g(x_{N-i+1}))(w - x_{N-i+1}), \quad \forall w \in [x_{N-i}, x_{N-i+1}). \tag{21}$$

end for

set

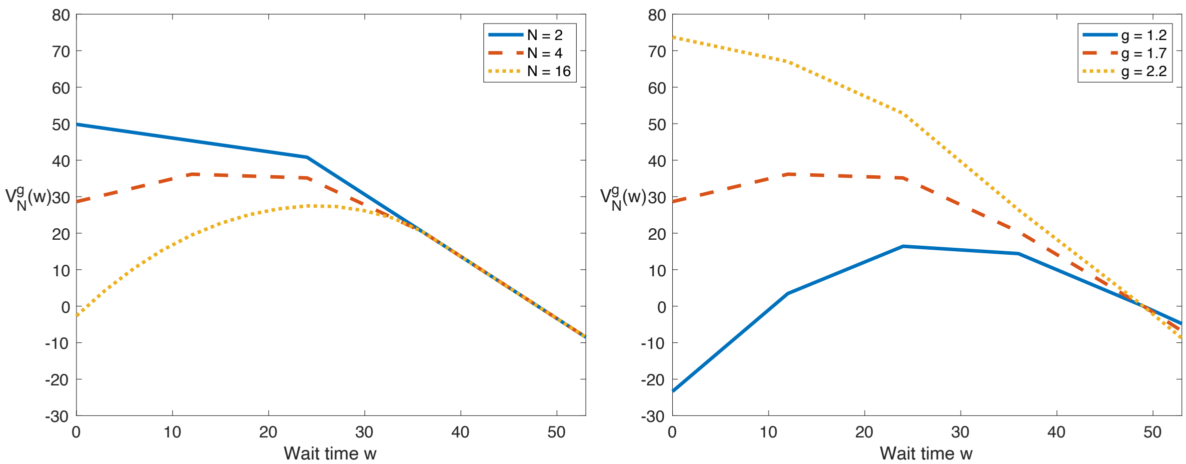
$$K_N^g(0) = \lambda \max\left\{ \max_{p\in[0,\Gamma(0)]} \mathbf{v}(p) + V_N^g\big(\mathtt{t}(p)\big) - V_N^g(0),\ 0 \right\}. \tag{22}$$

We first present the following proposition, demonstrating properties of the functions constructed in Algorithm 1. As we will show later, these properties still hold when we take limit as N approaches infinity.

PROPOSITION 4. *Given $g \geq 0$ and $N \in \mathbb{N}$, $V_N^g(w)$ is concave in w and $K_N^g(w)$ decreases in w for all $w \in [0, \bar{w}+C]$.*

Proposition 4 implies that given a parameter g and the number of steps N, the slope of the concave function $V_N^g(w)$ takes its lowest value, $-g$, at $w = \bar{w}$, and its highest value, $-g + K_N^g(0)$ at $w = 0$.

We intend to use function $V_N^g(w)$ from Algorithm 1 as an approximation to the solution $V(x)$ of (12)-(15). It is clear that boundary conditions (14) and (15) are satisfied in this approximation, following (16). We hope to identify a value g such that as N approaches infinity, the function $V_N^g(w)$ converges to $V(w)$ that satisfies (12). Furthermore, we need to establish that the slope of $V_N^g(w)$ at $w = 0$, which equals to $-g + K_N^g(0)$, approaches zero, in order to satisfy the boundary condition (13). A major challenge in this process is that for an arbitrary value g, the sequence of functions



(a) Lemma 2 (b) Lemma 3

Figure 1 Demonstration of Lemma 2 and Lemma 3. In these examples, $\lambda = 0.056$**,** $C = 20$**,** $U(t) = 68 \ln(1 + 0.15t)$**, and** $c(w) = 0.04w^2$**.**

$V_N^g(w)$ may not even converge as N approaches infinity. Therefore, we present the following two lemmas, which reveal the impact of changing N and g to the function $V_N^g(w)$.

First, we present the following lemma, which illustrates, for a fixed g, the impact of increasing N to function V_N^g.

Lemma 2. *Given $g \geq 0$ and any positive integer j, we have*

$$V_j^g(w) \geq V_{2j}^g(w) \text{ and } K_j^g(w) \leq K_{2j}^g(w), \quad \forall w \in [0, \bar{w} + C].$$

Lemma 2 states that for a fixed g, when N doubles, the corresponding $K_N^g(w)$ increases, leading to a smaller $V_N^g(w)$ at each point w. Figure 1a compares different $V_N^g(w)$ functions for $N \in \{2, 4, 16\}$ and $g = 1.7$. Note that the slope $-g + K_4^g(0)$ for $w = 0$ is already positive, so $g = 1.7$ does not solve the DDEs. Lemma 2 implies that as we further increase j, the slope at $w = 0$ cannot converge to 0, as required by (13). (In fact, if we keep increasing j, the slope $-g + K_{2^j}^g(0)$ diverges to infinity when g is sufficiently small.)

We next examine the impact of g on $V_N^g(w)$ for a fixed N. Lemma 3 formalizes the result that $K_N^g(w)$, as well as the slope $-g + K_N^g(w)$, decreases in g.

Lemma 3. *For any given N, and g_1 and g_2 such that $0 \leq g_1 < g_2$, we have*

$$K_N^{g_1}(w) \geq K_N^{g_2}(w), \quad \forall w \in [0, \bar{w} + C].$$

Figure 1b illustrates Lemmas 3 with $N = 4$ and $g \in \{1.2, 1.7, 2.2\}$. As we see from the figure, if we increase the starting slope $-g$ at $w = \bar{w}$ by decreasing g, the slope $-g + K_N^g(w)$ is getting larger for all w values. Hence, the function $V_N^g(w)$ is trending downwards more towards left, following (21).

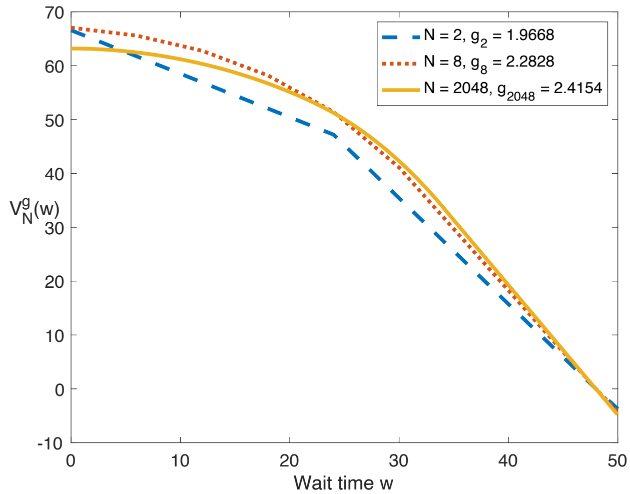


Figure 2 **Demonstration of Proposition 5. In these examples,** $\lambda = 0.056$, $C = 20$, $U(t) = 68\ln(1 + 0.15t)$**, and** $c(w) = 0.04w^2$**.**

Lemmas 2 and 3 demonstrate the intricacy of finding the right value g to meet the boundary condition (13), which implies that as N approaches infinity, $-g + K_N^g(0)$ has to approach zero. Following Lemma 2, as N increases, $-g + K_N^g(0)$ increases, and we can decrease $-g + K_N^g(0)$ by reducing g following Lemma 3. In order to fine-tune the value of g for $-g + K_N^g(0)$ to approach zero, we define the following quantity, g_N, for any finite N,

$$g_N := \min\{g \mid g \geq K_N^g(0)\}. \tag{23}$$

Here, g_N is the smallest value such that the slope at $w = 0$ is non-positive for a given N.

Next, we consider a sequence of g_N where N takes values 2^j for $j = 1, 2, ...$, and show that the sequence has a limit.

Proposition 5. *Following the definition (23), the sequence g_{2^j} for $j = 1, 2, \ldots$ is monotonically increasing in j, and is upper bounded by $U'(0)$. Therefore, the following limit exists,*

$$\bar{g} := \lim_{j\to\infty} g_{2^j}. \tag{24}$$

Figure 2 illustrates g_{2^j} for $j = 1, 3, 11$ when N increases from 2 to 2048. These g_{2^j} values are the smallest g such that the slope $-g_{2^j} + K_N^g(0)$ is non-positive. As shown, the g_{2^j} value increases and converges to $\bar{g} = 2.4154$.

With the construction of value $\bar{g}$, one may conjecture that the sequence of functions $V_N^{\bar{g}}$ converges as N approaches infinity. Indeed, we again let N take values 2^j while increasing j, and show the uniform convergence through point-wise convergence. The uniform convergence result also implies that the concavity property of V_N^g that we showed in Proposition 4 preserves in the limit, which is useful when we establish structures of the optimal control policy in the next section. The next Proposition 6 summarizes these results, as well as additional properties that are useful for proving that the convergent function satisfies the DDE (12).

PROPOSITION 6. *For any given $w \in [0, \bar{w}+C]$, the sequence of values $V_{2^j}^{\bar{g}}(w)$ for $j = 1, 2, \ldots$ has a limit, and we define*

$$V^{\bar{g}}(w) := \lim_{j\to\infty} V_{2^j}^{\bar{g}}(w).$$

Furthermore, we have:

(i) The sequence of functions $\{V_{2^j}^{\bar{g}}\}_{j=1}^{\infty}$ converges to the function $V^{\bar{g}}$ uniformly;

(ii) The function $V^{\bar{g}}(w)$ is concave and decreasing in w;

(iii) The function $V^{\bar{g}}(w)$ is Lipschitz continuous in the interval $w \in [0, \bar{w}+C]$.

So far we have constructed a value $\bar{g}$ and a function $V^{\bar{g}}$ in the previous two propositions. Finally, we show that the value $\bar{g}$ and function $V^{\bar{g}}$ satisfy (12)-(15) in the next two propositions. First, Proposition 7 states that $\bar{g}$ and $V^{\bar{g}}$ satisfy the boundary condition (13).

PROPOSITION 7. *We have*

$$\bar{g} = \lambda \max\left\{ \max_{p\in[0,\Gamma(0)]} [\mathbf{v}(p) + V^{\bar{g}}\big(\mathtt{t}(p)\big) - V^{\bar{g}}(0)],\ 0 \right\}.$$

To establish that $\bar{g}$ and $V^{\bar{g}}$ also satisfy (12), we need to show that $V^{\bar{g}}$ is differentiable, even though the function $V_N^{\bar{g}}$ from Algorithm 1 for any N is not differentiable on the grid points. The following proposition resolves this issue by showing that the left and right derivatives of the function $V_N^{\bar{g}}$ on the grid points converges to the be same as N approaches infinity.

PROPOSITION 8. *The function $V^{\bar{g}}(w)$ is differentiable on $w \in [0, \bar{w}]$. Moreover, we have*

$$\bar{g} = -V^{\bar{g}\prime}(w) + \lambda \max\left\{ \max_{p\in[0,\Gamma(w)]} [\mathbf{v}(p) - \alpha c(w) + V^{\bar{g}}\big(w + \mathtt{t}(p)\big) - V^{\bar{g}}(w)],\ 0 \right\},$$

for any $w \in [0, \bar{w}]$.

Propositions 7 and 8, together with Algorithm 1, directly imply the following result.

THEOREM 1. *The value $\mathtt{g} = \bar{g}$ and function $V = V^{\bar{g}}$ solve DDEs (12)-(15). Moreover, the relative value function $V(w)$ is decreasing and concave in w.*

4.2. The Optimality of Average Value Rate

Having constructed a solution $\mathtt{g}$ and V to DDEs (12)-(15) in the previous subsection, we are now ready to show that the value $\mathtt{g}$ is indeed the optimal value defined in (10).

First, we define a policy obtained from the relative value function V. In particular, following (12) and (13), define

$$\xi(p, w) := \mathbf{v}(p) - \alpha c(w) + V\big(w + \mathtt{t}(p)\big) - V(w),$$

and the following maximizer,

$$\bar{p}(w) := \max\left\{\arg\max_{p\in[0,\Gamma(w)]} \xi(p,w)\right\}. \tag{25}$$

Note that for any given w, there may exist multiple p values that maximize $\xi(p,w)$, or, the set $\arg\max_{p\in[0,\Gamma(w)]}\xi(p,w)$ may not be a singleton. With this set up, for any customer who arrives at time s facing a wait time state w_{s-}, define the pricing control policy $\{p_s^*\}$ as

$$p_s^* := \begin{cases} \bar{p}(w_{s-}), & \text{if } \xi\left(\bar{p}(w_{s-}), w_{s-}\right) \geq 0, \\ U'(0), & \text{if } \xi\left(\bar{p}(w_{s-}), w_{s-}\right) < 0. \end{cases} \tag{26}$$

Here the wait time process $\{w_s\}$ follows dynamics (9), in which the pricing control p_s is replaced with p_s^*. Therefore, at any point in time s, the control p_s^* is a function of w_{s-}, and solves the optimization problems in (12) and (13), in which state w takes value w_{s-}. When needed (for example, in the next lemma), we explicitly write the pricing control as $p_s^*(w_{s-})$. The long-run average revenue/social welfare rate under the control policy $\{p_s^*\}$ is $\mathcal{G}(\{p_s^*\})$ as defined in (10). In the rest of this section we show that the pricing control process $\{p_s^*\}$ defined above is optimal.

We first present the following result, which implies that when the queue is empty, the control policy $\{p_s^*\}$ allows the next arriving customer to join the system and choose a positive service time.

LEMMA 4. *We have* $\mathbf{t}\big(p_s^*(0),0\big) > 0$.

Proof. We prove this result by contradiction. Suppose $\mathbf{t}\big(p_s^*(0),0\big) = 0$ instead. Following Proposition 2, we have $\Gamma(0) = U'(0)$, and, therefore, $\mathbf{t}\big(p_s^*(0),0\big) = \mathtt{t}(p_s^*(0)) = 0$, which further implies that $p_s^*(0) = U'(0)$ according to (5). Following (V2), we have $\mathbf{v}\big(p_s^*(0)\big) - \alpha c(0) = 0$, and, therefore, $\xi(p_s^*(0),0) = 0$. For any $p \in [0,\Gamma(0)]$, we must have $\xi(p,0) \leq 0$, following the definition of p_s^* in (26). Therefore, the optimality condition (13) implies $\mathsf{g} = 0$, which further implies $V(w) = 0$ for all w due to (14), (15), and Theorem 1. Given condition (V2), however, $\mathsf{g} = 0$ and $V(w) = 0$ cannot satisfy (12). Therefore, we cannot find a solution to (12)-(15), contradicting Theorem 1. Q.E.D.

To this end, we call a time epoch s an "out of empty queue" time, or OEQ time, if $w_{s-} = 0$ and $w_s > 0$. That is, the wait time is zero right before s, and becomes positive at time s with a customer joining the queue. Denote $\mathcal{F}_s$-random time $\tau_k(\{p_s\})$ to represent the k-th OEQ time following any pricing policy $\{p_s\}$. Lemma 4 guarantees that under the control policy $\{p_s^*\}$, after one OEQ time $\tau_k(\{p_s^*\})$, the next OEQ time $\tau_{k+1}(\{p_s^*\})$ is finite with probability 1.

Proposition 9 below proves that the long-run average rate $\mathcal{G}\left(\{p_s^*\}\right)$ from the control policy $\{p_s^*\}$ is equal to the value g from the optimality conditions. Furthermore, both are equal to the expected revenue/social welfare accumulated in a cycle between two consecutive OEQ times divided by the expected cycle length following policy $\{p_s^*\}$. Therefore, Lemma 4 is useful here because it implies that the aforementioned expected cycle length is finite.

PROPOSITION 9. *We have, for all $k = 1, 2, \ldots,$*

$$\mathtt{g} = \mathcal{G}\left(\{p_s^*\}\right) = \frac{\mathbb{E}\left[\int_{\tau_k(\{p_s^*\})}^{\tau_{k+1}(\{p_s^*\})-} [\mathbf{v}(p_x^*) - \alpha c(w_{x-})] dN_x\right]}{\mathbb{E}\left[\tau_{k+1}(\{p_s^*\}) - \tau_k(\{p_s^*\})\right]}, \tag{27}$$

in which w_x is generated from (9) with x replacing s and p_x^* replacing p_x.

Proposition 9 indicates that regardless of how the system reaches an OEQ time, following control $\{p_s^*\}$, the average revenue/social welfare rate during the cycle until the next OEQ times equals the long run average rate of control $\{p_s^*\}$.

The next proposition further implies that the average revenue/social welfare rate in a cycle for a generic control $\{p_s\}$ is inferior to that of the control $\{p_s^*\}$, which further implies the optimality of $\{p_s^*\}$.

PROPOSITION 10. *For all k such that $\tau_k(\{p_s\})$ and $\tau_{k+1}(\{p_s\})$ are finite for a control $\{p_s\}$, we have*

$$\mathtt{g} \geq \frac{\mathbb{E}\left[\int_{\tau_k(\{p_s\})}^{\tau_{k+1}(\{p_s\})-} [\mathbf{v}(p_x) - \alpha c(w_{x-})] dN_x\right]}{\mathbb{E}\left[\tau_{k+1}(\{p_s\}) - \tau_k(\{p_s\})\right]}, \tag{28}$$

which further implies that

$$\mathtt{g} \geq \mathcal{G}(\{p_s\}), \ \forall \{p_s\} \in \Pi. \tag{29}$$

Propositions 9 and 10 together imply the following main result of this section.

THEOREM 2. *We have $g^* = \mathtt{g} = \mathcal{G}(\{p_s^*\})$. That is, the solution (12)-(15) yields the optimal long-run average revenue rate $\mathtt{g}$, as well as an optimal policy $\{p_s^*\}$, defined in (26).*

Theorem 2 establishes that the system of differential equations with boundary conditions (12)-(15) indeed provides the optimality conditions for the original problem. In particular, the value $\mathtt{g}$ in the solution is the optimal long-run average revenue/social welfare rate. Furthermore, the optimal pricing decision obtained from solving the problem in (12) and (13) yields a control policy that achieves the maximum long-run average rate. In the next section, we characterize the structure of the optimal control policy.

5. Structure of the Optimal Control Policy

Now we are ready to characterize the optimal control policy. Although we have monotonicity and concavity properties of the relative value function V, it is hard to tell if $V\big(w + \mathtt{t}(p)\big)$ in (12) as a function of p possesses properties that reveal structures of the optimal policy. However, given the correspondence between the price rate decision and service time decision $\mathtt{t}(p)$ in (5), we can transform optimality conditions (12) and (13) into ones that control service times rather than price rates. Such a transformation allows us to obtain structural results on the optimal control policies. The next result specifies the transformation, in which we use notations $\hat{w}$ and $L(w)$ defined in Proposition 1, and define function $\mathbf{f}(t,w) := \mathbf{v}(U'(t)) - \alpha c(w)$ for $w \in [0,\hat{w}]$ for simplicity of expression.

PROPOSITION 11. *The value* g *and function* V *that satisfy* (12)-(15) *also satisfy*

$$\mathsf{g} = -V'(w) + \lambda \max\left\{ \max_{t\in[L(w),C]} \left\{\mathbf{f}(t,w) + V(w+t) - V(w)\right\}, 0\right\}, \quad \forall w \in (0,\hat{w}], \tag{30}$$

$$\mathsf{g} = -V'(w) + \lambda \max\left\{ \max_{p\in[0,U'(C)]} \left\{\mathbf{v}(p) - \alpha c(w) + V(w+C) - V(w)\right\}, 0\right\}, \quad \forall w \in (\hat{w},\bar{w}], \tag{31}$$

$$\mathsf{g} = \lambda \max\left\{ \max_{t\in[L(0),C]} \{\mathbf{f}(t,0) + V(t) - V(0)\}, 0\right\}, \tag{32}$$

$$\mathsf{g} = -V'(w), \quad \forall w > \bar{w}, \tag{33}$$

$$0 = V(\bar{w}), \tag{34}$$

in which the function $\mathbf{f}(t,w)$ *is strictly concave in* t.

Conversely, any value g *and function* V *that satisfy* (30)-(34) *also solve* (12)-(15). *Furthermore, the function* $L(w)$ *is increasing in* w.

According to the transformation, we first separate the wait time at $\hat{w}$, below which optimizing over price rate is equivalent to optimizing over service time, as shown in (30). If the wait time w is above $\hat{w}$ but still no longer than $\bar{w}$, the only way to induce a customer to enter the queue is to set the service time to the maximum level C while charging a price rate p that is no larger than $[U(C) - c(w)]/C$, so that $U(c) - pC - c(w) \geq 0$.

In order to define the corresponding optimal policy from (30)-(34), similar to the previous section, we define

$$\hat{\xi}(t,w) := \mathbf{f}(t,w) + V(w+t) - V(w),$$

which is strictly concave in t for any $w \in [0,\hat{w}]$. Therefore, for any $w \in [0,\hat{w}]$, $\hat{\xi}(t,w)$ has a unique maximizer on the interval $[L(w),C]$, and we denote it as $t^*(w)$. Following (30), the optimal service time decision for any $w \in [0,\hat{w}]$ is

$$\mathcal{T}^*(w) := t^*(w)\mathbb{1}_{\left\{\hat{\xi}(t^*(w),w)\geq 0\right\}}, \tag{35}$$

and the corresponding pricing decision is

$$\mathcal{P}^*(w) := U'(\mathcal{T}^*(w)). \tag{36}$$

For $w \in (\hat{w}, \bar{w}]$, define

$$\mathcal{P}^*(w) := \frac{U(C)-c(w)}{C}, \quad \text{if } \mathbf{v}\left(\frac{U(C)-c(w)}{C}\right) - \alpha c(w) + V(w+C) - V(w) \geq 0, \tag{37}$$

$$\mathcal{P}^*(w) := U'(0), \quad \text{if } \mathbf{v}\left(\frac{U(C)-c(w)}{C}\right) - \alpha c(w) + V(w+C) - V(w) < 0, \text{ and} \tag{38}$$

$$\mathcal{T}^*(w) := \mathbf{t}(\mathcal{P}^*(w), w). \tag{39}$$

That is, the inner maximization of (31) sets the optimal price at the upper bound of the feasible region. This is because when the service time is fixed at C, it is easy to verify that the function $\mathbf{v}(p)$ is non-decreasing in p under either revenue or social welfare maximization.

We now present the main result of this section, before discussing the implications.

Theorem 3. *Consider pricing decisions $\mathcal{P}^*(w)$ and wait time decisions $\mathcal{T}^*(w)$ defined in (35)-(39). We have $\mathcal{P}^*(w) = p_s^*(w)$ for $w \in [0, \bar{w}]$, in which p_s^* is defined in (26). Furthermore, there exist thresholds W_1, W_2, and W_3, with $0 \leq W_1 \leq W_2 \leq \hat{w} \leq W_3 \leq \bar{w}$, such that*

- *$\mathcal{T}^*(w) = C$ for $w \in [0, W_1)$;*
- *$\mathcal{T}^*(w)$ is decreasing in w for $w \in [W_1, W_2)$;*
- *$\mathcal{T}^*(w) = L(w)$, which is increasing in w, for $w \in [W_2, \hat{w})$;*
- *$\mathcal{T}^*(w) = C$ for $w \in [\hat{w}, W_3)$;*
- *$\mathcal{T}^*(w) = 0$ for $w \in [W_3, \bar{w}]$.*

The corresponding pricing policy $\mathcal{P}^*$ is such that

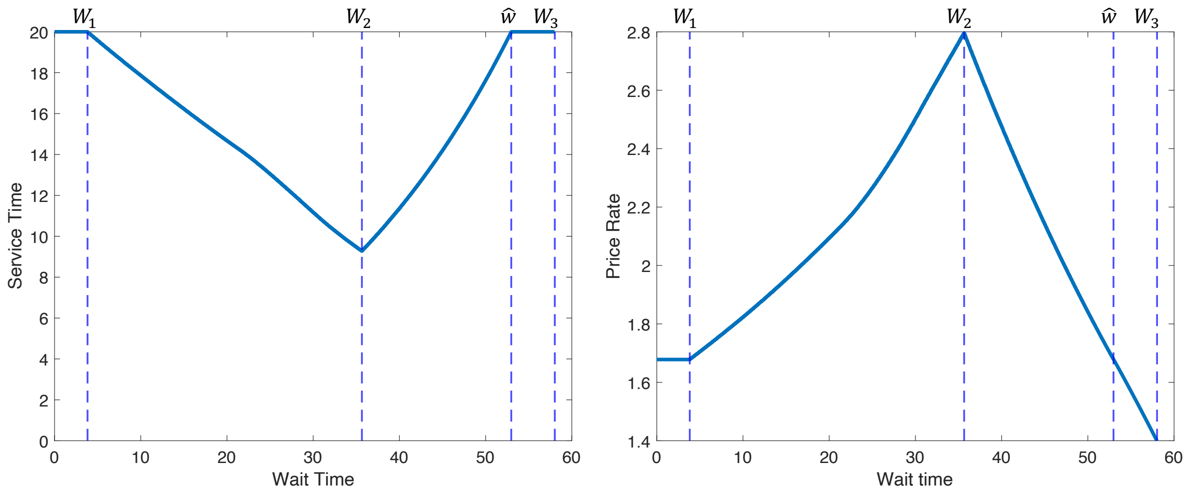
- *$\mathcal{P}^*(w) = U'(C)$ for $w \in [0, W_1)$;*
- *$\mathcal{P}^*(w)$ is increasing in w for $w \in [W_1, W_2)$;*
- *$\mathcal{P}^*(w)$ is decreasing in w for $w \in [W_2, \hat{w})$;*
- *$\mathcal{P}^*(w) = \dfrac{U(C)-c(w)}{C}$, which is decreasing in w, for $w \in [\hat{w}, W_3)$;*
- *$\mathcal{P}^*(w) = U'(0)$ for $w \in [W_3, \bar{w}]$.*

It is worth noting that due to the concavity of the function U (Assumption 1(i)), Eq.(36) implies that the monotonicity of $\mathcal{P}^*$ and $\mathcal{T}^*$ are opposite to each other, as reflected in Theorem 3. Furthermore, it is possible that some of the five wait time intervals in the theorem collapse into a single point, or do not exist, under certain model parameters.

Figure 3 illustrates the structure described in Theorem 3 using a specific example. In this example, we maximize the long-run average revenue rate, that is, $\mathbf{v}(p) = p\mathtt{t}(p)$ and $\alpha = 0$. Furthermore, we use the following customer usage utility function and wait time cost function:

$$U(t) = 44.9924 \ln(1 + 0.1468t), \text{ and } c(w) = 0.01w^2, \text{ respectively,}$$

and set the maximum service time $C = 20$ and the arrival rate $\lambda = 0.056$. Some of these model parameters are estimated from data, which will be explained in more details in the next section.



(a) Optimal Service Time $\mathcal{T}^*(w)$ (b) Optimal Price Rate Policy $\mathcal{P}^*(w)$

Figure 3 Illustration of the Optimal Policy

We use Algorithm 1 to obtain V_N^g with $N = 2048$ for any given g, and a binary search to identify g_N defined in (23) as an approximation for $\bar{g}$. More precisely, we stop the binary search when the absolute value of $-g + K_N^g(0)$ is less than 10^{-6}, and use the corresponding value g and function V_N^g to approximate the optimal value g and the relative value function V.

Figure 3a plots the optimal service time policy $\mathcal{T}^*(w)$ and Figure 3b the optimal price rate $\mathcal{P}^*(w)$. As we see, when the wait time w is lower than W_1, each arriving customer is offered a low price $U'(C) \approx 1.678$, which induces the customer to use the maximum service time $C = 20$. As the wait time increases in the interval $[W_1, W_2)$, the price rate increases, and the customer chooses a shorter service time in response. When the wait time w is in this interval, the maximizer $t^*(w)$ of $\hat{\xi}(t,w)$ is in the interior of the interval $[L(w), C]$, and $\hat{\xi}(t^*(w), w) \geq 0$. Furthermore, the maximizer is monotonically decreasing in w, as we show in the proof in the Appendix. The decrease in service time for incoming customers reflects the firm's desire to alleviate congestion. Hence, the monotonicity in this part is similar to the *congestion effect* that we often see in the queueing control literature. When the wait time further increases to the interval $[W_2, \hat{w})$, the lower bound $L(w)$ maximizes $\hat{\xi}(t,w)$, while $\hat{\xi}(L(w), w)$ is still non-negative. The monotonicity of $L(w)$ from Proposition 11 implies that the service time increases in w, while the price rate decreases in w. As a result, an arriving customer's utility from usage increases in the weight time w, which compensates the increasing wait time cost. This is why we call this phenomenon the *compensation effect*. If the wait time further increases in the interval $[\hat{w}, W_3]$, the only way to induce a customer to join the system is to offer the maximum service time C. If the objective is to maximize revenue, the optimal price rate should be the one that makes the customer indifferent between joining the queue or not, that is, $[U(C) - c(w)]/C$. If the objective is to maximize social welfare, on the other hand, any price

Charging Time (Minutes)	2.5	5	7.5	10	12.5	15	17.5	20	22.5	25	27.5	30
SOC (%)	12	23	33	41	48	53	58	63	67	70	72	73

Table 1 Charging time and the Corresponding State of Charge (SOC).

between 0 and $[U(C) - c(w)]/C$ induces the customer to join the queue for the maximum service time C, which yields the same social welfare. In Theorem 3, we only mention the upper bound of this interval, which is optimal for both the revenue and the social welfare cases, and consistent with (25). Finally, when the wait time is so high that is above W_3, the corresponding wait time cost is too high for the firm to attract a new customer to join the system, or, $\hat{\xi}(t,w) < 0$. Therefore, the firm blocks arriving customers with the highest possible price, which allows the wait time to decrease.

6. Numerical Study

We apply our model to fast-charging stations, aiming to reveal insights on managing such systems. Available data for fast-charging stations are sparse, as the technology is relatively new. As a result, we leverage some data sets for level-2 charging stations in the literature to calibrate some of our model parameters. Specifically, we use the data set in Motoaki and Shirk (2017) to estimate the usage utility function $U(t)$, and the data set in Lee et al. (2019) to estimate the arrival rate λ. For the other parameters, we consider wide ranges to conduct robustness tests and reveal insights.

Usage Utility. Based on Figure 1 of Motoaki and Shirk (2017), Table 1 contains the relationship between charging time and state of charge (SOC), which represents the percentage of charged battery level with respect to the full capacity. Here, we use SOC as a proxy for driving distance, which can be viewed as the usage utility in our model. Consequently, we use Table 1 to fit the following utility function for our numerical study:

$$\text{Increased SOC} = U(\text{Charging Time}) = U(t) = 44.99 \ln (1 + 0.15t).$$

Note that the utility function with a shift of constant 1 satisfies Assumption 1(i) and (ii). To model the fast-charging station which can charge a vehicle for 90% of SOC within 20 minutes, we set $C = 20$ minutes. We shall use this utility function throughout the entire section.

Arrival Rate. Lee et al. (2019) presents a data set that reports the daily transaction records of a level-2 charging station in CalTech University campus. There are 54 chargers in this charging station. The transaction data include the starting and ending times for each charged vehicle. According to the data, the probability of a vehicle being blocked because of no chargers available is close to zero. Thus, we treat the starting time as the arrival time, and use the total numbers of arrivals to estimate the uncensored arrival rate. In our numeral study, we assume that this arrival rate remains a constant because it represents the demand in this region. Given that the cost of

Time Interval	Arrival Rate (vehicles/min)
12:00 AM - 11:59 PM	0.016
4:00 AM - 9:00 PM	0.056
5:00 AM - 6:00 PM	0.098

Table 2 Arrival Rates for Different Time Intervals

installing a faster-charging station is about 29 times higher than that of a level-2 charger (Smith and Castellano 2015), we assume the number of level-3 chargers is $54/29 = 1.86$ chargers in our study. With this number, we are able to calculate the arrival rate for a single server assumed in our study. Because the arrival rate is non-stationary throughout a day, Table 2 considers the arrival rate (vehicle per minute) under three scenarios, full-day (low rate), normal hours (medium rate), and peak hours (high rate).

Finally, we assume the wait time cost function has a quadratic form:

$$c(w) = \beta w^2,$$

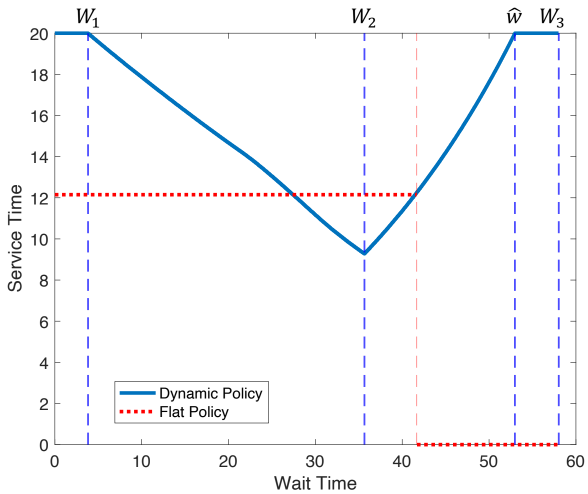
where β denotes the wait time sensitivity. This cost function satisfies Assumption 1(iii).

6.1. Dynamic vs. Flat Pricing Policy

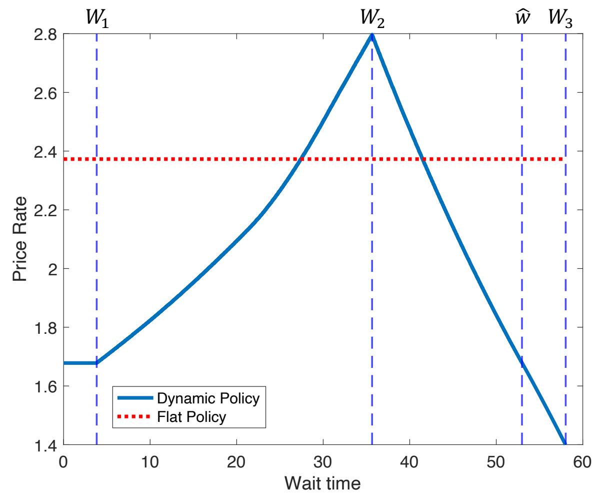
This section compares the optimal dynamic pricing policy with the best flat-rate pricing policy under the objective of maximizing the revenue rate for the fast-charging station. Our goal is to use the best flat-rate price as a benchmark and examine how our dynamic price rate reacts to the wait time. The reason why we choose the flat-rate pricing as the benchmark is because it is commonly used in practice. The flat-rate pricing policy charges customers a single price rate (\$/min) at any time without considering the system's status. Let p_f denote the best flat-rate price.

We can modify Algorithm 1 to find p_f. Specifically, for a large N and a given g, we can evaluate the K_N^g function for a fixed p (instead of optimizing over p) from (19), and use the shooting method to construct the resulting V_N^g function following (21). Then, for this particular price p, we need to search for a g value that satisfies the boundary condition $-g + K_N^g(0) = 0$. The resulting g value is the average revenue rate for the flat-rate price p. Thus, to find the best flat-rate price, p_f, we can conduct a search for $p \in [0, U'(0)]$ such that the resulting g is the largest one, denoted as g_f.

For this comparison, we assume the wait time sensitivity $\beta = 0.01$. The optimal dynamic policy achieves a long-run revenue rate of $g^* = \$1.67$/min, which is 6.04% higher than that of $g_f = \$1.58$/min generated by the best flat-rate policy. To understand why the optimal dynamic pricing policy outperforms the best flat-rate policy, Figure 4 illustrates the optimal charging time and the corresponding price rate under different wait times. We find that the dynamic policy induces a longer (shorter, respectively) charging time and a lower (higher, respectively) price rate than those of the flat-rate policy when the wait time is short (long, respectively). This suggests that



(a) Induced Service Time Policy Comparison



(b) Price Rate Policy Comparison

Figure 4 Optimal Policy Comparison

the dynamic pricing policy increases the utilization while shaping the load of electricity supply. Another interesting observation is that when the wait time exceeds 42.2 minutes, our dynamic pricing policy continues inducing customers to join the queue by reducing the price (and increasing the charging time). Consequently, this increases the consumer surplus, which, in turn, increases the social welfare for this particular example. In the next subsection, we further explore implications on social welfare from the dynamic pricing policy.

6.2. Revenue and Welfare Implications of Revenue-maximizing Policies

In the introduction, we mention that financial viability is a key hurtle for the development of fast-charging stations, most of which implement flat-rate pricing currently. Thus, in this section, we first explore the revenue generated by the dynamic pricing policy, and then quantify the differences between the dynamic and the flat-rate pricing policies in terms of revenue, consumer surplus and social welfare. We also characterize conditions under which our dynamic pricing policy yields the most revenue over the flat pricing one. Insights from studying our model provide guidance on when to implement dynamic pricing policies.

We conduct a numerical study with a test bed of 100 instances by varying the wait time sensitivity β and the arrival rate λ. Specifically, we take

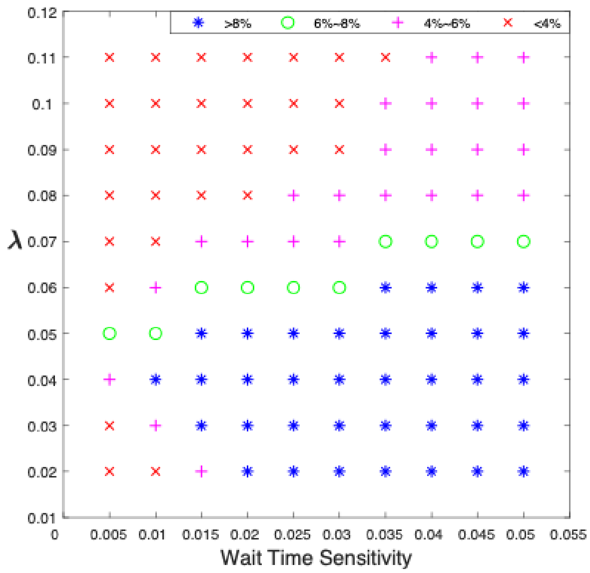
$$\beta \in \{0.005, 0.01, 0.015, 0.02, 0.025, 0.03, 0.035, 0.04, 0.045, 0.05\}, \text{ and}$$
$$\lambda \in \{0.02, 0.03, 0.04, 0.05, 0.06, 0.07, 0.08, 0.09, 0.1, 0.11\}.$$

The average optimal revenue rate for these 100 instances under the dynamic pricing policy is $g^* = \$1.70/\text{min}$, with a maximum rate of \$2.69/min and a minimum rate of \$0.63/min. We find that when the demand rate λ is high and the wait time sensitivity β is low (patient customers), dynamic pricing yields the highest revenue. This is intuitive because when customers are patient, they are less likely to balk. With a larger λ, the firm generates more revenue. The dynamic pricing policy generates the next highest revenue when both λ and β are high (impatient customers). In this case, dynamic pricing plays a significant role of regulating the price to ensure customers not to balk when the wait time is long. The dynamic pricing policy yields the least revenue when λ is small and β is high. In this case, while dynamic pricing can effectively induce the impatient customers to join the queue, the arrival rate λ is too small to generate much revenue.

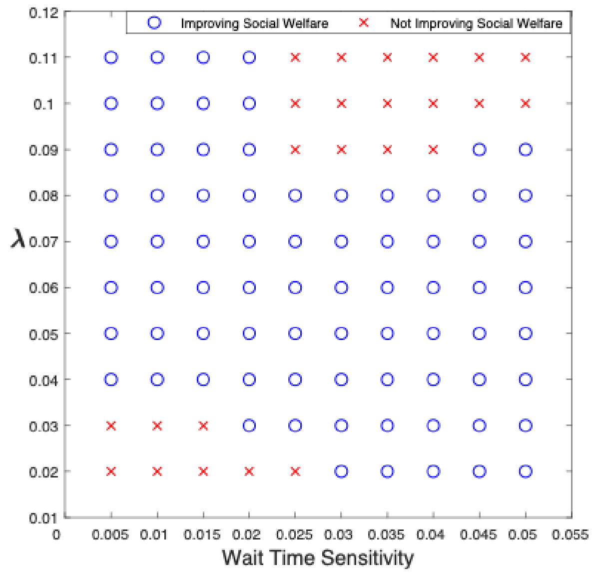
So far, we have focused on considering the dynamic pricing policy alone. Because most fast-charging stations implement flat pricing, it is of interest to understand under what conditions the dynamic pricing policy yields the largest improvement over flat pricing in revenue. First, the average revenue generated from the best flat pricing policy over the same 100 instances is $g_f = \$1.60$/min. In other words, the dynamic pricing policy increases the revenue over the flat pricing one by 6.1% on average, which translates to a revenue increase of $\$4,320$ per month.

Figure 5(a) shows the percentage increase of dynamic pricing over flat pricing for each of the 100 instances. We categorize the entire area into four regions according to the percentage increase. When the arrival rate λ is low and the wait time sensitivity β is high (impatient customers), the benefit of dynamic pricing over flat pricing is most significant. For example, considering the 30 instances with $(\lambda, \beta) \in \{0.02, 0.03, 0.04, 0.05, 0.06\} \times \{0.025, 0.03, 0.035, 0.04, 0.045, 0.05\}$, the average g^* is $1.17/min, which is 12.54% higher than the average g_f. (Note that this is the region where the dynamic pricing yields the least revenue.) This is because when customers are impatient, the dynamic pricing policy generates a significant value to the firm by providing flexibility on the charging time - when the system is not congested (wait time is short), which often is the case, the policy allows a customer to charge a longer period of time. On the other hand, when the system is congested, which happens less frequently, the dynamic pricing policy reduces the charging time to prevent future customers from balking. Following the same logic, when the arrival rate is high and customers are patient, the value of dynamic pricing over flat pricing is minimal. For example, for the instances $(\lambda, \theta) \in \{0.07, 0.08, 0.09, 0.10, 0.11\} \times \{0.005, 0.01, 0.015, 0.02\}$, the average g^* is $2.29/min, which is 3.61% higher than the $g_f = \$2.21$/min. This is because there is not much difference between dynamic pricing and flat pricing, since the system is almost always congested but customers are willing to wait. In such a situation, dynamic pricing does little on customer entry control, thus creating less value over flat pricing.

Maximizing revenue is important for the development of charging stations. From a perspective of public interest, on the other hand, it is also important to investigate the impact of switching from the flat pricing to revenue-maximizing dynamic pricing on the social welfare. Clearly, a revenue-maximizing policy does not necessarily lead to an improvement of social welfare. Our goal is to



(a) Percentage of Revenue Rate Improvement



(b) Instances with Improvement on Social Welfare

Figure 5 Comparison of Revenue and Social Welfare between Dynamic Pricing and Flat-Rate Pricing

investigate the conditions under which both revenue and social welfare are improved when switching from flat pricing to dynamic pricing. For this purpose, we compare the social welfare between the optimal dynamic pricing and the best flat pricing for the same 100 instances under the objective of maximizing the revenue. Among these 100 instances, the average improvement in social welfare is 4.53%. Figure 5(b) illustrates whether the dynamic policy improves social welfare over the flat rate policy. In this figure, a circle represents the case if the social welfare is improved, and a cross otherwise. It is clear that the improvement occurs either when the wait time sensitivity β is high and the demand rate λ is low, or when β is low and λ is high. In the former case, customers are impatient, and dynamic pricing is a powerful lever to increase consumer surplus by reducing service time when wait time becomes long. The increase of revenue and consumer surplus jointly improves the overall social welfare. For the latter case, when customers are patient and the arrival rate is high, dynamic pricing ensures that fewer customers balk from the system while increasing the revenue, which further increases the overall social welfare. Compared with Figure 5(a), it is clear that the dynamic pricing policy improves both the revenue and the social welfare over the flat pricing policy when the arrival rate is not too high and drivers are patient.

Table 3 is a summary of performance metrics under the dynamic and flat pricing policies from our numerical study. As expected, compared with the flat pricing policy, the revenue-maximizing dynamic policy improves not only the revenue over flat pricing policy, but also the social welfare as well as the consumer surplus. That is, dynamic pricing increases efficiency of the system so that the "total pie" is enlarged, which also benefits customers. This is consistent with the last row of Table 3, which shows that the system utilization improves. The only performance metric that

Metrics	Dynamic Pricing	Flat Pricing	Change of %
Revenue Rate	1.70	1.60	6.10%
Social Welfare Rate	2.39	2.28	4.53%
Customer Surplus Rate	0.69	0.68	1.47%
Average Service Time	12.99	10.59	22.63%
Average Wait Time	12.30	8.38	46.85%
Utilization	0.7192	0.6094	18.02%

Table 3 Summary of Performance Metrics for All Instances

deteriorates under dynamic pricing is the average wait time, which increases by a big margin. It is interesting that despite the increase in wait time, the average social welfare still improves, because the dynamic pricing policy often yields longer service times compared with the flat rate policy.

7. Concluding Remarks

This paper studies a dynamic pricing policy in a queueing system where customers arrive according to a Poisson process. The objective is to maximize either the average revenue or social welfare rate. The service provider announces a price rate and the system wait time to each arriving customer, who then decides whether to join the queue, and, if so, how long to be served. Our model is applicable to many discretionary services where people can decide the service time. We formulate this problem as a continuous-time optimal control model. Unlike the majority of queueing-pricing literature, in which system state is often the queue length, our model uses the wait time as the system state variable. This unique modeling feature translates the optimal condition into a set of delay differential equations. We provide an innovative approach to show the existence of the optimal solution by construction. We also characterize the optimal dynamic pricing policy which has a simple and intuitive structure. A numerical study based on real data obtained from a level-2 charging station reveals that when customers are impatient and the arrival rate is low, dynamic pricing can significantly improve revenue over flat pricing, and even improves the social welfare in most scenarios.

It is worth discussing why we study a continuous-time, long-run average model. The analysis for such systems are often more involved than discounted or discrete-time models, and, therefore, requires a justification. First, following Blackwell optimality, we expect that the five-section optimal policy structure continues to hold in a discounted system when the discount rate is close to 0 (or, in a discrete-time model, the discount factor close to 1). However, the corresponding value function is no longer concave, unlike our relative value function $V(w)$. This implies that one may need to devise a new, and potentially even more involved, proof technique to establish similar results in a discounted system. Similarly, some salient features of the optimal value function no longer holds in a discrete-time system, either. For example, the first order condition allows us to relate the price with the service time in a relationship $p = U'(t)$. In a discrete time setting, however, we would have to translate such a simple relationship into two inequalities, which further complicates results and analysis. Therefore, the continuous-time, long-run average model may be the best modeling choice that allows us to rigorously establish the most salient features in the optimal control policy.

Finally, we believe that there are many interesting extensions from our model for future studies. For example, one may consider slightly more richer, but still simple pricing policies, such as a two-part tariff policy, which charges a customer a fixed cost for entering the queue, in addition to the linear payment with respect to the service time. Alternatively, one can also consider combining a pricing rate (without the fixed charge) with an upper or lower bound on service time that dynamically changes over time to control the process. Furthermore, while we consider homogeneous customers, in many applications, customers may be heterogeneous over preferences towards the service. Considering heterogeneous customers naturally brings up the question of potential asymmetric information on customer types. With multiple types of customers who know their own preferences, for example, one may consider designing a menu of pricing decisions that changes according to the wait time. Furthermore, it is interesting to consider the situation where the customer utility function is not known, which allows empirical developments on estimating customer utilities, or algorithmic developments on learning and earning in our setting. Finally, we assume a single server in our model. It would be interesting to extend the analysis to multi-server systems.

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Appendix A: Proofs in Section 3

Proposition 1

We first show (i). Assumption 1(iii) indicates that $c(w)$ strictly increases in w. Because $c(w)$ increases in w from Assumption 1(iii), there is $t \in (0, C]$ such that $U(t) - U'(t)t - c(w) \geq 0$ for each $w \leq \hat{w}$. Moreover, due to $\frac{d}{dt}[U(t) - U'(t)t] > 0$ for all $t \in [0, C]$ from Assumption 1(i) and (ii),we have $U(t) - U'(t)t - c(w) < 0$ for all $t \in [0, C]$ when $w > \hat{w}$. We thus have a unique $\hat{w} > 0$ such that

$$U(C) - U'(C)C - c(\hat{w}) = 0, \tag{40}$$

as we have $U(C) - U'(C)C > 0$ from Assumption 1(i).

We next show (ii). From (1), the price $p = U'(t)$ can induce t (service time) chosen by the customer if the utility is non-negative. As a result, if the balking decision is not considered, the term $U'(t)t$ is the customer payment when service time t is induced. From Assumption 1(ii), we have

$$\frac{d}{dt}[U(t) - U'(t)t - c(w)] = -U''(t)t > 0, \quad \forall t \in [0, C]. \tag{41}$$

Combining (40) and (41), we thus have

$$U(C) - U'(C)C - c(w) \geq 0, \quad \forall w \in [0, \hat{w}].$$

As a result, we define a unique $L(w) \in [0, C]$ for each $w \in [0, \hat{w}]$ that satisfies

$$U(L(w)) - U'(L(w))L(w) - c(w) = 0, \tag{42}$$

due to the monotonicity in (41). Furthermore, from (41) and (42), we have

$$U(t) - U'(t)t - c(w) \geq 0, \quad \forall t \in [L(w), C], \tag{43}$$

for any $w \in [0, \hat{w}]$. Q.E.D.

Proposition 2

We first prove (i). Assumption 1(iii) indicates that $c(w)$ strictly increases in w. We have a unique $\bar{w} > 0$ and

$$U(C) - c(\bar{w}) = 0, \tag{44}$$

because we have $U(C) - c(0) > 0$ and that $c(w)$ increases in w from Assumption 1(i) and (iii).

We next prove (ii). We first show the equivalence in the statement. From (6), if $U(\mathtt{t}(p)) - p\mathtt{t}(p) - c(w) \geq 0$, we have $\mathbf{t}(p, w) = \mathtt{t}(p)$. On the other hand, if $\mathbf{t}(p, w) = \mathtt{t}(p)$, we have $\mathbb{1}_{\{U(\mathtt{t}(p)) - p\mathtt{t}(p) - c(w) \geq 0\}} = 1$, which implies $U(\mathtt{t}(p)) - p\mathtt{t}(p) - c(w) \geq 0$. Before showing the second part of (8), we first show the following: $\mathtt{t}'(p) \leq 0$. Given (5), the customer chooses $\mathtt{t}(p)$ that satisfies

$$U'(\mathtt{t}(p)) = p, \text{ if } p \in [U'(C), U'(0)], \text{ and } \mathtt{t}(p) = C, \text{ if } p \in [0, U'(C)). \tag{45}$$

Taking derivative with respect to p, we have

$$\mathtt{t}'(p) = \frac{1}{U''(\mathtt{t}(p))} < 0, \ \forall p \in [U'(C), U'(0)], \text{ and } \mathtt{t}'(p) = 0, \ \forall p \in [0, U'(0)), \tag{46}$$

because of Implicit Function Theorem. From (46), we have

$$\frac{d}{dp}\left[U\left(\mathtt{t}(p)\right)-p\mathtt{t}(p)\right]=U'\left(\mathtt{t}(p)\right)\mathtt{t}'(p)-\mathtt{t}(p)-p\mathtt{t}'(p)\leq\left[U'\left(\mathtt{t}(p)\right)-p\right]\mathtt{t}'(p)=0,\ \forall p\in[0,U'(0)]. \tag{47}$$

Fix $w\in[0,\bar{w}]$, we thus have $\Gamma(w)\geq 0$ that satisfies $U\left(\mathtt{t}\left(\Gamma(w)\right)\right)-\Gamma(w)\mathtt{t}\left(\Gamma(w)\right)-c(w)=0$, which implies

$$U\left(\mathtt{t}(p)\right)-p\mathtt{t}(p)-c(w)\geq 0,\ \forall p\in\left[0,\Gamma(w)\right],\ \text{and}\ U\left(\mathtt{t}(p)\right)-p\mathtt{t}(p)-c(w)<0,\ \forall p\in\left(\Gamma(w),U'(0)\right], \tag{48}$$

because of the monotonicity in (47).

We finally prove (iii). Given $w\in[0,\bar{w}]$ and (48), we have

$$\mathbf{t}(p,w)=\mathtt{t}(p),\ \forall p\leq\Gamma(w),\ \text{and}\ \Gamma(w_1)\geq\Gamma(w_2),\ \text{if}\ w_1\leq w_2,$$

because the function $c(w)$ is strictly decreasing from Assumption 1(iii). From Assumption 1(i) and (6), we have $\mathtt{t}\left(U'(0)\right)=0$ and $\mathbf{t}\left(U'(0),0\right)=0$, which imply $U\left(\mathtt{t}(U'(0))\right)-U'(0)\mathtt{t}(U'(0))-c(0)=0$. As a result, from (48), we have $\Gamma(0)=U'(0)$. Q.E.D.

Proposition 3

We first show (V1). Recall that $\mathbf{v}\left(p\right)=p\mathtt{t}(p)$ if the objective is maximizing the revenue, and $\mathbf{v}\left(p\right)=U\left(\mathbf{t}(p)\right)$ if maximizing the social welfare. From Assumption 1(i) and (6), it is easy to show that both expressions of $\mathbf{v}\left(p\right)\leq U'(0)\mathtt{t}(p),\ \ \forall\ p\in[0,U'(0)]$.

We now show (V2). From (48), we have

$$\mathbf{v}(p)-\alpha c(0)=\mathbf{v}(p)=0,\ \forall\ p\geq\Gamma(0)=U'(0). \tag{49}$$

Moreover, there exists a price $p\in(0,\Gamma(0)]$ which induces $\mathbf{t}(p,0)>0$ such that

$$\mathbf{v}(p)-\alpha c(0)=\mathbf{v}(p)>0, \tag{50}$$

because we have either $p\mathbf{t}(p,0)>0$ or $U(\mathbf{t}(p,0))-c(0)=U(\mathbf{t}(p,0))>0$ from Assumption 1(i) and (iii). Q.E.D.

Appendix B: Proofs in Section 4.1

Proposition 4

We prove the result by induction. For $i=1$, $x_{N-i}=x_{N-1}$, and we show that $V_N^g(w)$ is concave and $K_N^g(w)$ decreases in $w\in[x_{N-1},\bar{w}+C]$. From (19), we have

$$K_N^g\left(x_{N-i+1}\right)=K_N^g(\bar{w})=\lambda\max\left\{\max_{0\leq p\leq\Gamma(\bar{w})}\left\{\mathbf{v}(p)-\alpha c\left(\bar{w}\right)+V_N^g\left(\bar{w}+\mathtt{t}(p)\right)-V_N^g(\bar{w})\right\},0\right\}.$$

From (21), we have

$$V_N^g\left(x_{N-1}\right)=V_N^g\left(x_N-\epsilon_N\right)=V_N^g\left(\bar{w}-\epsilon_N\right)=V_N^g\left(\bar{w}\right)-\epsilon_N\left(-g+K_N^g\left(x_N\right)\right),$$

and

$$V_N^g(w)=V_N^g\left(x_{N-1}\right)+\left(w-x_{N-1}\right)\left(-g+K_N^g\left(x_N\right)\right),\quad\forall w\in\left[x_{N-1},x_N\right].$$

From (16), (17), (19), and (21), the function $V_N^g(w)$ is concave in $w \in [x_{N-1}, \bar{w}+C]$ because

$$K_N^g(\bar{w}) \geq K_N^g(w) = 0,\ \forall w \in (\bar{w}, \bar{w}+C].$$

As a result, we have

$$\begin{aligned} V_N^g\left(x_{N-1}+\mathtt{t}\left(p\right)\right)-V_N^g\left(x_{N-1}\right) &= V_N^g\left(x_N-\epsilon_N+\mathtt{t}\left(p\right)\right)-V_N^g\left(x_N-\epsilon_N\right) \\ &\geq V_N^g\left(x_N+\mathtt{t}\left(p\right)\right)-V_N^g\left(x_N\right), \quad \forall p \in [0, \Gamma(x_N)]. \end{aligned} \tag{51}$$

We thus have

$$\begin{aligned} K_N^g\left(x_{N-i}\right) =& K_N^g\left(x_{N-1}\right) = \lambda \max\left\{\max_{0\leq p\leq \Gamma\left(x_{N-1}\right)}\left\{\mathbf{v}\left(p\right)-\alpha c\left(x_{N-1}\right)+V_N^g\left(x_{N-1}+\mathtt{t}\left(p\right)\right)-V_N^g\left(x_{N-1}\right)\right\},0\right\} \\ &\geq \lambda \max\left\{\max_{0\leq p\leq \Gamma(x_N)}\left\{\mathbf{v}\left(p\right)-\alpha c\left(x_N\right)+V_N^g\left(x_N+\mathtt{t}\left(p\right)\right)-V_N^g\left(x_N\right)\right\},0\right\} = K_N^g\left(x_N\right) = K_N^g\left(\bar{w}\right). \end{aligned}$$

The inequality hods because $\Gamma(x_{N-1}) \geq \Gamma(x_N)$ from Proposition 2 and we have

$$\begin{aligned} &\mathbf{v}\left(p\right)-\alpha c\left(x_{N-1}\right)+V_N^g\left(x_{N-1}+\mathtt{t}(p)\right)-V_N^g\left(x_{N-1}\right) \\ &\qquad \geq \mathbf{v}\left(p\right)-\alpha c\left(x_N\right)+V_N^g\left(x_N+\mathtt{t}\left(p\right)\right)-V_N^g\left(x_N\right), \quad \forall p\in[0,\Gamma(x_N)], \end{aligned}$$

from (51). Therefore, combining (20) with that $K_N^g\left(x_{N-1}\right) \geq K_N^g\left(\bar{w}\right)$, we conclude that $K_N^g(x)$ decreases in $[x_{N-1}, \bar{w}+C]$ and thus the case $i=1$ holds.

We assume $i=n$ holds, i.e., the function $V_N^g(w)$ is concave in $w \in [x_{N-n}, \bar{w}+C]$ and the adjustment $K_N^g(w)$ decreases in $w \in [x_{N-n}, \bar{w}+C]$. For $i=n+1$, (21) indicates

$$V_N^g\left(x_{N-(n+1)}\right) = V_N^g\left(x_{N-n}-\epsilon_N\right) = V_N^g\left(x_{N-n}\right) - \epsilon_N\left(-g+K_N^g\left(x_{N-n}\right)\right),$$

and

$$V_N^g(w) = V_N^g\left(x_{N-(n+1)}\right) + \left(w - x_{N-(n+1)}\right)\left(-g+K_N^g\left(x_{N-n}\right)\right), \quad \forall w \in \left[x_{N-(n+1)}, x_{N-n}\right]. \tag{52}$$

From the induction assumption and (52), the function $V_N^g(w)$ is concave in $w \in [x_{N-n}, \bar{w}+C]$ and we have

$$\begin{aligned} V_N^g\left(x_{N-(n+1)}+\mathtt{t}\left(p\right)\right)-V_N^g\left(x_{N-(n+1)}\right) &= V_N^g\left(x_{N-n}-\epsilon_N+\mathtt{t}\left(p\right)\right)-V_N^g\left(x_{N-n}-\epsilon_N\right) \\ &\geq V_N^g\left(x_{N-n}+\mathtt{t}\left(p\right)\right)-V_N^g\left(x_{N-n}\right), \quad \forall p\in[0,\Gamma\left(x_{N-n}\right)]. \end{aligned} \tag{53}$$

Therefore, we have

$$\begin{aligned} K_N^g\left(x_{N-(n+1)}\right) =& \lambda \max\left\{\max_{0\leq p\leq \Gamma\left(x_{N-(n+1)}\right)}\left\{\mathbf{v}\left(p\right)-\alpha c\left(x_{N-(n+1)}\right)+V_N^g\left(x_{N-(n+1)}+\mathtt{t}\left(p\right)\right)-V_N^g\left(x_{N-(n+1)}\right)\right\},0\right\} \\ &\geq \lambda \max\left\{\max_{0<\leq p\leq \Gamma\left(x_{N-n}\right)}\left\{\mathbf{v}\left(p\right)-\alpha c\left(x_{N-n}\right)+V_N^g\left(x_{N-n}+\mathtt{t}\left(p\right)\right)-V_N^g\left(x_{N-n}\right)\right\},0\right\} = K_N^g\left(x_{N-n}\right), \end{aligned}$$

because we have $\Gamma\left(x_{N-(n+1)}\right) \geq \Gamma\left(x_{N-n}\right)$ from Proposition 2 and

$$\begin{aligned} &\mathbf{v}\left(p\right)-\alpha c\left(x_{N-(n+1)}\right)+V_N^g\left(x_{N-(n+1)}+\mathtt{t}\left(p\right)\right)-V_N^g\left(x_{N-(n+1)}\right) \\ &\qquad \geq \mathbf{v}\left(p\right)-\alpha c\left(x_{N-n}\right)+V_N^g\left(x_{N-n}+\mathtt{t}\left(p\right)\right)-V_N^g\left(x_{N-n}\right), \quad \forall p\in[0,\Gamma\left(x_{N-n}\right)], \end{aligned}$$

from (53). This completes the proof. Q.E.D.

Lemma 2

We prove the result by induction. We denote $A(g,j)$ as Algorithm 1 with given g and the number of steps j and denote $A(g,2j)$ as that with value g and the number of steps $2j$. We label grids in $A(g,j)$ as $\left\{x_0^j, x_1^j, ..., x_j^j\right\}$ with the corresponding step size ϵ_j which follows (18) and $A(g,2j)$ has twice the number of grids labeled as $\left\{x_0^{2j}, x_1^{2j}, ..., x_{2j}^{2j}\right\}$ with step size $\epsilon_{2j} = \frac{1}{2}\epsilon_j$. Grids for two algorithms coincide at $x_i^j = x_{2i}^{2j}$ for $i \in \{0,1,2,...,j\}$ and we use these grids $\{x_0^j, x_1^j, ...x_j^j\}$ in the below induction to show $V_j^g(w) \geq V_{N_{2j}}^g(w)$ and $K_j^g(w) \geq K_{N_{2j}}^g(w)$ for all $w \in [0, \bar{w}]$.

From (16) and (17), we have

$$V_j^g(w) = V_{2j}^g(w) = -(w - \bar{w})g, \quad \forall w \in [\bar{w}, \bar{w} + C], \text{ and } K_j^g(w) = K_{2j}^g(w) = 0, \quad \forall w \in (\bar{w}, \bar{w} + C]. \tag{54}$$

Moreover, from (19) and (54), we have

$$K_j^g(\bar{w}) = K_j^g\left(x_j^j\right) = K_{2j}^g\left(x_j^j\right) = K_{2j}^g(\bar{w}). \tag{55}$$

For $i = 1$, $x_{j-i}^j = x_{j-1}^j$, and we show that $V_j^g(w) \geq V_{2j}^g(w)$ and $K_j^g(w) \leq K_{2j}^g(w)$ for all $w \in [x_{j-1}^j, \bar{w} + C]$. First, we have

$$\begin{aligned}
V_{2j}^g\left(x_{j-i}^j\right) &= V_{2j}^g\left(x_{j-1}^j\right) = V_{2j}^g\left(\bar{w} - \epsilon_j\right) \\
&= V_{2j}^g\left(x_j^j - \epsilon_{2j}\right) - \epsilon_{2j}\left(-g + K_{2j}^g\left(\bar{w} - \epsilon_{2j}\right)\right) \\
&\leq V_j^g\left(x_j^j - \epsilon_{2j}\right) - \epsilon_{2j}\left(-g + K_{2j}^g\left(\bar{w}\right)\right) \\
&= V_j^g\left(x_j^j - \epsilon_{2j}\right) - \epsilon_{2j}\left(-g + K_j^g\left(\bar{w}\right)\right) \\
&= V_j^g\left(\bar{w}\right) - \epsilon_j\left(-g + K_j^g\left(\bar{w}\right)\right) \\
&= V_j^g\left(\bar{w} - \epsilon_j\right) = V_j^g\left(x_{j-1}^j\right).
\end{aligned}$$

The last equality holds because $V_j^g\left(\bar{w} - \epsilon_{2j}\right) = V_{2j}^g\left(\bar{w} - \epsilon_{2j}\right)$ from (21) and (55), $K_{2j}^g\left(\bar{w} - \epsilon_{2j}\right) \geq K_{2j}^g\left(\bar{w}\right)$ from Proposition 4, and (21). Moreover, we have

$$-g + K_{2j}^g(w) \geq -g + K_j^g(w), \quad \forall w \in \left(x_{j-1}^j, \bar{w} + C\right], \tag{56}$$

from (20), because Proposition 4 implies

$$K_{2j}^g(\bar{w} - \epsilon_{2j}) \geq K_j^g(\bar{w}) = K_{2j}^g(\bar{w}).$$

Combining (21) and (56), we have

$$V_{2j}^g\left(x_{j-1}^j + \mathtt{t}(p)\right) - V_{2j}^g\left(x_{j-1}^j\right) \geq V_j^g\left(x_{j-1}^j + \mathtt{t}(p)\right) - V_j^g\left(x_{j-1}^j\right), \quad \forall p \in \left[0, \Gamma\left(x_{j-1}^j\right)\right]. \tag{57}$$

Therefore, for the grid $x_{j-i}^j = x_{j-1}^j = \bar{w} - \epsilon_j$, (57) implies

$$\begin{aligned}
K_j^g\left(x_{j-1}^j\right) &= \lambda \max\left\{\max_{0 \leq p \leq \Gamma\left(x_{j-1}^j\right)}\left\{\mathbf{v}(p) - \alpha c\left(x_{j-i}^j\right) + V_j^g\left(x_{j-1}^j + \mathtt{t}(p)\right) - V_j^g\left(x_{j-i}^j\right)\right\}, 0\right\} \\
\leq K_{2j}^g\left(x_{j-1}^j\right) &= \lambda \max\left\{\max_{0 \leq p \leq \Gamma\left(x_{j-1}^j\right)}\left\{\mathbf{v}(p) - \alpha c\left(x_{j-1}^j\right) + V_{2j}^g\left(x_{j-1}^j + \mathtt{t}(p)\right) - V_{2j}^g\left(x_{j-1}^j\right)\right\}, 0\right\}.
\end{aligned}$$

As a result, from the above inequality, (20), and (21), the base step $i=1$ holds, i.e., $V_j^g(w) \geq V_{2j}^g(w)$ and $K_j^g(w) \leq K_{2j}^g(w)$ for all $w \in [x_{j-1}^j, \bar{w}+C]$.

We assume the step $i=n$ holds, i.e., both $V_j^g(w) \geq V_{2j}^g(w)$ and $K_j^g(w) \leq K_{2j}^g(w)$ hold for all $w \in \left[x_{j-n}^j, \bar{w}+C\right]$. For the induction step $i=n+1$, we have

$$\begin{aligned} V_{2j}^g\left(x_{j-(n+1)}^j\right) =& V_{2j}^g\left(x_{j-n}^j\right) - \epsilon_{2j}\left(-g+K_{2j}^g\left(x_{j-n}^j\right)\right) - \epsilon_{2j}\left(-g+K_{2j}^g\left(x_{j-n}^j-\epsilon_{2j}\right)\right) \\ \leq& V_j^g\left(x_{j-n}^j\right) - \epsilon_j\left(-g+K_j^g\left(x_{j-n}^j\right)\right) = V_j^g\left(x_{j-(n+1)}^j\right). \end{aligned}$$

This holds because we have $K_{2j}^g\left(x_{j-n}^j-\epsilon_{2j}\right) \geq K_{2j}^g\left(x_{j-n}^j\right) \geq K_j^g\left(x_{j-n}^j\right)$ due to the induction assumption and Proposition 4. Furthermore, from the induction assumption and (20), we have

$$-g+K_{2j}^g(w) \geq -g+K_j^g(w), \quad \forall w \in \left(x_{j-(n+1)}^j, \bar{w}+C\right].$$

From (21), we thus have

$$\begin{aligned} & V_{2j}^g\left(x_{j-(n+1)}^j+\mathtt{t}(p)\right) - V_{2j}^g\left(x_{j-(n+1)}^j\right) \\ & \qquad \geq V_j^g\left(x_{j-(n+1)}^j+\mathtt{t}(p)\right) - V_j^g\left(x_{j-(n+1)}^j\right), \quad \forall p \in \left[0, \Gamma\left(x_{j-(n+1)}^j\right)\right]. \end{aligned} \tag{58}$$

Therefore, from (58), we have

$$\begin{aligned} K_j^g\left(x_{j-(n+1)}^j\right) &= \lambda \max\left\{\max_{0\leq p\leq \Gamma\left(x_{j-(n+1)}^j\right)}\left\{\mathbf{v}(p) - \alpha c\left(x_{j-(n+1)}^j\right) + V_j^g\left(x_{j-(n+1)}^j+\mathtt{t}\left(x_{j-(n+1)}^j\right)\right) - V_j^g\left(x_{j-(n+1)}^j\right)\right\}, 0\right\} \\ \leq K_{2j}^g\left(x_{j-(n+1)}^j\right) &= \lambda \max\left\{\max_{0\leq p\leq \Gamma\left(x_{j-(n+1)}^j\right)}\left\{\mathbf{v}(p) - c\left(x_{j-(n+1)}^j\right) + V_{2j}^g\left(x_{j-(n+1)}^j+\mathtt{t}\left(x_{j-(n+1)}^j\right)\right) - V_{2j}^g\left(x_{j-(n+1)}^j\right)\right\}, 0\right\}, \end{aligned}$$

As a result, from the above inequality, (20), and (21), the induction step $i=n+1$ holds. This completes the proof. Q.E.D.

Lemma 3

We prove the result by induction. From (16), (19), and $g_2 > g_1 \geq 0$, we have

$$V_N^{g_1}(w) = -g_1(w-\bar{w}) \geq -g_2(w-\bar{w}) = V_N^{g_2}(w), \quad \forall w \in [\bar{w}, \bar{w}+C],$$

and

$$K_N^{g_1}(x_N) = K_N^{g_1}(\bar{w}) \geq K_N^{g_2}(\bar{w}) = K_N^{g_2}(x_N). \tag{59}$$

For the base step $i=1$, we consider the grid $x_{N-i} = x_{N-1}$ and show that $K_N^{g_1}(w) \geq K_N^{g_2}(\mathrm{w})$ holds for all $w \in [x_{N-1}, \bar{w}+C]$. Equation (59) implies

$$V_N^{g_1}(x_{N-1}+\mathtt{t}(p)) - V_N^{g_1}(x_{N-1}) \geq V_N^{g_2}(x_{N-1}+\mathtt{t}(p)) - V_N^{g_2}(x_{N-1}), \quad \forall p \in [0, \Gamma(x_{N-1})],$$

from (21) and $K_N^{g_1}(w) \geq K_N^{g_2}(w)$ for all $w \in (x_{N-1}, \bar{w}]$ from (20). We thus have

$$\begin{aligned} K_N^{g_1}(x_{N-i}) &= K_N^{g_1}(x_{N-1}) \\ &= \lambda \max\left\{\max_{0\leq p\leq \Gamma\left(x_{N-1}\right)}\left\{\mathbf{v}(p) - \alpha c(x_{N-1}) + V_N^{g_1}(x_{N-1}+\mathtt{t}(p)) - V_N^{g_1}(x_{N-1})\right\}, 0\right\} \\ &\geq \lambda \max\left\{\max_{0\leq p\leq \Gamma\left(x_{N-1}\right)}\left\{\mathbf{v}(p) - \alpha c(x_{N-1}) + V_N^{g_2}(x_{N-1}+\mathtt{t}(p)) - V_N^{g_2}(x_{N-1})\right\}, 0\right\} \\ &= K_N^{g_2}(x_{N-1}). \end{aligned}$$

As a result, the base step $i=1$ holds, i.e., $K_N^{g_1}(w) \geq K_N^{g_2}(w)$ for all $w \in [x_{N-1}, \bar{w}]$.

We assume the property holds from the base step $i=1$ to the step $i=n$, i.e., $K_N^{g_1}(w) \geq K_N^{g_2}(\mathrm{w})$ holds for all $w \in [x_{N-n}, \bar{w}+C]$.

For the induction step $i=n+1$, we have

$$V_N^{g_1}\left(x_{N-(n+1)}+\mathtt{t}(p)\right)-V_N^{g_1}\left(x_{N-(n+1)}\right) \\ \geq V_N^{g_2}\left(x_{N-(n+1)}+\mathtt{t}(p)\right)-V_N^{g_2}\left(x_{N-(n+1)}\right), \quad \forall p \in \left[0, \Gamma\left(x_{N-(n+1)}\right)\right], \tag{60}$$

due to (21) and $K_N^{g_1}(w) \geq K_N^{g_2}(w)$ for all $w \in (x_{N-(n+1)}, \bar{w}+C]$ from the induction assumption and (20). From (60), we thus have

$$\begin{aligned} K_N^{g_1}\left(x_{N-(n+1)}\right) =& \lambda \max\left\{\max_{0\leq p\leq \Gamma\left(x_{N-(n+1)}\right)}\left\{\mathbf{v}(p)-\alpha c\left(x_{N-(n+1)}\right)+V_N^{g_1}\left(x_{N-(n+1)}+\mathtt{t}(p)\right)-V_N^{g_1}\left(x_{N-(n+1)}\right)\right\}, 0\right\} \\ \geq& \lambda \max\left\{\max_{0\leq p\leq \Gamma\left(x_{N-(n+1)}\right)}\left\{\mathbf{v}(p)-\alpha c\left(x_{N-(n+1)}\right)+V_N^{g_2}\left(x_{N-(n+1)}+\mathtt{t}(p)\right)-V_N^{g_2}\left(x_{N-(n+1)}\right)\right\}, 0\right\} \\ =& K_N^{g_2}\left(x_{N-(n+1)}\right). \end{aligned}$$

As a result, the induction step $i=n+1$ holds. This completes the proof. Q.E.D.

Proposition 5

To prove this proposition, we need **Lemma** 2, **Lemma** 3, and the following Lemma 5.

LEMMA 5. *The slope adjustment* $K_N^{U'(0)}(w)=0$ *for all* $w \in [0, \bar{w}+C]$ *and for all* $N \in \mathbb{N}$.

We first prove Lemma 5 by induction. For the base step $i=1$, we consider the grid $x_{N-i}=x_{N-1}$ and show that $K_N^{U'(0)}(w)=0$ for all $w \in [x_{N-1}, \bar{w}+C]$. From (16) and (19), we have

$$K_N^{U'(0)}(x_{N-i+1})=K_N^{U'(0)}(\bar{w})=\lambda \max\left\{\max_{0\leq p\leq \Gamma(\bar{w})}\left\{\mathbf{v}(p)-\alpha c(\bar{w})+(-U'(0))\,\mathtt{t}(p)\right\}, 0\right\}=0.$$

Following (V1), we have

$$\mathbf{v}(p)-\alpha c(w)+(-U'(0))\,\mathtt{t}(p) \leq 0, \;\; \forall\; p \in [0, U'(0)], \;\; w \in [0, \bar{w}]. \tag{61}$$

Therefore, we have $K_N^{U'(0)}(x_{N+i-1})=K_N^{U'(0)}(\bar{w})=0$ and

$$V_N^{U'(0)}(w)=-(w-\bar{w})\,U'(0), \quad \forall w \in [x_{N-1}, \bar{w}], \tag{62}$$

from (21). Given (62), the calculation of $K_N^{U'(0)}(x_{N-1})$ in (19) is similar to that of $K_N^{U'(0)}(x_N)$ with a larger feasible set $[0, \Gamma(x_{N-1})]$ from Proposition 2. Therefore,

$$K_N^{U'(0)}(x_{N-i})=K_N^{U'(0)}(x_{N-1})=\lambda \max\left\{\max_{0\leq p\leq \Gamma\left(x_{N-1}\right)}\left\{\mathbf{v}(p)-\alpha c(x_{N-1})+(-U'(0))\,\mathtt{t}(p)\right\}, 0\right\}=0.$$

The term $U'(0)\mathtt{t}(p)$ is still greater than that of the reward $\mathbf{v}(p)-\alpha c(w)$ as shown in (61) from (V1), so it is not optimal to induce any service time with any price rate $p \in [0, \Gamma(x_{N-1})]$. As a result, the base step $i=1$ holds, i.e., $K_N^{U'(0)}(w)=0$ for all $w \in [x_{N-1}, \bar{w}+C]$.

We assume the step $i = n$ holds, i.e., $K_N^{U'(0)}(w) = 0$ for all $w \in [x_{N-n}, \bar{w} + C]$. For the induction step $i = n+1$, we have

$$V_N^{U'(0)}(x_{N-i} + \mathtt{t}(p)) - V_N^{U'(0)}(x_{N-i}) = V_N^{U'(0)}\left(x_{N-(n+1)} + \mathtt{t}(p)\right) - V_N^{U'(0)}\left(x_{N-(n+1)}\right) = -\mathtt{t}(p)(U'(0)), \quad \forall p \in [0, \Gamma\left(x_{N-(n+1)}\right)]. \tag{63}$$

from (20), (21), and the induction assumption. Similarly, following (19), (61) and (63), we have

$$K_N^{U'(0)}(x_{N-i}) = K_N^{U'(0)}\left(x_{N-(n+1)}\right) = \lambda \max\left\{\max_{0 \le p \le \Gamma\left(x_{N-(n+1)}\right)}\left\{\mathbf{v}(p) - \alpha c\left(x_{N-(n+1)}\right) + (-U'(0))\,\mathtt{t}(p)\right\}, 0\right\} = 0,$$

As a result, from (20), we have $K_N^{U'(0)}\left(x_{N-(n+1)}\right) = 0$ for all $w \in \left[x_{N-(n+1)}, \bar{w} + C\right]$. The induction step $i = n+1$ holds. This completes the proof of Lemma 5. ■

Now we are ready to show Proposition 5. Lemma 5 implies a feasible solution $U'(0)$ to the problem of (23) for all positive integer j. Therefore, following (22) and (23), we have $g_j \ge 0$ and $g_j \le U'(0)$, for all $j \in \mathbb{N}$. On the other hand, **Lemma** 2 implies

$$K_{2j}^{g}(x_0) = K_{2j}^{g}(0) \ge K_j^{g}(x_0) = K_j^{g}(0), \ \forall g \in [0, U'(0)]. \tag{64}$$

From (64) and **Lemma** 3, any feasible solution in the problem of (23) that determines g_{2j} is also feasible for the problem that determines g_j. That is, we have $g_{2j} \ge g_j$. Consequently, define $\bar{g} := \lim_{j\to\infty} g_{2^j}$, which is well defined because $\{g_{2^j}\}_{j=1}^{\infty}$ is an increasing sequence in a bounded compact set $[0, U'(0)]$.   Q.E.D.

Proposition 6

To prove Proposition 6, we first prove the following Lemma 6.

LEMMA 6. *Given any positive integer j, we have*

1. $V_{2^j}^{\bar{g}}(w)$ *decreases and is concave in w on the interval* $[0, \bar{w} + C]$.
2. $V_{2^j}^{\bar{g}}(w)$ *is Lipschitz continuous in w with Lipschitz constant $\bar{g}$.*

The concavity of $V_{2^j}^{\bar{g}}(w)$ in w follows Proposition 4. Moreover, Proposition 5 and (23) imply

$$-\bar{g} + K_{2^j}^{\bar{g}}(w) \le 0, \quad \forall w \in [0, \bar{w} + C], \quad j \in \mathbb{N}.$$

As a result, the function $V_{2^j}^{\bar{g}}(w)$ decreases in w on the interval $[0, \bar{w} + C]$ from (21).

To show the Lipschitz continuity, we let x be greater than y on the interval $[0, \bar{w} + C]$ without loss of generality. Following (21), we have

$$\left|V_{2^j}^{\bar{g}}(x) - V_{2^j}^{\bar{g}}(y)\right| = \left|\int_y^x \left(-\bar{g} + K_{2^j}^{\bar{g}}(w)\right) dw\right| \le \left|\int_y^x \left(-\bar{g} + K_{2^j}^{\bar{g}}(0)\right) dw\right| \le \bar{g}|x - y|.$$

The inequality holds because we have $K_{2^j}^{\bar{g}}(0) \ge K_{2^j}^{\bar{g}}(w)$ from Proposition 4 and $\bar{g} \ge \left|-\bar{g} + K_{2^j}^{\bar{g}}(0)\right| \ge 0$ from Proposition 5 and (23). As a result, the function $V_{2^j}^{\bar{g}}(w)$ is Lipschitz continuous in w on the interval $[0, \bar{w} + C]$ with Lipschitz constant $\bar{g}$ for all $j \in \mathbb{N}$. This completes the proof of Lemma 6. ■

We now are ready to show Proposition 6. From (16) and Lemma 6, we have

$$-(\bar{w}+C)\bar{g} \le V^{\bar{g}}_{2^j}(w) \le \bar{w}\bar{g}, \quad \forall w \in [0, \bar{w}+C], \quad \forall j \in \mathbb{N}.$$

Let

$$V^{\bar{g}}(w) := \lim_{j\to\infty} V^{\bar{g}}_{2^j}(w), \tag{65}$$

which is well-defined because the sequence $\left\{V^{\bar{g}}_{2^j}(w)\right\}_{j=1}^{\infty}$ decreases in j from Lemma 2 and the sequence is bounded by the compact set $[-(\bar{w}+C)\bar{g}, \bar{w}\bar{g}]$.

We first show part (ii). The function $V^{\bar{g}}(w)$ decreases and is concave in $w \in [0, \bar{w}+C]$. We consider two states $x, y \in [0, \bar{w}+C]$ and assume $x > y$ without loss of generality. From Lemma 6, we have $V^{\bar{g}}_{2^j}(x) \le V^{\bar{g}}_{2^j}(y)$, for all $j \in \mathbb{N}$. This inequality further implies

$$V^{\bar{g}}(x) = \lim_{j\to\infty} V^{\bar{g}}_{2^j}(x) \le \lim_{j\to\infty} V^{\bar{g}}_{2^j}(y) = V^{\bar{g}}(y).$$

To show the concavity, we consider $\lambda \in [0,1]$. From Proposition 4, we have

$$V^{\bar{g}}_{2^j}(\lambda x + (1-\lambda) y) \ge \lambda V^{\bar{g}}_{2^j}(x) + (1-\lambda) V^{\bar{g}}_{2^j}(y).$$

Therefore, we have

$$V^{\bar{g}}(\lambda x + (1-\lambda) y) = \lim_{j\to\infty} \left[V^{\bar{g}}_{2^j}(\lambda x + (1-\lambda) y)\right] \ge \lim_{j\to\infty} \left[\lambda V^{\bar{g}}_{2^j}(x) + (1-\lambda) V^{\bar{g}}_{2^j}(y)\right] = \lambda V^{\bar{g}}(x) + (1-\lambda) V^{\bar{g}}(y).$$

We next show part (iii). We claim the function $V^{\bar{g}}(w)$ is Lipschitz continuous in w on the interval $[0, \bar{w}+C]$ with constant $\bar{g}$. Given two states w and $w' \in [0, \bar{w}+C]$, we have

$$\begin{aligned} |V^{\bar{g}}(w) - V^{\bar{g}}(w')| &= \left|\lim_{j\to\infty} V^{\bar{g}}_{2^j}(w) - \lim_{j\to\infty} V^{\bar{g}}_{2^j}(w')\right| \\ &= \left|\lim_{j\to\infty} \left(V^{\bar{g}}_{2^j}(w) - V^{\bar{g}}_{2^j}(w')\right)\right| \le \bar{g}\,|(w-w')|, \end{aligned} \tag{66}$$

from the Lipschitz continuity proven in Lemma 6. As a result, the function $V^{\bar{g}}(w)$ is Lipschitz continuous in w with Lipschitz constant $\bar{g}$ on the closed interval $[0, \bar{w}+C]$.

Lastly, we show part (i). Each function in the sequence of functions $\left\{V^{\bar{g}}_{2^j}(w)\right\}_{j=1}^{\infty}$ is continuous in w on the compact set $[0, \bar{w}+C]$ from Lemma 6. Moreover, the sequence of functions $\{V^{\bar{g}}_{2^j}(w)\}_{j=1}^{\infty}$ decreases in the index j because we have

$$V^{\bar{g}}_{2^{j+1}}(w) \le V^{\bar{g}}_{2^j}(w),$$

from Lemma 2. From (66), $V^{\bar{g}}(w)$ is continuous in w on the compact set $[0, \bar{w}+C]$. As a result, the convergence of the sequence of function $\left\{V^{\bar{g}}_{2^j}(w)\right\}_{j=1}^{\infty}$ is uniform convergence because sufficient conditions of Dini's theorem are satisfied. Q.E.D.

Proposition 7

To prove Proposition 7, we first show the following Lemma 7.

LEMMA 7.

$$\bar{g} = \lim_{j\to\infty} K^{\bar{g}}_{2^j}(0).$$

From (19) and (21) in Algorithm 1, $V_N^g(w)$ and $K_N^g(w)$ are continuous in g for any state $w \in [0, \bar{w}]$ and any number of steps $N \in \mathbb{N}$. As a result, the constraint in (23) is binding and we have

$$g_{2^j} = K_{2^j}^{g_{2^j}}(0), \quad \forall j \in \mathbb{N}. \tag{67}$$

Given (67) and the definition of $\bar{g}$ in Proposition 5, we have

$$\bar{g} := \lim_{j\to\infty} g_{2^j} = \lim_{j\to\infty} K_{2^j}^{g_{2^j}}(0). \tag{68}$$

For a small $\epsilon > 0$, the monotonicity of g_{2_j} in Proposition 5 and the convergence in (68) imply there is an index $j_\epsilon \in \mathbb{N}$ such that $\bar{g} - g_{2^j} < \epsilon$, and $\bar{g} - K_{2^j}^{g_{2^j}}(0) < \epsilon$, for all $j > j_\epsilon$. Furthermore, these two inequalities imply

$$K_{2^j}^{\bar{g}-\epsilon}(0) \geq K_{2^j}^{g_{2^j}}(0) \geq K_{2^j}^{\bar{g}}(0), \quad \forall j > j_\epsilon,$$

because $K_{2^j}^g(w)$ decreases in g from Lemma 3. As a result, because $\epsilon > 0$ is arbitrary, we have

$$\lim_{j\to\infty} K_{2^j}^{g_{2^j}}(0) = \lim_{j\to\infty} K_{2^j}^{\bar{g}}(0). \tag{69}$$

Combining (69) with (68), we have

$$\bar{g} = \lim_{j\to\infty} K_{2^j}^{\bar{g}}(0).$$

■

We now are ready to show Proposition 7. From the Lemma 7, we have $\bar{g} = \lim_{j\to\infty} K_{2^j}^{\bar{g}}(0)$. From Proposition 6, we have

$$\begin{aligned}
\bar{g} &= \lim_{j\to\infty} K_{2^j}^{\bar{g}}(0) \\
&= \lim_{j\to\infty} \lambda \max\left\{ \max_{0\leq p\leq \Gamma(0)} \left\{ \mathbf{v}(p) + V_{2^j}^{\bar{g}}(\mathtt{t}(p)) - V_{2^j}^{\bar{g}}(0) \right\}, 0 \right\} \\
&= \lim_{j\to\infty} \left\{ \lambda \max\left\{ \max_{0\leq p\leq \Gamma(0)} \left\{ \mathbf{v}(p) + V_{2^j}^{\bar{g}}(\mathtt{t}(p)) \right\}, V_{2^j}^{\bar{g}}(0) \right\} - V_{2^j}^{\bar{g}}(0) \right\} \\
&= \lambda \max\left\{ \max_{0\leq p\leq \Gamma(0)} \left\{ \mathbf{v}(p) + V^{\bar{g}}(\mathtt{t}(p)) \right\}, V^{\bar{g}}(0) \right\} - V^{\bar{g}}(0) \\
&= \lambda \max\{ \max_{0\leq p\leq \Gamma(0)} \left\{ \mathbf{v}(p) + V^{\bar{g}}(\mathtt{t}(p)) - V^{\bar{g}}(0) \right\}, 0 \}.
\end{aligned}$$

Q.E.D.

Proposition 8

To prove Proposition 8, we first show Lemmas 8-12 below.

LEMMA 8. *Given $g \geq 0$, any positive integer j, and $0 \leq a < b \leq \bar{w}$, we have*

$$V_{2j}^g(b) - V_{2j}^g(a) \geq V_j^g(b) - V_j^g(a).$$

We first prove Lemma 8. We consider grids $\{x_0, x_1, x_2, \ldots x_j\}$ in (18) for the algorithm $A(g, j)$ and denote $b \in [x_m, x_{m+1}]$ and $a \in [x_n, x_{n+1}]$ for $m \geq n$. Following (21), $(V_j^g(b) - V_j^g(a))$ is the difference of piece-wise linear sections:

$$\begin{aligned}
V_j^g(b) - V_j^g(a) = &(x_{n+1} - a)\left(-g + K_j^g(x_n)\right) + (b - x_m)\left(-g + K_j^g(x_m)\right) \\
&+ \sum_{s=n+1}^{m} (\epsilon_j)\left(-g + K_j^g(x_s)\right).
\end{aligned} \tag{70}$$

Thus, we have

$$
\begin{aligned}
V_{2j}^{g}(b)-V_{2j}^{g}(a) \geq & (x_{n+1}-a)\left(-g+K_{2j}^{g}(x_n)\right)+(b-x_m)\left(-g+K_{2j}^{g}(x_m)\right) \\
& + \sum_{s=n+1}^{m}(\epsilon_j)\left(-g+K_{2j}^{g}(x_s)\right) \\
\geq & (x_{n+1}-a)\left(-g+K_{j}^{g}(x_n)\right)+(b-x_m)\left(-g+K_{j}^{g}(x_m)\right) \\
& + \sum_{s=n+1}^{m}(\epsilon_j)\left(-g+K_{j}^{g}(x_s)\right) \\
= & V_j^g(b)-V_j^g(a).
\end{aligned}
$$

This holds because we have $K_{2j}^{g}(w) \geq K_j^g(w)$ for all $w \in [0,\bar{w}]$ from Lemma 2. ■

Lemma 9.

$$\lim_{j\to\infty} V_{2^j}^{\bar{g}}(w+\epsilon_{2^j}) = \lim_{j\to\infty} V_{2^j}^{\bar{g}}(w-\epsilon_{2^j}) = V^{\bar{g}}(w), \quad \forall w \in (0, \bar{w}+C).$$

We prove Lemma 9. We choose a small $\delta>0$ and fix $w\in(0,\bar{w}+C)$. There is a positive integer N_δ such that for all $j>N_\delta$,

$$\left|V_{2^j}^{\bar{g}}(w+\epsilon_{2^j})-V_{2^j}^{\bar{g}}(w)\right| \leq \bar{g}\epsilon_{2^j} < \frac{\delta}{2}, \text{ and } \left|V_{2^j}^{\bar{g}}(w-\epsilon_{2^j})-V_{2^j}^{\bar{g}}(w)\right| \leq \bar{g}\epsilon_{2^j} < \frac{\delta}{2},$$

because of Lipschitz continuity in Lemma 6. For the same δ, there is another positive integer M_δ such that for all $j>M_\delta$,

$$\left|V_{2^j}^{\bar{g}}(w)-V^{\bar{g}}(w)\right| < \frac{\delta}{2},$$

because of the uniform convergence in Proposition 6. We let the index $j>O_\delta=\max\{N_\delta,M_\delta\}$, and we thus have

$$
\begin{aligned}
\left|V_{2^j}^{\bar{g}}(w+\epsilon_{2^j})-V^{\bar{g}}(w)\right| \leq & \left|V_{2^j}^{\bar{g}}(w+\epsilon_{2^j})-V_{2^j}^{\bar{g}}(w)\right| \\
& + \left|V_{2^j}^{\bar{g}}(w)-V^{\bar{g}}(w)\right| < \frac{\delta}{2}+\frac{\delta}{2}=\delta,
\end{aligned}
$$

and

$$
\begin{aligned}
\left|V_{2^j}^{\bar{g}}(w-\epsilon_{2^j})-V^{\bar{g}}(w)\right| \leq & \left|V_{2^j}^{\bar{g}}(w-\epsilon_{2^j})-V_{2^j}^{\bar{g}}(w)\right| \\
& + \left|V_{2^j}^{\bar{g}}(w)-V^{\bar{g}}(w)\right| < \frac{\delta}{2}+\frac{\delta}{2}=\delta.
\end{aligned}
$$

These inequalities imply

$$
\begin{aligned}
&\lim_{j\to\infty} V_{2^j}^{\bar{g}}(w+\epsilon_{2^j}) = V^{\bar{g}}(w), \quad \forall w\in(0,\bar{w}+C), \\
&\lim_{j\to\infty} V_{2^j}^{\bar{g}}(w-\epsilon_{2^j}) = V^{\bar{g}}(w), \quad \forall w\in(0,\bar{w}+C).
\end{aligned}
$$

■

Lemma 10.

$$
\begin{aligned}
&\lim_{j\to\infty} \max_{0\leq p\leq\Gamma(w)} \left\{\mathbf{v}(p)-\alpha c(w+\epsilon_{2^j})+V_{2^j}^{\bar{g}}(w+\epsilon_{2^j}+\mathtt{t}(p))\right\} \\
&= \max_{0\leq p\leq\Gamma(w)} \left\{\mathbf{v}(p)-\alpha c(w)+V^{\bar{g}}(w+\mathtt{t}(p))\right\}, \ \forall w\in(0,\bar{w}).
\end{aligned}
\tag{71}
$$

We fix $w \in (0, \bar{w})$, define

$$\begin{aligned} p_{2^j} := \max \quad & p \\ s.t. \quad & 0 \le p \le \Gamma(w) \\ & \mathbf{v}(p) - \alpha c\left(w + \epsilon_{2^j}\right) + V^{\bar{g}}_{2^j}\left(w + \epsilon_{2^j} + \mathtt{t}\left(p\right)\right) \ge \mathbf{v}(p') - \alpha c\left(w + \epsilon_{2^j}\right) + V^{\bar{g}}_{2^j}\left(w + \epsilon_{2^j} + \mathtt{t}\left(p'\right)\right), \\ & \forall p' \in [0, \Gamma(w)], \end{aligned} \tag{72}$$

and define

$$\begin{aligned} p^* := \max \quad & p \\ s.t. \quad & 0 \le p \le \Gamma(w) \\ & \mathbf{v}(p) - \alpha c(w) + V^{\bar{g}}\left(w + \mathtt{t}(p)\right) \ge \mathbf{v}(p') - \alpha c\left(w\right) + V^{\bar{g}}\left(w + \mathtt{t}(p')\right) \quad \forall p' \in [0, \Gamma(w)]. \end{aligned} \tag{73}$$

That is, p_{2^j} is the maximizer of the problem of index j in the left-hand side of (71), and p^* is the maximizer of the problem in the right-hand side of (71).

We show $p_{2^{j+1}} \le p_{2^j}$. Considering $p_{2^{j+1}}$ from (72), we have

$$\begin{aligned} & \mathbf{v}\left(p_{2^{j+1}}\right) - \alpha c\left(w + \epsilon_{2^{j+1}}\right) + V^{\bar{g}}_{2^{j+1}}\left(w + \epsilon_{2^{j+1}} + \mathtt{t}\left(p_{2^{j+1}}\right)\right) \\ & \qquad \ge \mathbf{v}(p) - \alpha c\left(w + \epsilon_{2^{j+1}}\right)) + V^{\bar{g}}_{2^{j+1}}\left(w + \epsilon_{2^{j+1}} + \mathtt{t}\left(p\right)\right), \quad \forall p \in [0, p_{2^{j+1}}]. \end{aligned} \tag{74}$$

Moreover, we have

$$\begin{aligned} & V^{\bar{g}}_{2^{j+1}}\left(w + \epsilon_{2^{j+1}} + \mathtt{t}\left(p\right)\right) - V^{\bar{g}}_{2^{j+1}}\left(w + \epsilon_{2^{j+1}} + \mathtt{t}\left(p_{2^{j+1}}\right)\right) \\ & \ge V^{\bar{g}}_{2^j}\left(w + \epsilon_{2^{j+1}} + \mathtt{t}\left(p\right)\right) - V^{\bar{g}}_{2^j}\left(w + \epsilon_{2^{j+1}} + \mathtt{t}\left(p_{2^{j+1}}\right)\right) \\ & \ge V^{\bar{g}}_{2^j}\left(w + \epsilon_{2^j} + \mathtt{t}\left(p\right)\right) - V^{\bar{g}}_{2^j}\left(w + \epsilon_{2^j} + \mathtt{t}\left(p_{2^{j+1}}\right)\right), \quad \forall p \in [0, p_{2^{j+1}}], \end{aligned} \tag{75}$$

where the first inequality follows Lemma 8 and the second inequality follows the concavity in Proposition 6. Combining (74) and (75), we have

$$\begin{aligned} \mathbf{v}\left(p_{2^{j+1}}\right) - \alpha c\left(w + \epsilon_{2^{j+1}}\right) - \left(\mathbf{v}(p) - \alpha c\left(w + \epsilon_{2^{j+1}}\right)\right) & \ge V^{\bar{g}}_{2^{j+1}}\left(w + \epsilon_{2^{j+1}} + \mathtt{t}\left(p\right)\right) - V^{\bar{g}}_{2^{j+1}}\left(w + \epsilon_{2^{j+1}} + \mathtt{t}\left(p_{2^{j+1}}\right)\right) \\ & \ge V^{\bar{g}}_{2^j}\left(w + \epsilon_{2^j} + \mathtt{t}\left(p\right)\right) - V^{\bar{g}}_{2^j}\left(w + \epsilon_{2^j} + \mathtt{t}\left(p_{2^{j+1}}\right)\right), \quad \forall p \in [0, p_{2^{j+1}}]. \end{aligned}$$

By rearranging the above inequality and substituting $c\left(w + \epsilon_{2^{j+1}}\right)$ by $c\left(w + \epsilon_{2^j}\right)$, we have

$$\mathbf{v}\left(p_{2^{j+1}}\right) - \alpha c\left(w + \epsilon_{2^j}\right) + V^{\bar{g}}_{2^j}\left(w + \epsilon_{2^j} + \mathtt{t}\left(p_{2^{j+1}}\right)\right) \ge \mathbf{v}\left(p\right) - \alpha c\left(w + \epsilon_{2^j}\right) + V^{\bar{g}}_{2^j}\left(w + \epsilon_{2^j} + \mathtt{t}\left(p\right)\right), \quad \forall p \in [0, p_{2^{j+1}}].$$

Considering this inequality with (72), we have $p_{2^j} \ge p_{2^{j+1}}$. Moreover, we can show $p_{2^j} \ge p^*$ for all positive integer j with a similar argument. As a result, we have

$$p_{2^j} \ge p_{2^{j+1}} \ge 0, \quad \forall j \in \mathbb{N}, \tag{76}$$

and $\lim_{j\to\infty} p_{2^j}$ is thus well-defined.

Next, we show that $\lim_{j\to\infty} p_{2^j} = p^*$ by contradiction. We assume

$$p^* < \bar{p} = \lim_{j\to\infty} p_{2^j}. \tag{77}$$

From (72), (76), and (77), we have

$$\mathbf{v}(\bar{p}) - \alpha c(w+\epsilon_{2^j}) + V_{2^j}^{\bar{g}}(w+\epsilon_{2^j}+\mathtt{t}(\bar{p})) \geq \mathbf{v}(p) - \alpha c(w+\epsilon_{2^j}) + V_{2^j}^{\bar{g}}(w+\epsilon_{2^j}+\mathtt{t}(p)), \quad \forall p\in[0,\bar{p}], \quad j\in\mathbb{N}.$$

By taking j to infinity on both sides, we further have

$$\begin{aligned}\mathbf{v}(\bar{p}) - \alpha c(w) + V^{\bar{g}}(w+\mathbf{t}(\bar{p})) &= \lim_{j\to\infty}\left[\mathbf{v}(\bar{p}) - \alpha c(w+\epsilon_{2^j}) + V_{2^j}^{\bar{g}}(w+\epsilon_{2^j}+\mathbf{t}(\bar{p}))\right] \\ &\geq \lim_{j\to\infty}\left[\mathbf{v}(p) - \alpha c(w+\epsilon_{2^j}) + V_{2^j}^{\bar{g}}(w+\epsilon_{2^j}+\mathtt{t}(p))\right] \\ &= \mathbf{v}(p) - \alpha c(w) + V^{\bar{g}}(w+\mathtt{t}(p)), \quad \forall p\in[0,\bar{p}]. \end{aligned} \tag{78}$$

From the definition of p^* in (73), we have that $p^* \geq \bar{p}$ from (78), which contradicts the assumption (77). By contradiction, we have

$$\lim_{j\to\infty} p_{2^j} = p^*. \tag{79}$$

Combining (79) and Lemma 9 with Proposition 6, we have

$$\begin{aligned}&\lim_{j\to\infty}\left|\max_{0\leq p\leq\Gamma(w)}\left\{\mathbf{v}(p) - \alpha c(w+\epsilon_{2^j}) + V_{2^j}^{\bar{g}}(w+\epsilon_{2^j}+\mathtt{t}(p))\right\} - \max_{0\leq p\leq\Gamma(w)}\left\{\mathbf{v}(p) - \alpha c(w) + V^{\bar{g}}(w+\mathtt{t}(p))\right\}\right| \\ &= \lim_{j\to\infty}\left|\mathbf{v}(p_{2^j}) - \alpha c(w+\epsilon_{2^j}) + V_{2^j}^{\bar{g}}(w+\epsilon_{2^j}+\mathtt{t}(p_{2^j})) - (\mathbf{v}(p^*) - \alpha c(w) + V^{\bar{g}}(w+\mathtt{t}(p^*)))\right| \\ &\leq \lim_{j\to\infty}\left|\mathbf{v}(p_{2^j}) - \alpha c(w+\epsilon_{2^j}) - (\mathbf{v}(p^*) - \alpha c(w))\right| + \lim_{j\to\infty}\left|V_{2^j}^{\bar{g}}(w+\epsilon_{2^j}+\mathtt{t}(p_{2^j})) - V_{2^j}^{\bar{g}}(w+\mathtt{t}(p_{2^j}))\right| \\ &+ \lim_{j\to\infty}\left|V_{2^j}^{\bar{g}}(w+\mathtt{t}(p_{2^j})) - V^{\bar{g}}(w+\mathtt{t}(p_{2^j}))\right| + \lim_{j\to\infty}\left|V^{\bar{g}}(w+\mathtt{t}(p_{2^j})) - V^{\bar{g}}(w+\mathtt{t}(p^*))\right| = 0.\end{aligned}$$

■

LEMMA 11.

$$\begin{aligned}&\lim_{j\to\infty}\max_{0\leq p\leq\Gamma(w+\epsilon_{2^j})}\left\{\mathbf{v}(p) - \alpha c(w+\epsilon_{2^j}) + V_{2^j}^{\bar{g}}(w+\epsilon_{2^j}+\mathtt{t}(p))\right\} \\ &= \max_{0\leq p\leq\Gamma(w)}\left\{\mathbf{v}(p) - \alpha c(w) + V^{\bar{g}}(w+\mathtt{t}(p))\right\}, \ \forall w\in(0,\bar{w}).\end{aligned} \tag{80}$$

We fix the state $w\in(0,\bar{w})$ to prove the result. Maximizers p_{2^j} and p^* are defined in (72) and (73). We further define the maximizer of the problem of index j in the left-hand side of (80):

$$\begin{aligned}p_{2^j}^r := \max \quad & t \\ s.t. \quad & 0\leq p\leq\Gamma(w+\epsilon_{2^j}) \\ & \mathbf{v}(p) - \alpha c(w+\epsilon_{2^j}) + V_{2^j}^{\bar{g}}(w+\epsilon_{2^j}+\mathtt{t}(p)) \geq \mathbf{v}(p') - \alpha c(w+\epsilon_{2^j}) + V_{2^j}^{\bar{g}}(w+\epsilon_{2^j}+\mathtt{t}(p')), \ \forall p'\in[0,\Gamma(w+\epsilon_{2^j})].\end{aligned} \tag{81}$$

We have $p_{2^j}^r \leq p_{2^j}$ hold for all positive integer j since the feasible solution to (72) is also feasible for (81) because of Proposition 2(iii) showing that $\Gamma(w)\geq\Gamma(w+\epsilon_{2^j})$.

We show the sequence $\{p_{2^j}^r\}_{j=1}^{\infty}$ is convergent to p^* by considering two cases: $p^*=\Gamma(w)$ and $p^*<\Gamma(w)$. For the first case, i.e., $p^*=\Gamma(w)$, we have $p_{2^j}=\Gamma(w)$ from (72) and (76) in Lemma 10. This further implies

$$\begin{aligned}\mathbf{v}(p_{2^j}) - \alpha c(w+\epsilon_{2^j}) + V_{2^j}^{\bar{g}}(w+\epsilon_{2^j}+\mathtt{t}(p_{2^j})) &= \mathbf{v}(\Gamma(w)) - \alpha c(w+\epsilon_{2^j}) + V_{2^j}^{\bar{g}}(w+\epsilon_{2^j}+\mathtt{t}(\Gamma(w))) \\ &\geq \mathbf{v}(p') - \alpha c(w+\epsilon_{2^j}) + V_{2^j}^{\bar{g}}(w+\epsilon_{2^j}+\mathtt{t}(p')), \quad \forall p'\in[0,\Gamma(w)],\end{aligned} \tag{82}$$

due to the definition in (72). Combining (82) with (81), we have $p^r_{2^j} \in [\Gamma(w), \Gamma(w+\epsilon_{2^j})]$. Therefore, we have

$$\lim_{j\to\infty} p^r_{2^j} = \lim_{j\to\infty} \Gamma(w+\epsilon_{2^j}) = \Gamma(w) = p^*.$$

For the second case, i.e., $p^* < \Gamma(w)$, we have an index j' such that

$$\Gamma(w+\epsilon_{2^j}) \geq p^r_{2^j} = p_{2^j} \geq p^*, \quad \forall j > j',$$

because p_{2^j} decreases in j from (76) and $\Gamma(w+\epsilon_{2^j})$ increases in j from Proposition 2. Therefore, we have $\lim_{j\to\infty} p^r_{2^j} = \lim_{j\to\infty} p_{2^j} = p^*$. As a result, we conclude from both cases that

$$\lim_{j\to\infty} p^r_{2^j} = p^*. \tag{83}$$

Combining (83) and Lemma 9 with Proposition 6, we have

$$\begin{aligned}
&\lim_{j\to\infty}\left|\max_{0\le p\le\Gamma\left(w+\epsilon_{2^j}\right)}\left\{\mathbf{v}(p)-\alpha c(w+\epsilon_{2^j})+V^{\bar{g}}_{2^j}(w+\epsilon_{2^j}+\mathtt{t}(p))\right\}-\max_{0\le p\le\Gamma(w)}\left\{\mathbf{v}(p)-\alpha c(w)+V^{\bar{g}}(w+\mathtt{t}(p))\right\}\right|\\
&=\lim_{j\to\infty}\left|\mathbf{v}(p^r_{2^j})-\alpha c(w+\epsilon_{2^j})+V^{\bar{g}}_{2^j}(w+\epsilon_{2^j}+\mathtt{t}(p^r_{2^j}))-\left(\mathbf{v}(p^*)-\alpha c(w)+V^{\bar{g}}(w+\mathtt{t}(p^*))\right)\right|\\
&\le\lim_{j\to\infty}\left|\mathbf{v}(p^r_{2^j})-\alpha c(w+\epsilon_{2^j})-(\mathbf{v}(p^*)-\alpha c(w))\right|+\lim_{j\to\infty}\left|V^{\bar{g}}_{2^j}(w+\epsilon_{2^j}+\mathtt{t}(p^r_{2^j}))-V^{\bar{g}}_{2^j}(w+\mathtt{t}(p^r_{2^j}))\right|\\
&+\lim_{j\to\infty}\left|V^{\bar{g}}_{2^j}(w+\mathtt{t}(p^r_{2^j}))-V^{\bar{g}}(w+\mathtt{t}(p^r_{2^j}))\right|+\lim_{j\to\infty}\left|V^{\bar{g}}(w+\mathtt{t}(p^r_{2^j}))-V^{\bar{g}}(w+\mathtt{t}(p^*))\right|=0.
\end{aligned}$$

This completes the proof. ■

Lemma 12.

$$\begin{aligned}
&\lim_{j\to\infty}\max_{0\le p\le\Gamma\left(w-\epsilon_{2^j}\right)}\left\{\mathbf{v}(p)-\alpha c(w-\epsilon_{2^j})+V^{\bar{g}}(w-\epsilon_{2^j}+\mathtt{t}(p))\right\}\\
&=\max_{0\le p\le\Gamma(w)}\left\{\mathbf{v}(p)-\alpha c(w)+V^{\bar{g}}(w+\mathtt{t}(p))\right\},\ \forall w\in(0,\bar{w}).
\end{aligned} \tag{84}$$

The proof is similar to that of Lemma 11. We omit it for brevity. A complete proof is available from the authors. ■

Now we are ready to prove Proposition 8. We show the function $V^{\bar{g}}(w)$ is differentiable in the interval $(0, \bar{w}+C]$. Given the concavity in Proposition 6, both right derivative and left derivative of $V^{\bar{g}}(w)$ are well defined. Therefore, to show the function is differentiable, we have to show that the right derivative of $V^{\bar{g}}(w)$ is equal to the left derivative of $V^{\bar{g}}(w)$ at any $w \in (0, \bar{w}+C]$. To construct the proof, we first show that the function $V^{\bar{g}}(w)$ is differentiable at w when w is a grid point as (18) in Algorithm 1. Then, we complete the proof by showing that $V^{\bar{g}}(w)$ is also differentiable at those non-grid points.

Case 1: $V^{\bar{g}}$ is Differentiable at Gird Points. We consider the sequence of steps $\{2^j\}_{j=1}^{\infty}$. If the state w is the grid point in (18) for an index $j' \in \mathbb{N}$, then w is labeled as the grid point in Algorithm 1 for all number of steps 2^j greater than that of $2^{j'}$. We now show that the function $V^{\bar{g}}$ is differentiable at these grid points.

We define a decreasing sequence $\{h_i\}_{i=1}^{\infty}$ and $\lim_{i\to\infty} h_i = 0$. Following Proposition 6, we have the slope $\frac{V^{\bar{g}}(w+h_i)-V^{\bar{g}}(w)}{h_i}$ decrease in the index i and

$$\left|\frac{V^{\bar{g}}(w+h_i)-V^{\bar{g}}(w)}{h_i}\right| \le \bar{g}, \quad \forall i \in \mathbb{N}.$$

We then define the right derivative of $V^{\bar{g}}(w)$ at the grid point w:

$$V_{+}^{\bar{g}}(w) := \lim_{i\to\infty} \frac{V^{\bar{g}}(w+h_i) - V^{\bar{g}}(w)}{h_i}.$$

Given (65), the right derivative $V_{+}^{\bar{g}}(w)$ is further expressed as

$$V_{+}^{\bar{g}}(w) = \lim_{i\to\infty}\lim_{j\to\infty} \frac{V_{2^j}^{\bar{g}}(w+h_i) - V_{2^j}^{\bar{g}}(w)}{h_i}. \tag{85}$$

Following (21) and the concavity in Proposition 6, we have

$$\frac{V_{2^j}^{\bar{g}}(w+\epsilon_{2^j}) - V_{2^j}^{\bar{g}}(w)}{\epsilon_{2^j}} = \lim_{i\to\infty} \frac{V_{2^j}^{\bar{g}}(w+h_i) - V_{2^j}^{\bar{g}}(w)}{h_i} \geq \frac{V_{2^j}^{\bar{g}}(w+h_i) - V_{2^j}^{\bar{g}}(w)}{h_i}, \quad \forall i\in\mathbb{N}. \tag{86}$$

Plugging (86) into (85), we have

$$V_{+}^{\bar{g}}(w) = \lim_{i\to\infty}\lim_{j\to\infty} \frac{V_{2^j}^{\bar{g}}(w+h_i) - V_{2^j}^{\bar{g}}(w)}{h_i} \leq \lim_{i\to\infty}\lim_{j\to\infty} \frac{V_{2^j}^{\bar{g}}(w+\epsilon_{2^j}) - V_{2^j}^{\bar{g}}(w)}{\epsilon_{2^j}} = \lim_{j\to\infty} \frac{V_{2^j}^{\bar{g}}(w+\epsilon_{2^j}) - V_{2^j}^{\bar{g}}(w)}{\epsilon_{2^j}}. \tag{87}$$

On the other hand, Lemma 8 implies

$$\frac{V^{\bar{g}}(w+h_i) - V^{\bar{g}}(w)}{h_i} \geq \frac{V_{2^j}^{\bar{g}}(w+h_i) - V_{2^j}^{\bar{g}}(w)}{h_i}, \quad \forall i\in\mathbb{N}, \quad j\in\mathbb{N}.$$

By taking the index i to infinity, we thus have

$$V_{+}^{\bar{g}}(w) = \lim_{i\to\infty} \frac{V^{\bar{g}}(w+h_i) - V^{\bar{g}}(w)}{h_i} \geq \lim_{i\to\infty} \frac{V_{2^j}^{\bar{g}}(w+h_i) - V_{2^j}^{\bar{g}}(w)}{h_i} = \frac{V_{2^j}^{\bar{g}}(w+\epsilon_{2^j}) - V_{2^j}^{\bar{g}}(w)}{\epsilon_{2^j}}.$$

Because the above inequality holds for all index j, we have

$$V_{+}^{\bar{g}}(w) \geq \lim_{j\to\infty} \frac{V_{2^j}^{\bar{g}}(w+\epsilon_{2^j}) - V_{2^j}^{\bar{g}}(w)}{\epsilon_{2^j}}. \tag{88}$$

Combining (87) with (88), we conclude

$$V_{+}^{\bar{g}}(w) = \lim_{j\to\infty} \frac{V_{2^j}^{\bar{g}}(w+\epsilon_{2^j}) - V_{2^j}^{\bar{g}}(w)}{\epsilon_{2^j}}, \tag{89}$$

for the grid point w. Following from (21) and Lemma 11, we have

$$\begin{aligned}
V_{+}^{\bar{g}}(w) &= \lim_{j\to\infty} \frac{V_{2^j}^{\bar{g}}(w+\epsilon_{2^j}) - V_{2^j}^{\bar{g}}(w)}{\epsilon_{2^j}} \\
&= \lim_{j\to\infty}\left[-\bar{g} + \lambda \max\left\{\max_{0\leq p\leq \Gamma\left(w+\epsilon_{2^j}\right)} \left\{\mathbf{v}(p) - \alpha c(w+\epsilon_{2^j}) + V_{2^j}^{\bar{g}}(w+\epsilon_{2^j}+\mathtt{t}(p)) - V_{2^j}^{\bar{g}}(w+\epsilon_{2^j})\right\}, 0\right\}\right] \\
&= \lim_{j\to\infty}\left[-\bar{g} + \lambda \max\left\{\max_{0\leq p\leq \Gamma\left(w+\epsilon_{2^j}\right)} \left\{\mathbf{v}(p) - \alpha c(w+\epsilon_{2^j}) + V_{2^j}^{\bar{g}}(w+\epsilon_{2^j}+\mathtt{t}(p))\right\}, V_{2^j}^{\bar{g}}(w+\epsilon_{2^j})\right\} - V_{2^j}^{\bar{g}}(w+\epsilon_{2^j})\right] \\
&= -\bar{g} + \lim_{j\to\infty} \lambda \max\left\{\max_{0\leq p\leq \Gamma\left(w+\epsilon_{2^j}\right)} \left\{\mathbf{v}(p) - \alpha c(w+\epsilon_{2^j}) + V_{2^j}^{\bar{g}}(w+\epsilon_{2^j}+\mathtt{t}(p))\right\}, V_{2^j}^{\bar{g}}(w+\epsilon_{2^j})\right\} - V^{\bar{g}}(w) \\
&= -\bar{g} + \lambda \max\left\{\lim_{j\to\infty}\max_{0\leq p\leq \Gamma\left(w+\epsilon_{2^j}\right)} \left\{\mathbf{v}(p) - \alpha c(w+\epsilon_{2^j}) + V_{2^j}^{\bar{g}}(w+\epsilon_{2^j}+\mathtt{t}(p))\right\}, \lim_{j\to\infty} V_{2^j}^{\bar{g}}(w+\epsilon_{2^j})\right\} - V^{\bar{g}}(w) \\
&= -\bar{g} + \lambda \max\left\{\max_{0\leq p\leq \Gamma(w)} \left\{\mathbf{v}(p) - \alpha c(w) + V^{\bar{g}}(w+\mathtt{t}(p))\right\}, V^{\bar{g}}(w)\right\} - V^{\bar{g}}(w) \\
&= -\bar{g} + \lambda \max\left\{\max_{0\leq p\leq \Gamma(w)} \left\{\mathbf{v}(p) - \alpha c(w) + V^{\bar{g}}(w+\mathtt{t}(p)) - V^{\bar{g}}(w)\right\}, 0\right\}.
\end{aligned} \tag{90}$$

As a result, we have

$$V_{+}^{\bar{g}}(w) = -\bar{g} + \lambda \max\left\{ \max_{0 \le p \le \Gamma(w)} \left\{ \mathbf{v}(p, w) + V^{\bar{g}}(w + \mathtt{t}(p)) - V^{\bar{g}}(w) \right\}, 0 \right\}. \tag{91}$$

Now, we show the left derivative $V_{-}^{\bar{g}}(w)$ of the grid w which is

$$V_{-}^{\bar{g}}(w) := \lim_{i \to \infty} \frac{V^{\bar{g}}(w) - V^{\bar{g}}(w - h_i)}{h_i}.$$

Given the concavity in Proposition 6, we have

$$V_{+}^{\bar{g}}(w') \ge V_{-}^{\bar{g}}(w) \ge V_{+}^{\bar{g}}(w),$$

for any grid point w' smaller than the grid point w. We consider the step size ϵ_{2^j} and thus have

$$\lim_{j \to \infty} V_{+}^{\bar{g}}(w - \epsilon_{2^j}) \ge V_{-}^{\bar{g}}(w) \ge V_{+}^{\bar{g}}(w).$$

Combining (90) and Lemma 12, we have

$$\begin{aligned}
\lim_{j \to \infty} V_{+}^{\bar{g}}(w - \epsilon_{2^j}) &= \lim_{j \to \infty} \left[-\bar{g} + \lambda \max\left\{ \max_{0 \le p \le \Gamma\left(w - \epsilon_{2^j}\right)} \left\{ \mathbf{v}(p) - \alpha c(w - \epsilon_{2^j}) + V^{\bar{g}}(w - \epsilon_{2^j} + \mathtt{t}(p)) - V^{\bar{g}}(w - \epsilon_{2^j}) \right\}, 0 \right\} \right] \\
&= -\bar{g} + \lambda \max\left\{ \max_{0 \le p \le \Gamma(w)} \left\{ \mathbf{v}(p) - \alpha c(w) + V^{\bar{g}}(w + \mathtt{t}(p)) - V^{\bar{g}}(w) \right\}, 0 \right\} \\
&\ge V_{-}^{\bar{g}}(w) \ge V_{+}^{\bar{g}}(w) \\
&= -\bar{g} + \lambda \max\left\{ \max_{0 \le p \le \Gamma(w)} \left\{ \mathbf{v}(p) - \alpha c(w) + V^{\bar{g}}(w + \mathtt{t}(p)) - V^{\bar{g}}(w) \right\}, 0 \right\}.
\end{aligned}$$

We conclude that $V^{\bar{g}}(w)$ is differentiable at grid points and has the derivative

$$V^{\bar{g}\prime}(w) = -\bar{g} + \lambda \max\left\{ \max_{0 \le p \le \Gamma(w)} \left\{ \mathbf{v}(p) - \alpha c(w) + V^{\bar{g}}(w + \mathtt{t}(p)) - V^{\bar{g}}(w) \right\}, 0 \right\}. \tag{92}$$

Case 2: $V^{\bar{g}}$ is Differentiable at Non-girds. We next show that $V^{\bar{g}}$ is also differentiable at the non-grid point on the interval $(0, \bar{w}]$ where w is not a grid point in Algorithm 1 for any number of step 2^j for all positive integer j. To prove this result, we fix a non-grid point w and we let $x_{2^j}^i$ be the i-th grid in (18) when the number of steps is 2^j for the algorithm. We then define a sequence of grids $\left\{x_j^w\right\}_{j=1}^{\infty}$ that

$$\begin{aligned}
x_j^w := \max_{i \in \left\{1, 2, \ldots, 2^j\right\}} \quad & x_{2^j}^i \\
s.t. \quad & x_{2^j}^i = \bar{w} - i\epsilon_{2^j} \\
& x_{2^j}^i < w.
\end{aligned}$$

That is, this sequence includes the grid point closest to w from the left for each number of steps 2^j. We thus have the sequence of grids $\left\{x_j^w\right\}_{j=1}^{\infty}$ increase in j and converge to w, and have another sequence of grids $\left\{x_j^w + \epsilon_{2^j}\right\}_{j=1}^{\infty}$ decrease in j and converge to w. From the concavity of $V^{\bar{g}}$ in Proposition 6,

$$\frac{V^{\bar{g}}\left(x_j^w + h\right) - V^{\bar{g}}\left(x_j^w\right)}{h} \ge \frac{V^{\bar{g}}(w + h) - V^{\bar{g}}(w)}{h} \ge \frac{V^{\bar{g}}\left(x_j^w + \epsilon_{2^j} + h\right) - V^{\bar{g}}\left(x_j^w + \epsilon_{2^j}\right)}{h}, \quad \forall h > 0.$$

Taking the limit $h \to 0$ and knowing grids points are differentiable, we have

$$V^{\bar{g}\prime}\left(x_j^w\right) \ge \lim_{h \to 0} \frac{V^{\bar{g}}(w + h) - V^{\bar{g}}(w)}{h} \ge V^{\bar{g}\prime}\left(x_j^w + \epsilon_{2^j}\right).$$

Following (92), Lemma 11 and Lemma 12, we have

$$\begin{aligned}
\lim_{j\to\infty} V^{\bar g\prime}\left(x_j^w\right) &= -\bar g + \lim_{j\to\infty} \lambda \max\{\max_{0\le p\le \Gamma\left(x_j^w\right)} \{\mathbf{v}(p) - \alpha c\left(x_j^w\right) + V^{\bar g}\left(x_j^w + \mathtt{t}(p)\right) - V^{\bar g}\left(x_j^w\right)\}, 0\} \\
&= -\bar g + \lambda \max\{\max_{0\le p\le \Gamma(w)} \{\mathbf{v}(p) - \alpha c(w) + V^{\bar g}(w + \mathtt{t}(p)) - V^{\bar g}(w)\}, 0\} \\
&\ge \lim_{h\to 0} \frac{V^{\bar g}(w+h) - V^{\bar g}(w)}{h} \\
&\ge -\bar g + \lambda \max\{\max_{0\le p\le \Gamma(w)} \{\mathbf{v}(p) - \alpha c(w) + V^{\bar g}(w + \mathtt{t}(p)) - V^{\bar g}(w)\}, 0\} \\
&= -\bar g + \lim_{j\to\infty} \lambda \max\{\max_{0\le p\le \Gamma\left(x_j^w + \epsilon_{2^j}\right)} \{\mathbf{v}(p) - \alpha c\left(x_j^w + \epsilon_{2^j}\right) + V^{\bar g}\left(x_j^w + \epsilon_{2^j} + \mathtt{t}(p)\right) - V^{\bar g}\left(x_j^w + \epsilon_{2^j}\right)\}, 0\} \\
&= \lim_{j\to\infty} V^{\bar g\prime}\left(x_j^w + \epsilon_{2^j}\right).
\end{aligned}$$

As a result, we have

$$V_+^{\bar g}(w) := \lim_{h\to 0} \frac{V^{\bar g}(w+h) - V^{\bar g}(w)}{h} = -\bar g + \lambda \max\{\max_{0\le p\le \Gamma(w)} \{\mathbf{v}(p) - \alpha c(w) + V^{\bar g}(w + \mathtt{t}(p)) - V^{\bar g}(w)\}, 0\},\ \forall w \in (0, \bar w].$$

For the left derivative, from the concavity of $V^{\bar g}$ in Proposition 6, we have

$$\frac{V^{\bar g}\left(x_j^w\right) - V^{\bar g}\left(x_j^w - h\right)}{h} \ge \frac{V^{\bar g}(w) - V^{\bar g}(w-h)}{h} \ge \frac{V^{\bar g}\left(x_j^w + \epsilon_{2^j}\right) - V^{\bar g}\left(x_j^w + \epsilon_{2^j} - h\right)}{h}, \quad \forall h > 0.$$

Taking the limit $h \to 0$, we have

$$V^{\bar g\prime}\left(x_j^w\right) \ge \lim_{h\to 0} \frac{V^{\bar g}(w) - V^{\bar g}(w-h)}{h} \ge V^{\bar g\prime}\left(x_j^w + \epsilon_{2^j}\right).$$

Following (92), Lemma 11 and Lemma 12, we have

$$\begin{aligned}
\lim_{j\to\infty} V^{\bar g\prime}\left(x_j^w\right) &= -\bar g + \lim_{j\to\infty} \lambda \max\left\{\max_{0\le p\le \Gamma\left(x_j^w\right)} \left\{\mathbf{v}(p) - \alpha c\left(x_j^w\right) + V^{\bar g}\left(x_j^w + \mathtt{t}(p)\right) - V^{\bar g}\left(x_j^w\right)\right\}, 0\right\} \\
&= -\bar g + \lambda \max\left\{\max_{0\le p\le \Gamma(w)} \{\mathbf{v}(p) - \alpha c(w) + V^{\bar g}(w + \mathtt{t}(p)) - V^{\bar g}(w)\}, 0\right\} \\
&\ge \lim_{h\to 0} \frac{V^{\bar g}(w) - V^{\bar g}(w-h)}{h} \\
&\ge -\bar g + \lambda \max\left\{\max_{0\le p\le \Gamma(w)} \{\mathbf{v}(p) - \alpha c(w) + V^{\bar g}(w + \mathtt{t}(p)) - V^{\bar g}(w)\}, 0\right\} \\
&= -\bar g + \lim_{j\to\infty} \lambda \max\left\{\max_{0\le p\le \Gamma\left(x_j^w + \epsilon_{2^j}\right)} \left\{\mathbf{v}(p) - \alpha c\left(x_j^w + \epsilon_{2^j}\right) + V^{\bar g}\left(x_j^w + \epsilon_{2^j} + \mathtt{t}(p)\right) - V^{\bar g}\left(x_j^w + \epsilon_{2^j}\right)\right\}, 0\right\} \\
&= \lim_{j\to\infty} V^{\bar g\prime}\left(x_j^w + \epsilon_{2^j}\right).
\end{aligned}$$

Therefore, we have

$$V_-^{\bar g}(w) := \lim_{h\to 0} \frac{V^{\bar g}(w) - V^{\bar g}(w-h)}{h} = -\bar g + \lambda \max\left\{\max_{0\le p\le \Gamma(w)} \{\mathbf{v}(p) - \alpha c(w) + V^{\bar g}(w + \mathtt{t}(p)) - V^{\bar g}(w)\}, 0\right\},\ \forall w \in (0, \bar w].$$

and thus conclude

$$V_+^{\bar g}(w) = V_-^{\bar g}(w) = -\bar g + \lambda \max\left\{\max_{0\le p\le \Gamma(w)} \{\mathbf{v}(p) - \alpha c(w) + V^{\bar g}(w + \mathtt{t}(p)) - V^{\bar g}(w)\}, 0\right\}.$$

We have shown both cases that the function $V^{\bar g}(w)$ is differentiable in the interval $(0, \bar w]$ with the derivative

$$V^{\bar g\prime}(w) = -\bar g + \lambda \max\left\{\max_{0\le p\le \Gamma(w)} \{\mathbf{v}(p) - \alpha c(w) + V^{\bar g}(w + \mathtt{t}(p)) - V^{\bar g}(w)\}, 0\right\}.$$

Q.E.D.

Theorem 1

Theorem 1 is an immediate result from Propositions 7, Proposition 8, and (16). Q.E.D.

Appendix C: Proofs in Section 4.2

For simplicity of notations, we define

$$\hat{\mathbf{v}}\left(w_{s-}\right)=\mathbf{v}\left(p_s^*(w_{s^-})\right)-\alpha c\left(w_{s-}\right), \text{and } \mathbf{t}^*(w_{s^-})=\mathbf{t}\left(p_s^*(w_{s^-}),w_{s^-}\right). \tag{93}$$

For this Appendix, we also define

$$G\left(\{p_s\},T\right):=\frac{1}{T}\mathbb{E}\left[\int_0^T\left[\mathbf{v}\left(p_s\right)-\alpha c\left(w_{s-}\right)\right]dN_s\right], \text{ such that } \mathcal{G}\left(\{p_s\}\right)=\liminf_{T\to\infty}G\left(\{p_s\},T\right). \tag{94}$$

Furthermore, for a control policy $\{p_s\}$ and time T, define an $\mathcal{F}_T$-measurable random variable $K_T\left(\{p_s\}\right)$ to be the number of OEQ times before time epoch T following policy $\{p_s\}$, that is,

$$K_T\left(\{p_s\}\right):=\max\{k \mid \tau_k\left(\{p_s\}\right)\le T\}.$$

Finally, define

$$\varDelta\left(\{p_s\},T\right):=\mathbb{E}\left[\int_{\tau_{K_T(\{p_s\})}}^T\left[\mathbf{v}\left(p_s\right)-\alpha c\left(w_{s-}\right)\right]dN_s\right], \text{ and } \varTheta\left(\{p_s\},T\right):=\mathbb{E}\left[\tau_1\left(\{p_s\}\right)+T-\tau_{K_T(\{p_s\})}\left(\{p_s\}\right)\right].$$

Proposition 9

To prove Proposition 9, we first show the following lemma.

LEMMA 13. *We have $\mathbb{E}\left[K_T\left(\{p_s^*\}\right)\right]$ approaching infinity as T approaches infinity, and*

$$0\le\varDelta\left(\{p_s^*\},T\right)<\delta_1, \text{ and } 0\le\varTheta\left(\{p_s^*\},T\right)<\delta_2, \quad \forall T\ge0,$$

for some positive constants δ_1 and δ_2, which implies that

$$\lim_{T\to\infty}\frac{\varDelta\left(\{p_s^*\},T\right)}{\mathbb{E}\left[K_T\left(\{p_s\}\right)\right]-1}=0, \text{ and } \lim_{T\to\infty}\frac{\varTheta\left(\{p_s^*\},T\right)}{\mathbb{E}\left[K_T\left(\{p_s^*\}\right)\right]-1}=0. \tag{95}$$

Consider a Poisson process with arrival rate λ. The probability that there is no arrival during a period of $\bar{w}$ is $e^{-\lambda\bar{w}}$. Wait time must have reached 0 at least once whenever there is no arrival during a period of $\bar{w}$. Divide a large enough T, into $K_e(T):=\max\{k \mid 2k\bar{w}\le T\}$ number of intervals, each with length $2\bar{w}$. In each such interval, the probability of wait time reaching zero during the first half is at least $e^{-\lambda\bar{w}}$. After emptying the queue, Lemma 4 implies that the next arrival time is an OEQ time. The probability that there is at least one arrival in the second half of the interval is $1-e^{\lambda\bar{w}}$. Therefore, the probability of having an OEQ time during each interval of length $2\bar{w}$ is at least $p_e:=e^{-\lambda\bar{w}}\left(1-e^{\lambda\bar{w}}\right)>0$. Consequently, $\mathbb{E}\left[K_T\left(\{p_s^*\}\right)\right]\ge p_eK_e(T)$. Since $K_e(T)$ approaches infinity with T, we have $\mathbb{E}\left[K_T\left(\{p_s^*\}\right)\right]$ approaching infinity regardless of the control policy $\{p_s^*\}$.

Next, the discussion above demonstrates that $T-\tau_{K_T(\{p_s^*\})}$ must be finite with probability 1 for any T, and, hence, $\varTheta\left(\{p_s^*\},T\right)$ is bounded. Furthermore, the function value $\mathbf{v}(p_s)$ is bounded from (V1) and w is in a compact set. This implies that $\varDelta\left(\{p_s^*\},T\right)$ is bounded. Therefore, (95) follows naturally. ■

We are ready to prove Proposition 9. We first show the second equality in (27). For simplicity of notations, let $\tau_k=\tau_k\left(\{p_s^*\}\right)$ and $K_T=K_T\left(\{p_s^*\}\right)$. We have

$$G\left(\{p_s^*\},T\right)=\frac{\mathbb{E}\left[\sum_{k=1}^{K_T-1}\int_{\tau_k}^{\tau_{k+1}-}\hat{\mathbf{v}}(w_{s-})dN_s\right]+\varDelta\left(\{p_s^*\},T\right)}{\mathbb{E}\left[\sum_{k=1}^{K_T-1}\left(\tau_{k+1}-\tau_k\right)\right]+\varTheta\left(\{p_s^*\},T\right)}=\frac{\mathbb{E}\left[\sum_{k=1}^{K_T-1}\mathbb{E}\left[\int_{\tau_k}^{\tau_{k+1}-}\hat{\mathbf{v}}(w_{s-})dN_s \;\Big|\; \mathcal{F}_{\tau_k}\right]\right]+\varDelta\left(\{p_s^*\},T\right)}{\mathbb{E}\left[\sum_{k=1}^{K_T-1}\mathbb{E}\left[\tau_{k+1}-\tau_k \mid \mathcal{F}_{\tau_k}\right]\right]+\varTheta\left(\{p_s^*\},T\right)},$$

where the first equality follows $w_{0-}=0$, and hence $\mathbb{E}\left[\int_0^{\tau_1-}\hat{\mathbf{v}}(w_{s-})dN_s\right]=0$, and the second equality from straightforward derivation. For any OEQ time $\hat{s}$ (which is a $\mathcal{F}_s$-random time), define a random variable $\tau(\{p_s\},\hat{s})$ to be the time epoch of the next OEQ time after $\hat{s}$ under control policy $\{p_s\}$, that is,

$$\tau(\{p_s\},\hat{s}) := \min\{\bar{s} \mid w_{\bar{s}-}=0, w_{\bar{s}}>0, \bar{s}>\hat{s},\ w_s \text{ is induced by } \{p_s\} \text{ following (9) for } s\geq\hat{s}\}. \tag{96}$$

Because w_s is a Markov process under control $\{p_s^*\}$, and τ_k is a OEQ time, following Theorem T40 of Brémaud (1981) (page 291), we know that

$$\mathbb{E}\left[\int_{\tau_k}^{\tau_{k+1}-}\hat{\mathbf{v}}(w_{s-})dN_s \;\middle|\; \mathcal{F}_{\tau_k}\right]=\mathbb{E}\left[\int_{\hat{s}}^{\tau(\{p_s^*\},\hat{s})-}\hat{\mathbf{v}}(w_{s-})dN_s \;\middle|\; w_{\hat{s}-}=0, w_{\hat{s}}>0\right],\ \forall k,$$
$$\text{and}\ \ \mathbb{E}\left[\tau_{k+1}-\tau_k \;\middle|\; \mathcal{F}_{\tau_k}\right]=\mathbb{E}\left[\tau(\{p_s^*\},\hat{s})-\hat{s} \;\middle|\; w_{\hat{s}-}=0, w_{\hat{s}}>0\right].$$

Therefore,

$$G(\{p_s^*\},T)=\frac{\mathbb{E}\left[\int_{\hat{s}}^{\tau(\{p_s^*\},\hat{s})-}\hat{\mathbf{v}}(w_{s-})dN_s \;\middle|\; w_{\hat{s}-}=0, w_{\hat{s}}>0\right]+\dfrac{\Delta(\{p_s^*\},T)}{\mathbb{E}[K_T]-1}}{\mathbb{E}\left[\tau(\{p_s^*\},\hat{s})-\hat{s} \;\middle|\; w_{\hat{s}-}=0, w_{\hat{s}}>0\right]+\dfrac{\Theta(\{p_s^*\},T)}{\mathbb{E}[K_T]-1}}.$$

Following (95) of Lemma 13, we have

$$\lim_{T\to\infty} G(\{p_s^*\},T)=\frac{\mathbb{E}\left[\int_{\hat{s}}^{\tau(\{p_s^*\},\hat{s})-}\hat{\mathbf{v}}(w_{s-})dN_s \;\middle|\; w_{\hat{s}-}=0, w_{\hat{s}}>0\right]}{\mathbb{E}\left[\tau(\{p_s^*\},\hat{s})-\hat{s} \;\middle|\; w_{\hat{s}-}=0, w_{\hat{s}}>0\right]}.$$

Further following (94), we have the second equality in (27).

Now we show the first equality in (27). Following (12), (13), (35), and (93), we have

$$-V'(w_{s-}) = \mathsf{g}-\lambda\left[\hat{\mathbf{v}}(w_{s-})+V\big(w_{s-}+\mathbf{t}^*(w_{s-})\big)-V(w_{s-})\right],\ \text{and} \tag{97}$$
$$0 = \mathsf{g}-\lambda\left[\hat{\mathbf{v}}(0)+V\big(\mathbf{t}^*(0)\big)-V(0)\right]. \tag{98}$$

Denote time τ^0 to be the first time after $\hat{s}$ that the wait time becomes 0. That is,

$$\tau^0 := \min\{s \mid w_s=0, s>\hat{s}\} < \tau(\{p_s^*\},\hat{s}).$$

Furthermore, $w_s=0$, and, therefore, $\mathcal{T}^*(0)dN_s=0$ for $s\in[\tau^0,\tau(\{p_s^*\},\hat{s}))$ following the dynamic derived from (9),

$$dw_s=-ds\mathbb{1}_{\{w_{s-}>0\}}+\mathbf{t}^*(w_{s-})dN_s. \tag{99}$$

We have,

$$\begin{aligned}
V(0) =& V\left(w_{\tau(\{p_s^*\},\hat{s})-}\right)\\
=& V(w_{\hat{s}-})+\int_{\hat{s}}^{\tau^0}\left(-V'(w_{s-})\right)ds+\int_{\hat{s}}^{\tau(\{p_s^*\},\hat{s})-}\left[V\left(w_{s-}+\mathbf{t}^*(w_{s-})\right)-V(w_{s-})\right]dN_s\\
=& V(0)+\int_{\hat{s}}^{\tau^0}\left\{\mathsf{g}-\lambda\left[\hat{\mathbf{v}}(w_{s-})+V\left(w_{s-}+\mathbf{t}^*(w_{s-})\right)-V(w_{s-})\right]\right\}ds\\
&+\int_{\hat{s}}^{\tau(\{p_s^*\},\hat{s})-}\left[V\left(w_{s-}+\mathbf{t}^*(w_{s-})\right)-V(w_{s-})\right]dN_s+\int_{\tau^0}^{\tau(\{p_s^*\},\hat{s})-}\left\{\mathsf{g}-\lambda\left[\hat{\mathbf{v}}(0)+V\big(\mathbf{t}^*(0)\big)-V(0)\right]\right\}ds\\
=& V(0)+\mathsf{g}\left(\tau(\{p_s^*\},\hat{s})-\hat{s}\right)-\int_{\hat{s}}^{\tau(\{p_s^*\},\hat{s})-}\hat{\mathbf{v}}(w_{s-})dN_s\\
&+\int_{\hat{s}}^{\tau^0}\left[\hat{\mathbf{v}}(w_{s-})+V\left(w_{s-}+\mathbf{t}^*(w_{s-})\right)-V(w_{s-})\right]\left[dN_s-\lambda ds\right],
\end{aligned}$$

where the first equality follows the definition of $\tau(\{p_s^*\},\hat{s})$ as an OEQ time, the second from (99) and Ito's formula for jump processes (see Theorem 17.5 in Bass (2011)), the third from (97), (98), and the fourth straightforward derivations. Therefore, we have

$$\begin{aligned}&\mathtt{g}\mathbb{E}\left[\tau(\{p_s^*\},\hat{s})-\hat{s}\;\middle|\;w_{\hat{s}-}=0,w_{\hat{s}}>0\right]-\mathbb{E}\left[\int_{\hat{s}}^{\tau(\{p_s^*\},\hat{s})^-}\hat{\mathbf{v}}(w_{s-})dN_s\;\middle|\;w_{\hat{s}-}=0,w_{\hat{s}}>0\right]\\&=-\mathbb{E}\left[\int_{\hat{s}}^{\tau^0}\left[\hat{\mathbf{v}}(w_{s-})+V(w_{s-}+\mathbf{t}^*(w_{s-}))-V(w_{s-})\right][dN_s-\lambda ds]\;\middle|\;w_{\hat{s}-}=0,w_{\hat{s}}>0\right],\end{aligned}$$

the right hand side of which is zero. This is because of the Doob's Optional Stopping Theorem ($\tau(\{p_s^*\},\hat{s})$ is an $\mathcal{F}_s$ stopping time conditioning on $\mathcal{F}_{\hat{s}}$), and the fact that

$$|\hat{\mathbf{v}}(w_{s-})+V(w_{s-}+\mathbf{t}^*(w_{s-}))-V(w_{s-})|\le\max\{(U'(0)+\mathtt{g})C,U(C)+\mathtt{g}\mathtt{C}\}<\infty,$$

in which we use $\mathbf{v}(p)\le U'(0)\mathtt{t}(p)$ following (V1), $\mathcal{T}^*(w_{s-})\le C$ from (35), and $|V'(w)|\le\mathtt{g}$ from concavity of V and (13). This completes the proof for the first equality of (27). Q.E.D.

Proposition 10

Following the definition (93) and the optimality conditions (12) and (13), we have,

$$V'(w_{s-})\ge-\mathtt{g}+\lambda\left[\mathbf{v}(p_s)-\alpha c(w_{s-})+V(w_{s-}+\mathbf{t}(p_s,w_{s-}))-V(w_{s-})\right],\text{ and}\tag{100}$$

$$0\ge-\mathtt{g}+\lambda\left[\mathbf{v}(p_s)+V(0+\mathbf{t}(p_s,0))-V(0)\right].\tag{101}$$

Similar to the proof of Proposition 9, denote time τ^0 to be the first time after $\tau_k(\{p_s\})$ that the wait time becomes 0. That is, $\tau^0:=\min\{s\mid w_s=0,s>\tau_k(\{p_s\})\}<\tau_{k+1}(\{p_s\})$. Note that conditional on $\tau_k(\{p_s\})$ being finite and $\mathcal{F}_{\tau_k(\{p_s\})}$, τ^0 is finite with probability 1. However, $\tau_{k+1}(\{p_s\})$ could be infinite ($\tau_{k+1}(\{p_s\})=\infty$) with positive probability for certain control $\{p_s\}$.

Furthermore, $w_s=0$, and, therefore, $\mathbf{t}(p_s,0)dN_s=0$ for $s\in[\tau^0,\tau_{k+1}(\{p_s\}))$ following (9). We have,

$$\begin{aligned}V(0)=&V\left(w_{\tau(\{p_s\},\tau_k(\{p_s\}))-}\right)\\=&V(w_{\tau_k(\{p_s\})-})+\int_{\tau_k(\{p_s\})}^{\tau^0}\left(-V'(w_{s-})\right)ds+\int_{\tau_k(\{p_s\})}^{\tau_{k+1}(\{p_s\})^-}\left[V(w_{s-}+\mathbf{t}(p_s,w_{s-}))-V(w_{s-})\right]dN_s\\\le&V(0)+\int_{\tau_k(\{p_s\})}^{\tau^0}\left\{\mathtt{g}-\lambda\left[\mathbf{v}(p_s)-\alpha c(w_{s-})+V(w_{s-}+\mathbf{t}(p_s,w_{s-}))-V(w_{s-})\right]\right\}ds\\&+\int_{\tau_k(\{p_s\})}^{\tau_{k+1}(\{p_s\})^-}\left[V(w_{s-}+\mathbf{t}(p_s,w_{s-}))-V(w_{s-})\right]dN_s\\&+\int_{\tau^0}^{\tau_{k+1}(\{p_s\})^-}\left\{\mathtt{g}-\lambda\left[\mathbf{v}(p_s)+V\big(\mathbf{t}(p_s,0)\big)-V(0)\right]\right\}ds\\=&V(0)+\mathtt{g}\left(\tau_{k+1}(\{p_s\})-\tau_k(\{p_s\})\right)-\int_{\tau_k(\{p_s\})}^{\tau_{k+1}(\{p_s\})^-}\left[\mathbf{v}(p_s)-\alpha c(w_{s-})\right]dN_s\\&+\int_{\tau_k(\{p_s\})}^{\tau^0}\left[\mathbf{v}(p_s)-\alpha c(w_{s-})+V(w_{s-}+\mathbf{t}(p_s,w_{s-}))-V(w_{s-})\right][dN_s-\lambda ds],\end{aligned}$$

where the first equality follows the definition of $\tau(\{p_s\},\tau_k(\{p_s\}))$ in (96) as an OEQ time, the second equality from (9) and Ito's formula for jump processes (see Theorem 17.5 in Bass (2011)), and the inequality follows (100) and (101). Following the same logic as in the proof of Proposition 9, we have

$$\mathsf{g}\mathbb{E}\left[\tau_{k+1}(\{p_s\})-\tau_k(\{p_s\})\right]\geq\mathbb{E}\left[\int_{\tau_k(\{p_s\})}^{\tau_{k+1}(\{p_s\})^-}\mathbf{v}(p_s)-\alpha c(w_{s^-})\,dN_s\right],\tag{102}$$

which establishes (28).

Finally, to show (29), we notice that (102) implies

$$\mathsf{g}\cdot\mathbb{E}_{K_T(\{p_s\})}\left[\sum_{k=1}^{K_T(\{p_s\})-1}\mathbb{E}\left[\tau_{k+1}(\{p_s\})-\tau_k(\{p_s\})\right]\right]\geq\mathbb{E}\left[\sum_{k=1}^{K_T(\{p_s\})-1}\mathbb{E}\left[\int_{\tau_k(\{p_s\})}^{\tau_{k+1}(\{p_s\})^-}\mathbf{v}(p_s)-\alpha c(w_{s^-})\,dN_s\right]\right],$$

or, equivalently,

$$\mathsf{g}\cdot\mathbb{E}\left[\tau_{K_T(\{p_s\})}(\{p_s\})-\tau_1(\{p_s\})\right]\geq\mathbb{E}\left[\int_{\tau_1(\{p_s\})}^{\tau_{K_T(\{p_s\})}(\{p_s\})^-}\left[\mathbf{v}(p_s)-\alpha c(w_{s^-})\right]dN_s\right].\tag{103}$$

We consider two different cases. First, $\mathbb{E}[K_T(\{p_s\})]$ approaches infinity with T. In this case, both $\Delta(\{p_s\},T)$ and $\Theta(\{p_s\},T)$ are finite, following the same logic as the proof of Lemma 13. For T large enough such that $\mathbb{E}[K_T(\{p_s\})]>1$, (103) implies that

$$\mathsf{g}\cdot\frac{1}{\mathbb{E}[K_T(\{p_s\})]-1}\mathbb{E}\left[\tau_{K_T(\{p_s\})}(\{p_s\})-\tau_1(\{p_s\})\right]\geq\frac{1}{\mathbb{E}[K_T(\{p_s\})]-1}\mathbb{E}\left[\int_{\tau_1(\{p_s\})}^{\tau_{K_T(\{p_s\})}(\{p_s\})^-}\left[\mathbf{v}(p_s)-\alpha c(w_{s^-})\right]dN_s\right],$$

which implies

$$\begin{aligned}\mathsf{g}&\geq\liminf_{T\to\infty}\frac{\frac{1}{\mathbb{E}[K_T(\{p_s\})]-1}\mathbb{E}\left[\int_{\tau_1(\{p_s\})}^{\tau_{K_T(\{p_s\})}(\{p_s\})^-}\left[\mathbf{v}(p_s)-\alpha c(w_{s^-})\right]dN_s\right]}{\frac{1}{\mathbb{E}[K_T(\{p_s\})]-1}\mathbb{E}\left[\tau_{K_T(\{p_s\})}(\{p_s\})-\tau_1(\{p_s\})\right]}\\&=\liminf_{T\to\infty}\frac{\frac{1}{\mathbb{E}[K_T(\{p_s\})]-1}\mathbb{E}\left[\int_{\tau_1(\{p_s\})}^{\tau_{K_T(\{p_s\})}(\{p_s\})^-}\left[\mathbf{v}(p_s)-\alpha c(w_{s^-})\right]dN_s\right]+\frac{1}{\mathbb{E}[K_T(\{p_s\})]-1}\Delta(\{p_s\},T)}{\frac{1}{\mathbb{E}[K_T(\{p_s\})]-1}\mathbb{E}\left[\tau_{K_T(\{p_s\})}(\{p_s\})-\tau_1(\{p_s\})\right]+\frac{1}{\mathbb{E}[K_T(\{p_s\})]-1}\Theta(\{p_s\},T)}\\&=\liminf_{T\to\infty}\frac{\mathbb{E}\left[\int_{\tau_1(\{p_s\})}^{\tau_{K_T(\{p_s\})}(\{p_s\})^-}\left[\mathbf{v}(p_s)-\alpha c(w_{s^-})\right]dN_s\right]+\Delta(\{p_s\},T)}{\mathbb{E}\left[\tau_{K_T(\{p_s\})}(\{p_s\})-\tau_1(\{p_s\})\right]+\Theta(\{p_s\},T)}=\mathcal{G}(\{p_s\}).\end{aligned}$$

Hence we have established (29) for this case.

Next, consider the case that $\mathbb{E}[K_T(\{p_s\})]$ is finite when T approaches infinity. In this case $\Delta(\{p_s\},T)$ is still bounded for all T. This is because absence of an OEQ must be due to not allowing any customer into an empty queue under control p_s. When wait time remains zero over a period of time, (V2) implies that the accumulated objective value is also zero. In contrast, $\Theta(\{p_s\},T)$ must approach infinity as T approaches infinity in this case. Therefore, from (103), we have that for T large enough,

$$\mathsf{g}\cdot\left(\mathbb{E}\left[\tau_{K_T(\{p_s\})}(\{p_s\})-\tau_1(\{p_s\})\right]+\Theta(\{p_s\},T)\right)\geq\mathbb{E}\left[\int_{\tau_1(\{p_s\})}^{\tau_{K_T(\{p_s\})}(\{p_s\})^-}\left[\mathbf{v}(p_s)-\alpha(w_{s^-})\right]dN_s\right]+\Delta(\{p_s\},T),$$

or, equivalently,

$$\mathsf{g}\geq G(\{p_s\},T)$$

which further implies (29). Q.E.D.

Appendix D: Proofs in Section 5

Proposition 11

Following from (4) in Proposition 1, the firm can induce service time $t \in [L(w), C]$ by setting the price $p = U'(t) \in [0, \Gamma(w)]$ when $w \le \hat{w}$. On the other hand, when $w > \hat{w}$, the firm can set the price $p \in [0, \Gamma(w)] = [0, (U(C) - c(w))/C]$ to induce the customer to choose C service time because we have

$$U'(C) = \frac{U(C) - c(\bar{w})}{C}, \tag{104}$$

which follows (3) and (4). We thus can transform (12) into

$$\mathsf{g} = -V'(w) + \lambda \max\left\{\max_{L(w)\le t\le C}\left\{\mathbf{v}\left(U'(t)\right) - \alpha c\left(w\right) + V\left(w + \mathtt{t}(p)\right) - V\left(w\right)\right\}, 0\right\}, \quad \forall w \in (0, \hat{w}], \tag{105}$$

and

$$\mathsf{g} = -V'(w) + \lambda \max\left\{\max_{p\in[0,U'(C)]}\{\mathbf{v}\left(p\right) - \alpha c\left(w\right) + V\left(w + C\right) - V\left(w\right)\}, 0\right\}, \quad \forall w \in (\hat{w}, \bar{w}]. \tag{106}$$

Similarly, we have the boundary condition

$$\mathsf{g} = -V'(w) + \lambda \max\left\{\max_{L(0)\le t\le C}\left\{\mathbf{v}\left(U'(t)\right) + V\left(\mathtt{t}(p)\right) - V(0)\right\}, 0\right\}. \tag{107}$$

The remaining two boundary conditions (14) and (15) do not involve the firm's decision, so they are unchanged in the service time control expression. We directly establish the equivalence between (12)-(15) and (30)-(34). Therefore, if g and V solve (12)-(15), they also satisfy (30)-(34) and vice versa.

Now we show $\mathbf{f}(t, w)$ is strictly concave in $t \in [L(w), C]$. Considering two expressions of $\mathbf{v}(p) - \alpha c(w)$ in Lemma 1, we have

$$\mathbf{v}(U'(t)) - \alpha c(w) = tU'(t) \text{ or } \mathbf{v}(U'(t)) - \alpha c(w) = U(t) - c(w),\ \forall t \in [L(w), C], w \in [0, \hat{w}]. \tag{108}$$

This implies $\mathbf{v}(U'(t)) - \alpha c(w)$ is strictly concave in $t \in [L(w), C]$ from Assumption 1(i), (ii), and (iii).

Lastly, we show $L(w)$ increases in w. We let $w_1 < w_2 \le \hat{w}$ without loss of generality. From Proposition 1 and Assumption 1(iii), we have

$$U\left(L(w_1)\right) - U'(L(w_1))L(w_1) - c(w_1) = 0, \text{ and } U\left(L(w_1)\right) - U'(L(w_1))L(w_1) - c(w_2) < 0. \tag{109}$$

The later inequality implies that $L(w_2) > L(w_1)$, which completes the proof. Q.E.D.

Theorem 3

We show the policy obtained from (12)-(15) and Theorem 1 by considering the service time control problem in Proposition 11. We define

$$\hat{\xi}(t, w) := \mathbf{v}\left(U'(t)\right) - \alpha c\left(w\right) + V(w + t) - V(w), \tag{110}$$

which is strictly concave in t for any $w \in [0, \hat{w}]$ from Proposition 11 and Theorem 1. Thus,

$$t^*(w) \in \underset{t\in[L(w),C]}{\arg\max}\, \hat{\xi}(t, w), \tag{111}$$

is the unique maximizer for any $w \in [0, \hat{w}]$.

We define the service time $t^u(w)$ as the solution to Eq. (110) that satisfies

$$\mathbf{v}'\left(U'(t^u(w))\,U''(t^u(w))+V'\left(w+t^u(w)\right)=0.$$

By Implicit Function Theorem and considering two types of objective from (108), the change in $t^u(w)$ varying w as the following:

$$\text{Revenue maximization: } \frac{d}{dw}t^u(w)=\frac{-V''\left(w+t^u(w)\right)}{V''\left(w+t^u(w)\right)+\frac{d^2}{dt^2}\left[t^u(w)U'\left(t^u(w)\right)\right]}<0, \tag{112}$$

$$\text{Social welfare maximization: } \frac{d}{dw}t^u(w)=\frac{-V''\left(w+t^u(w)\right)}{V''\left(w+t^u(w)\right)+U''\left(t^u(w)\right)}<0. \tag{113}$$

This monotonicity holds because $V(w)$ is concave in w from Theorem 1, $U''(t)$ is strictly concave from Assumption 1(i), and $tU'(t)$ is strictly concave from Assumption 1(ii). This further implies the constrained service time decision $t^*(w)$ in Eq. (111) decreases in w if the lower bound is not binding.

Because of (112), (113), and $t^*(w)\in[L(w),C]$, the unique threshold $W_C\geq 0$ exist that

$$t^u(w)>t^*(w)=C,\quad \forall w\in[0,W_C)$$

where the unconstrained solution $t^u(w)$ exceeds the upper bound C. The uniqueness of W_C is implied by (112) and (113). Next, we have another threshold $W_I\geq 0$ such that $W_I\geq W_C$ and

$$t^u(w)=t^*(w)\geq L(w),\quad \forall w\in[W_C,W_I),$$

where the unconstrained solution satisfies the constraint of feasible region. Equations (112) and (113) imply that $t^*(w)$ decreases in w. The uniqueness of W_I further follows that the lower bound $L(w)$ increases in w from Proposition 11.

Because $t^u(w)$ decreases in w, we have

$$t^u(w)<t^*(w)=L(w),\quad \forall w\in[W_I,\hat{w}].$$

This follows that the unconstrained solution is smaller than the lower bound $L(w)$ while $L(w)$ increases in w from Proposition 11. As a result, $t^*(w)$ increases in $w\in[W_I,\hat{W}]$.

From (31), we have

$$t^*(w)=C,\quad \forall w\in[\hat{w},\bar{w}],$$

by implementing the price $p^*(w)=\frac{U(C)-c(w)}{C}$.

The decision $t^*(w)$ does not include the situation that the firm rejects the customer away for the state w. We thus include the optimal rejection decision for the firm when

$$\hat{\xi}\left(t^*(w),w\right)=\mathbf{v}\left(U'\left(t^*(w)\right),w\right)+V\left(w+t^*(w)\right)-V(w)<0,$$

by setting the price rate as $U'(0)$ to induce balking decision from the customer. Given the concavity of V in Theorem 1, strict concavity of $\mathbf{v}(U'(t),w)$, and (112), we have $\hat{\xi}\left(t^*(w),w\right)$ decreases in w. As a result, there is a unique rejection threshold $W_3\geq 0$ such that

$$\hat{\xi}(t^*(w),w)<0,\quad \forall w\geq W_3.$$

Now, we are able to characterize the optimal policy where $\mathcal{T}^*$ denotes the optimal service time policy and p^* is the corresponding optimal price rate policy. The policy is composed of at most five section where $W_1 = W_C > 0$, $W_2 = W_I,$ and $W_3 > \hat{w}$. That is, the firm offers full service time when the wait time is short, decreases the induced service time as the wait time increases, provides full service time to the arrival when the wait time exceeds $\hat{w}$ and stops the service when the wait time reaches W_3. Therefore, the complete control policy can be described as following:

- $\mathcal{T}^*(w) = C$ and $p^*(w) = U'(C)$ for $w \in [0, W_1)$;
- $\mathcal{T}^*(w) = t^*(w)$ and $p^*(w) = U'(t^*(w))$ w for $w \in [W_1, W_2)$; (decreasing service time)
- $\mathcal{T}^*(w) = L(w)$ and $p^*(w) = U'(L(w))$ for $w \in [W_2, \hat{w})$; (increasing service time)
- $\mathcal{T}^*(w) = C$ and $p^*(w) = \frac{U(C) - c(w)}{C}$ for $w \in [\hat{w}, W_3)$; (full service time with decreasing price)
- $\mathcal{T}^*(w) = 0$ and $p^*(w) = U'(0)$ for $w \geq W_3$.

This completes the proof. Q.E.D.