# Dynamic Moral Hazard with Adverse Selection

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We study dynamic contracts that incentivize an agent to exert effort to increase the arrival rate of a Poisson arrival (breakthrough), where both the effort cost and the effort level at any time are the agent's private information. The principal needs to offer a menu of contracts such that each type of agent (with a different effort cost) chooses the corresponding item on the menu. Each item specifies a deadline for the contract and the payment process over time. We focus on two types of agents and fully characterize the optimal menu of contracts. Specifically, the principal will not provide any contracts when the revenue of the breakthrough is low enough. When the revenue is at the medium level, only the good agent is worth hiring. Hence, the principal provides a pay-to-leave contract to the bad agent, which pays a lump-sum payment to the agent without asking the agent to work. At the same time, the good agent is provided a so-called IC-binding contract that provides exactly enough incentives for him to work. When the revenue is high enough, the bad agent obtains an IC-binding contract, while the good agent gets a so-called screening contract such that he is not willing to mimic the bad agent. Overall, we are able to obtain the closed-form expression of the optimal menu, and the optimal contracts are easy to describe, compute, and implement.

*Key words*: dynamic, moral hazard, optimal control, jump process, adverse selection

## 1. Introduction

Research breakthrough projects refer to initiatives or endeavors that aim to achieve significant advancements or discoveries in a particular field of study. These projects typically involve pushing the boundaries of existing knowledge, technologies, or understanding to attain novel and transformative outcomes. Breakthroughs in research projects often lead to groundbreaking innovations, solutions to long-standing problems, or the development of new theories and methodologies.

To achieve environmental sustainability, projects on climate change mitigation technologies are developed across the globe. A total of 243 projects (USD 13.5 billion, 51.9 billion including co-financing) are funded by the Green Climate Fund (GCF) to deliver transformative climate actions. In pharmaceutical development, any medical breakthrough that is an advancement to cure a currently incurable disease, such as cancer or Alzheimer's, can be enormously lucrative. Leading pharmaceutical companies have their R&D spending, which can be as high as 30% of their yearly revenue. Research breakthrough projects are common in technology companies. The R&D spending in Amazon was as high as $ 26.6 Billion, accounting for 13% of its total revenue. Yet the return once a research project succeeds is also enormous: Open AI brings Microsoft a revenue of $1.3 Billion in the year 2023. It is vital that the fundings are used properly and effectively. However, there is news showing improper usage of funding by researchers. A professor is accused of cheating NASA out of millions of Space research institutes at the University of Florida. The NASA grant, which is supposed to be spent on high-tech engineering research, was privately spent on cars and condos.[1]. United States Office of Special Counsel has overseen another investigation, which revealed that hundreds of millions of dollars were not used for their intended purpose. Funds for the Biomedical Advanced Research and Development Authority (BARDA) were intended for the development of drugs, therapies, and vaccines. However, a large fraction of the funds have been used for unrelated purposes, such as salaries and the removal of office furniture.[2].

There are some common features among those motivating examples. First, these projects are usually structured within an agency relationship due to the need for both high technical expertise and significant capital inputs (funding). The skilled individuals execute the project under some level of separation from the entity supplying financial support. Additionally, the preferences of the agent (entrepreneur, contractor, or researcher) in terms of investment timing, intensity, and direction are not likely to perfectly align with the preferences of the principal (financier, end user, or institution). Second, moral hazard issues exist because of the misalignment of incentives. The agent may have the incentive to misuse funding. They may divert the funding for their private benefits rather than to the proper uses to develop the breakthroughs. Therefore, the principal has to provide sufficient incentive for the agent to be willing to exert effort. Third, the funding levels that are required may vary across different types of agents. Scientific teams who work on such projects are heterogeneous: they vary in their approaches/methods to tackle problems. Moreover, due to professional barriers, how much funding a project requires remains private information to the agent. The project may be terminated if no success has been observed after a certain period. Despite its high potential and enormous returns, project cancellation and terminations are sometimes necessary. An example of project cancellation is the V.C. Summer project, which was intended to build nuclear plant in South Carolina. The project was cancelled in 2017, after more than $9 billion was invested.[3]

These common ingredients give rise to this paper's research question: when sponsoring research-breakthrough projects, the principal faces two dimensions of disadvantages in information. First, she does not know what levels of funding are sufficient; second, the principal cannot observe whether the agent is misusing the funding. To overcome such informational disadvantages, how should she design optimal contracts? Notice that the agent has a great amount of latitude for misconduct: he may ask for funding that is more than necessary; he may also divert the funding for his private benefits, and he may even do both. The principal, thus, has to be meticulous in designing contracts that prevent such misbehaviors.

We model the problem to be the optimal design of the principal's contract (when there exists both adverse selection and dynamic moral hazard). The principal delegates the development of a research project to an agent. With the agent's continued effort, the success of the project may arrive randomly. Once it succeeds, a fixed amount of revenue goes to the principal. The misalignment of interest between the principal and the agent lies in the fact that the revenue is enjoyed by the principal while the effort cost is paid by the agent. Therefore, incentives must be created to induce the agent's effort. Being the residual claimer of the project, the principal also needs to provide the funding to reimburse for the agent's cost. However, the agent's cost, which is his private information, is uncertain.

Following standard results in mechanism design in Laffont and Martimort (2009), the principal should provide a menu of contracts, such that an agent with a specific cost chooses a particular contract from this menu. The application of menu of contracts in real life is common. When applying for research funding, a principal investigator (PI) drafts proposals with clear budget and intentions on spending the budget, which is a way to self-select a contract with a funding provider such as NSF. Following the revelation principle (Myerson 1981), it is without loss of generality for us to consider direct mechanisms. In our setting, however, after the agent reports the operating cost, the two players still face a dynamic moral hazard game. That is, the agent can choose his effort at any point in time, which depends on the payment process designed by the principal. Therefore, the principal needs to optimize over the menus of dynamic incentive-compatible contracts. Furthermore, one type of agent should not have an incentive to choose a dynamic contract for another type. Consequently, the optimal design problem can no longer be formulated as a classic dynamic program.

The major contributions of this paper are threefold. First, to solve the optimal contract design problem, we develop a solution approach based on deterministic continuous-time optimal control. Specifically, we construct an easy-to-solve upper-bound optimization problem for the original contract design problem. Fortunately, the optimal solution to the optimization problem provides a menu of contracts where the performance achieves the upper bound. Second, we fully characterize the optimal menu of contracts that can be expressed in closed form, and the optimal contracts are easy to describe, compute, and implement. Third, we summarize the structures of the optimal contracts in the space of key parameters, which provide managerial implications on how the principal should address the problem of dynamic moral hazard and adverse selection at the same time.

### 1.1. Summary of main results

Our paper focuses on a two-type case, where the effort cost of the agent can either be high or low. For simplicity, the low-cost agent is called a good agent, and the high-cost agent is called a bad agent. Therefore, the principal offers the agent a menu of two contracts, with each item in the menu designed specifically for the good agent and the bad agent, respectively. In each contract, the principal specifies the deadline of the project and the payments to the agent upon success. The contract is terminated if the agent is not able to bring the arrival before the deadline. We show that to implement the optimal menu of contracts; we only need to consider three types of contracts: *Pay-to-leave contract*, *IC-binding contract*, and *Screening contract*. Specifically, the pay-to-leave contract pays a lump-sum payment to the agent without asking the agent to work. The IC-binding contract keeps the agent's incentive constraints always binding and is proved to be optimal in the benchmark case when the agent's cost is known by the principal. In the screening contract, the incentive constraint is not always binding. In the screening contract, the good agent exerts full effort, while the bad agent does not have the incentive to work. Therefore, anyone who manages to deliver an arrival in the screening contract has to be the good agent.

Our analysis reveals that the optimal menu of contracts demonstrates three possibilities depending on model parameters, as illustrated in the three regions of Figure 2 later in the paper. First, when the revenue $R$ is low enough, no contract will be provided. That is, it is not worth hiring any agent at all. When the revenue $R$ is medium, only the good agent is worth hiring. Hence, the principal offers the bad agent a pay-to-leave contract and offers the good agent an IC-binding contract. The pay-to-leave contract is for the bad agent to truthfully report his type because otherwise, he could have pretended to be the good agent and collected the funding in the good agent's contract. This medium case only occurs when the good agent takes up a sufficiently high fraction of the population and the revenue from the success is medium.

Finally, when the revenue $R$ is high, the principal allows both types of agents to work. As a result, the bad agent will be offered an IC-binding contract, while the good agent will be offered a screening contract. The screening contract creates a discrepancy in the expected payoff between the good and the bad agent. Notice that whenever the principal wants to induce the bad agent to work, the good agent could always pretend to be the bad agent and become better off than the bad agent. Therefore, a utility gap naturally exists between the good and bad agents. The screening contract manifests an interesting synergy between adverse selection and moral hazard: the existence of multiple types of agents makes the principal shift the optimal contract from the IC-binding contract in our benchmark to a novel screening contract.

Note to Feifan himself, will provide further explanations of these results later.

Finally, we discuss the welfare implications of unknown costs by comparing the optimal contracts in the unknown cost scenario to that in the scenario when the principal knows the agent's cost. We find that the unknown cost situation hurts the good agent and benefits the bad agent if and only if the revenue $R$ is low enough.

## 2. Literature Review

There have been previous attempts on the problem with both adverse selection and dynamic moral hazard. Ma (1991) focuses on renegotiation and actions with long term effects, whereas we give the principal full commitment power on contracting and hence the issue of renegotiation does not exist. Gershkov and Perry (2012) studies moral hazard with both persistent and repeated adverse selection in a discrete and finite horizon. In every period, the agent receives a task and different types of agent differ in the probability of success, while in our model types differ in the cost of exerting effort. Furthermore, our payment contract has an arguably cleaner and simpler structure thanks to the continuous-time setting. Mayer (2020) and Rong et al. (2021) both consider dynamic moral hazard problems with adverse selection where an agent is hired to exert effort to reach a single breakthrough, which is similar to ours. However, the adverse selection in their model comes from the information about the arrival (timing of the arrival or the status of the arrival) but not a characteristic of the agent (capability of the agent in our model). Similarly, Chen et al. (2018) considers an infinite horizon Poisson model where the adverse selection also comes from the feature of the arrivals. Foarta and Sugaya (2021) studies the optimal contract design under both moral hazard and unknown agent's effort cost. The optimal contract asks the agent to truthfully report their types, and will assign him to the corresponding position. However, their contract is a static contract, as opposed to our dynamic moral hazard contract setting. Second, they made simplification assumptions on the implementable contracts by giving the low type agent a contract under perfect monitoring. Our paper considers a much more sophisticated setting where both types of the agent receives a dynamic contracts with imperfect monitoring through breakthrough arrivals.

Our adverse selection and moral hazard setting is also related to the one studied in Cvitanić and Zhang (2007). Their underlying stochastic process is Brownian motion over a finite time horizon. They suggest a relaxation-based procedure to obtain contracts that "are not necessarily optimal." In comparison, we are able to derive closed form optimal contracts thanks to our Poisson setting. Cvitanić et al. (2013) and Santibáñez et al. (2020) study continuous-time moral hazard problems in infinite horizon with adverse selection under Brownian and Poisson stochasticity, respectively. To solve the adverse selection problem, they adopt the methodology of a credible set regarding the agents' continuation and temptation values. Rather than resorting to their method, which involves stochastic differential equations with variational inequalities, we formulate our optimization problem with a deterministic optimal control approach. Our formulation enables us to provide closed-form solutions of optimal menu of contracts with intuitive implementations, such as *screening contract* and *pay-to-leave* contract.

Our paper implements optimal contracts that screens out different types of agent, which relates to the literature with screening. Curello and Sinander (2020) relates to our work by analyzing screening contracts in research breakthroughs. In their paper, the agent privately observes the arrival of a breakthrough of technology and decides when to disclose it to the principal. There is conflict of interest between the principal and the agent because once the breakthrough is revealed, the principal prefers to lower the agent's utility under the new technology. Though similar in considering truthful reporting and breakthroughs, our paper differ in the following two dimensions. First, their breakthrough is an intermediate process, which is the origin of the agent's heterogeneity, while our agent's type is given in the beginning of the game and our breakthrough marks the end of the game. Second, we offer a menu of contracts to solve the adverse selection issue, while they use a single contract to induce truthful reports of arrivals.

The dynamic moral hazard dimension of our model is based on Green and Taylor (2016), in which all model parameters (including the operating cost) are known. In Green and Taylor (2016), the agent's effort level is binary while we allow for a continuum of effort. Similar dynamic moral hazard models based on Poisson arrivals include Sun and Tian (2018), Chen et al. (2020a), Tian et al. (2021), Cao et al. (2022, 2023). The key difference between our paper and these papers is the addition of the adverse selection component into the dynamic moral hazard model. The adverse-selection extension brings this line of work much closer to reality, because the agent's capability is often not transparent in real-world settings. The analysis and results are also strikingly different. First, due to adverse selection, our design involves a *menu* of contracts rather than a single contract as in each of the other papers. Second, the individual contract received by each type of agent may have different structures including pay- to-leave contract and screening contract, which do not appear in the aforementioned papers. The promised utility formulation of continuous time moral hazard problem originates from Sannikov (2008), which provides a martingale representation of incentive compatibility constraint with an underlying Brownian motion uncertainty. This framework has been further applied to Poisson settings by Biais et al. (2010), in which the agent is hired to decrease the arrival rate of "bad news." Increasing the arrival rate of "good news," as in our model, has been studied in a stream of recent papers, (see, for example, Green and Taylor 2016, Shan 2017, Sun and Tian 2018, etc.), although without an adverse selection component.

Note: Need to search for project management + innovation + adverse selection literature.

Dynamic contracting problems have been the subject of recent Operations Research studies (see, for example, Zhang 2012a,b, Li et al. 2013). A particularly relevant stream of papers applies dynamic incentive design to project management settings (Kwon et al. 2010, Chen et al. 2015, Dawande et al. 2019). In these papers, a project manager designs short-term contracts for multiple independent agents (contractors). In comparison, we focus on designing contracts for a single agent. Further, including both moral hazard and adverse selection on the agent's effort cost distinguishes our paper from the aforementioned literature. It is worth mentioning two recent papers, which also consider both moral hazard and adverse selection issues. Chen et al. (2020b) studies a principal who periodically offers agents "limited-term" non-monetary rewards in order to induce agents' effort over the long-run. The reward's value to each agent is the agent's private information. The paper is focused on designing near-optimal "limited-term" stationary policies. Zorc et al. (2019) considers a delegated search model where the agent's search effort and the findings from the search process are private information. They adopt the framework of Plambeck and Zenios (2000), which considers a risk averse agent who can borrow money. In comparison, we study a risk-neutral agent who can exert partial effort when cash-constrained.

Another related strand of literature combines dynamic moral hazard with learning. Unlike our private information setting where the principal can elicit truthful information, under their setting with learning, the true state is unobservable to either party, and hence the contract has to update both parties' belief (see, for example, Bhaskar 2012, Kwon 2011). Halac et al. (2016) further considers long-term contracting problems that involve adverse selection, moral hazard, and learning. In another paper, Bimpikis et al. (2019) studies when and what information a contest designer should disclose in a contest design setting, which affects how much participants learn about the feasibility of the innovation over time.

The remainder of the paper is organized as follows. We introduce the model in Section 3. In Section 4, we present three contracts that are candidates for optimal contracts when the agent type is unknown by the principal. In Section 5, we summarize the mian results of the paper. Section 6 contains detailed steps to derive the optimal menu of contracts, how to show their optimality. To deliver further managerial implications, we summarize the main properties of the optimal menu of contracts in Subsection 7.1, and in Subsection 7.2, we discuss the welfare implications of unknown cost by comparing the optimal menu of contracts between the unknown cost and the known cost scenarios. Finally, Section 8 concludes the paper with future directions.

## 3. Model

A principal contracts an agent to increase the instantaneous arrival rate of a Poisson process over a potentially infinite time horizon. At any point of time $t$, the agent can privately choose an effort level $\nu_t \in [0, \mu]$, which incurs a flow cost, and generates an arrival (breakthrough) with an instantaneous rate $\nu_t$. The arrival yields a revenue $R$ to the principal, and is observable to both the principal and the agent. We denote the arrival time as $\tau$.

The agent's capability, reflected in the operating cost, is linear in the effort level. We denote $c$ to represent the operating cost when the agent chooses the highest possible effort level $\mu$. Therefore, the per arrival rate ~~per unit time unit~~ flow operating cost is

$$\beta_c := c/\mu.$$

Hence, if the agent chooses effort level $\nu_t \in [0, \mu]$, the corresponding flow operating cost is $\beta_c \cdot \nu_t$. That is, a more capable agent can generate the arrival with a lower operating cost. In this paper, we use "capability" and "cost" interchangeably. The operating cost is the agent's private information, and stays the same throughout the time horizon. The common prior distribution of the operating cost has a support $\mathcal{C}$. In this paper we consider a binary set $\mathcal{C} = \{g, b\}$ with $g < b$. The operating cost $c$ is also referred to as the agent's *type*. The prior probabilities of types $g$ and $b$ are $p$ and $1-p$, respectively. We refer to the type $g$ agent, who has a lower cost $g$, as the *good* agent and $b$ as the *bad* agent. In this paper, we require the following assumption.

Assumption 1.

$$\beta_g \leq R, \ \textit{or, equivalently,} \ \mu R \geq g.$$

This assumption guarantees that the good agent is efficient, which means that this agent generates a positive societal value whenever exerting effort. With this assumption, we exclude the trivial case where both agents are inefficient because it is obviously dominant for the principal to offer a null contract with no payment and immediate termination in that trivial case. Note that we have not made any assumption on the bad agent's cost $b$ yet. Indeed, the bad agent can either be efficient ($\beta_b \leq R$) or inefficient ($\beta_b > R$), which leads to somewhat different optimal contract structures, as will be discussed later in Section 6.

We assume that the principal needs to reimburse the flow operating cost in real time, because the agent has limited liability and is cash constrained, a standard assumption in the dynamic contracting literature. In particular, at any point in time, the agent's effort choice is constrained by the flow reimbursement provided by the principal. Because the agent knows the operating cost in the beginning, following the *Revelation Principle*, it is without loss of generality to consider direct mechanisms (see, for example, Myerson 1986, Pavan et al. 2014). In our context, the principal designs a menu of contracts $\Gamma_{\mathcal{C}} = \{\gamma^c\}_{c \in \mathcal{C}}$, such that type $c$ agent chooses contract $\gamma^c$. It is natural to assume that contract $\gamma^c$ provides the type $c$ agent a flow reimbursement $c$ before the termination of the contract.(This situation is fairly common in contexts such as R&D and lobbying, where the principal has to provide a continuous flow of payments for the agent to operate. It may take the form of retainers in the case of lobbyists or repetitive payments in the case of R&D contracts.)[4] Generally speaking, if the agent chooses an effort level strictly less than $\mu$, and therefore does not use up the operating payment, the part not being used can be diverted as a shirking benefit to the agent.[5]

Any contract $\gamma^c = (I^c, T^c)$ includes a payment process $I^c$, and a time duration $T^c$ (deadline). When it is not necessary in the context to stress the operating cost $c$, we also use the notation $\gamma = (I, T)$ without superscripts to represent a generic contract. Specifically, for a payment process $I = \{I_t\}_{t\geq 0}$, at each time epoch $t \geq 0$, $I_t$ represents the instantaneous payment for success at time $t$, and $T$ is the time at which the contract is terminated absent an arrival. Therefore, the contract ends at time $T \wedge \tau$. The agent's *limited liability* (LL) and being cash constrained imply that payment is from the principal to the agent but not the other way around. That is,

$$I_t^c \geq 0, \quad \forall t \geq 0, c \in \mathcal{C}. \tag{LL}$$

**Agent utility** Given a dynamic contract $\gamma^c = (I^c, T^c)$ and an effort process $\nu$, the expected utility of the agent with an operating cost $c'$ is

$$u(\gamma^c, \nu; c') := \mathbb{E}^\nu \left[ I_\tau^c \cdot \mathbb{1}_{\tau < T^c} + \int_0^{T^c \wedge \tau} (c - \nu_t \cdot \beta_{c'}) \mathrm{d}t \right], \tag{1}$$

in which the expectation $\mathbb{E}^\nu$ is taken with respect to arrival rates

probabilities generated from the effort process $\nu$.[6]

Denote $\mathcal{N}$ to be the set of all effort processes $\nu$ that satisfy the condition $\nu_t \in [0, \mu]$. Further denote $\mathfrak{N}(\gamma^c, c) \subseteq \mathcal{N}$ to represent the set of *best-response* (BR) effort processes, when the type $c$ agent truthfully reports and chooses contract $\gamma^c$. That is,

$$u\big(\gamma^c, \nu; c\big) \geq u(\gamma^c, \nu'; c), \ \forall \nu \in \mathfrak{N}(\gamma^c, c) \text{ and } \nu' \in \mathcal{N}. \tag{BR}$$

Next, it is worth discussing the scenario when a type $c$ agent pretends to be of type $c'$ in the following remark.

Remark 1. When the type $g$ agent pretends to be of type $b$, he can choose any effort process in $\mathcal{N}$, because the flow payment $b$ is able to fully cover his effort cost $g$. On the other hand, if the type $b$ agent pretends to be type $g$, then this agent is only reimbursed with an effort cost $g$. Consequently, the agent's effort level can not be higher than $\mu g/b$. We denote the set of effort processes that can be chosen by the bad type when mimicking the good one as

$$\mathcal{N}' := \{\nu \mid \nu_t \leq \mu g/b, \ \forall t\}.$$

If the agent chooses an effort level strictly less than $\mu g/b$, he diverts the rest flow reimbursement as a shirking benefit. Q.E.D.

The revelation principle implies that we can focus on direct mechanisms. Therefore, we need the following *Incentive-Compatibility* (IC) constraints on the menu $\Gamma_{\mathcal{C}}$, which ensures that an agent with operating cost $c$ indeed chooses contract $\gamma^c$ from the menu, that is,

$$u(\gamma^g,\nu^g;g) \geq u\left(\gamma^b,\nu;g\right),\ \forall \nu^g \in \mathfrak{N}(\gamma^g,g), \nu \in \mathcal{N}, \tag{ICg}$$

$$u(\gamma^b,\nu^b;b) \geq u\left(\gamma^g,\nu;b\right),\ \forall \nu^b \in \mathfrak{N}(\gamma^b,b), \nu \in \mathcal{N}'. \tag{ICb}$$

It is standard to consider the agent's continuation utility (also called the *promised utility*, see, for example, Biais et al. 2010) at time $t$, defined as,[7]

$$W_t(\gamma^c,\nu;c) = \mathbb{E}^{\nu}\left[I^c_\tau \cdot \mathbb{1}_{\tau<T^c} + \int_{t+}^{T^c\wedge\tau} (c - \nu_s \cdot \beta_c)\mathrm{d}s\right] \mathbb{1}_{t<T^c\wedge\tau}, \tag{2}$$

for the type $c$ agent who exerts an effort process $\nu$ under contract $\gamma^c$. Following standard assumptions in the dynamic contracting literature, the principal has the commitment power to issue a long term contract, while the agent can choose to walk away from the contract at any time with a zero utility. That is, we need the following *Individual Rationality* (IR) constraint to guarantee the agent's participation before contract termination,

$$W_t(\gamma^c,\nu;c) \geq 0,\ \forall t \in [0, T^c\wedge\tau],\ c \in \mathcal{C}. \tag{IR}$$

**Principal utility.** Denote $U(\gamma^c,\nu^c)$ to represent the principal's total expected utility from a contract $\gamma^c$ for a type $c$ agent who exerts effort according to a process $\nu^c \in \mathfrak{N}(\gamma^c,c)$. That is,

$$U(\gamma^c,\nu^c) := \mathbb{E}^{\nu^c}\left[(R - I^c_\tau)\,\mathbb{1}_{\tau<T^c}\right]. \tag{3}$$

Now we define $\mathcal{U}(\Gamma_{\mathcal{C}}) := \mathbb{E}\left[U(\gamma^c,\nu^c)\right]$ to represent the principal's total expected utility from the *menu* of contracts $\Gamma_{\mathcal{C}}$. The principal's contract design problem is

$$\begin{aligned} \mathcal{Z}(\mathcal{C}) := \sup_{\Gamma_{\mathcal{C}}}\quad & \mathcal{U}(\Gamma_{\mathcal{C}}) = p\cdot U(\gamma^g,\nu^g) + (1-p)\cdot U(\gamma^b,\nu^b) \\ & \text{s.t. (LL), (BR), (IR), (ICg) and (ICb).} \end{aligned} \tag{4}$$

The objective function value $\mathcal{Z}(\mathcal{C})$ is the principal's optimal expected utility. Note that the constraints (LL), (BR), and (IR) are for each $c \in \mathcal{C}$. The constraints (ICg) and (ICb) imply that the maximization problem (4) cannot be decoupled in $c$.

## 4. Implementable Contracts

In this section, we present all possible contract forms that will appear in an optimal menu of contracts before rigorously deriving them in the next section. Note that the space of the dynamic contracts could be enormous. Here, we greatly narrow down the possibilities to three structures: *pay-to-leave* contract, *IC-binding* contract, and *screening* contract, all of which are easy to compute and implement. In later sections, we will formally show how to derive the optimal contracts and verify that these three contract structures suffice.

### 4.1. Pay-to-leave contract

First, we introduce the so-called *pay-to-leave* contract. In particular, we allow the principal to pay a lump-sum payment without asking the agent to work.

DEFINITION 1. For a lump-sum payment $B \geq 0$, define a *pay-to-leave contract* $\check{\gamma}(B)$, which pays the agent $I_0 = B$ at time 0, and then set the time duration as $T = 0$.

One can imagine that when the bad type is inefficient ($\mu R \leq b$), hiring the agent hurts the principal. To resolve the adverse selection issue, the principal can provide the bad type a pay-to-leave contract that asks the bad type to leave without working. Later in Section 5, we show that a pay-to-leave contract may also appear when the bad type is efficient ($\mu R > b$) but the revenue $R$ is not large enough.

### 4.2. IC-binding contract

Second, we introduce the so-called *IC-binding* contract.

DEFINITION 2. For any deadline $T \geq 0$, define an *IC-binding contract* $\hat{\gamma}^c(T) = (I^c, T^c)$, which generates a promised utility process $\{W_t^c\}$ following $W_0^c = c \cdot T$, as well as a payment process $\{I_t^c\}$ and time duration $T^c = T$, such that

$$W_t^c = c(T^c - t), \text{ and} \tag{5}$$

$$I_t^c = W_t^c + \beta_c. \tag{6}$$

Note that (5) and (6) implies that time duration $T$ fully specifies $I_t^c$. That is why $\hat{\gamma}^c(T^c)$ only has one parameter $T$. Consider a small time interval $[t, t+\delta]$, the agent's expected gain is $\nu_t \cdot (I_t - W_t)$, and expected cost is $\nu_t \cdot \beta_c$. Hence, he chooses his effort $\nu_t$ to maximize $\nu_t \cdot (I_t - W_t - \beta_c)$. Hence, the agent's optimal effort choice is $\nu_t = \mu$ if $I_t - W_t \geq \beta_c$, and $\nu_t = 0$ if $I_t - W_t < \beta_c$. Hence, the IC-binding contract keeps the incentive constraint always binding. Therefore, the agent is always indifferent between exerting full effort and exerting zero effort. That is why the agent's continuation utility at time $t$ is $c(T - t)$, which equals the operating cost from time $t$ to the end of duration $T$. Not surprisingly, $\hat{\gamma}^c(T)$ is the optimal contract when the principal knows the agent's type. We formally show this result in Section 4.4. In the contract design problem with adverse selection, we show that the bad type is provided either a pay-to-leave contract or an IC-binding contract. However, in the optimal menu of contracts, the good type may obtain a contract in which the incentive constraint is not always binding. That is, providing more incentives to the good type may help with screening out good from bad.

### 4.3. Screening contract

As mentioned above, a contract where the incentive constraint is not always binding is helpful in our context. We introduce this so-called *Screening* contract for type $g$.

DEFINITION 3. For any deadline $T \geq 0$, and a transition time $t_1 \in [0,T]$, define a *screening contract* $\bar{\gamma}(t_1,T) = (I,T)$, which generates a promised utility process $\{W_t\}$ according to

$$W_t = \begin{cases} g(T-t) + (\beta_b - \beta_g)\left(1 - e^{-\mu(T-t)}\right), & \text{for } t \in [t_1, T], \\ g(t_1 - t) + W_{t_1}, & \text{for } t \in [0, t_1), \end{cases} \tag{7}$$

as well as a payment process $\{I_t\}$ and time duration $T$,

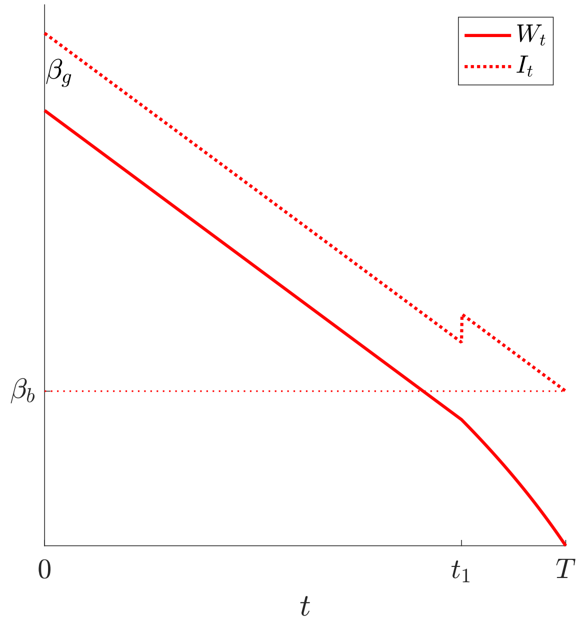
$$I_t = \begin{cases} g(T-t) + \beta_b, & \text{for } t \in [t_1, T], \\ W_t + \beta_g, & \text{for } t \in [0, t_1). \end{cases} \tag{8}$$

In particular, the good type agent's incentive constraint is not always binding in contract $\bar{\gamma}$. Specifically, there exists $t_1$ such that before $t_1$, the good type agent's incentive constraint is set to be binding, and after $t_1$, the bad type agent's incentive constraint is binding. To see this, (7) and (8) imply that $I_t - W_t = \beta_g$ for $t \in [0,t_1)$ and $I_t - g(T-t) = \beta_b$ for $t \in [t_1, T]$. Here, $g(T-t)$ can be interpreted as the bad agent's continuation utility by mimicking the good agent since the bad agent weakly prefers not working, and his continuation utility at time $t$ equals the rest of the reimbursement cost he can collect from time $t$. As a result, it delivers the good agent a utility that is higher than the utility obtained from an IC-binding contract with the same deadline, i.e., $u(\bar{\gamma}(t_1,T), \bar{\nu}; g) \geq g \cdot T = u(\hat{\gamma}^g(T), \bar{\nu}; g)$, where $\bar{\nu} := \{\bar{\nu}_t\}_{t \geq 0}$ represents the *always exerting effort process* such that $\bar{\nu}_t = \mu$ for all $t$ before contract termination. In Section 4.4, we formally show that always exerting effort is the good type's best response to the contract $\bar{\gamma}$. It is worth noting that $\bar{\gamma}$ becomes the IC-binding contract when $t_1 = T$.

This contract provides incentives high enough for the good agent to work while not so high as to make the bad agent exert any effort (even partial effort). An immediate result of implementing such a contract is that only the good agent is willing to work, which may eventually result in a breakthrough. Hence, it screens out the good agent from the bad, making it a *Screening* contract. Because the bad agent does not work in the contract, it also reduces the bad agent's information rent by mimicking the good agent. However, the good agent can still enjoy a higher promised utility from the extra incentives higher than that in the IC-binding case. Finally, we plot $I_t$ and $W_t$ of $\bar{\gamma}$ in Figure 1. It is worth noting that $W_t$ is continuous in $t$ on interval $[0,T]$ and $I_t$ may not be continuous at $t = t_1$. This is because $I_t - W_t = \beta_g$ for $t \in [0,t_1)$ and $I_t - W_t > \beta_g$ for $t_1 \in [t_1, T]$.

Note to Feifan: There needs to be a discussion explain why the jump at $t_1$ helps, intuitively.

**Figure 1** Sample trajectories of the screening contract $\bar{\gamma}(t_1, T)$.




*Note.* In this figure, $W_t$ and $I_t$ follow definition 3. $g=1$, $b=2$, $\mu=1$, and $t_1=4, T=5$.

### 4.4. Principal and agent utilities

To conclude this section, we present results on agent and principal utilities under the aforementioned contract structures. First, we formally establish the agent's utility under the three contract structures.

Proposition 1. *(i) For any $T \geq 0$, we have $\bar{\nu} \in \mathfrak{N}(\hat{\gamma}^c(T), c)$, and*

$$u\Big(\hat{\gamma}^c(T), \bar{\nu}; c\Big) = c \cdot T.$$

*(ii) For any $T \geq 0$ and $g < b$, we have $\nu^0 \in \mathfrak{N}(\hat{\gamma}^g(T), b)$, $\bar{\nu} \in \mathfrak{N}(\hat{\gamma}^b(T), g)$*

$$u\Big(\hat{\gamma}^g(T), \nu^0; b\Big) = g \cdot T, u\Big(\hat{\gamma}^b(T), \bar{\nu}; g\Big) = b \cdot T + (\beta_b - \beta_g)\left(1 - e^{-\mu T}\right),$$

*where $\nu^0 := \{\nu_t^0\}_{t \geq 0}$ represents the* always shirking process *such that $\nu_t^0 = 0$ for all $t$ before contract termination.*

*(iii) For any $0 \leq t_1 \leq T$, we have $\bar{\nu} \in \mathfrak{N}(\bar{\gamma}(t_1, T), g)$, $\nu^0 \in \mathfrak{N}(\bar{\gamma}(t_1, T), b)$, and*

$$u\Big(\bar{\gamma}(t_1, T), \bar{\nu}; g\Big) = g \cdot T + (\beta_b - \beta_g)\left(1 - e^{-\mu(T - t_1)}\right), u\Big(\bar{\gamma}(t_1, T), \nu^0; b\Big) = g \cdot T. \tag{9}$$

First, part (i) of Proposition 1 shows that an IC-binding contract with duration $T$ just provides the minimum incentive for the agent to always exert full effort, and the agent's utility is exactly

$c \cdot T$. Second, part (ii) of Proposition 1 shows that the bad type is not willing to work in a good type's IC binding contract and will get $g \cdot T$, which is equal to the operating cost collected from the entire duration. On the other hand, not surprisingly, the good type is willing to exert full effort in a bad type's IC-binding contract, and his utility equals the bad type's utility under the IC-binding contract $(b \cdot T)$ plus the expected operation cost difference between $b$ and $g$. Third, part (iii) of Proposition 1 shows that the good type is willing to exert full effort in the screening contract, which confirms that the screening contract can screen the good type from the bad.

In the following, we derive the principal utilities under the above contract forms. First, we consider the IC-binding contract $\hat{\gamma}^c(T)$. Define a societal value function, $S_c(w)$, as a function of the promised utility $w$,

$$S_c(w) = \begin{cases} \left(R - \dfrac{c}{\mu}\right)\left(1 - e^{-\mu w/c}\right), & \text{if } \mu R - c > 0, \\ 0, & \text{if } \mu R - c \leq 0, \end{cases} \tag{10}$$

and the principal's value function is $F_c(w) = S_c(w) - w$. The next proposition shows that the function $F_c(w)$ is indeed the principal's value under an IC-binding contract with duration $T = w/c$.

Proposition 2. *If $R \geq \beta_c$, $F_c(w) = U(\hat{\gamma}^c(w/c), \bar{\nu})$. Furthermore, $\mathcal{Z}(\{c\}) = \max_{w \geq 0} F_c(w)$.*

Therefore, contract $\hat{\gamma}^c(w_c^*)$ is an optimal solution to (4) when the set $\mathcal{C}$ of operating costs is reduced to a singleton $\{c\}$, in which $w_c^*$ is the unique maximizer of the strictly concave function $F_c(w)$. If $R \leq \beta_c$, it is optimal for the principal not to hire the agent from the beginning. One can think of the scenario when $\mathcal{C}$ is a singleton as a benchmark setting of our contract design problem. This benchmark setting is similar to, although slightly more general than, the model in Green and Taylor (2016).[8]

Proposition 3. *(i) For any $w \geq 0$ and $\nu \in \mathcal{N}$, we have $U(\check{\gamma}(w), \nu) = -w$.*
*(ii) For any $0 \leq t_1 \leq T$ and $g < \mu R$, we have*

$$U(\bar{\gamma}(t_1, T), \bar{\nu}) = S_g(g \cdot T) - u(\bar{\gamma}(t_1, T), \bar{\nu}).$$

Hence, part (i) of Proposition 3 shows that, under a pay-to-leave contract which pays the agent $w$, the principal obtains $-w$. Part (ii) of Proposition 3 shows that under the screening contract, the principal's utility can be calculated by the societal value function $S_g$. Both parts are somewhat straightforward since the principal's utility is just the societal value minus the agent's utility.

## 5. Summary of main results

In this section, we present our main result in the next theorem, and leave its justification to the next section of the paper.

Theorem 1. *Given model parameters $\mu$ and $b > g$, the optimal menu of contracts demonstrate the following three possible pairs.*

*(i) If the revenue $R \leq \underline{R}(p)$, in which*

$$\underline{R}(p) = \begin{cases} \dfrac{(2+p)b - pg}{bp + (1-p)g}\beta_g, & p \leq \dfrac{g}{b}, \\ \dfrac{1+p}{p}\beta_g, & p > \dfrac{g}{b}, \end{cases} \tag{11}$$

*the optimal menu of contracts is $\underline{\Gamma}_{\{g,b\}} := \{\check{\gamma}(0), \ \check{\gamma}(0)\}$. That is, it is optimal not to hire the agent at all.*

*(ii) If the revenue $\underline{R}(p) < R \leq \bar{R}(p)$, in which*

$$\bar{R}(p) = \begin{cases} \dfrac{(2+p)b - pg}{bp + (1-p)g}\beta_g, & p \leq \dfrac{g}{b}, \\ \dfrac{b - pg}{(1-p)\mu}, & p > \dfrac{g}{b}, \end{cases} \tag{12}$$

*the optimal menu of contracts is $\Gamma^*_{\{g,b\}} := \left\{\hat{\gamma}^g\left(T^*_g\right), \ \check{\gamma}\left(g \cdot T^*_g\right)\right\}$ with*

$$T^*_g = \frac{1}{\mu}\log\left(\frac{(\mu R - g)\cdot p}{g}\right) \tag{13}$$

*(iii) If the revenue $R > \bar{R}(p)$, the optimal menu of contract is $\Gamma^{**}_{\{g,b\}} := \left\{\bar{\gamma}\left(t^*_1, T^1_g\right), \ \hat{\gamma}^b\left(g \cdot T^1_g/b\right)\right\}$ where $T^1_g$ solves*

$$p(\mu R - g)e^{-\mu T^1_g} + g/b \cdot [(1-p)(\mu R - b) - p(b-g)]\, e^{-\mu \cdot g/b \cdot T^1_g} = g, \tag{14}$$

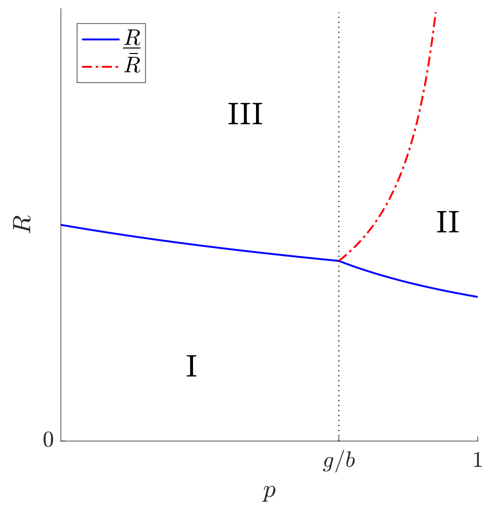
*and $t^*_1 = \left(1 - \dfrac{g}{b}\right) T^1_g$.*

Figure 2 demonstrates the optimal menu of contracts. First, if the revenue $R$ is low, it is not worth hiring the agent (Area I). Second, if the revenue $R$ is above the threshold $\underline{R}(p)$ but below the thresold $\bar{R}(p)$ (Area II), only the good type agent is worth hiring. Hence, the good type is provided an IC-binding contract, while the bad type is provided a pay-to-leave contract. When $R$ is above the threshold $\bar{R}(p)$ (Area III), both types are worth hiring, which leads to the result where the good type is provided a screening contract and the bad type is provided an IC-binding contract.

## 6. Analysis of Theorem 1

In this section, we present the main steps to prove Theorem 1. First, we construct an optimization problem, which provides an upper bound for the original contract design problem (4), in subsection 6.1. Then, in Subsection 6.2-6.4, we construct a menu of contracts based on the optimal solution to the upper bound optimization problem and show that this menu of contracts achieves the upper bound, and therefore is indeed the optimal menu.

**Figure 2** Partition of the $(p, R)$ Plane Based on Optimal Contract Menus




*Note.* In this figure, $g = 1, b = 1.5, \mu = 1$.

### 6.1. Upper Bound Optimization

In this subsection, we present a new optimization problem, which provides an upper bound to the original contract design problem (4). In the following result, we use functions $S_g$ and $S_b$ as defined in (10) for $c \in \{g, b\}$.

PROPOSITION 4. *The following optimization problem yields the upper bound of the optimal value of the contract design problem* (4). *That is,* $\mathcal{Y} \geq \mathcal{Z}(\{g, b\})$, *where*

$$\max_{w_g, w_b, T_g} p\left[S_g\left(g \cdot T_g\right) - w_g\right] + (1-p) \min\left\{F_b(w_b), \frac{w_g - w_b}{b - g}\left(\mu R - b\right)^+ - w_b\right\}, \tag{15}$$

$$s.t. w_g \geq w_b \geq g \cdot T_g, \tag{16}$$

$$T_g \geq 0. \tag{17}$$

It is instructive to explain the terms in the optimization problem (15)-(17). First, the decision variables $w_g$ and $w_b$ represent the utilities of type $g$ and $b$ agent under their respective contracts. The decision variable $T_g$ is the time duration for the type $g$ agent. The value $S_g\left(g \cdot T_g\right) - w_g$ represents the principal's expected utility facing a type $g$ agent. The value right after $1 - p$ represents the principal's expected utility facing a type $b$ agent.

The constraint (16) states that the good agent's utility, $w_g$, needs to be as good as or better than the bad agent's $w_b$. Furthermore, the last inequality in (16) states that $w_b$ needs to be no less than the total discounted expected operating cost that the agent would receive by pretending to be a good agent. This is because receiving the operating cost $g$ without working yields a utility $g \cdot T_g$.

Constraint (16) reveals why the optimal value $\mathcal{Y}$ is only an upper bound. This is because the constraint itself (or any other constraint in this optimization) does not provide sufficient incentive for the bad agent not to work facing the good agent's contract. Restricting the bad agent's response (to zero effort) allows the principal to obtain a higher utility than reality, in which the agent may be able to obtain a utility higher than the right-hand-side of (16) with some effort.

Clearly, $S_g(g \cdot T_g) - w_g$ is the upper bound of the principal's expected utility facing a type $g$ agent when the good agent's utility is $w_g$. Next, we focus on the value right after $1-p$. First, the first part of the value states that the principal's utility facing a type $b$ agent is upper bounded by $F_b(w_b)$ when offering the type $b$ agent a promised utility $w_b$, consistent with Proposition 2. Finally, the second part of the value ensures a type $g$ agent does not pretend to be of type $b$, which is elaborated in the following remark.

Remark 2. Should the type $g$ agent receive the type $b$ contract, the agent is able to exert effort and receive the same trajectory of payments as a type $b$ agent. In addition to receiving the $w_b$ reward, the type $g$ agent also collects the extra operating cost $b-g$ for the duration of the contract. This duration can be calculated as the societal utility, $U(\gamma^b, \nu^b) + w_b$, divided by the societal utility rate, $\mu R - b$, if $\mu R > b$. This implies the following inequality,

$$w_g \geq w_b + (b-g)\frac{U(\gamma^b,\nu^b)+w_b}{\mu R - b}, \text{ or, equivalently, } U(\gamma^b,\nu^b)+w_b \leq \frac{w_g - w_b}{b-g}(\mu R - b). \tag{18}$$

If $\mu R \leq b$, on the other hand, the societal value of hiring the agent is negative, and, therefore, $U(\gamma^b, \nu^b) + w_b \leq 0$. The value right after $1-p$ captures both cases of $\mu R > b$ and $\mu R \leq b$. □

So far, we have provided intuitive interpretations of various components of the optimization problem (15)-(17). This optimization plays a central role in our contract design problem since solving it obtains a menu of contracts based on its optimal solution. We further show that the performance of such a menu of contracts indeed achieves the upper bound $\mathcal{Y}$ of $\mathcal{Z}(\{g,b\})$. Therefore, this menu of contracts is optimal. In our construction, each contract in the menu has a simple form.

**6.1.1. Optimal Solution to the Upper Bound Optimization** First, if the bad type is inefficient, i.e., $b \geq \mu R$. Then, upper bound optimization problem (4) becomes

$$\max_{w_g, w_b, T_g} p\{S_g(g \cdot T_g) - w_g\} - (1-p)w_b, \tag{19}$$

$$s.t. w_g \geq w_b \geq g \cdot T_g, \tag{20}$$

$$T_g \geq 0. \tag{21}$$

Clearly, the objective (19) is decreasing in both $w_g$ and $w_b$. Hence, the optimal choice of $w_g$ and $w_b$ are $w_g = w_b = g \cdot T_g$. After plugging in $w_g, w_b$, the objective function is concave in $T_g$; it is straightforward to use the first-order condition to find the optimal $T_g$. We summarize the optimal solution in the following proposition.

Proposition 5. *If $\beta_b \geq R$, the optimal solution to the optimization problem* (15)-(17) *is*

$$T_g = \begin{cases} 0, & \text{if } R \leq \frac{(1+p)}{p}\beta_g, \\ T_g^*, & \text{if } R > \frac{(1+p)}{p}\beta_g, \end{cases} \tag{22}$$

*where $T_g^*$ is defined in* (13) *and $w_g = w_b = g \cdot T_g$.*

Next, we focus on the case when the bad type is also efficient, i.e., $b < \mu R$. We present the optimal solution to the optimization problem (15).

Proposition 6. *(i) If $b < \mu R$ and $p \cdot b \geq g$, the optimal solution to the optimization problem* (15)-(17) *is*

$$T_g = \begin{cases} 0, & R \leq \frac{1+p}{p}\beta_g, \\ T_g^*, & R \in \left(\frac{1+p}{p}\beta_g, \frac{b-pg}{(1-p)\mu}\right], \\ T_g^1, & R > \frac{b-pg}{(1-p)\mu}. \end{cases} \tag{23}$$

*where $T_g^*$ is defined in* (13) *and $T_g^1$ is defined in (iii) of Theorem 1.*

$$w_g = \begin{cases} 0, & R \leq \frac{1+p}{p}\beta_g, \\ g \cdot T_g^*, & R \in \left(\frac{1+p}{p}\beta_g, \frac{b-pg}{(1-p)\mu}\right], \\ g \cdot T_g^1 + (\beta_b - \beta_g)\left(1 - e^{-\mu \cdot g/b \cdot T_g^1}\right), & R > \frac{b-pg}{(1-p)\mu}. \end{cases} \tag{24}$$

*and $w_b = g \cdot T_g$.*

*(ii) If $b < \mu R$ and $p \cdot b < g$, the optimal solution to the optimization problem* (15)-(17) *is*

$$T_g = \begin{cases} 0, & R \leq \frac{(2+p)b-pg}{bp+(1-p)g}\beta_g, \\ T_g^1, & R > \frac{(2+p)b-pg}{bp+(1-p)g}\beta_g. \end{cases} \tag{25}$$

*and*

$$w_g = \begin{cases} 0, & R \leq \frac{(2+p)b-pg}{bp+(1-p)g}\beta_g, \\ g \cdot T_g^1 + (\beta_b - \beta_g)\left(1 - e^{-\mu \cdot g/b \cdot T_g^1}\right), & R > \frac{(2+p)b-pg}{bp+(1-p)g}\beta_g. \end{cases} \tag{26}$$

*and $w_b = g \cdot T_g$.*

Finally, we combine the optimal solutions in Propositions 5 and 6.

Corollary 1. *(i) If $R \leq \underline{R}(p)$, then the optimal solution to the optimization problem* (15)-(17) *is*

$$w_g = w_b = T_g = 0 \tag{27}$$

*(ii)* *If* $\underline{R}(p) < R \leq \underline{R}(p)$*, then the optimal solution to the optimization problem* (15)-(17) *is*

$$w_g = w_b = g \cdot T_g^*, T_g = T_g^*, \tag{28}$$

*where* $T_g^*$ *follows* (13)*.*

*(iii)* *If* $R > \bar{R}(p)$*, then the optimal solution to the optimization problem* (15)-(17) *is*

$$w_g = g \cdot T_g^1 + (\beta_b - \beta_g)\left(1 - e^{-\mu \cdot g/b \cdot T_g^1}\right), w_b = g \cdot T_g^1, T_g = T_g^1, \tag{29}$$

*where* $T_g^1$ *is defined in (iii) of Theorem 1.*

### 6.2. Low Revenue $R$ (Area I of Figure 2)

Following point (i) of Corollary 1, we have

$$0 = \mathcal{Y} \geq \mathcal{Z}(\{g, b\}).$$

Clearly, not hiring the agent at all achieves the upper bound. Hence, if revenue $R$ is low enough, i.e., $R < \underline{R}(p)$, the principal cannot do better than just firing the agent. We formally summarize the results in the following.

Proposition 7. *If* $R \leq \underline{R}(p)$*, then the menu of contracts* $\underline{\Gamma}_{\{g,b\}}$ *defined in (i) of Theorem 1 satisfies* (LL)*,* (BR)*,* (IR)*,* (ICg)*, and* (ICb)*. Furthermore, we have* $\mathcal{U}(\underline{\Gamma}_{\{g,b\}}) = \mathcal{Y} = 0$*, in which* $\mathcal{Y}$ *is defined in* (15)-(17)*. Therefore, we have* $\mathcal{U}(\underline{\Gamma}_{\{g,b\}}) = \mathcal{Z}(\{g, b\})$*, or, the menu of contract* $\underline{\Gamma}_{\{g,b\}}$ *solves the optimal contract design problem* (4) *if* $R \leq \underline{R}(p)$*.*

### 6.3. Medium Revenue $R$ (Area II of Figure 2)

Following point (ii) of Corollary 1, in the optimal solution $w_g = w_b$, which implies that the principal obtains $-w_b$ from the bad type. Furthermore, $w_g = g \cdot T_g$ implies that the incentive constraints in the good agent's contract should always be binding. Hence, the bad type is always given a pay-to-leave contract defined in Definition 1, where the lump-sum payment $g \cdot T_g^*$ is as high as the benefit from mimicking the good type. The good type is always given an IC-binding contract defined in Definition 2, with the contract duration $T_g^*$, which increases in revenue $R$, probability of the good type $p$, and decreases in good type's cost $g$. In the following, we formally prove that this menu of contract satisfies all the constraints for the contract design problem (4) and it delivers the principal a value that achieves the optimal value of the upper bound optimization (15)-(17).

Proposition 8. *If* $\underline{R}(p) < R \leq \bar{R}(p)$*, then the menu of contracts* $\Gamma^*_{\{g,b\}}$ *defined in (ii) of Theorem 1 satisfies* (LL)*,* (BR)*,* (IR)*,* (ICg)*, and* (ICb)*. Furthermore, we have* $\mathcal{U}(\Gamma^*_{\{g,b\}}) = \mathcal{Y}$*, in which* $\mathcal{Y}$ *is defined in* (15)-(17)*. Therefore, we have* $\mathcal{U}(\Gamma^*_{\{g,b\}}) = \mathcal{Z}(\{g, b\})$*, or, the menu of contract* $\Gamma^*_{\{g,b\}}$ *solves the optimal contract design problem* (4) *if* $\underline{R}(p) < R \leq \bar{R}(p)$*.*

It is worth reflecting incentives around the optimal menu of contracts $\Gamma^*_{\{g,b\}}$. In the case when revenue $R$ is in the medium level, the initial lump-sum payment $w_b^*$ to the bad agent equals the total operating cost $g$ that the agent can collect by pretending to be the good agent while shirking until the end of the contract duration. This initial lump-sum payment mitigates the bad agent's incentive to lie about the high cost. It is also worth noting that the (ICg) (truth-telling constraint for the good agent) is also binding. That is, the good agent's promised utility $w_g^*$ under the IC-binding contract is the same as the lump-sum payment to the bad agent $w_b^*$.

### 6.4. High Revenue $R$ (Area III of Figure 2)

The optimal solution presented in (iii) of Corollary 1 shows that $w_g > w_b = g \cdot T_g$, which implies that the good type's utility is higher than the bad type's utility and the utility of the good type that can receive from an IC-binding contract with the same deadline $T_g$. Hence, the good type is provided a screening contract $\bar{\gamma}(t_1, T_g)$, where $t_1$ is chosen such that

$$u(\bar{\gamma}(t_1,T_g),\bar{\nu};g) = gT_g^1 + (\beta_b - \beta_g)\left(1 - e^{-\mu(T_g^1 - t_1)}\right) = w_g = g \cdot T_g^1 + (\beta_b - \beta_g)\left(1 - e^{-\mu \cdot g/b \cdot T_g^1}\right),$$

where the first equality follows from (9), and the last equality follows from (29). Furthermore, (29) implies that

$$\begin{aligned}\frac{w_g - w_b}{b-g}(\mu R - b) - w_b &= \frac{(\beta_b - \beta_g)\left(1 - e^{-\mu \cdot g/b \cdot T_g^1}\right)}{b-g}(\mu R - b) - g \cdot T_g \\ &= (R - \beta_b)\left(1 - e^{-\mu \cdot g/b \cdot T_g^1}\right) - g \cdot T_g = S_b(g \cdot T_g) - g \cdot T_g = F_b(g \cdot T_g),\end{aligned}$$

where the third equality follows from (10). That is, under the optimal solution, the two terms in the minimum of objective (15) are exactly equal. Hence, the principal's value obtained from the bad agent can be achieved by an IC-binding contract to type $b$ with deadline $g \cdot T_g / b$. We are now ready to present the main result of this section.

PROPOSITION 9. *If $R > \bar{R}(p)$, then the menu of contracts $\Gamma^{**}_{\{g,b\}}$ defined in (iii) of Theorem 1 satisfies* (LL), (BR), (IR), (ICg), *and* (ICb). *Furthermore, we have $\mathcal{U}(\Gamma^{**}_{\{g,b\}}) = \mathcal{Y}$, in which $\mathcal{Y}$ is defined in* (15)-(17). *Therefore, we have $\mathcal{U}(\Gamma^{**}_{\{g,b\}}) = \mathcal{Z}(\{g,b\})$, or, the menu of contract $\Gamma^{**}_{\{g,b\}}$ solves the optimal contract design problem* (4) *if $R > \bar{R}(p)$.*

When revenue $R$ is high enough, both types are worth hiring. To screen out the good type from the bad, the good type is provided a screening contract, which delivers the good agent a higher continuation utility than an IC-binding contract. It is worth noting that the truth-telling constraint for the bad agent, (ICb), is always binding and that for the good agent, (ICg), is not binding. Izak

## 7. Managerial Implications

In Subsection 7.1, we further summarize the main properties of the optimal menu of contracts. Finally, in Subsection 7.2, we discuss the welfare implications of unknown cost by comparing the optimal menu of contracts between the unknown cost and the known cost scenarios.

### 7.1. Main properties of the optimal contracts

In this subsection, we highlight the following properties of the optimal menu of contracts.

**Property 1:**

$$\bar{\nu} \in \mathfrak{N}\Big(\gamma^g,\ g\Big),\ \text{and}\ \bar{\nu} \in \mathfrak{N}\Big(\gamma^b,\ b\Big)$$

The optimal menu of contracts induces both types of agents to exert full effort in their own contract.

**Property 2:**

$$\nu^0 \in \mathfrak{N}\Big(\gamma^g,\ b\Big)$$

The property states that a bad agent who pretends to be good prefers shirking until the end.

**Property 3:**

$$I_t^b - W_t^b = \beta_b, \forall t > 0.$$

This property indicates that under the optimal menu, the incentive constraint is binding in the bad agent's contract the entire time. For the bad agent, arrivals do not resolve adverse selection because the good agent is able to mimic bad agents and generate arrivals. Therefore, the principal always offers a dynamically efficient contract, i.e., the IC binding contract to the bad agent, and adjusts other parameters in the menu to achieve optimality. By contrast, the incentive constraint may not always be binding in the good agent's contract.

### 7.2. Welfare implications of unknown cost

In this section, we present how unknown cost affects the welfare of the principal and the agent, compared to the situations with known cost. We show that unknown cost always hurts the principal, but may hurt or benefit the agent, depending on whether or not the bad agent is efficient.

Denote $\bar{\mathcal{Y}}$ to represent the principal's expected payoff when cost is observable, and either takes value $g$ with probability $p$, or $b$ with probability $1-p$. That is,

$$\bar{\mathcal{Y}} := p\, \mathcal{Z}(\{g\}) + (1-p)\, \mathcal{Z}(\{b\}), \tag{30}$$

in which $\mathcal{Z}(\{g\})$ and $\mathcal{Z}(\{b\})$ are the principal's optimal utility earned from the good agent and the bad agent, respectively, following (4). Proposition 2 has shown that the IC-binding contract is optimal for the case the agent's cost is known by the principal. It is clear that the principal is always better off knowing the cost of the agent before issuing the contract, or, $\mathcal{Y} \leq \bar{\mathcal{Y}}$. Intuitively, this conclusion follows from the basic idea of value of information. In particular, with known cost, the principal does not need to pay the information rent associated with unknown cost.

Now, we consider the agent's utility in the following two different cases. Define $w_*^g$ and $w_*^b$ to be the maximizers of functions $F_g(w)$ and $F_b(w)$, respectively. Following Proposition 2, we know that they are the good and bad agents' utilities when the cost is observable under the respective optimal contracts. First, consider the situation that the bad agent is not efficient, or $\beta_b \geq R$. In this case, the good agent is worse off, and the bad agent is better off in the unknown cost situation, compared with the known cost one, as stated in the following result.

PROPOSITION 10. *If $\beta_b \geq R$, we have*

$$w_g^* \leq w_*^g, \quad \textit{and} \quad w_b^* \geq w_*^b = 0, \tag{31}$$

*where $w_g^*$ and $w_b^*$ are from the optimal solution of* (19).

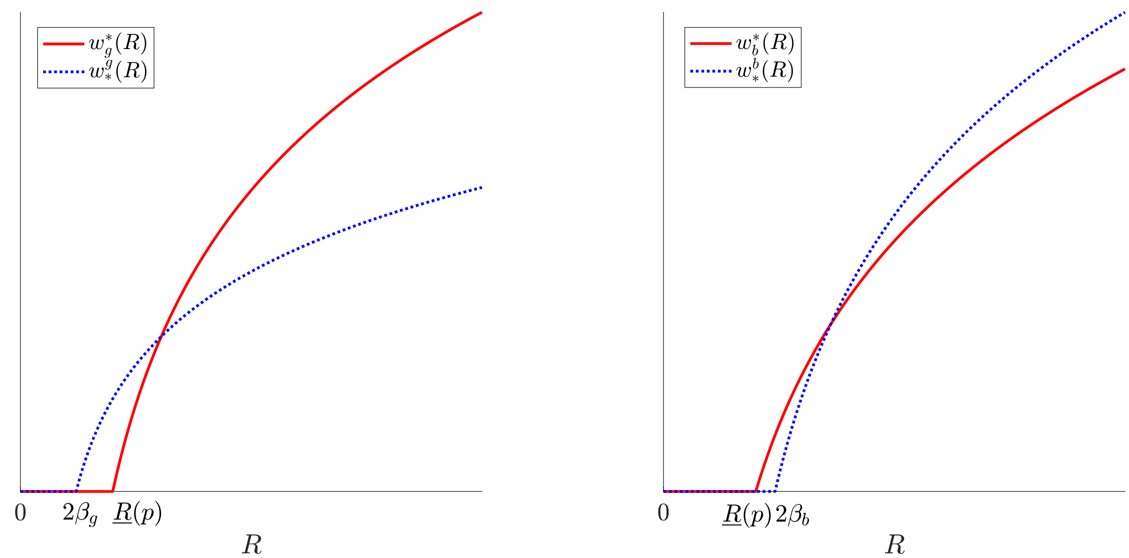
Apparently, the bad agent can earn some information rent if the cost is unknown. Such information rent does not exist if the cost is known. Therefore, the bad agent is better off with unknown costs. The good agent is worse off because, with unknown costs, the bad agent could mimic the good agent, triggering the principal to curtail the good agent's payoff to prevent paying the bad agent too much information rent.

If the bad agent is efficient, or $\beta_b < R$, then either agent can be better or worse off with unknown cost. The following proposition shows that whether the agent is better or worse off is monotone in revenue $R$.

PROPOSITION 11. *Fix $p \in (0,1)$ and $b > g$, there exists two thresholds $R_1, R_2 > \beta_b$ such that $w_g^* \geq w_*^g$ if and only if $R \geq R_1$ and $w_b^* \leq w_*^b$ if and only if $R \geq R_2$.*

Proposition 11 shows that when revenue $R$ is large enough, the bad agent is worse off, and the good agent is better off with unknown cost. The agent's *information rent* arises from his ability to mimic the other type. Notice that once such information rent exists, the principal would tweak the contract of whomever is mimicked to reduce the information rent. As a result, the imitator becomes better off, and the imitatee becomes worse off. Since both types of agent may have the incentive to mimic the other type, whether the agent becomes better off or worse off depends on which effect dominates. Firstly, when $R$ is small, only the bad agent wants to mimic the good agent, and thus the good agent becomes worse off and the good agent is better off; however, as $R$ increases, the good agent can earn information rent by mimicking the bad agent, and thus he becomes better off and the bad agent, as the imitatee, becomes worse off. Finally, Figure 3 shows how good and bad agent's utilities change as revenue $R$ changes under known cost and unknown cost scenarios.

**Figure 3** Agent's utilities under optimal contracts as $R$ varies




(a) Good agent's utility under known cost, and unknown cost scenarios

(b) Bad agent's utility under known cost, and unknown cost scenarios

*Notes. In this figure,* $\mu = 1, g = 1, b = 2, p = 0.3, R \in [0, 16.5]$.

## 8. Conclusion

Note to Feng himself:

1. Strengthen our contribution on providing a way to solve the dynamic moral hazard with adverse selection problem (Upper bound optimization). Luckily, it leads to the optimal solution. In a more complex setting, the upper bound optimization may not lead to an implementable optimal results. However, the solution provides good idea of creating implementable contracts.

2. Mention future extension to infinite arrivals

## Endnotes

1. https://www.chron.com/news/houston-texas/article/Prof-accused-of-spending-NASA-grants-on-cars-1722521.php

2. https://www.newsweek.com/fund-meant-vaccine-research-misused-least-145m-unrelated-expenses-almost-decade-1564954

3. See https://www.chooseenergy.com/news/article/failed-v-c-summer-nuclear-project-timeline/.

4. The charging of retainers by lobbyists is common, see for example, https://lobbyit.com/pricing/, https://arnoldpublicaffairs.com/faq/ and https://lobbying101.wordpress.com/about-lobbyists/how-much-do-they-charge/. Furthermore, it is common that R&D projects are funded for long durations of time and may not bring any results

in the end.

5. Shirking and misuse of research funds are surprisingly common in R&D settings, see, for example, https://www.chron.com/news/houston-texas/article/Prof-accused-of-spending-NASA-grants-on-cars-1722521.php, https://www.nbcnews.com/news/us-news/philadelphia-professor-accused-spending-185-000-grant-funds-strip-clubs-n1118571, https://www.newsweek.com/fund-meant-vaccine-research-misused-least-145m-unrelated-expenses-almost-decade-1564954, and https://www.theguardian.com/higher-education-network/2015/mar/27/research-grant-money-spent.

6. In equation (1), if $c' = c$, then the agent truthfully reports his type. If $c' \neq c$, then the agent misreports his type and mimic the other type.

7. It is worth noting that the integral is from $t+$. Therefore, any instantaneous payment at time $t$ (for example, potential lump-sum payment at time 0) is not included in the promised utility. We use notation $W_{t-} := \lim_{s\uparrow t} W_t$, which includes the potential upwdard jump at time $t$.

8. In our paper, the agent can choose any effort level in $[0, \mu]$, while in Green and Taylor (2016), the agent can only choose between $\mu$ or 0.

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# Appendix

## A. Summary of Notations

## B. Proof in Section 4

### B.1. Formal incentive compatibility constraints

The following result, which can be adapted from Proposition 1 in Biais et al. (2010), depicts the dynamic of the process $W_t$, and provides an equivalent condition to a best-response effort process.

LEMMA 1. *For any contract $\gamma^c$, effort process $\nu$, and operating cost $c$, we have*

$$\mathrm{d}W_t(\gamma^c,\nu;c) = \{\nu_t[W_{t-}(\gamma^c,\nu;c) - I^c_t + \beta_c]\mathrm{d}t - c\mathrm{d}t\}\,\mathbb{1}_{0\le t<T\wedge\tau}. \tag{PK}$$

*Furthermore, the following defined effort process is a best response to contract $\gamma^c$, or, $\{\nu_t\}_{t\in[0,T\wedge\tau]}\in\mathfrak{N}(\gamma,c)$, in which*

$$\nu_t = \begin{cases} \mu, & \text{if } I_t - W_{t-}(\gamma^c,\nu;c) > \beta_c,\\ \nu\in[0,\mu], & \text{if } I_t - W_{t-}(\gamma^c,\nu;c) = \beta_c,\\ 0, & o.w.. \end{cases} \tag{ICw}$$

**Proof:** Denote a right-continuous counting process $N=\{N_t\}_{t\ge0}$ to record the number of arrivals up to time $t$, which generates a filtration $\mathcal{F}^N=\{\mathcal{F}^N_t\}_{t\ge0}$. Therefore, the instantaneous arrival rate of the counting process at time $t$ is $\nu_t$, and the left-continuous effort process $\nu=\{\nu_t\}_{t\ge0}$ is $\mathcal{F}^N$-predictable. Hence, the arrival time can be expressed as $\tau=\inf\{t|\mathrm{d}N_t=1\}$. Furthermore, $W_t(\gamma^c,\nu;c)$ defined in (2) can also be expressed as

$$W_t(\gamma^c,\nu;c) = \mathbb{E}^\nu\left[\int_{t+}^{T^c\wedge\tau} I^c_s\mathrm{d}N_s + (c-\nu_s\cdot\beta_c)\mathrm{d}s\right]\mathbb{1}_{t<T\wedge\tau}. \tag{32}$$

To characterize how the agent's continuation utility evolves over time, it is useful to consider his lifetime expected utility, evaluated conditionally upon the information available at time $t$

$$u_t(\gamma^c,\nu;c) = \mathbb{E}^{\nu}\left[\int_0^{T^c\wedge\tau}(c-\nu_s\beta_c)\,\mathrm{d}s + I_s^c\mathrm{d}N_s\,\middle|\,\mathcal{F}_t^N\right]$$
$$= \int_0^{t\wedge T^c\wedge\tau-}(c-\nu_s\beta_c)\,\mathrm{d}s + I_s^c\mathrm{d}N_s + W_t(\gamma,\nu;c) \tag{33}$$

Since $u_t(\gamma,\nu;c)$ is the expectation of a given random variable conditional on $\mathcal{F}_t^N$, the process $\mathbf{u}(\gamma^c,\nu;c) = \{u_t(\gamma^c,\nu;c)\}_{t\geq 0}$ is an martingale under the probability measure $\mathbf{P}^{\nu}$. Relying on this martingale property, we now offer an alternative representation of $\mathbf{u}(\gamma^c,\nu;c)$. Consider the process $M^{\nu} = \{M_t^{\nu}\}_{t=\geq 0}$ defined by

$$M_t^{\nu} = N_t - \int_0^t \nu_s\mathrm{d}s \tag{34}$$

for all $t\geq 0$. The martingale representation theorem for point processes implies that the martingale $\mathbf{u}(\gamma,\nu;c)$ satisfies

$$u_t(\gamma,\nu;c) = u_0(\gamma,\nu;c) + \int_0^{t\wedge\tau} H_s(\gamma^c,\nu;c)\mathrm{d}M_s^{\nu} \tag{35}$$

for all $t\geq 0$, $\mathbf{P}^{\nu}$-almost surely, for some $\mathcal{F}^N$-predictable process $H(\gamma,\nu;c) = \{H_t(\gamma,\nu;c)\}_{t\geq 0}$. By (33) and (35), we obtain that

$$\mathrm{d}u_t(\gamma^c,\nu;c) = (c-\nu_t\beta_c)\mathrm{d}t + I_t^c\mathrm{d}N_t + \mathrm{d}W_t(\gamma^c,\nu;c)$$
$$= H_t(\gamma^c,\nu;c)\left(\mathrm{d}N_t - \nu_t\mathrm{d}t\right),$$

which further implies that

$$\mathrm{d}W_t(\gamma^c,\nu;c) = \nu_t[-H_t(\gamma,\nu;c)+\beta_c]\mathrm{d}t - c\mathrm{d}t + H_t\mathrm{d}N_t - I_t^c\mathrm{d}N_t. \tag{36}$$

Finally, $W_{T\wedge\tau}=0$ and $\mathrm{d}W_t = H_t - I_t^c$ at $\mathrm{d}N_t = 1$ implies that $H_t = I_t^c - W_t$ for any $t$, which verifies (PK).

In the following, we show that $\{\nu_t\}_{t\in[0,\tau]}$ defined in (ICw) is a best response to contract $\gamma^c$. Let $\tilde{u}_t$ denote the agent's lifetime expected payoff, given the information available at date $t$, when he acts according to $\nu' = \{\nu_t'\}_{t\geq 0}$ until date $t$ and then reverts to $\nu = \{\nu_t\}_{t\geq 0}$:

$$\tilde{u}_t = \int_0^{t\wedge T^c\wedge\tau^-}(c-\nu_s'\beta_c)\,\mathrm{d}s + I_s^c\mathrm{d}N_s + W_t(\gamma^c,\nu;c). \tag{37}$$

In fact, we have $\tilde{u}_{0-} = W_{0-} = u(\gamma^c,\nu;c)$. We finish the proof in three steps.

*Step 1:* We show that if $\tilde{u} = \{\tilde{u}_t\}_{t\geq 0}$ is an $\mathcal{F}^N$-submartingale under $\mathbf{P}^{\nu}$ that is not a martingale, then $\nu$ is suboptimal for the agent. Indeed, in that case there exists some $t>0$ such that

$$u(\gamma^c,\nu;c) = u_{0-}(\gamma^c,\nu;c) = \tilde{u}_{0-} < \mathbb{E}^{\nu'}[u_t'] \tag{38}$$

where $u_{0-}(\gamma^c,\nu;c)$ and $\tilde{u}_{0-}$ correspond to unconditional expected payoffs at date 0. By (37), the agent is then strictly better off acting according to $\nu'$ until date $t$ and then reverting to $\nu$. The first claim follows.

*Step 2:* We show that if $u'$ is a $\mathcal{F}^N$-supermartingale under $\mathbf{P}^{\nu'}$, then $\nu$ is at least as good as $\nu'$ for the agent. Following (33) and (37), we have

$$\tilde{u}_t = u_t(\gamma^c,\nu;c) + \int_0^{t\wedge T^c\wedge\tau}(\nu_s - \nu_s')\beta_c\mathrm{d}s \tag{39}$$

for all $t\geq 0$. Hence, since $u_t(\gamma^c,\nu;c)$ is right-continuous with left-hand limits, so is $u'$. Moreover, since $u'$ is non-negative, then the limit $\lim_{t\to T^c\wedge\tau} u_t'$ exists. Hence, by the optional sampling theorem (Dellacherie and Meyer (2011), Chapter VI, Theorem 10)),

$$u(\gamma^c,\nu;c) = \tilde{u}_{0-} \geq \mathbb{E}^{\nu'}[u_{T^c\wedge\tau}'] = u_{0-}(\gamma^c,\nu';c) = u(\gamma^c,\nu';c), \tag{40}$$

where again $u_{0-}(\gamma^c,\nu';c)$ is an unconditional expected payoff at date 0, and the first equality follows from (37). Hence, the second claim follows.

*Step 3:* We apply the above two claims to complete the proof. For each $t \geq 0$,

$$\begin{aligned}
\tilde{u}_t &= u_t(\gamma,\nu;c) + \int_0^{t\wedge T^c\wedge\tau} (\nu_s - \nu'_s)\beta_c \mathrm{d}s \\
&= u_0(\gamma,\nu;c) + \int_0^{t\wedge T^c\wedge\tau} (I^c_s - W_s(\gamma^c,\nu;c)) dM^{\nu}_s + \int_0^{t\wedge T^c\wedge\tau} (\nu_s - \nu'_s)\beta_c \mathrm{d}s \\
&= u_0(\gamma,\nu;c) + \int_0^{t\wedge T^c\wedge\tau} (I^c_s - W_s(\gamma^c,\nu;c)) \mathrm{d}M^{\nu'}_s + \int_0^{t\wedge T^c\wedge\tau} (I^c_s - W_s(\gamma^c,\nu;c))(\nu'_s - \nu_s)\mathrm{d}s + \int_0^{t\wedge T^c\wedge\tau} (\nu_s - \nu'_s)\beta_c \mathrm{d}s \\
&= u_0(\gamma,\nu;c) + \int_0^{t\wedge T^c\wedge\tau} (I^c_s - W_s(\gamma^c,\nu;c)) \mathrm{d}M^{\nu'}_s + \int_0^{t\wedge T^c\wedge\tau} (\nu'_s - \nu_s)\left[I^c_s - W_s(\gamma^c,\nu;c) - \beta_c\right] \mathrm{d}s. \qquad (41)
\end{aligned}$$

Since $M^{\nu'}$ is an $\mathcal{F}^N$-martingale under $P^{\nu'}$, the drift of $\tilde{u}$ has the same sign as

$$(\nu'_t - \nu_t)\left[I^c_t - W_t(\gamma^c,\nu;c) - \beta_c\right]$$

for all $t \in [0, T^c \wedge \tau)$. If the effort process $\nu$ satisfies (ICw), then this drift remains negative for all $t \in [0, T^c \wedge \tau)$ and all choices of $\nu'_t \in [0,\mu]$. This implies that for any effort process $\nu'$, $\tilde{u}$ is an $\mathcal{F}^N$-supermartingale under $P^{\nu'}$ and, thus, that $\nu$ is at least as good as $\nu'$ for the agent. This completes the proof.

If (ICw) does not hold for the effort process $\nu$, then choose $\nu'$ such that for each $t \in [0, T^c \wedge \tau)$, $\nu'_t = \mu$ if $I^c_t - W_t(\gamma^c,\nu;c) \geq \beta_c$ and $\nu'_t = 0$ if $I^c_t - W_t(\gamma^c,\nu;c) < \beta_c$. The drift of $\tilde{u}$ is then everywhere non-negative and strictly positive over a set of $P^{\nu'}$-strictly positive measure. As a result of this, $\tilde{u}$ is an $\mathcal{F}^N$-submartingale under $P^{\nu'}$ that is not a martingale and, thus, $\nu$ is suboptimal for the agent.

## B.2. Useful Definitions and Technical Lemma

**Effective Cumulated Effort**:

$$\bar{T}(\gamma,\nu) := \mathbb{E}^{\nu}\left[\int_0^{T\wedge\tau} \nu_t \mathrm{d}t\right], \qquad (42)$$

which measures the agent's expected effective cumulated effort under contract $\gamma$ when the agent chooses the effort process $\nu$.

**Societal Value**:

$$S(\gamma,\nu;c) = \mathbb{E}^{\nu}\left[R\mathbb{1}_{T\leq\tau} - \int_0^{T\wedge\tau} \nu_t \cdot \beta_c \mathrm{d}t\right], \qquad (43)$$

which measures the expected total value net of cost produced with effort $\nu$ when the agent's cost is $c$.

LEMMA 2. *The societal value produced is proportional to the working duration, i.e.,* $S(\gamma,\nu;c) = (R - \beta_c)\bar{T}(\gamma,\nu)$.

*Proof:*

$$S(\gamma,\nu;c) = \mathbb{E}^{\nu}\left[R\mathbb{1}_{T\leq\tau} - \int_0^{T\wedge\tau} \nu_t \cdot \beta_c \mathrm{d}t\right] = \mathbb{E}^{\nu}\left[\int_0^{T\wedge\tau} R\cdot\nu_t \mathrm{d}t - \nu_t\cdot\beta_c\mathrm{d}t\right] = (R-\beta_c)\bar{T}(\gamma,\nu). \qquad (44)$$

Hence, for each moment the agent exerts effort $\nu_t$, he produces an expected revenue of $(R-\beta_c)\nu_t$. Q.E.D.

**Generalized incentive compatibility constraints:** We generalize the incentive compatibility constraints presented in (BR) and subsection B.1 to the scenario when type $c'$ agent mimic type $c$ agent by taking the contract $\gamma^c$. First, we generalize the set $\mathfrak{N}$ in (BR). Denote $\mathfrak{N}(\gamma^c, c')$ to represent the set of best-response effort processes, when the type $c'$ agent chooses contract $\gamma^c$. That is,

$$u\big(\gamma^b,\nu;g\big) \geq u(\gamma^b,\nu';g),\ \forall \nu \in \mathfrak{N}(\gamma^b, g) \text{ and } \nu' \in \mathcal{N}. \qquad \text{(ICg)}$$

$$u\big(\gamma^g,\nu;b\big) \geq u(\gamma^g,\nu';b),\ \forall \nu \in \mathfrak{N}(\gamma^g, b) \text{ and } \nu' \in \mathcal{N}', \qquad \text{(ICb)}$$

where $\mathcal{N}'$ is defined in Remark 1. We can further generalize the definition of continuation utility in (2) to

$$W_t(\gamma^c,\nu;c') = \mathbb{E}^{\nu}\left[I^c_\tau \cdot \mathbb{1}_{\tau<T^c} + \int_{t+}^{T^c\wedge\tau} (c - \nu_s\cdot\beta_{c'})\mathrm{d}s\right]\mathbb{1}_{t<T^c\wedge\tau}, \qquad (45)$$

for tyeo $c$ with effort process $\nu$ under contract $\gamma^c$. Similar to Lemma 1, we have the following results which describe the dynamics of the process $W_t(\gamma^g,\nu;b)$ and $W_t(\gamma^b,\nu;g)$, respectively, and provides an equivalent condition to a best-response effort process.

Lemma 3. *For any contract $\gamma^b$, effort process $\nu$, and operating cost $g$, we have*

$$\mathrm{d}W_t(\gamma^b,\nu;g)=\left\{\nu_t[W_{t-}(\gamma^b,\nu;g)-I_t^b+\beta_g]\mathrm{d}t-b\mathrm{d}t\right\}\mathbb{1}_{0\le t<T\wedge\tau}. \tag{PKg}$$

*Furthermore, the following defined effort process is a best response to contract $\gamma^b$, or, $\{\nu_t\}_{t\in[0,T\wedge\tau]}\in\mathfrak{N}(\gamma^b,g)$, in which*

$$\nu_t=\begin{cases}\mu, & \text{if } I_t-W_{t-}(\gamma^b,\nu;g)>\beta_g,\\ \nu\in[0,\mu], & \text{if } I_t-W_{t-}(\gamma^b,\nu;g)=\beta_g,\\ 0, & o.w..\end{cases} \tag{ICgw}$$

*For any contract $\gamma^g$, effort process $\nu$, and operating cost $b$, we have*

$$\mathrm{d}W_t(\gamma^g,\nu;b)=\{\nu_t[W_{t-}(\gamma^g,\nu;b)-I_t^c+\beta_b]\mathrm{d}t-g\mathrm{d}t\}\,\mathbb{1}_{0\le t<T\wedge\tau}. \tag{PKb}$$

*Furthermore, the following defined effort process is a best response to contract $\gamma^g$, or, $\{\nu_t\}_{t\in[0,T\wedge\tau]}\in\mathfrak{N}(\gamma^g,b)$, in which*

$$\nu_t=\begin{cases}\mu\cdot g/b, & \text{if } I_t-W_{t-}(\gamma^g,\nu;b)>\beta_b,\\ \nu\in[0,\mu\cdot g/b], & \text{if } I_t-W_{t-}(\gamma^g,\nu;b)=\beta_b,\\ 0, & o.w..\end{cases} \tag{ICbw}$$

The proof of Lemma 3 can be adapted from the proof of Lemma 1.

### B.3. Proof of Proposition 1

Part (i): Since $I_t^c=c(T-t)+\beta_c$, we can verify that $\mathrm{d}W_t(\hat\gamma^c(T),\bar\nu;c)=-c\mathrm{d}t$, $W_{t-}(\hat\gamma^c(T),\bar\nu;c)=c(T-t)$, and $I_t^c-W_{t-}(\hat\gamma^c(w),\bar\nu;c)=\beta_c\ge\beta_c$, Hence, following Lemma 1, we have $\bar\nu\in\mathfrak{N}(\gamma^c,c)$. Clearly, $u\Big(\hat\gamma^c(T),\bar\nu;c\Big)=W_{0-}(\hat\gamma^c(w),\bar\nu;c)=c\cdot T$.

Part (ii): In the contract $\hat\gamma^g(T)$, we have $I_t^g=g(T-t)+\beta_g$. Following definition (45) and Lemma 3, we can verify that $W_{t-}(\hat\gamma^g(T),\nu^0;b)=g(T-t)$, and $I_t^g-W_{t-}(\hat\gamma^g(T),\nu^0;b)=\beta_g<\beta_b$. Hence, following Lemma 3, we have $\nu^0\in\mathfrak{N}(\gamma^g(T),b)$. Therefore, we have $u\Big(\hat\gamma^g(T),\nu^0;b\Big)=g\cdot T$.

In the contract $\hat\gamma^b(T)$, we have $I_t^b=b(T-t)+\beta_b$. Following definition (45) and Lemma 3, we can verify that

$$\begin{aligned}W_{t-}(\hat\gamma^b(T),\bar\nu;g)&=b(T-t)+\frac{b-g}{\mu}\left(1-e^{-\mu(T-t)}\right),\\ I_t^b-W_{t-}(\hat\gamma^b(T),\bar\nu;g)&=\beta_b-(\beta_b-\beta_g)\left(1-e^{-\mu(T-t)}\right)>\beta_b-(\beta_b-\beta_g)=\beta_g,\forall t\in[0,T].\end{aligned}$$

Hence, we have $\bar\nu\in\mathfrak{N}(\gamma^b,g)$. It is straightforward to verify that $u\Big(\hat\gamma^b(T),\bar\nu;g\Big)=W_{0-}(\hat\gamma^b(T),\bar\nu;g)=b\cdot T+(\beta_b-\beta_g)\left(1-e^{-\mu T}\right)$.

Part (iii): Following definition (45) and Lemma 3, we can verify that $W_{t-}(\bar\gamma(t_1,T),\nu^0;b)=g(T-t)$, and

$$I_t-W_{t-}(\bar\gamma(t_1,T),\nu^0;b)=\begin{cases}\beta_b-(\beta_b-\beta_g)\left(1-e^{-\mu(T-t_1)}\right)<\beta_b, & \text{for } t\in[0,t_1),\\ \beta_b\le\beta_b, & \text{for } t\in[t_1,T].\end{cases}$$

Hence, following Lemma 3, we have $\nu^0\in\mathfrak{N}(\bar\gamma(t_1,T),b)$. Clearly, we have $u\Big(\bar\gamma(t_1,T),\nu^0;b\Big)=g\cdot T$.

Again, following Lemma 1, we can verify that

$$W_{t-}(\bar\gamma(t_1,T),\bar\nu;g)=W_t,$$

where $W_t$ follows (7), and

$$I_t-W_{t-}(\bar\gamma(t_1,T),\bar\nu;g)=\begin{cases}\beta_g\ge\beta_b, & \text{for } t\in[0,t_1),\\ \beta_b>\beta_g, & \text{for } t\in[t_1,T].\end{cases}$$

Hence, following Lemma 1, we have $\bar\nu=\mathfrak{N}(\bar\gamma(t_1,T),g)$. And it is straightforward to verify that $u\Big(\bar\gamma(t_1,T),\bar\nu;g\Big)=W_{0-}=g\cdot T+(\beta_b-\beta_g)\left(1-e^{-\mu(T-t_1)}\right)$.

### B.4. Proof of Proposition 2

Proposition 12. *If $\mu R - c > 0$, then the maximium societal value function, which represents the expected sum of principal and agent's value subject to the "promise-keeping" constraint that $u(\gamma^c, \nu^c; c) = w$ with $\nu^c \in \mathfrak{N}(\gamma^c, c)$, is $S_c(w)$.*

Fixing any $w \geq 0$, we first show that the maximium societal value function is smaller or equal to $S_c(w)$. Since $u(\gamma^c, \nu^c; c) = w$, incentive and promise keeping constraint requires that $T \leq w/c$. If $\mu R > c$, then the total surplus is maximized by setting $T = w/c$ and $\nu_t = \mu$ for all $t \in [0, w/c]$. That is

$$\max_{\nu^c \in \mathfrak{N}(\gamma^c, c)} S(\gamma^c, \nu^c; c) \leq \max_{\nu, T \leq w/c} \left[ R \cdot \mathbb{1}_{\tau \leq T} - \int_0^{T \wedge \tau} c dt \right] = (R - \beta_c)\left(1 - e^{-\mu T}\right) = S_c(w).$$

Next, following Proposition 1, we have $\bar{\nu} \in \mathfrak{N}(\hat{\gamma}^c(w/c), c)$, which implies that $S_c(w)$ is achieved by $\hat{\gamma}^c(w/c)$.
Similarly, if $\mu R - c \leq 0$, then the total surplus is 0 and is achieved by setting $T = 0$. Q.E.D.
We are prepared to prove Proposition 2. Under contract $\hat{\gamma}^c(w/c)$, we have

$$U(\hat{\gamma}^c(w/c), \bar{\nu}) = S(\gamma^c, \nu^c; c) - u(\hat{\gamma}^c(w/c), \bar{\nu}; c) = S_c(w) - w = F_c(w), \tag{46}$$

where the second equality follows from Proposition 1 and 12. Furthermore, Proposition 12 implies that

$$\max_{\nu^c \in \mathfrak{N}(\gamma^c, c)} U(\gamma^c, \nu^c) = \max_{\nu^c \in \mathfrak{N}(\gamma^c, c)} S(\gamma^c, \nu^c) - u(\gamma^c, \nu^c; c) \leq S_c(w) - w.$$

Hence, we have $\mathcal{Z}(\{c\}) = \max_{w \geq 0} F_c(w)$.

### B.5. Proof of Proposition 3

Part 1: Under pay-to-leave contract, clearly, the principal's utility is $-w$.
Part 2:

$$U(\bar{\gamma}(t_1, T), \bar{\nu}) = S(\bar{\gamma}(t_1, T), \bar{\nu}; g) - u(\bar{\gamma}(t_1, T), \bar{\nu}; g) = S_g(g \cdot T) - u(\bar{\gamma}(t_1, T), \bar{\nu}; g).$$

where the second equality follows from Part 3 of Proposition 1 and the contract duration is $T$.

### B.6. Proof of Proposition 4

For any pair of contracts $(\gamma^g, \gamma^b)$ that satisfy (LL), (BR), (IR), (ICg) and (ICb), we create a vector $(w_g, w_b, T_g)$ such that, they satisfy the constraints (16) - (17), and

$$p\left\{S_g\left(g \cdot T_g\right) - w_g\right\} + (1-p)\min\left\{S_b(w_b) - w_b, \frac{w_g - w_b}{b - g}\left(\mu R - b\right)^+ - w_b\right\} \geq p \cdot U(\gamma^g, \nu^g) + (1-p) U(\gamma^b, \nu^b) \tag{47}$$

Then, we have $\mathcal{Y} \geq \mathcal{Z}(\{g, b\})$. We let $w_g := u(\gamma^g, \nu^g; g)$, $w_b := u(\gamma^b, \nu^b; b)$, $T_g$ be the time duration of $\gamma^g$.
**Step 1:** We check the constraints (16) - (17). First, (ICg) implies that

$$w_g \geq \max_{\nu} u(\gamma^b, \nu; g) \geq u(\gamma^b, \nu^b; g) = w_b + (b-g)\bar{T}(\gamma^b, \nu^b) \geq w_b, \tag{48}$$

where $\bar{T}$ is defined in (42). Second, (ICb) implies that

$$w_b \geq \max_{\nu} u(\gamma^g, \nu; b) \geq u(\gamma^g, \nu^0; b) = g \cdot T_g. \tag{49}$$

Hence, constraints (16)-(17) are satisfied.
**Step 2:** The inequality (47) clearly follows from

$$U(\gamma^g, \nu^g) = S(\gamma^g, \nu^g; g) - w_g \leq S_g(g \cdot T_g) - w_g,$$

where the inequality follows from the Proposition 2 and,

$$U(\gamma^b, \nu^b) \leq \min\left\{\frac{(w_g - w_b)}{b - g}\max\{\mu R - b, 0\} - w_b, S_b(w_b) - w_b\right\},$$

which will be shown in the following. If $R > \beta_b$, then following (48), we have

$$w_g \geq w_b + (b-g)\bar{T}(\gamma^b, \nu^b) = w_b + \frac{(b-g)S(\gamma^b, \nu^b; b)}{\mu R - b} = w_b + \frac{(b-g)(U(\gamma^b, \nu^b) + w_b)}{\mu R - b} \tag{50}$$

where the first equality follows from Lemma 2 and the last equality follows from $S(\gamma^b, \nu^b; b) = U(\gamma^b, \nu^b) + u(\gamma^b, \nu^b; b)$. Rearrange (50), we have

$$U(\gamma^b, \nu^b) \leq \frac{(w_g - w_b)(\mu R - b)}{b-g} - w_b, \text{ if } R > \beta_b. \tag{51}$$

Further, following Proposition 2, we have

$$U(\gamma^b, \nu^b) \leq S_b(w_b) - w_b. \tag{52}$$

On the other hand, if $R \leq \beta_b$, then

$$U(\gamma^b, \nu^b) \leq 0 - w_b = -w_b, \text{ if } R \leq \beta_b. \tag{53}$$

Hence, following (51) - (53), we have

$$U(\gamma^b, \nu^b) \leq \min\left\{\frac{(w_g - w_b)}{b-g}\max\{\mu R - b, 0\} - w_b, S_b(w_b) - w_b\right\}. \tag{54}$$

## C. Proof in Section 5

### C.1. Proof of Theorem 1

Theorem 1 directly follows from Proposition 7-9.

## D. Proof in Section 6

### D.1. Proof of Proposition 5

Clearly, the objective (19) is decreasing in both $w_g$ and $w_b$. Hence, the optimal choice of $w_g$ and $w_b$ are $w_g = w_b = g \cdot T_g$. After plugging in $w_g, w_b$, the objective becomes

$$pS_g(g\cdot T_g) - g\cdot T_g = p\left(R - g/\mu\right)\left(1 - e^{-\mu T_g}\right) - g \cdot T_g$$

Since the objective function is concave in $T_g$, it is straightforward to use the first-order condition to find the optimal $T_g$, and the optimal choice of $T_g$ follows (22).

### D.2. Proof of Proposition 6

Since the objective (15) is strictly decreasing in $T_g$, the optimal solution should satisfy $w_b = g \cdot T_g$.

$$\max_{w_g, T_g} p\left[S_g\left(g\cdot T_g\right) - w_g\right] + (1-p)\min\left\{S_b(g\cdot T_g) - g\cdot T_g, \frac{w_g - g\cdot T_g}{b-g}\left(\mu R - b\right) - g \cdot T_g\right\} \tag{55}$$

$$s.t. w_g \geq g \cdot T_g \tag{56}$$

$$T_g \geq 0 \tag{57}$$

If $-p + (1-p)\dfrac{\mu R - b}{b-g} \leq 0$ (which is equivalent to $R \leq \dfrac{b - pg}{(1-p)\mu}$), for any fixed $T_g \geq 0$, the objective is linear decreasing in $w_g$. Hence, the optimal solution should satisfy $w_g = g\cdot T_g$. Furthermore, the optimal choice of $T_g$ should follow (22).

If $-p + (1-p)\dfrac{\mu R - b}{b-g} > 0$, for any fixed $T_g \geq 0$, the objective increases in $w_g$ for $w_g \in \left[g\cdot T_g, g\cdot T_g + \dfrac{b-g}{\mu}e^{-\mu\cdot g/b\cdot T_g}\right]$, and then decreases in $w_g$ for $w_g > g\cdot T_g + \dfrac{b-g}{\mu}e^{-\mu\cdot g/b\cdot T_g}$. Hence, the optimal solution should satisfy $w_g = g\cdot T_g + \dfrac{b-g}{\mu}e^{-\mu\cdot g/b\cdot T_g}$. Plugging in $w_g$ to the obejective, the optimization problem becomes

$$\begin{aligned} &p\left[S_g\left(g\cdot T_g\right) - g\cdot T_g - \frac{b-g}{\mu}e^{-\mu\cdot g/b\cdot T_g}\right] + (1-p)\left[S_b(g\cdot T_g) - g\cdot T_g\right] \\ =&p\left[\left(R - \frac{g}{\mu}\right)\left(1 - e^{-\mu T_g}\right) - g\cdot T_g - \frac{b-g}{\mu}e^{-\mu\cdot g/b\cdot T_g}\right] + (1-p)\left[\left(R - \frac{b}{\mu}\right)\left(1 - e^{-\mu\cdot g/b\cdot T_g}\right) - g\cdot T_g\right] \end{aligned}$$

$$=p\left(R-\frac{g}{\mu}\right)\left(1-e^{-\mu T_g}\right)+\left[(1-p)\left(R-\frac{b}{\mu}\right)-\frac{p(b-g)}{\mu}\right]\left(1-e^{-\mu\cdot g/b\cdot T_g}\right), \tag{58}$$

which is a concave function of $T_g$ and the optimal choice of $T_g$ is

$$T_g=\begin{cases}0, & R\le \frac{(2+p)b-pg}{bp+(1-p)g}\beta_g,\\ T_g^1, & R>\frac{(2+p)b-pg}{bp+(1-p)g}\beta_g.\end{cases} \tag{59}$$

where $T_g^1$ solves

$$p(\mu R-g)e^{-\mu T_g^1}+g/b\cdot\left[(1-p)(\mu R-b)-p(b-g)\right]e^{-\mu\cdot g/b\cdot T_g^1}=g. \tag{60}$$

Therefore, if $p\cdot b\ge g$, then

$$\frac{b-pg}{(1-p)\mu}\ge\frac{(2+p)b-pg}{bp+(1-p)g}\beta_g\ge\frac{1+p}{p}\beta_g, \tag{61}$$

and the optimal choice of $T_g$ follows (23) and the optimal choice of $w_g$ follows (24). If $p\cdot b<g$, then

$$\frac{b-pg}{(1-p)\mu}<\frac{(2+p)b-pg}{bp+(1-p)g}\beta_g<\frac{1+p}{p}\beta_g, \tag{62}$$

and the optimal choice of $T_g$ follows (25) and the optimal choice of $w_g$ follows (26).

### D.3. Proof of Corollary 1

For any $g<b$, we have

$$\beta_b<\frac{b-pg}{(1-p)\mu}. \tag{63}$$

If $p<g/b$, then we have

$$\beta_b<\frac{1+p}{p}\beta_g,\beta_b<\frac{(2+p)b-pg}{bp+(1-p)g}\beta_g \tag{64}$$

Part (i): If $p<g/b$, then following Propositions 5 and 6, we have if $R\le\min\left\{\beta_b,\frac{1+p}{p}\beta_g\right\}$ or $R\in\left(\beta_b,\frac{(2+p)b-pg}{bp+(1-p)g}\beta_g\right]$, then optimal $T_g=0$. Furthermore, following (62) and (64), we have if $R\le\frac{(2+p)b-pg}{bp+(1-p)g}\beta_g$, we have optimal $T_g=0$. Furthermore, $w_g=w_b=0$.

If $p\ge g/b$, then following Propositions 5 and 6, we have if $R\le\min\left\{\beta_b,\frac{1+p}{p}\beta_g\right\}$ or $R\in\left(\beta_b,\frac{1+p}{p}\beta_g\right]$, then optimal $T_g=0$. Hence, if $R\le\frac{1+p}{p}\beta_g$, then we have optimal $T_g=0$. Furthermore, $w_g=w_b=0$.

Part (ii): If $p<g/b$, then following Propositions 5 and 6, we have if $R\in\left(\frac{1+p}{p}\beta_g,\beta_b\right]$, then optimal $T_g=T_g^*$. Following (64), the above set is empty.

If $p\ge g/b$, then following Propositions 5 and 6, we have if $R\in\left(\frac{1+p}{p}\beta_g,\beta_b\right]$ or $R\in\left(\max\left\{\beta_b,\frac{1+p}{p}\beta_g\right\},\frac{b-pg}{(1-p)\mu}\right]$, then optimal $T_g=T_g^*$. Hence, following (61) and (63), we have if $R\in\left(\frac{1+p}{p}\beta_g,\frac{b-pg}{(1-p)\mu}\right]$, then we have optimal $T_g=T_g^*$. Furthermore, $w_g=w_b=g\cdot T_g^*$.

Part (iii): If $p<g/b$, then following Proposition 6, we have if $R>\max\left\{\beta_b,\frac{(2+p)b-pg}{bp+(1-p)g}\beta_g\right\}$, then optimal $T_g=T_g^1$. By (64), we have if $R>\frac{(2+p)b-pg}{bp+(1-p)g}\beta_g$, then optimal $T_g=T_g^1$. Furthermore, $w_g=g\cdot T_g^1+(\beta_b-\beta_g)\left(1-e^{-\mu\cdot g/b\cdot T_g^1}\right),w_b=g\cdot T_g^1$.

If $p \geq g/b$, then following Proposition 6, we have if $R > \max\left\{\frac{1+p}{p}\beta_g, \frac{b-pg}{(1-p)\mu}\right\}$, then optimal $T_g = T_g^1$. Hence, following (61), we have if $R > \frac{b-pg}{(1-p)\mu}$, then optimal $T_g = T_g^1$. Furthermore, $w_g = g \cdot T_g^1 + (\beta_b - \beta_g)\left(1 - e^{-\mu \cdot g/b \cdot T_g^1}\right)$, $w_b = g \cdot T_g^1$.

### D.4. Proof of Proposition 7

Following Corollary 1 and Proposition 4, we have if $R \leq \underline{R}(p)$, then

$$0 = \mathcal{Y} \geq \mathcal{Z}(\{g,b\}).$$

Clearly, $\underline{\Gamma}_{\{g,b\}}$ defined in (i) of Theorem 1 satisfies (LL), (BR), (IR), (ICg), and (ICb). Furthermore, $\mathcal{U}(\underline{\Gamma}_{\{g,b\}}) = 0$, which completes the proof.

### D.5. Proof of Proposition 8

First, by definition, contract $\Gamma^*_{\{g,b\}}$ satisfies (LL), (BR), (IR). Second, we verify (ICg) and (ICb).

$$\max_\nu u(\check{\gamma}(g \cdot T_g^*), \nu; b) = g \cdot T_g^* = u(\hat{\gamma}^g(g \cdot T_g^*), \nu^0; b) = \max_\nu u(\hat{\gamma}^g(g \cdot T_g^*), \nu; b),$$

where the last equality follows from Part (ii) of Proposition 1. Furthermore,

$$\max_\nu u(\hat{\gamma}^g(g \cdot T_g^*), \nu; g) = u(\hat{\gamma}^g(g \cdot T_g^*), \bar{\nu}; g) = g \cdot T_g^* = u(\check{\gamma}(g \cdot T_g^*), \nu^0; g) = \max_\nu u(\check{\gamma}(g \cdot T_g^*), \nu; g).$$

Finally, if $\underline{R}(p) < R \leq \underline{R}(p)$

$$\begin{aligned}\mathcal{U}(\Gamma^*_{\{g,b\}}) &= pU(\hat{\gamma}^g(T_g^*), \bar{\nu}) + (1-p)U(\check{\gamma}^g(g \cdot T_g^*), \bar{\nu}) \\ &= p\left(S_g(g \cdot T_g^*) - g \cdot T_g^*\right) - (1-p)g \cdot T_g^* = \mathcal{Y},\end{aligned}$$

where the second equality follows from Proposition 2, and the last equality follows from part (ii) of Corollary 1, which completes the proof.

### D.6. Proof of Proposition 9

First, by definition, contract $\Gamma^{**}_{\{g,b\}}$ satisfies (LL), (BR), and (IR). Second, we verify (ICg), and (ICb).

$$\max_\nu u(\hat{\gamma}^b(g \cdot T_g^1/b), \nu; b) = u(\hat{\gamma}^b(g \cdot T_g^1/b), \bar{\nu}; b) = g \cdot T_g^1 = u(\bar{\gamma}(t_1^*, T_g^1), \nu^0; b) = \max_\nu u(\bar{\gamma}(t_1^*, T_g^1), \nu; b),$$

where the last equality follows from Part (iii) of Proposition 1. Furthermore,

$$\begin{aligned}\max_\nu u(\bar{\gamma}(t_1^*, T_g^1), \nu; g) &= u(\bar{\gamma}(t_1^*, T_g^1), \bar{\nu}; g) = g \cdot T_g^1 + (\beta_b - \beta_g)\left(1 - e^{-\mu(T_g^1 - t_1^*)}\right) \\ &= g \cdot T_g^1 + (\beta_b - \beta_g)\left(1 - e^{-\mu \cdot g/b \cdot T_g^1}\right) \\ &= w_b^* + (\beta_b - \beta_g)\left(1 - e^{-\mu \cdot g/b \cdot T_g^1}\right) = u(\hat{\gamma}^b(g \cdot T_g^1/b), \bar{\nu}; g) = \max_\nu u(\hat{\gamma}^b(g \cdot T_g^1/b), \nu; g),\end{aligned}$$

where the third equality follows from $t_1^* = (1 - g/b) \cdot T_g^1$, and the sixth equality follows from Part (ii) of Proposition 1. Finally, if $R > \bar{R}(p)$, then

$$\begin{aligned}\mathcal{U}(\Gamma^*_{\{g,b\}}) &= pU(\bar{\gamma}(t_1^*, T_g^1), \bar{\nu}) + (1-p)U(\hat{\gamma}^b(g \cdot T_g^1/b), \bar{\nu}) \\ &= p\left(S_g(g \cdot T_g^1) - \left(g \cdot T_g^1 + (\beta_b - \beta_g)\left(1 - e^{-\mu \cdot g/b \cdot T_g^1}\right)\right)\right) + (1-p)\left(S_b(g \cdot T_g^1) - g \cdot T_g^1\right) = \mathcal{Y},\end{aligned}$$

where the second equality follows from Proposition 2 and 3, the last equality follows from part (iii) of Corollary 1, which completes the proof.

### D.7. Proof of Proposition 10

LEMMA 4.

$$w_*^c = \begin{cases} 0, & R \le 2\beta_c, \\ \frac{1}{\mu}\log\left(\frac{\mu R - c}{c}\right), & R > 2\beta_c. \end{cases}$$

Following Proposition 2, it is straightforward to verify that $w_*^c$ maximizes $F_c(w)$. Q.E.D.

We are prepared to prove the statement. First, if $R \le \beta_b$, following Lemma 4, we have $w_*^b = 0$. Hence, we have $w_b^* \ge 0 = w_*^b$.

Second, following Corollary 1, we have

$$w_g^* = \begin{cases} 0, & R \le \underline{R}(p), \\ T_g^*, & R \in (\underline{R}(p), \bar{R}(p)]. \end{cases} \tag{65}$$

If $p < g/b$, following (64), we have $\beta_b < \underline{R}(p)$, hence, $w_g^* = 0 \le w_*^g$.

If $p \ge g/b$ and $R \le 2\beta_g < \underline{R}(p)$, then $w_g^* = w_*^g = 0$. If $p \ge g/b$ and $R \in (2\beta_g, \underline{R}(p)]$, then $0 = w_g^* < w_*^g$. If $p \ge g/b$ and $R \in (\underline{R}(p), \beta_b]$, then

$$w_g^* = g \cdot T_g^* = \frac{g}{\mu}\log\left(\frac{p(\mu R - c)}{c}\right) < \frac{g}{\mu}\log\left(\frac{\mu R - c}{c}\right) = w_*^g,$$

which completes the proof.

### D.8. Proof of Proposition 11

For any $g < b$, we have

$$\frac{(2+p)b - pg}{bp + (1-p)g}\beta_g < 2\beta_b \tag{66}$$

In the following, we first show that there exists $R_2 > \beta_b$ such that $w_b^* \le w_*^b$ if and only if $R \ge R_2$ in two cases.

1. If $p \le g/b$, following (66), Corollary 1 and Lemma 4, we have if $R \le \underline{R}(p)$, $w_*^b = w_b^* = 0$. If $R \in (\underline{R}(p), 2\beta_b]$, $w_b^* > w_*^b = 0$. If $R > 2\beta_b$, $w_b^* = g \cdot T_g^1$, and $w_*^b = \frac{b}{\mu}\log\left(\frac{\mu R - b}{b}\right) > 0$. Hence, we only need to focus on the last case. We claim that there exists $R_2 > 2\beta_b$ such that $w_b^* \le w_*^b$ if and only if $R \ge R_2$. Since $T_g^1$ solves equation (14), and the left-hand side of equation (14) strictly decreases in $T_g^1$, we have $f_1\left(w_*^b/g\right) > 0$ if and only if $w_b^* > w_*^b$, where

$$f_1(T_g^1) := p(\mu R - g)e^{-\mu T_g^1} + g/b \cdot [(1-p)(\mu R - b) - p(b-g)]\, e^{-\mu \cdot g/b \cdot T_g^1} - g \tag{67}$$

Next, we define

$$\begin{aligned} f_1(w_*^b/g) &= p(\mu R - g)\left(\frac{b}{\mu R - b}\right)^{b/g} + (1-p)g - \frac{pg(b-g)}{\mu R - b} - g, \\ &= pg\left[\frac{\mu R - g}{g}\left(\frac{b}{\mu R - b}\right)^{b/g} - \frac{b-g}{\mu R - b} - 1\right] := pg \cdot f_2(R) \end{aligned} \tag{68}$$

and, the claim is true if we can show that there exists $R_2$ such that $f_2(R) \le 0$ if and only if $R \ge R_2$. Since $b > g$, we have $\lim_{R\to\infty} f_2(R) < 0$.

To facilitate our analysis, we further denote $k := b/g > 1$, $x := (\mu R - b)/b - 1$, hence, we have

$$f_2(R) = \frac{1}{x^k}\left[(x+1)k - 1 - \frac{k-1}{k}x^{k-1} - x^k\right] := f_3(x)/x^k, \tag{69}$$

and the claim is true if we can show that $f_3'(x) < 0$ for $x > 1$ and $k > 1$. We can verify that

$$f_3'(x) = k - \frac{(k-1)^2}{k}x^{k-2} - k \cdot x^{k-1} < 0,$$

for $x > 1$ and $k > 1$.

2. If $p > g/b$, then we have $\underline{R}(p) < 2\beta_b$, which implies that $w_b^* \geq w_*^b = 0$ if $R \leq 2\beta_b$. If further $p \geq b/(2b-g)$ (which is equivalent to $2\beta_b \geq \bar{R}(p)$), then for $R > 2\beta_b$, $w_b^* = g \cdot T_g^1$, and $w_*^b = \frac{b}{\mu}\log\left(\frac{\mu R - b}{b}\right) > 0$. Following the above case 1, we can show that there exists $R_2 > 2\beta_b$ such that $w_b^* \leq w_*^b$ if and only if $R \geq R_2$. In the following, we consider the case $p \in (g/b, b/(2b-g))$.
   If $R \in [2\beta_b, \bar{R}(p)]$, $w_b^* = g \cdot T_g^* = \frac{g}{\mu}\log\left(\frac{p(\mu R - g)}{g}\right)$ and $w_*^b = \frac{b}{\mu}\log\left(\frac{\mu R - b}{b}\right)$. Clearly $w_b^* - w_*^b$ strictly decreases in $R$. Either $w_b^* - w_*^b > 0$ at $R = \bar{R}(p)$ or $w_b^* - w_*^b \leq 0$ at $R = \bar{R}(p)$. In the former case, we have $w_b^* > w_*^b$ for $R \in [2\beta_b, \bar{R}(p)]$. When $R > \bar{R}(p)$, we have $w_b^* = g \cdot T_g^1$, and $w_*^b = \frac{b}{\mu}\log\left(\frac{\mu R - b}{b}\right) > 0$. It is straightforward to show that $w_b^*$ is continuous in $R$ at $R = \bar{R}(p)$. Hence, following the above case 1, we can show that there exists $R_2 > \bar{R}(p)$ such that $w_b^* \leq w_*^b$ if and only if $R \geq R_2$.
   In the latter case, there exists $R_2 \in (2\beta_b, \bar{R}(p)]$ such that $w_b^* \leq w_*^b$ if and only if $R \in [R_2, \bar{R}(p)]$. Again, since $w_b^*$ is continuous in $R$ at $R = \bar{R}(p)$, following the proof of case 1, we can show that for any $R > \bar{R}(p)$, we have $w_b^* < w_*^b$, which completes the proof.

In the following, we first show that there exists $R_1 > \beta_b$ such that $w_g^* \geq w_*^g$ if and only if $R \geq R_1$ in two cases.

1. If $p \leq g/b$, then $\bar{R}(p) > 2\beta_g$. Hence, if $R \leq 2\beta_g$, then $w_*^g = w_g^* = 0$. If $R \in (2\beta_g, \bar{R}(p)]$, then $w_*^g > w_g^* = 0$. If $R > \bar{R}(p)$, we have $w_g^* = g \cdot T_g^1 + (\beta_b - \beta_g) \cdot \left(1 - e^{-\mu \cdot g/b \cdot T_g^1}\right)$ and $w_*^g = \frac{g}{\mu}\log\left(\frac{\mu R - g}{g}\right)$. We claim that there exists $R_1 > \bar{R}(p) > \beta_b$ such that $w_g^* \geq w_*^g$ if and only if $R \geq R_1$. Since $T_g^1$ solves equation (14), and the left-hand side of equation (14) strictly decreases in $T_g^1$, we have $f_1(T) > 0$ if and only if $w_g^* > w_*^g$, where $T$ solves
$$g \cdot T + (\beta_b - \beta_g) \cdot \left(1 - e^{-\mu \cdot g/b \cdot T}\right) - \frac{g}{\mu}\log\left(\frac{\mu R - g}{g}\right) = 0 \tag{70}$$
   and $f_1$ follows (67). Clearly, $T$ is a function of $R$ and we have
$$\frac{\partial T}{\partial R} = \frac{1}{1 + (\beta_b - \beta_g) \cdot (1 - e^{-\mu \cdot g/b \cdot T})} \frac{1}{\mu R - g}. \tag{71}$$
   Therefore, the claim follows from
$$\begin{aligned}\frac{\partial f_1(T)}{\partial R} &= p \cdot \mu e^{-\mu T} + p(\mu R - g) \cdot (-\mu)\frac{\partial T}{\partial R} e^{-\mu T} + \frac{g}{b}(1-p)\mu e^{-\mu \cdot g/b \cdot T} + \frac{g}{b}(1-p)(\mu R - b)\frac{-\mu g}{b}\frac{\partial T}{\partial R} e^{-\mu \cdot g/b \cdot T} \\ &= p\mu e^{-\mu T}\left(1 - \frac{1}{1 + (b-g)/b \cdot e^{-\mu \cdot g/b \cdot T}}\right) + \frac{g}{b}(1-p)\mu e^{-\mu \cdot g/b \cdot T}\left(1 - \frac{\mu R/b - 1}{\mu R/g - 1}\frac{1}{1 + (b-g)/b \cdot e^{-\mu \cdot g/b \cdot T}}\right) > 0.\end{aligned}$$
   where the second inequality follows from (71) and if $R \to \infty$, then $T \to \infty$ and
$$\lim_{R \to \infty} f_1(T) = pge^{(\beta_b - \beta_g)/\beta_g} + g/b \cdot (1-p) \cdot ge^{\mu(1-g/b)T + (\beta_b - \beta_g)/\beta_g} - g > 0.$$
2. If $p > g/b$, then $2\beta_g < \underline{R}(p) < \bar{R}(p)$. Hence, if $R \leq 2\beta_g$, then $w_*^g = w_g^* = 0$. If $R \in (2\beta_g, \underline{R}(p)]$, then $w_*^g > w_g^* = 0$. If $R \in (\underline{R}(p), \bar{R}(p)]$, then $w_*^g = \frac{g}{\mu}\log\left(\frac{\mu R - g}{g}\right) > \frac{g}{\mu}\log\left(\frac{(\mu R - g)p}{g}\right) = w_g^*$. If $R > \bar{R}(p)$, then $w_g^* = g \cdot T_g^1 + (\beta_b - \beta_g) \cdot \left(1 - e^{-\mu \cdot g/b \cdot T_g^1}\right)$ and $w_*^g = \frac{g}{\mu}\log\left(\frac{\mu R - g}{g}\right)$. Since $w_g^*$ is continuous at $R = \bar{R}(p)$, following the above case 1, we can show that there exists $R_1 > \bar{R}(p)$ such that $w_g^* \geq w_*^g$ if and only if $R \geq R_1$.