Dynamic Pricing and Release Time Control for Service Systems

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This paper studies a mechanism design problem for a single-server queue in which customers are served in a first-come-first-serve (FCFS) manner. Customers are heterogeneous and have private information on their valuations for immediate service and wait time sensitivity. Namely, impatient (patient, respectively) customers are more (less, respectively) sensitive to waiting and value immediate service higher (lower, respectively). The service provider designs a history-dependent menu such that each menu item consists of a price and a release time for each type of arriving customer, aiming to maximize the long-run average revenue rate. We demonstrate that the optimal menu depends on history only through the system completion time, and includes four strategies: delaying patient customers, pooling all customers, screening customers (serving one type and rejecting the other), and rejecting all customers. In particular, we show that it is optimal for the provider to strategically delay the release time of a patient customer beyond the service completion time if and only if the completion time is shorter than a threshold. The delay in the release time allows the provider to charge a premium on the impatient customers and thus increase revenue. Interestingly, the delayed release time is set to a fixed value, regardless of the exact completion time state, which makes the mechanism easy to implement. Additionally, the decision to serve either the more patient or impatient customers under the screening strategy hinges on an intuitive trade-off between a customer's utility from joining the service and the externality from a more congested system.

Key words: queueing system, mechanism design, dynamic control, strategic delay.

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1. Introduction

Price discrimination has been a commonly applied approach to increase revenue for firms facing a diverse customer base. In practice, firms may also exercise operational levers, beyond setting prices, to elevate their performance. Specifically, when managing a service system, the firm can strategically adjust waiting times to account for differences in customers' patience levels. When wielded adeptly, this operational lever has the potential to augment the efficacy of price discrimination, further improving the firm's financial performance.

In this paper, we explore a firm's strategic use of delaying the release time of a completed job to distinguish between customers with different levels of patience. To be precise, we define the *release time* as the time duration between when a service request arrives and when a completed service is released back to the customer. Before delving into our model, we first examine the rationale behind this operational lever to understand its potential benefits. When customers exhibit various patience levels, the firm can tailor its pricing and release time strategies accordingly. Specifically, the firm may opt to delay the release time of a completed product or service for a customer who self-identifies as a patient type. This deliberate delay for patient customers enables the firm to impose a higher price on impatient ones who prefer shorter wait through immediate release. Essentially, the extended release time for patient customers serves as a screening device in addition to pricing.

Afeche (2013) and Afeche and Pavlin (2016) are some of the earliest works studying price discrimination and strategic delay in queueing systems. They study queueing systems in a steady state. Although the aforementioned intuition has been revealed in these papers, determining the optimal prices and release times *dynamically* in environments where the firm faces limited processing capacity and fluctuating workloads can be challenging. This paper explores this complexity and examines how a firm should dynamically fine-tune these control parameters for customers with private information on their patience levels in a queueing system.

The strategy of differentiating prices and release times is applicable in various make-to-order customization manufacturing or service systems. For instance, automobile manufacturers offer customization options in their online quotation systems. Depending on specific options, which may include colors, equipment, and other features, customers sometimes need to wait for weeks or months before product delivery. In settings such as these, the actual processing time often remains undisclosed to customers, and firms may opt to charge an expedited fee for faster release while strategically delaying the release time if the customer chooses a lower price option. Similarly, online photo-editing services and more general professional services on gig platforms can offer tiered pricing and release timing options to cater to customers with different patience levels. Ride-hailing companies or delivery businesses can also apply similar pricing and delay strategies.

We focus on investigating a first-come-first-serve (FCFS) single-server system where customer arrivals follow a Poisson process. Assume service times are deterministic for all requests. When a new customer arrives, the firm can accurately assess the total time required to complete all pending requests in the queue. We call this total time the *completion time* of the system. As we show in the paper, the completion time serves as the system's state for optimal control. The release time cannot be sooner than the current system completion time plus the deterministic service time. *Strategic delay* means releasing the request after this earliest possible release time.

Customers are heterogeneous regarding their immediate service valuation (perceived value of the service with no delay) and their patience levels (cost per unit of time). In our base model, we consider two types of customers: patient and impatient ones. Patient customers have a lower immediate service valuation and wait-time sensitivity, and both values are higher for impatient customers. A customer's overall utility is the immediate service valuation minus the wait-time sensitivity times the release time. The firm is aware of the distribution of customer types in the population but not the specific type of each arriving customer. Upon a new arrival, the firm presents the customer a (history-dependent) menu of options, each specifying a price and release time, for the customer to choose from. The objective of the firm is to maximize the long-run average revenue rate by dynamically adjusting the menu of prices and release times.

We formulate this problem as a continuous-time control model and show that the optimal menu depends on history only through the completion time. We characterize the optimal mechanism from the resulting Hamilton-Jacob-Bellman equation. Depending on the system completion time, the optimal mechanism demonstrates up to four strategies. When the completion time is sufficiently short, the firm should delay the release time offered to a patient customer, and charge a premium price to an impatient one. The optimal delayed release time is set to a fixed value, regardless of the current completion time state. This simple structure allows for easy implementation of real-time control. When the system becomes more congested (completion time becomes longer), the firm may either pool customers into a single class and offer them the same price and release time, or only serve one of the two types. Finally, when the completion time is sufficiently long, the firm should reject all incoming customers.

We further extend our base model along two separate directions. First, we allow customer types to be drawn from a continuous interval, instead of from a binary set. The structure of the optimal mechanism for continuous-type customers echoes that of the base model. Specifically, strategic delay is implemented for less patient customer types when the completion time is below a threshold, and the optimal delayed release time remains a fixed value for all types and all completion time states. The optimal menu contains at most two options at each point in time, which can be implemented easily.

In the second extension, we allow the firm to decide the service time for each arriving customer. Shorter service time costs the provider more. The problem can be decomposed into a sequence of single-variable optimization problems, which is easy to solve. Strategic delay still occurs in the less patient type when the completion time is low.

To quantify the value of strategic delay in the optimal policy, we compare it with a simpler mechanism, which does not involve strategic delay, so that each request is released at its completion time. By varying the arrival rate and type distribution, we observe that allowing strategic delay generates an average of 11% improvements in the long-run revenue rate in the base-case model. The maximum improvement can be as high as 49%. For the continuous-type model and the endogenous service time model, we observe consistent improvements, although not as prominent. These findings underscore the potential benefits of using strategic delay as a dynamic control lever.

The rest of the paper is organized as follows: Section 2 provides a comprehensive literature review. Section 3 introduces the base model with two customer types, and characterizes the structure of the optimal policy. Section 4 studies two extensions: a continuous-type setting and an endogenous service time setting, respectively. We present numerical results in Section 5. Finally, Section 6 concludes the paper. All proofs are presented in the online supplement.

2. Literature Review

This paper is related to a large body of literature on queueing control. Aligning with our research question, we categorize related works into three pertinent areas. For a comprehensive overview of the field, we direct the reader to Hassin and Haviv (2003), Keskinocak and Tayur (2004), and Zang et al. (2024), the references therein.

Price/lead-time quotation in queueing systems

This stream of studies is focused on obtaining optimal or near-optimal pricing and lead-time quotation decisions without considering information asymmetry issues. Duenyas (1995) investigated a single-server queue with various customer classes, each having distinct preferences for lead-time and price. Simulation results indicated that policies accounting for customer preferences yield significantly greater improvements compared to those that do not. Extending this, Duenyas and Hopp (1995) derived a closed-form expression for optimal lead-time quotation and established conditions for processing jobs in the earliest due date order for a queueing system with infinite capacity. In the context of managing multi-product make-to-order systems, Maglaras (2006) addressed dynamic pricing control with sequencing decisions. The paper established the optimality of the $c\mu$-rule using a fluid model and showed that the optimal pricing decisions can be decoupled from sequencing and solely depend on the total workload. Çelik and Maglaras (2008) examined a multiclass make-to-order system in which the demand rate depends on a menu of prices and lead-times. They derived near-optimal dynamic pricing, lead-time quotation, sequencing, and expediting policies under diffusion approximation. Besbes and Maglaras (2009) expanded on this framework by considering stochastically varying market sizes. They developed an asymptotic analysis based on a fluid model, which yields near-optimal policies. Kim and Randhawa (2018) assessed the value of dynamic pricing to maximize revenues in queueing systems with price- and delay-sensitive customers. They established the superiority of dynamic over static pricing schemes in large customer market size and capacity scenarios and demonstrated that using only two prices can capture the most dynamic pricing benefits. The main difference between our paper and this stream of research is that we explicitly address information asymmetry, i.e., customers have private information about their preferences.

Mechanism design in queueing systems

This line of research investigates adverse selection or moral hazard issues caused by information asymmetry between the service provider and customers. Typically, there are two modeling approaches. One approach examines the mechanism through steady-state analysis, while the other considers dynamic settings under approximation, for example, using fluid or heavy traffic models. For the steady-state analysis, Mendelson and Whang (1990) developed a pricing mechanism assuming customers' private information on expected service times and delay costs. Ha (1998) extended this by examining a single-server queue where customers can adjust service times by exerting effort. Building on this, Ha (2001) introduced multiple customer classes with diverse demand and cost structures and showed how a simple pricing mechanism can effectively manage the system in a steady state. Van Mieghem and Van Mieghem (2000) explored service differentiation through service grades and pricing, focusing on how these mechanisms influence customer choices under different information scenarios. Afeche and Mendelson (2004) further investigated how a service provider can utilize pricing and priority adjustment decisions to maximize either revenue or social welfare. Although these studies all consider private information issues, their incentive compatibility designs are generally based on the steady state of the system, instead of real-time information in dynamic settings.

Regarding dynamic models, Plambeck (2004) examined how to manage two customer types using a diffusion approximation to derive a near-optimal price/lead-time policy for a queueing system in which customer arrival rates depend on both price and delay. Ata and Olsen (2013) explored a queueing model where two types of customers compete for the same resource. The firm offers incentive-compatible pricing and lead-time options, dynamically quoted at customer arrival without prior knowledge of customer types. Using a heavy-traffic queueing regime, the paper establishes asymptotically optimal policies that aim to maximize revenue. Akan et al. (2012) investigated dynamic lead-time quotation to maximize social welfare using a fluid approximation model. Our model distinguishes itself by considering real-time incentive compatibility control and deriving the exact optimal policy, instead of using approximation. Furthermore, modeling deterministic service time allows us to use the total completion time as the state variable, as opposed to using queue length.

Strategic delay in queueing systems

These papers are most relevant to our work. Afeche (2013) demonstrated the effectiveness of using a price/lead-time menu to screen patient customers with extended release times so that the provider can charge a high price for impatient customers in order to maximize revenue. Afeche and Pavlin (2016) extended this by considering a continuous spectrum of customer types, showing that the best strategy is to serve the extremes of the customer types, while pooling different types into the same service option. Our paper differentiates from Afeche (2013) and Afeche and Pavlin (2016) in several key ways. First, the steady-state analysis allows these studies to utilize an achievable region method without a specified scheduling policy. We demonstrate that strategic delay still occurs in the optimal dynamic control under the first-come-first-serve schedule. Second, their papers use a steady-state design approach, which is proper in settings where the customers do not observe queue length and the service provider is not able to dynamically adjust the pricing and delay decisions depending on real-time congestion levels. In comparison, our dynamic control specifies when to implement strategic delay, pooling, or screening based on the system completion time, which can be applied when the technology allows real-time control.

3. Two-type Model

In this section, we first introduce the base model with two types of customers and then analyze the optimal mechanism.

3.1. Model Formulation

We consider a firm managing a single-server, make-to-order system in which customers arrive according to a Poisson process with a rate λ. The customers are served according to the first-come-first-serve (FCFS) rule.[1] Each customer takes a constant service time of B to complete. Customers are heterogeneous and belong to either one of two types: impatient type h and patient type l. The impatient (patient, respectively) customer has high (low, respectively) immediate service valuation ν_h (ν_l, respectively) and high (low, respectively) wait-time sensitivity c_h (c_l, respectively). We assume $\nu_l < \nu_h$ and $c_l < c_h$. That is, impatient customers (type h) value the service higher. Denote a binary set $\mathbf{\Theta} := \{h, l\}$ to be the set of types. Each customer's type is independent of others and is private information known only to the customer. The firm only knows the proportion of impatient (type-h) customers $\alpha_h \in [0,1]$, and the proportion of patient (type-l) customers $\alpha_l = 1 - \alpha_h$.

Whenever a customer arrives, the firm decides whether to admit the customer into the queue and the corresponding price to charge. If a customer request is admitted, the firm also decides

[1] Due to the strategic delay of release time, FCFS does not imply first-in-first-out (FIFO).

when to release it back to the customer – either release it as soon as the service is completed, or delay to a later time.

To this end, define w as the completion time of all these requests already in the system, or "completion time" for simplicity. This value equals to the service time B times the number of jobs in the queue plus the remaining service time for the one in service. The completion time represents how much the request needs to wait in queue before its service can start. Denote $\bar{r}$ as a customer's *release time*, which is the duration between when the customer request arrives and when the fulfilled service is released back to the customer. Releasing the request as soon as the service is completed corresponds to $\bar{r} = w + B$, while delaying the release to a later time means $\bar{r} > w + B$. As explained earlier, holding the finished service requests potentially allows the firm to better screen between the two customer types.

The waiting cost for a type-θ customer is defined as $c_\theta \mathtt{s}(\bar{r})$ for $\theta \in \mathbf{\Theta}$, in which the function $\mathtt{s} : [B, \infty) \to \mathbb{R}^+$ is strictly increasing and taking non-negative values. For simplification of exposition, we set $\mathtt{s}(\bar{r}) = \bar{r}$. All our results can be generalized to other increasing $\mathtt{s}$ functions.

Denote $q \in \{0, 1\}$ to represent whether to admit the customer into the queue and $\bar{p}$ the price for receiving the service. Further introduce the following notations:

$$r := q\bar{r}, \text{ and } p := q\bar{p}.$$

The expected utility of a type-θ customer is

$$\mathrm{u}(\theta, q, p, r) := q\,(\nu_\theta - c_\theta \bar{r} - \bar{p}) = \nu_\theta q - c_\theta r - p, \tag{1}$$

where the term $\nu_\theta q - c_\theta r$ represents the utility of using the service, or *service utility* in short. Define quantities

$$r^* := \frac{\nu_h - \nu_l}{c_h - c_l} \text{ and } \bar{\nu} = \frac{c_h \nu_l - c_l \nu_h}{c_h - c_l}. \tag{2}$$

The service utilities for the l-type and h-type of customers both equal to $\bar{\nu}$ at release time r^* when $q = 1$, as shown in Figure 1.

We make the following assumptions.

Assumption 1. *(1)* $r^* \geq B$;
(2) $\dfrac{c_h}{c_l} \geq \dfrac{\nu_h}{\nu_l}$;
(3) $\nu_h - c_h B > 0$; *and*
(4) $\alpha_h \geq \dfrac{c_l}{c_h}$.

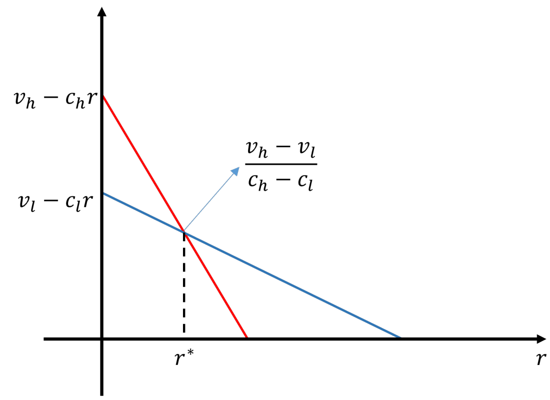


Figure 1 **The illustration of $\nu_\theta - c_\theta r$ under Assumption 1.**

The first two assumptions ensure that both $w^* := r^* - B$ and $\bar{\nu}$ are non-negative. The assumption $\nu_h - c_h B > 0$ ensures that the service utility for the type-h customer is positive when the customer does not need to wait. Together with the second assumption, it also implies that $\nu_l - c_l B > 0$. The last one $\alpha_h \geq c_l/c_h$ assumes that the proportion of the type-h customers is sufficiently high, which allows strategic delay to occur in the optimal control. Figure 1 illustrates both types' service utility functions when Assumption 1 holds.

We use subscript $t \in [0, \infty)$ to represent the time epoch. Denote a counting process $\{N_t\}$ to represent the total number of arrivals up to time t. Together with each arrival's type $\theta_t \in \Theta$, they generate a filtration $\{\mathcal{F}_t\}$. In terms of the probability space, the arrival rate of the counting process $\{N_t\}$ is λ, and θ_t takes value θ with probability α_θ for all $\theta \in \Theta$ and $t \geq 0$.

To screen between the two types of customers, the firm announces a menu (of two options) for each arriving customer, $\big\{\big(q_t(\theta), p_t(\theta), r_t(\theta)\big)\big\}_{\theta\in\Theta}$, adapted to the filtration $\{\mathcal{F}_t\}$. Following the revelation principal (Myerson 1979), a direct mechanism requires that a type-h customer chooses $\big(q_t(h), p_t(h), r_t(h)\big)$, and a type-$l$ customer chooses $\big(q_t(l), p_t(l), r(l)_t\big)$, which translates to the following constraints,

$$\mathrm{u}\big(\theta, q_t(\theta), p_t(\theta), r_t(\theta)\big) \geq \mathrm{u}\big(\theta, q_t(\theta'), p_t(\theta'), r_t(\theta')\big), \ \forall \theta, \theta' \in \mathbf{\Theta}, \text{ and} \tag{IC}$$

$$\mathrm{u}\big(\theta, q_t(\theta), p_t(\theta), r_t(\theta)\big) \geq 0, \ \forall \theta \in \mathbf{\Theta}. \tag{IR}$$

The (IC) constraint guarantees that a type-θ customer has no incentive to mimic any other type-θ' customer. The (IR) constraint further ensures that the customer receives a non-negative utility and, therefore, is willing to participate. Moreover, the payment and the release time offered by the firm must satisfy the following feasibility constraints:

$$q_t(\theta) \in \{0, 1\}, \ p_t(\theta) \geq 0, \text{ and } r_t(\theta) \geq q_t(\theta)(w_t + B), \tag{FE}$$

for any completion-time time w. Together with the definition (1), (IR) implies that when $q_t(\theta)=0$, we must have $p_t(\theta)=0$ and $r_t(\theta)=0$.[2]

Because the service time is deterministic, the completion time w_t at time t is also adapted to $\{\mathcal{F}_t\}$, and follows

$$\mathrm{d}w_t = -\mathbb{1}_{w_{t-}>0}\mathrm{d}t + B\cdot q_t(\theta_t)\cdot \mathrm{d}N_t. \tag{3}$$

Here we use notation $w_{t-} := \lim_{s\uparrow t} w_s$ to represent the completion time right before a customer arrival time t.

The firm's objective is to maximize the long-run average revenue rate, defined as

$$\begin{aligned} g^* &:= \sup_{\{q_t(\theta),p_t(\theta),r_t(\theta)\}_{\theta\in\Theta,t\ge 0}\in\Pi} \mathcal{G}\left(\{q_t(\theta),p_t(\theta),r_t(\theta)\}\right), \\ &\text{in which } \mathcal{G}\left(\{q_t(\theta),p_t(\theta),r_t(\theta)\}_{\theta\in\Theta,t\ge 0}\right) := \liminf_{T\to\infty}\frac{1}{T}\mathbb{E}\left[\int_0^T p_t(\theta_t)\mathrm{d}N_t\right], \end{aligned} \tag{4}$$

and the set of admissible policies Π is defined by constraints (IC), (IR), and (FE).

3.2. Optimality Equations

We now provide the optimality equation for this problem. We first restrict the control policy to be time-stationary given the completion-time state w, and later verify that the stationary policy is indeed optimal among all possible control policies. Hence we drop the time epoch t in the subscript of state w_t and decision variables. We also use superscripts "l" and "h" to represent customer types to simplify the notation. Following the standard heuristic derivation for continuous-time control problems, we obtain the optimality equations below (see Appendix A for the details):

$$\mathsf{g} + V'(w) = \lambda\Phi(w)\ \forall w>0, \text{ with boundary condition } V'(0)=0, \text{ and } V(0)=0, \tag{HJB}$$

in which $\Phi(w)$ is defined as the following integer linear optimization problem

$$\begin{aligned} \Phi(w) := \max_{q^\theta,p^\theta,r^\theta} & \sum_{\theta\in\boldsymbol{\Theta}} \alpha_\theta\left\{p^\theta + q^\theta\big[V(w+B)-V(w)\big]\right\}, \\ \text{s.t. } & \nu_h q^h - c_h r^h - p^h \ge \nu_h q^l - c_h r^l - p^l, \quad \nu_h q^h - c_h r^h - p^h \ge 0, \\ & \nu_l q^l - c_l r^l - p^l \ge \nu_l q^h - c_l r^h - p^h, \quad \nu_l q^l - c_l r^l - p^l \ge 0, \\ & r^h \ge q^h(w+B), \quad r^l \ge q^l(w+B), \quad p^h\ge 0, \quad p^l \ge 0, \\ & q^h, q^l \in\{0,1\}. \end{aligned} \tag{5}$$

The next result establishes the connection between the aforementioned optimality equations and the original optimization problem (4), following the result in Lin et al. (2024).

[2] We can implement $q_t(\theta)=0$ in the direct mechanism by setting a price $\bar{p}$ to be sufficiently large in the menu so that the customer chooses not to join the queue.

Proposition 1. *1. There exists a unique value* g *and a non-increasing and concave function* V *that solve* (HJB)*, in which* Φ *is defined in* (5)*.*

2. Furthermore, $\mathsf{g} = g^*$*, in which* g^* *is the optimal long-run average revenue rate defined in* (4)*.*

3. Finally, denote $(\{q^\theta(w), p^\theta(w), r^\theta(w)\}_{\theta\in\Theta})$ *to be an optimal solution of* (5) *for any* $w \geq 0$*. Define* $\{q_t^*(\theta), p_t^*(\theta), r_t^*(\theta)\}_{\theta\in\Theta, t\geq 0}$*, such that* $q_t^*(\theta) := q^\theta(w_{t-})$*,* $p_t^*(\theta) := p^\theta(w_{t-})$ *and* $r_t^*(\theta) := r^\theta(w_{t-})$*. We have,* $g^* = \mathcal{G}\left(\{q_t^*(\theta), p_t^*(\theta), r_t^*(\theta)\}_{\theta\in\Theta, t\geq 0}\right)$*. That is,* $\{q_t^*(\theta), p_t^*(\theta), r_t^*(\theta)\}_{\theta\in\Theta, t\geq 0}$ *is the optimal control.*

Proposition 1 shows that the optimal revenue rate g^* in (4) can be obtained by solving the HJB equation, and the stationary policy that depends on the wait time is indeed optimal among all possible policies.

We now study the solution of (5) to understand the optimal pricing and strategic delay decisions. To facilitate the subsequent discussion, we define

$$\Delta(w) := V(w+B) - V(w), \tag{6}$$

which represents the externality to the system's value function of accepting a customer into the service. The monotonicity and concavity of function V implies that $\Delta(w)$ is non-positive and non-increasing in w. Therefore, we can interpret $-\Delta(w)$ as the (positive) loss of admitting a customer into the system when the completion time is w.

Recall r^* defined in Assumption (2) and the function u defined in (1). If the release time $r < r^*$, we have $\mathrm{u}(h,1,p,r) > \mathrm{u}(l,1,p,r)$ for any price p. That is, the impatient customer receives a higher utility when the release time r is shorter than r^*. On the flip side, if $r > r^*$, we have $\mathrm{u}(h,1,p,r) < \mathrm{u}(l,1,p,r)$. As it turns out, the optimal solution to problem $\Phi(w)$ is closely related to the critical release time level r^*. We define the following critical completion time,

$$w^* := r^* - B. \tag{7}$$

To present the first main results, we also introduce the following three functions of the completion time w,

$$u_l(w) := \nu_l - c_l(w+B), \tag{8}$$

$$u_h(w) := \nu_h - c_h(w+B), \text{ and} \tag{9}$$

$$u_2(w) := u_h(w) + \frac{\alpha_l}{\alpha_h}\left(u_h(w) - u_l(w)\right). \tag{10}$$

Here, $u_l(w)$ and $u_h(w)$ are service utilities for the type-l and type-h customers, respectively, when the completion time is w. The following technical lemma guarantees that various thresholds in the main theorem that follows are well-defined.

LEMMA 1. *Functions $u_l(w)$, $u_h(w)$ and $u_2(w)$ are decreasing in w. Furthermore, we have*

$$u_l(w^*) = u_h(w^*) = u_2(w^*) = \frac{c_h\nu_l - c_l\nu_h}{c_h - c_l}, \text{ and} \tag{11}$$

$$u_h(w) - u_l(w) = (c_h - c_l)(w^* - w), \tag{12}$$

which implies that

$$u_l(w) > u_2(w), \ \forall w > w^*. \tag{13}$$

Define the quantity $\bar{\nu} := u_l(w^*) = u_h(w^*)$, which is the identical service utility for both types when the completion time is w^*. The next theorem is the first main result of the paper.

THEOREM 1. *Suppose $\bar{\nu} + \Delta(w^*) \geq 0$. Define thresholds*

$$\bar{w}_{ps} := \min\{w \in [w^*, \infty) |\ u_2(w) + \Delta(w) \leq 0\}, \tag{14}$$

$$\bar{w}_{sr}^l := \min\{w \in [w^*, \infty) |\ u_l(w) + \Delta(w) \leq 0\}. \tag{15}$$

We have $\bar{w}_{ps} \leq \bar{w}_{sr}^l$, and the following cases for the optimal solution to $\Phi(w)$:

(1) for $w \in [0, w^*)$,

$$q^h = q^l = 1,\ p^l = \nu_l - c_l r^*,\ p^h = p^l + c_h(w^* - w),\ r^h = w + B,\ r^l = r^* > w + B, \tag{16}$$

that is, the firm should admit both types and delay the release time of the type-l customer;

(2) for $w \in [w^*, \bar{w}_{ps})$,

$$q^h = q^l = 1,\ p^h = p^l = \nu_h - c_h(w + B) = u_h(w),\ r^h = r^l = w + B, \tag{17}$$

that is, the firm should admit both types without delay;

(3) for $w \in [\bar{w}_{ps}, \bar{w}_{sr}^l]$,

$$q^l = 1,\ p^l = \nu_l - c_l(w + B) = u_l(w),\ r^l = w + B,\ q^h = p^h = r^h = 0, \tag{18}$$

that is, the firm should admit only a type-l customer;

(4) for $w > \bar{w}_{sr}$,

$$q^h = q^l = p^h = p^l = r^h = r^l = 0, \tag{19}$$

that is, the firm should reject both types.

Furthermore, we have $\mathrm{u}(h, q^h, p^h, r^h) = \mathrm{u}(l, q^l, p^l, r^l) = 0$ in general, except in Case (2), where $\mathrm{u}(l, q^l, p^l, r^l) = u_l(w) - u_h(w) = \nu_l - \nu_h + (c_h - c_l)(w + B) > 0$.

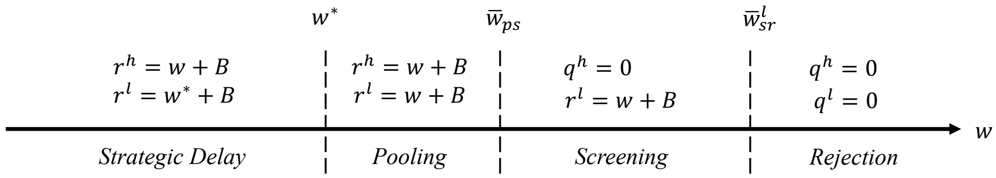


Figure 2 $\bar{\nu} + \Delta(w^*) \geq 0$.

In this example, $\lambda = 0.03$, $B = 5$, $\nu_h = 300$, $\nu_l = 200$, $c_h = 0.01$, $c_l = 0.004$, **and** $\alpha = 0.7$, **which implies that** $w^* = 1667$, $\bar{w}_{ps}^l = 2429$, **and** $\bar{w}_{sr} = 3988$.

The condition $\bar{\nu} + \Delta(w^*) \geq 0$ in Theorem 1 indicates that the value generated from serving a customer ($\bar{\nu}$) is higher than the loss of admitting a customer into the queue ($-\Delta(w^*)$) when the completion time is w^*. This condition implies that when the completion time $w > w^*$, the firm should still admit customers. Given that $\Delta(w)$ decreases in w, this condition further implies that the firm should admit customers when $w \leq w^*$. This interpretation is helpful when we explain each of the cases below.

Figure 2 illustrates different cases of the completion time and the corresponding optimal decisions described in the theorem. The threshold $\bar{w}_{ps}$ separates the "*p*ooling" region (Case (2)) with the "*s*creening" region (Case (3)). Similarly, the threshold $\bar{w}_{sr}^l$ separates the "*s*creening" region (Case (3)) with the "*r*ejection" region (Case (4)).

Case (1) indicates that when the completion time w is less than the threshold $\bar{w}_{ps} = w^*$, the firm should delay the release of a type-l request to $r^* = w^* + B$, instead of immediately releasing it after completion at $w + B$. The delay imposed on a type-l request reduces the incentive for a type-h customer to mimic type-l, because a type-h customer has a higher wait time sensitivity. The price and release time expressions in (16) imply that both types' utilities are zero. That is, by using the delay as a deterrence, the firm can charge a higher price to a type-h customer, effectively mitigating potential information rent from both types.

Strategic delay occurs only in Case (1) when the completion time w is short (less than w^*), and to type-l customers but never to type-h ones. Furthermore, the firm never strategically delays the release when only serving one type (type-l, as in Case (3)). This makes intuitive sense. First of all, the delay is used to screen between the two types of requests and, hence, should never occur when the firm only serves one type of customer. Second, when the firm serves both types of customers, the delay should be only used on type-l, which is less sensitive to long completion time. Third, a delay occurs when the completion time w is short. This is because if the current completion time is already longer than w^*, the impatient type's service utility is so low (lower than the patient type's, see Figure 1) that the firm has no room to extract premium by delaying the release of the patient type.

Despite the aforementioned intuitive features of the result, it is interesting, and may not be obvious *ex-ante*, that the release time is fixed at r^*, independent of the exact completion time w, as long as it is shorter than w^*. This simple "delay-up-to" structure makes the delay easy to implement in practice.

When the completion time w increases to the interval for Case (2), the waiting cost even without delay is high enough such that adding the delay would reduce a type-l customer's utility too much to squeeze out a good revenue. This is why both types should be released immediately ($r = w + b$). In this case, the prices offered to them need to be the same (pooling) to prevent one type from mimicking the other. Interestingly, the service provider can extract all surplus from type h while leaving some rent $u(l,1,p^l,r^l) > 0$ to type l. This is because $w > w^*$, or, equivalently, $r = w + B > r^*$, implying that the service utility of type-l customer is higher than that of the type-h customer, following Figure 1.

This intuition also explains why the only type being admitted is type-l customers for Case (3) in the screening region. We can also intuitively explain the expression for $\bar{w}_{ps}$, the boundary between Cases (2) and (3). The difference in the value-to-go between serving both types (see (17)) and only type l (see (18)) is

$$u_h(w) + \Delta(w) - \alpha_l\left[u_l(w) + \Delta(w)\right] = \alpha_h\left[u_2(w) + \Delta(w)\right],$$

which decreases in w following Lemma 1 and Proposition 1. The switching point from positive to negative is $\bar{w}_{ps}$ defined in (14).

As w further increases, the revenue from admitting a type-l customer shifts from positive to negative at $\bar{w}^l_{sr}$ defined in (15), leading to the rejection of the type-l customer when $w > \bar{w}^l_{sr}$.

In the proof of Theorem 1 (as well as Theorem 2 that comes next), we solve the linear relaxation of the mixed-integer linear program. The linear optimization solution satisfies the binary constraints for q^h and q^l, hence optimal to the original formulation.

We next consider the other case when $\bar{\nu} + \Delta(w^*) < 0$.

THEOREM 2. *Suppose $\bar{\nu} + \Delta(w^*) < 0$. Define thresholds*

$$\bar{w}_{ds} := \min\left\{w \in [0, w^*) \mid \bar{\nu} + \Delta(w) \le 0\right\}, \tag{20}$$

$$\bar{w}^h_{sr} := \min\left\{w \in [\bar{w}_{ds}, w^*) \mid u_h(w) + \Delta(w) \le 0\right\}. \tag{21}$$

(1) for $w \in [0, \bar{w}_{ds}]$, the optimal solution to $\Phi(w)$ follows (16);

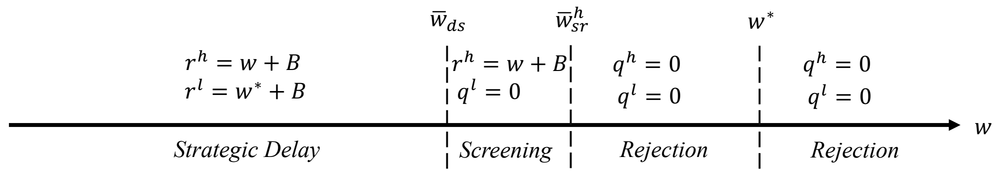


Figure 3 $\bar{\nu} + \Delta(w^*) < 0$.

In this example, $\lambda = 0.03$**,** $B = 15$**,** $\nu_h = 400$**,** $\nu_l = 100$**,** $c_h = 0.01$**,** $c_l = 0.002$**, and** $\alpha = 0.5$**, which implies that** $\bar{w}_{ds} = 2443$**,** $\bar{w}^h_{sr} = 2875$**, and** $w^* = 3750$**.**

(2) for $w \in (\bar{w}_{ds}, \bar{w}^h_{sr}]$*, the optimal solution to* $\Phi(w)$ *is*

$$q^h = 1,\ p^h = \nu_h - c_h(w+B),\ r^h = w + B,\ q^l = p^l = r^l = 0, \tag{22}$$

(3) for $w > \bar{w}_{sr}$*, the optimal solution to* $\Phi(w)$ *follows* (19)*.*

Furthermore, we have $\mathrm{u}(h, q^h, p^h, r^h) = \mathrm{u}(l, q^l, p^l, r^l) = 0$*.*

Finally, if $\bar{\nu} + \Delta(w^*) = 0$*, we have* $\bar{w}_{ps} = w_{ds} = \bar{w}^l_{sr} = \bar{w}^h_{sr} = w^*$*.*

Figure 3 depicts optimal release times described in Theorem 2. The threshold $\bar{w}_{ds}$ separates the "*d*elay" region with the "*s*creening" region, and the threshold $\bar{w}^h_{sr}$ separates the "*s*creening" region with the "*r*ejection" region.

Similar to Theorems 1, strategic delay only occurs when the completion time is shorter than a threshold $\bar{w}_{ds}$ in Theorems 2. However, unlike Theorem 1, when $\bar{\nu} + \Delta(w^*) < 0$ the threshold $\bar{w}_{ds}$ is no longer at w^*, but smaller. In fact, if $\bar{\nu} + \Delta(0) < 0$, the threshold $\bar{w}_{ds} = 0$ and strategic delay does not happen regardless of how short the completion time is.

Comparing Theorems 1 and 2, it is interesting to see that the admitted types in the "screening regions" differ between the two theorems. In fact, when $\bar{\nu} + \Delta(w^*) < 0$, the screening region occurs when $w < w^*$. This is because under the condition $\bar{\nu} + \Delta(w^*) < 0$, the firm should not admit any type of customer when the completion time w is at or above w^*. When $w < w^*$, the impatient customer (type h) derives higher service utility than the patient type ($\nu_l - c_l(w+B) < \nu_h - c_h(w+B)$).

It is also interesting that the pooling region disappears in Theorem 2. This is because, as we demonstrate in the proofs, Assumption 1(4) implies that when $w < w^*$, delaying type l always outperforms pooling both types, regardless of $\bar{\nu} + \Delta(w^*)$ being positive or negative. Without a pooling region, the firm extracts the entire surplus from both types, as indicated in Theorem 2.

Finally, if $\bar{\nu} + \Delta(w^*) = 0$, both the pooling and screening regions disappear from both figures. In this case, the delay region is connected with the rejection region at w^*.

4. Extensions

We present two extensions of the base model. In Section 4.1, we extend the two-type model into a continuous-type setting. In Section 4.2, we allow the firm to decide on the service time, in addition to admission, pricing, and release time decisions.

4.1. Continuous-Type Customers

In this model, customers still arrive according to a Poisson process with rate λ, each taking a constant time B to complete. To generalize the two-type model while maintaining its tractability, we keep the space of types in a single dimension. In particular, we use θ to represent a customer's type, corresponding to the release-time sensitivity. Each customer has its valuation ν, which has a one-to-one correspondence with θ. Following Afeche and Pavlin (2016), we assume the relationship is linear. That is, there exist values $\bar{\nu} > 0$ and $r^* > 0$ (with a slight abuse of notations) such that a customer with sensitivity θ values the immediate service at

$$\nu(\theta) := \bar{\nu} + r^* \cdot \theta.$$

Assume that each customer's type θ is drawn from a support $\boldsymbol{\Theta} := [c_l, c_h]$ (with a slight abuse of notation once again), and follows a common-prior distribution with a density function $f(\cdot)$ and a cumulative function $F(\cdot)$. The two-type model discussed in the previous section can be perceived as a special case of this model, with the a type-type distribution on θ and

$$\bar{\nu} = \frac{c_h \nu_l - c_l \nu_h}{c_h - c_l}, \text{ and } r^* = \frac{\nu_h - \nu_l}{c_h - c_l}, \tag{23}$$

so the definitions of $\bar{\nu}$ and r^* is consistent with (11) and Assumption 1, respectively.

The utility of a type θ customer subject to the admission decision q, the pricing decision p, and the release time decision r still follows (1), or,

$$\mathrm{u}\big(\theta, q, p, r\big) = \nu(\theta) q - \theta r - p = \bar{\nu} q + \theta\big(r^* q - r\big) - p. \tag{24}$$

We can use the same expressions for the (IC) and (IR) constraints from the two-type case.

Similar to Proposition 1 for the base two-type model, we have

PROPOSITION 2. *There exists a unique value* g *and non-increasing and concave function* V *that solves*

$$\mathsf{g} + V'(w) = \lambda \Upsilon(w) \ \forall w > 0, \text{ with boundary condition } V'(0) = 0, \text{ and } V(0) = 0, \tag{HJBc}$$

in which function Υ *is defined as*

$$\Upsilon(w) := \max_{\{q(\theta),p(\theta),r(\theta)\}_{\theta\in\Theta}\in\Pi} \int_{\theta\in\Theta} [p(\theta) + q(\theta)\Delta(w)] f(\theta)\mathrm{d}\theta. \tag{25}$$

Here, the feasible set Π *is defined by* (FE) *and the following constraints:*

$$\bar{\nu}q(\theta) + \theta\left[r^* q(\theta) - r(\theta)\right] - p(\theta) \geq \bar{\nu}q(\theta') + \theta\left[r^* q(\theta') - r(\theta')\right] - p(\theta'),\ \forall \theta, \theta' \in \Theta, \tag{26}$$

$$\bar{\nu}q(\theta) + \theta\left[r^* q(\theta) - r(\theta)\right] - p(\theta) \geq 0,\ \forall \theta \in \Theta. \tag{27}$$

Furthermore, $\mathsf{g} = g^*$*, in which* g^* *is the optimal long-run average revenue rate defined in* (4)*. Finally, one can obtain the optimal control policy from the optimal solution of* (25) *in the sense of Proposition 1.*

To analyze (25), we proceed with several steps to simplify the expression. First, we follow Myerson (1981) to simplify the (IC) constraint, such that the payment $p(\theta)$ is expressed with the other decision variables. Next, we further reduce the optimization problem (25) by introducing two thresholds $\check{c}$ and $\hat{c}$ as decision variables. The aforementioned simplification, together with properties of the value function $V(w)$, allows us to obtain the structure of the optimal control policy.

The next lemma follows standard techniques of Myerson (1981).

Lemma 2. *Define*

$$U(\theta) := \mathrm{u}(\theta, q(\theta), p(\theta), r(\theta)).$$

For any $q(\theta)$*,* $p(\theta)$*, and* $r(\theta)$ *that satisfy* (IC)*,* $U(\theta)$ *is a convex function of* θ*,*

$$U'(\theta) = r^* q(\theta) - r(\theta), \text{ and} \tag{28}$$

$$r^* q(\theta) - r(\theta) \text{ is non-decreasing in } \theta. \tag{29}$$

Conversely, for any $q(\theta)$ *and* $r(\theta)$ *that satisfy* (29)*, there exist a value* $\bar{u}$ *and a particular* $\hat{\theta} \in [c_l, c_h]$*, such that*

$$p(\theta) := -\bar{u} + \bar{\nu}q(\theta) + \theta[r^* q(\theta) - r(\theta)] - \int_{\hat{\theta}}^{\theta} [r^* q(x) - r(x)]\mathrm{d}x, \tag{30}$$

where $q(\theta)$*,* $p(\theta)$*, and* $r(\theta)$ *satisfy* (IC)*, and* $U(\hat{\theta}) = \bar{u}$*.*

With Lemma 2, we can replace the constraint (IC) in (25) with (29), and replace $p(\theta)$ with (30) in the objective function of (25). It is worth noting a key difference between the standard mechanism design model of Myerson (1981) and our setting. Even though $q(\theta)$ and $r(\theta)$ are non-negative, the derivative of the function $U(\theta)$, which is $r^* q(\theta) - r(\theta)$, could be either positive or negative. Thus, unlike in standard mechanism design, our U function may not be monotone. Generally speaking, the convex function $U(\theta)$ could be at its minimum (or, $r^*q(\theta)-r(\theta)=0$) for an interval of θ values within the support $[c_l, c_h]$. Consider $\check{c}$ and $\hat{c}$ to be the two ends of the interval, such that

$$c_l \le \check{c} \le \hat{c} \le c_h, \tag{31}$$

$$r^*q(\theta)-r(\theta) < 0,\ \forall \theta \in [c_l, \check{c}), \tag{32}$$

$$r^*q(\theta)-r(\theta) = 0,\ \forall \theta \in [\check{c}, \hat{c}], \tag{33}$$

$$r^*q(\theta)-r(\theta) > 0,\ \forall \theta \in (\hat{c}, c_h]. \tag{34}$$

The interval $(\check{c}, \hat{c})$ in general depends on the system completion-time state w. The following result converts the original optimization problem (25) into one without decision variables $p(\theta)$ but with $\check{c}$ and $\hat{c}$ as decision variables. Note that, following (30), there needs to be another decision variable $\bar{u}$. The next result implies that the optimal $\bar{u}$ is in fact 0.

Lemma 3. *The optimization problem in* (25) *is equivalent to the following formulation,*

$$\begin{aligned}\Upsilon(w) = \max_{q(\theta), r(\theta), \check{c}, \hat{c} \in \Pi'} & \int_{c_l}^{\check{c}} \left\{ q(\theta)\left[\bar{\nu}+\Delta(w)\right)\right] + \left[r^*q(\theta)-r(\theta)\right]\left[\theta + \frac{F(\theta)}{f(\theta)}\right]\right\} f(\theta)\mathrm{d}\theta \\ & + \int_{\check{c}}^{\hat{c}} q(\theta)\left[\bar{\nu}+\Delta(w)\right)\right] f(\theta)\mathrm{d}\theta \\ & + \int_{\hat{c}}^{c_h} \left\{ q(\theta)\left[\bar{\nu}+\Delta(w)\right)\right] + \left[r^*q(\theta)-r(\theta)\right]\left[\theta - \frac{1-F(\theta)}{f(\theta)}\right]\right\} f(\theta)\mathrm{d}\theta,\end{aligned} \tag{35}$$

in which the feasible region Π' *is defined by constraints* (FE), (27), (29), (31), (32), (33), *and* (34).

The approach of finding an optimal solution to (35) is to first ignore the monotonicity constraint (29), so that the maximization problem becomes separable in θ, following Myerson (1981). To guarantee the optimal solution of this relaxed problem satisfies (29), we make the following assumption, which is the same as the one in Afeche and Pavlin (2016).[3]

Assumption 2. *The distribution of* θ *has the following properties:*
(1) $\theta - \dfrac{1-F(\theta)}{f(\theta)}$ *is increasing in* θ*;*
(2) $\theta + \dfrac{F(\theta)}{f(\theta)}$ *is increasing in* θ*.*

[3] As stated in Afeche and Pavlin (2016), "[Assumption 2] holds for many common probability distributions, including those with log-concave density function (see Bagnoli and Bergstrom 2005). Examples include the uniform, normal, logistic, Laplace, and power function distributions, and the gamma and Weibull distributions with shape parameter ≥ 1."

Assumption 2(1) allows us to define

$$\bar{c} := \min\left\{\theta \in \mathbf{\Theta} \;\middle|\; \theta - \frac{1-F(\theta)}{f(\theta)} \geq 0\right\}. \tag{36}$$

If no θ satisfies the inequality condition in (36), we follow the convention and let $\bar{c}$ be positive infinity. Recall that $w^* := r^* - B$. Similar to the results in the basic binary type model, the optimal policy structure depends on whether the completion time w is higher or lower than w^*.

REMARK 1. If the completion time $w < w^*$, then a release time $r = w + B$ yields a service utility $\nu(\theta) - \theta r(\theta)$ that is increasing in θ. The more patient a customer is, the less service utility there is. This generalizes the two-type case in which the type-h customer has a higher service utility when $w < w^*$. On the other hand, if $w > w^*$, the service utility $\nu(\theta) - \theta r(\theta)$ is decreasing in θ, which is also consistent with the type-l customer having a higher service utility in the previous section.

Now we present the main result for the continuous-type model.

THEOREM 3. *(1) Suppose $\bar{\nu} + \Delta(w^*) \geq 0$.*

i. When $w \leq w^*$, the firm serves all types of customers and delays the release time to $w^* + B = r^*$ when the customer types $\theta < \bar{c}$. That is, the optimal solution to (25) is

$$q^*(\theta) = 1, \forall \theta \in \Theta,\; p^*(\theta) = \begin{cases} \bar{\nu}, & \theta < \bar{c} \\ \bar{\nu} + \bar{c}(w^* - w), & \theta \geq \bar{c} \end{cases},\; \text{and } r^*(\theta) = \begin{cases} r^*, & \theta < \bar{c} \\ w + B, & \theta \geq \bar{c} \end{cases}. \tag{37}$$

ii. When $w > w^*$, the firm only serves a customer type θ satisfying $\theta \leq \check{c}(w)$ with $r(\theta) = w + B$, in which

$$\check{c}(w) := \max\left\{\max\left\{\theta \in \mathbf{\Theta} \;\middle|\; \bar{\nu} + \Delta(w) + [r^* - (w+B)]\left[\theta + \frac{F(\theta)}{f(\theta)}\right] \geq 0\right\},\; c_l\right\}, \tag{38}$$

is non-increasing in w. That is, the optimal solution to (25) is

- *for $\theta < \check{c}(w)$, $q^*(\theta) = 1$, $r^*(\theta) = w + B$ and $p^*(\theta) = \bar{\nu} + \check{c}(w)(w^* - w)$;*
- *for $\theta \geq \check{c}(w)$, on the other hand, $q^*(\theta) = r^*(\theta) = p^*(\theta) = 0$.*

(2) If $\bar{\nu} + \Delta(w^*) < 0$, on the other hand, define a completion time threshold w_D, which satisfies $w_D < w^*$, as

$$w_D := \max\left\{\max\{w \geq 0 \mid \bar{\nu} + \Delta(w) \geq 0\},\; 0\right\}. \tag{39}$$

i. When $w \leq w_D$, the optimal solution to (25) follows (37).

ii. When $w > w_D$, on the other hand, the firm only serves a customer if the completion time w and type c satisfy $c \geq \hat{c}(w)$ with no delay, in which $\hat{c}(w)$ is defined as

$$\hat{c}(w) := \min\left\{\min\left\{\theta \in \mathbf{\Theta} \;\middle|\; \bar{\nu} + \Delta(w) + [r^* - (w+B)]\left[\theta - \frac{1-F(\theta)}{f(\theta)}\right]^+ \geq 0\right\},\; c_h\right\}, \tag{40}$$

which is non-decreasing in w. That is, the optimal solution to (25) is

∗ *for* $\theta > \hat{c}(w)$*,* $q^*(\theta) = 1$*,* $r^*(\theta) = w + B$ *and* $p^*(\theta) = \bar{\nu} + \hat{c}(w)(w^* - w)$*;*

∗ *for* $\theta \le \hat{c}(w)$*, on the other hand,* $q^*(\theta) = r^*(\theta) = p^*(\theta) = 0$*.*

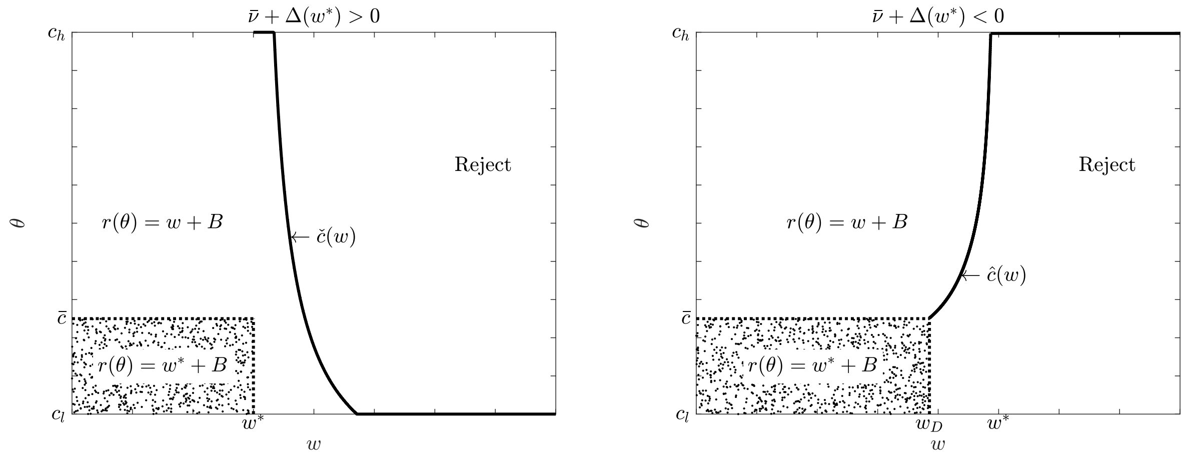
The proof of the theorem is presented in Appendix B. The idea is to first establish the optimal solution of (35), and then use (30) to obtain the optimal payment p^* from the optimal admission decisions q^* and r^*. In the process of obtaining the optimal solution of (35), we solve a linear relaxation of the binary constraint (FE). Assumption (2) guarantees that optimal q^* is indeed binary.

Although the optimization problem (35) has two decision threshold variables $\check{c}$ and $\hat{c}$, Theorem 3 demonstrates that for any given w, the optimal solution only involves one of them. The optimal $\check{c}$ in (35) corresponds to $\check{c}(w)$ defined in (38) for Case (1) in the theorem. In this case, the optimal $\hat{c} = c_h$. Similarly, the optimal $\hat{c}$ in Case (2) corresponds to $\hat{c}(w)$ from (40). The optimal $\check{c}$ in this case is c_l.

Here we explain Theorem 3 and connect it with the intuition obtained from Theorems 1 and 2 for the binary-type model. First consider $\nu + \Delta(w^*) \ge 0$ and $w \le w^*$. In this case, it is optimal to admit all customer types ($q(\theta) = 1$ for all $\theta \in \Theta$), but strategic delay the release type to $r^*(\theta) = r^*$ for customer whose delay sensitivity is below $\bar{c}$. More delay-sensitive customers ($\theta > \bar{c}$) are guaranteed immediate release ($r^*(\theta) = w + B$) but need to pay more ($p^*(\theta) = \bar{\nu} + \bar{c}(w^* - w) > \bar{\nu}$). This is similar to the result from Theorem 1, in which the type l corresponds to $\theta \le \bar{c}$ here. If the system is more congested ($w > w^*$), the firm only serves customer types that are less sensitive to delay ($\theta \le \check{c}(w)$) without delaying the release time ($r^*(\theta) = w + B$). These types again correspond to type l in Theorem 1.

Now consider the second case, with $\bar{\nu} + \Delta(w) < 0$. In this case, the firm should admit all customer types only if $w \le w_D < w^*$. Once again, a strategic delay occurs to less delay-sensitive types, similar to Theorem 2, if we interpret type l there as $\theta \le \bar{c}$ here. If the completion time w is above the threshold w_D, then the firm should only admit customers whose delay-sensitivity is higher than $\hat{c}(w)$, once again similar to Theorem 2.

It is instructive to illustrate the results in Theorem 3 in Figures 4a and 4b. The two figures represent two cases with $\bar{\nu} + \Delta(w^*) > 0$ and $\bar{\nu} + \Delta(w^*) < 0$, respectively. As we can see, the strategic delay region is marked by the shaded rectangle in the lower-left corner of both figures. The intuition is the same as for the two-type cases: the firm only strategically delays the release time of patient customers (low θ) when the completion time w is low. This is similar to the difference between Figures 2 and 3. In particular, $\bar{\nu}$ corresponds to $\nu_h - c_h r^* = \nu_l - c_l r^*$ in the two type case, according to (23).



(a) $\bar{\nu}=500$, $d=300$, and $w^*=260$

(b) $\bar{\nu}=300$, $d=500$, and $w^*=460$

Figure 4 **In both examples,** $\lambda=0.01$**,** $B=40$**,** $s(w)=w$ **and** $c\sim$ **Uni**$[c_l=1,c_h=3]$**, which implies that** $\bar{c}=1.5$**.**

An interesting comparison between the two figures is that the separator function $\check{c}(w)$ of Figure 4a is decreasing in w, while $\hat{c}(w)$ of Figure 4b is increasing. In fact, Figure 4a confirms Theorem 3 that function $\check{c}(w)$ is strictly decreasing when $w>w^*$. In comparison, $\hat{c}(w)$ in Figure 4a is strictly increasing when $w<w^*$. Following Remark 1, when the completion time is long ($w>w^*$), more patient customers derive higher net utilities. This explains why less patient customers (higher θ) are rejected for a given w in Figure 4a. Similarly, when $w<w^*$, more patient customers derive less net utilities than impatient customers. That is why more patient (lower θ) customers are rejected for a given w in Figure 4b.

4.2. Endogenous Service Times

In this subsection, we extend the base two-type model to allow the firm to decide the service time for each type of customer. We shall demonstrate that the delay strategy for the type-l customer persists in the optimal policy.

The model setup remains unchanged unless otherwise specified. For each type $\theta\in\mathbf{\Theta}=\{h,l\}$ customer arriving at time t, the firm decides the processing time $b_t(\theta)\in[0,\infty)$, in addition to decisions $q_t(\theta)$, $p_t(\theta)$ and $r_t(\theta)$ described in Section 3. The corresponding processing cost $k\left(b_t(\theta)\right)$ is convex and decreasing in the processing time $b_t(\theta)$ because it is costly for the firm to shorten the service time. The base case model is a special case of this one, with

$$k(b)=\begin{cases}\infty, & b<B,\\ 0, & b\geq B.\end{cases}$$

The corresponding optimal profit rate is defined as

$$g^*:=\sup_{\{q_t(\theta),p_t(\theta),r_t(\theta),b_t(\theta)\}_{\theta\in\{h,l\},t\geq 0}\in\Pi_b}\liminf_{T\to\infty}\frac{1}{T}\mathbb{E}\left[\int_0^T\left(p_t(\theta_t)-k(b_t(\theta_t))\right)\mathrm{d}N_t\right],\tag{41}$$

in which the set of admissible policies Π_b is defined by constraints (IC), (IR), and (FE), except we revise $r_t(\theta) \geq q_t(\theta)(w_t + B)$ to $r_t(\theta) \geq q_t(\theta)(w_t + b(\theta))$. The corresponding completion time dynamics changes from (3) to

$$\mathrm{d}w_t = -\mathbb{1}_{w_{t-}>0}\mathrm{d}t + b_t(\theta_t) \cdot q_t(\theta_t)\mathrm{d}N_t. \tag{42}$$

We have the following result on the optimality conditions, similar to Propositions 1 and 2.

Proposition 3. *There exists a unique value* g *and a non-increasing and concave function* V *that solve*

$$\mathsf{g} + V'(w) = \lambda\Psi(w)\ \forall w > 0,\ \textit{with boundary condition } V'(0) = 0,\ \textit{and } V(0) = 0, \tag{HJBb}$$

in which function Ψ *is defined as*

$$\begin{aligned}\Psi(w) := \max_{q^\theta, r^\theta, p^\theta, b^\theta} \ & \sum_{\theta \in \boldsymbol{\Theta}} \alpha_\theta \left\{p^\theta + q^\theta\left[V(w + b^\theta) - V(w) - k(b^\theta)\right]\right\}, \\ \textit{s.t.}\ & \nu_h q^h - c_h r^h - p^h \geq \nu_h q^l - c_h r^l - p^l, \quad \nu_h q^h - c_h r^h - p^h \geq 0, \\ & \nu_l q^l - c_l r^l - p^l \geq \nu_l q^h - c_l r^h - p^h, \quad \nu_l q^l - c_l r^l - p^l \geq 0, \\ & r^h \geq q^h(w + b^h), \quad r^l \geq q^l(w + b^l), \quad p^h \geq 0, \quad p^l \geq 0, \\ & b^h, b^l \geq 0,\ q^h, q^l \in \{0, 1\}.\end{aligned} \tag{43}$$

Furthermore, $\mathsf{g} = g^*$*, in which* g^* *is the optimal long-run average profit rate defined in* (41)*. Finally, the optimal decisions from* (43) *yields the optimal control policy in the sense of Proposition 1.*

The maximization problem defined in (43) is a mixed-integer nonlinear optimization problem with eight decision variables. The following result shows that it can be decomposed into a collection of single-variable concave maximization problems. This decomposition makes the problem much easier to solve than the original one.

Proposition 4. *For* $w \leq r^*$*, we have*

$$\Psi(w) = \max\{Q_1(w), Q_2(w), Q_3(w), Q_4(w), 0\}, \tag{44}$$

in which

$$\begin{aligned}Q_1(w) := & \nu_l + (\alpha_h c_h)(r^* - w) - c_l r^* - V(w) + \alpha_h \max_{b^h \geq 0} V(w + b^h) - k(b^h) - c_h b^h \\ & + \alpha_l \max_{b^l \in [0, r^* - w]} V(w + b^l) - k(b^l)\end{aligned} \tag{45}$$

$$\begin{aligned} Q_2(w) :=& \nu_h - c_h w - V(w) + \max_{b^h \geq 0} \ (\alpha_l c_l - c_h) b^h + \alpha_h \left(V(w+b^h) - k(b^h) \right)) \\ &+ \alpha_l \max_{b^l \geq r^* - w} V(w+b^l) - k(b^l) - c_l b^l, \end{aligned} \tag{46}$$

$$Q_3(w) := \alpha_h(\nu_h - c_h w - V(w)) + \alpha_h \max_{b^h \in [0, r^* - w]} V(w+b^h) - k(b^h) - c_h b^h, \tag{47}$$

$$Q_4(w) := \alpha_l \left(\nu_l - c_l w - V(w) \right) + \alpha_l \max_{b^l \geq r^* - w} V(w+b^l) - k(b^l) - c_l b^l. \tag{48}$$

For $w > r^*$, on the other hand, we have

$$\Psi(w) = \max\{Q_2'(w), Q_4'(w), 0\}, \tag{49}$$

in which

$$\begin{aligned} Q_2'(w) :=& \nu_h - c_h w - V(w) + \max_{b^h \geq 0} \ (\alpha_l c_l - c_h) b^h + \alpha_h \left(V(w+b^h) - k(b^h) \right)) \\ &+ \alpha_l \max_{b^l \geq 0} V(w+b^l) - k(b^l) - c_l b^l, \end{aligned} \tag{50}$$

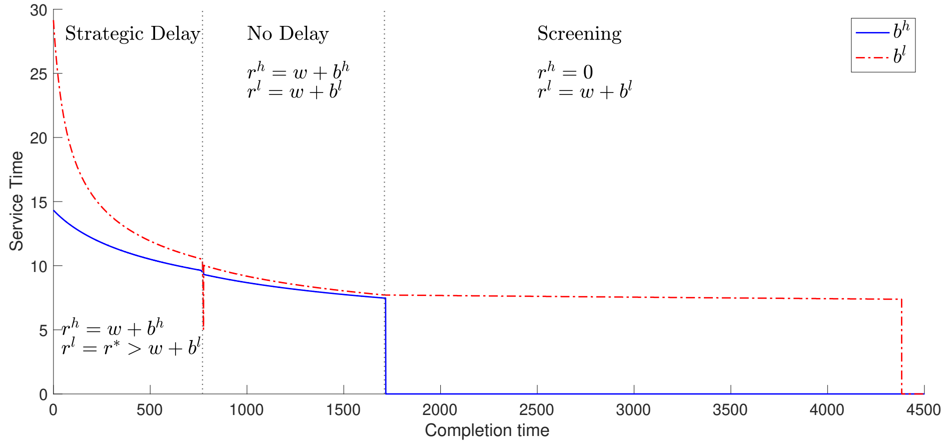
$$Q_4'(w) := \alpha_l(\nu_l - c_l w - V(w)) + \alpha_l \max_{b^l \geq 0} V(w+b^l) - k(b^l) - c_l b^l. \tag{51}$$

Furthermore, the optimal release time r^θ in (43) is $w + b^\theta$ for all cases except when $\Psi(w) = Q_1(w)$, in which case the optimal $r^l = r^*$.

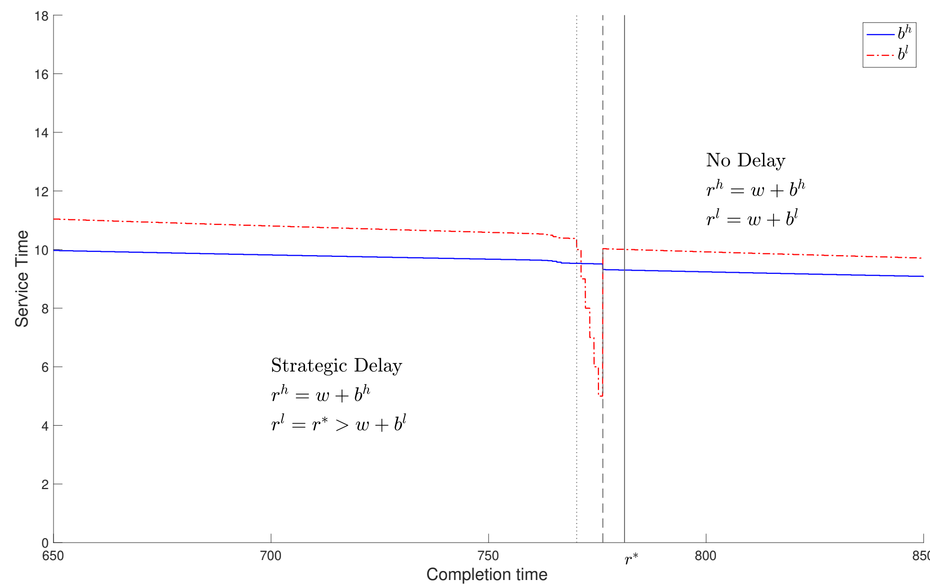
The derivation of Proposition 4 is to consider different strategies established by setting $q^\theta \in \{0, 1\}$ for $\theta \in \{l, h\}$ under different completion time states. When $w < r^*$, the firm needs to consider five strategies: admitting both types with delay, admitting both types without delay, admitting type-h only, admitting type-l, and rejecting both types, represented in (44) by $Q_1(w)$, $Q_2(w)$, $Q_3(w)$, $Q_4(w)$ and 0, respectively. Specifically, given $w < r^*$, the strategy of only serving type-l in $Q_4(w)$ is feasible when the release time $r^l = w + b^l$ is greater than r^* where type-l has a higher service utility. On the other hand, when $w > r^*$, the feasible strategies are admitting both types, admitting type-l only and rejecting all customers, represented by $Q_2^{'}(w)$, $Q_4^{'}(w)$, and 0, respectively, in (49). (There are no corresponding $Q_1^{'}(w)$ and $Q_3^{'}(3)$ when $w > r^*$ because there is no feasible service time.) Depending on the completion time, the firm can identify the optimal decision by solving these six optimization problems involving only single-dimensional concave maximization.

Unlike the two-type model in Section 3, there is no condition to ensure that the optimal decision involves strategic delay in $Q_1(w)$. Thus, strategic delay may or may not happen. Nevertheless, we find that the delay strategy does occur in our numerical study.

Figure 5 demonstrates one example in which the optimal control involves strategic delay. As shown in Figure 5(a), when w increases, the optimal strategy involves first delaying the type-l customer ($Q_1(w)$), followed by no delay while serving both types ($Q_2(w)$), screening the type-h



(a) Illustration of the release time policy.



(b) Close-up view of the completion time region between 650 and 850.

Figure 5 **In this example,** $\lambda = 0.005$, $\nu_h = 1000$, $\nu_l = 750$, $c_h = 0.48$, $c_l = 0.16$, $\alpha_h = 0.7, \alpha_l = 0.3$, $r^* = 781.25$, **and** $k(b) = \frac{600}{b^2}$.

customer $(Q_4^{'}(w))$, and finally rejecting all customers when w becomes too large. The vertical axis shows the corresponding optimal service times. Note that the optimal service times values b^h and b^l are different in the case of $\Psi(w) = Q_2(w)$. Therefore, we do not call this region "pooling." The two dotted vertical lines separate the different regions.

The left vertical dotted line in Figure 5(a) represents the completion time at $w = 770.25$ where the delay strategy ends (and no-delay starts). This threshold is close to r^*, and the service time above this threshold sharply decreases. In order to better understand this region, Figure 5(b) zooms in around this completion time interval. The separation between the delay and no-delay regions is still marked by the dotted vertical line at $w = 770.25$. It is worth mentioning that $\Psi(w) = Q_1(w)$ when the completion time w is to the left of the (new) dashed line at $w = 776.26$, which is to the right of the dotted line. Whenever Q_1 dominates ($\Psi(w) = Q_1(w)$), the optimal release time for the type-l customer is always set to $r^* = 781.25$. In the delay region (left of the dotted line), the constraint $b^l \leq r^* - w$ is not binding. This constraint becomes binding in between of the dotted and the dashed lines. In this region, the optimal solution no longer involves strategic delay, and b_l keeps decreasing with increasing w because $b_l = r^* - w$. When w moves to the right of the vertical dashed line, problem $Q_1(w)$ yields its dominance to $Q_2(w)$ ($\Psi(w) = Q_2(w)$). In this case, the optimal release time r_l is set to $w + b^l$, staying in the no-delay region. Finally, r^* is further to the right of the vertical line, as marked in the figure.

5. Numerical Study

We examine the benefits of implementing strategic delay by comparing our optimal control with the best control without allowing delay.

5.1. Two-type Model

This section assesses the effectiveness of implementing strategic delay in the base two-type model. We compare performances of the optimal policies presented in Theorems 1 and 2 with the performance of the optimal policy with no-delay, which offers all customers the shortest release time. This corresponds to solving (HJB) while replacing the constraints $r^h \geq q^h(w+B)$ and $r^l \geq q^l(w+B)$ with

$$r^h = q^h(w+B) \text{ and } r^l = q^l(w+B), \text{ respectively.} \tag{52}$$

Denote g^{n} to represent the corresponding long-run average revenue rate without strategic delay. We present the relative improvement $\dfrac{\mathsf{g}^* - \mathsf{g}^n}{\mathsf{g}^n} \times 100\%$ of allowing strategic delay throughout the numerical studies in this section.

We set the following parameter values for the customer's utility: $\nu_h = 200$ and $\nu_l = 100$ for the immediate service valuations and $c_h = 0.03$ and $c_l = 0.01$ for the release-time sensitivities. The deterministic service time is set as $B = 20$. We conduct the numerical study using a test bed of 969 instances, varying the ratio α_h and the arrival rate λ. Specifically, we consider a range of α_h from 0.4 to 0.9 with an increment of 0.01, and a range of λ from 0.01 to 0.1 with an increment of 0.005.

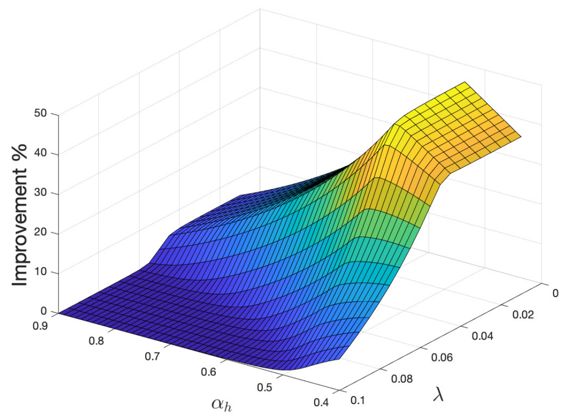


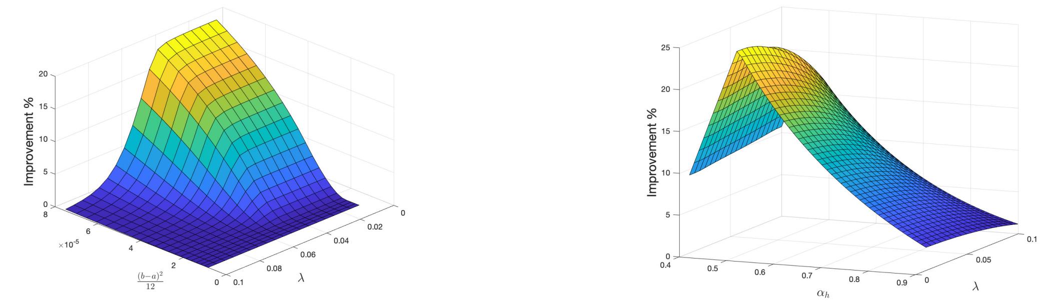
Figure 6 Revenue Comparison When Varying α_h and λ

All of these instances satisfy $\alpha_h c_h > c_l$, and strategic delay will occur. Across all instances, the long-run revenue rate generated from the optimal policy is 7.16, while that of the no-delay policy is 6.45. This represents an 11% increase in the long-run average revenue achieved by implementing the delay strategy, with the most significant improvement being 49.73%.

Figure 6 also illustrates that the benefit of the delay strategy over the no-delay strategy decreases in the arrival rate λ. The intuition is that as the arrival rate λ increases, the firm becomes more congested and w often exceeds w^*. In this case, it is less likely for the firm to exercise the delay strategy frequently, making both policies behave similarly.

5.2. Continuous-Type Model

This section presents a numerical study to assess the benefit of implementing strategic delay in the continuous-type model. Similar to the previous subsection, we still use the no-delay (always release immediately) heuristic policy as the benchmark policy for comparison with the optimal policy. In this study, we set the customer's utility parameter as $\bar{\nu} = 50$, $r^* = 5000$, which are consistent with the corresponding values in in Section 5.1. Also consistent with the previous section, we set the deterministic service time as $B = 20$. The completion time sensitivity θ follows a uniform distribution $\mathcal{U}(a,b)$ with a mean of $(a+b)/2 = (c_h + c_l)/2 = 0.02$. We vary the width $b-a$ of the support of the uniform distribution, considering values from $0.01, 0.011, 0.012, \ldots, 0.03$ where the variance of the distribution is $\frac{(b-a)^2}{12}$. The variation in the support of uniform distribution allows us to capture the sparsity of customers' patience levels. We conduct a test bed consisting of 399 instances, where the support of the distribution ranges from $[0.015, 0.025]$ to $[0.005, 0.035]$ with an increment and decrement of 0.005 for the endpoints. The arrival rate λ ranges from 0.01 to 0.1 with an increment of 0.005.



(a) Varying Distribution Variance and λ (b) Varying α_h and λ

Figure 7 Revenue comparison between the optimal policy and no-delay policy: (a) The continuous-type model and (b) the endogenous service time model.

On average, the long-run revenue rate generated by the optimal policy is 4.98, while that of the no-delay policy is 4.85. The maximum improvement in revenue percentage is 19.76%. The impact of varying arrival rates on the improvement percentage is consistent with the findings observed in the two-type customer model. Specifically, revenue improvement is most significant when the arrival rate is relatively low, indicating that strategic delay is more beneficial to the firm when the system is less congested.

Figure 7a illustrates that revenue improvement is more pronounced when θ is more variable. This is because when θ is more variable, the firm faces more extreme types of customers, and it is more likely the firm can exercise the delay strategy to yield higher revenue. This effect is more significant when the arrival rate is low, because the system more likely to be in a low completion time state when it is optimal to strategically delay the release.

5.3. Endogenous Service Time Model

We examine the benefit of implementing strategic delay when the firm can decide on the service time. Similar to the previous subsections, we compare the optimal cost with the no-delay policy, which sets the release time as the completion time plus the determined service time. For this study, we let the customer's utility parameters be $\nu_h = 200$, $\nu_l = 100$, $c_h = 0.03$, $c_l = 0.01$, and the service cost function is $k(b) = 800/b^2$ for $b \geq 0$. We consider a range of α_h from 0.4 to 0.9 with an increment of 0.01, and a range of λ from 0.01 to 0.1 with an increment of 0.01, resulting in a test bed of 510 instances. Across all instances, the long-run revenue rate generated from the optimal policy is 7.72, while that of the no-delay policy is 7.05. This represents a 9.5% increase in the long-run average profit achieved by implementing strategic delay, with the most significant improvement being 24.50%.

Figure 7b illustrates that the profit increase is substantial when the arrival rate is low, and the proportion α_h is around 0.5. The findings of varying λ illustrated in Figure 7b echo those in Figures 6. That is, the benefit of implementing strategic delay is not monotonic and significant only when α_h is moderately high. It is worth noting that improvements in Figure 7b from high α_h is higher than those in Figure 6. This increment comes from the flexibility of service times.

The effect of varying λ illustrated in Figure 7b differs from what we observe in Figure 6. When the service times are decision variables, the benefit of implementing strategic delay is less sensitive to the arrival rate, compared to the base two-type case. This is because under both policies, the firm can utilize the flexibility of service times to manage different arrival rates, which reduces the impact of the difference in arrival rates. Furthermore, when the arrival rate is high, which means the strategic delay rarely occurs as discussed in Figure 6, the firm can use flexible service times to reduce congestion, which boosts the benefit of strategic delay. This is why strategic delay performs relatively well even when the arrival rate is high.

6. Concluding Remarks

This paper studies a dynamic price/release-time mechanism for a firm facing customers who do not know the completion time of the requested service. The firm offers a menu of price and release-time options for each incoming customer, whose immediate service valuations and sensitivity to release time are their private information. Assuming a deterministic service time, we depart from the conventional dynamic queueing control models in two aspects. First, instead of using queue length, we use the completion time as the state variable in our continuous-time optimal control model. Second, we allow the firm to strategically delay the release time as an operational lever to screen customers.

Our analysis indicates that the firm should strategically delay the release time for patient customers when the completion time falls below a certain threshold. Furthermore, the delayed release time is a constant regardless of the completion time state, as long as it is below the threshold.

There are several potential extensions of our model for future research. First, our model assumes a first-come-first-serve (FCFS) system. It is natural to extend the control policy to allow scheduling/sequencing of tasks. Such an extension, however, appears to be quite complex and requires additional assumptions. For example, whether or not to allow preemption leads to different models and analyses. Moreover, altering the customer sequence would lead to a more complex state space representation, rendering the model potentially intractable. We believe that the control strategy studied in this paper, although not considering more complex sequencing decisions, is easy to compute and implement, and hence provides practical value. Second, we assume that the utility of an outside option is utility 0. In reality, outside options may be heterogeneous and customers' private information. Although incorporating additional private information is valuable, mechanism design with higher dimensional private information, even in static settings, is quite challenging and often intractable. Good approximation algorithms may be necessary to tackle these problems. Third, while we focus on a single-server environment, extending the analysis to multi-server systems offers an intriguing direction for future investigation. Finally, it is worth going beyond the assumption that customers do not know the exact completion time, and exploring customers' psychological responses when offered delayed release times. A deeper understanding of human emotions and behaviors when facing intentional delay may provide important guidance on the design of such policies.

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Appendix A: Proofs in Section 3

Heuristic Derivation of (HJB)

Given a sufficiently small time duration $\delta > 0$, we can write the discrete-time Bellman equation as:

$$\mathsf{g}\delta + V(w) = (1-\lambda\delta)V(w-\delta) + \lambda\hat{\Phi}(w), \ \forall w > \delta \tag{53}$$

and

$$\mathsf{g}\delta + V(0) = (1-\lambda\delta)V(0) + \lambda\hat{\Phi}(0). \tag{54}$$

The function $\hat{\Phi}(w)$ is defined as the following integer linear optimization problem

$$\begin{aligned}
\hat{\Phi}(w) := \max_{q^\theta, r^\theta, p^\theta} & \sum_{\theta\in\boldsymbol{\Theta}} \alpha_\theta \left\{ p^\theta + q^\theta V(w+B-\delta) + (1-q^\theta)V(w-\delta) \right\}, \\
\text{s.t. } & \nu_h q^h - c_h r^h - p^h \geq \nu_h q^l - c_h r^l - p^l, \quad \nu_h q^h - c_h r^h - p^h \geq 0, \\
& \nu_l q^l - c_l r^l - p^l \geq \nu_l q^h - c_l r^h - p^h, \quad \nu_l q^l - c_l r^l - p^l \geq 0, \\
& r^h \geq q^h(w+B), \quad r^l \geq q^l(w+B), \quad p^h \geq 0, \quad p^l \geq 0, \\
& q^h, q^l \in \{0,1\}.
\end{aligned} \tag{55}$$

By rearranging terms and divided by δ in (53) and (54), we have

$$\mathsf{g} + \frac{V(w) - V(w-\delta)}{\delta} = \lambda\bar{\Phi}(w), \ \forall w > \delta \ \text{ and} \tag{56}$$

$$\mathsf{g} = \lambda\bar{\Phi}(0), \tag{57}$$

where

$$\begin{aligned}
\bar{\Phi}(w) := \max_{q^\theta, r^\theta, p^\theta} & \sum_{\theta\in\boldsymbol{\Theta}} \alpha_\theta \left\{ p^\theta + q^\theta \big[V(w+B-\delta) - V(w-\delta)\big] \right\}, \\
\text{s.t. } & \nu_h q^h - c_h r^h - p^h \geq \nu_h q^l - c_h r^l - p^l, \quad \nu_h q^h - c_h r^h - p^h \geq 0, \\
& \nu_l q^l - c_l r^l - p^l \geq \nu_l q^h - c_l r^h - p^h, \quad \nu_l q^l - c_l r^l - p^l \geq 0, \\
& r^h \geq q^h(w+B), \quad r^l \geq q^l(w+B), \quad p^h \geq 0, \quad p^l \geq 0, \\
& q^h, q^l \in \{0,1\}.
\end{aligned} \tag{58}$$

Therefore, by taking the limit $\delta \to 0$ for , we heuristically derive (HJB). Specifically, the equation (57) implies $V'(0) = 0$ as the boundary condition.

Proposition 1

The proof of Proposition 1 follows immediately from Theorem 1 in Lin et al. (2024), which shows properties and the optimality for a general wait-time based queueing model. That is, Proposition 1 is established by verifying the model setting satisfying Assumption 1 (i)-(v) in Lin et al. (2024). To make the proof self-contained, we present Assumption 1 in Lin et al. (2024) as the following Assumption 3:

Assumption 3.

(i) The lead-time $w+\tau(\theta,\vec{a},w)$ is non-decreasing in w for any given θ and $\vec{a}$.

(ii) The feasible set $A(w)$ satisfies $A(w') \subseteq A(w)$ for $w' > w$.

(iii) For any $w \geq 0$, there exist a control $\vec{a}_0 \in A(w)$ such that $\rho(\theta,\vec{a}_0)=0$ for all $\theta \in \Theta$.

(iv) The reward $R(\theta,\vec{a},w)$ is non-increasing in w, upper-bounded by a value B, and continuous in w. Moreover, there exists $\bar{w} < \infty$ such that $R(\theta,\vec{a},w) \leq 0$, for all $w \geq \bar{w}$, $\theta \in \Theta$, and $\vec{a} \in A(w)$.

(v) There exists a control $\vec{a}_+ \in A(0)$ and a type $\hat{\theta} \in \Theta$ such that

$$\rho(\hat{\theta},\vec{a}_+) > 0, \ \tau(\hat{\theta},\vec{a}_+,0) > 0, \ \text{and } R(\hat{\theta},\vec{a}_+,0) > 0. \tag{59}$$

We define the control $\vec{a} := \{q(\theta),p(\theta),r(\theta)\}_{\theta\in\{h,l\}}$, the admission probability $\rho(\theta,\vec{a}) = q(\theta)$, the service time $\tau(\theta,\vec{a},w) = B$, and the reward

$$R(\theta,\vec{a},w) := \begin{cases} p(\theta)/q(\theta), & \text{if } q(\theta)=1, \\ 0, \ \text{if } q(\theta)=0. \end{cases}$$

First, the lead time $w+\tau(\theta,\vec{a},w) = w+B$ increases in w, which satisfies Assumption 3(i). Second, the feasible set $A(w)$ consists of (IC), (IR) and (FE). Only (FE) involves w in the constraint $r \geq w+B$. It is clear that Assumption 3(ii) is thus satisfied.

To show that Assumption 3(iii) holds, consider control $\vec{a}_0$ with $q(\theta)=p(\theta)=r(\theta)=0$ for all $\theta\in\Theta$ and $w \geq 0$. Such a control $\vec{a}_0$ satisfies (IC), (IR) and (FE), and $\rho(\theta,\vec{a}_0)=q(\theta)=0$ for all $\theta\in\Theta$.

Next, consider Assumption 3(iv). The reward function $R(\theta,\vec{a},w)$ is independent of w, so the monotonicity of the function is satisfied. Furthermore, the $R(\theta,\vec{a},w)$ is upper bounded by ν_h for all w from (IR). Finally, define

$$\bar{w} := \frac{\nu_l}{c_l}.$$

Following (FE) and (IR), the only admissible control is to set $p(\theta)=q(\theta)=0$, when $w \geq \bar{w}$. This implies that the $R(\theta,\vec{a},w)$ function is non-positive when $w \geq \bar{w}$.

Finally, define a control $\vec{a}_+ := \{q(h),r(h),p(h),q(l),r(l),p(l)\}$ such that

$$q(h)=1, \quad r(h)=B, \quad p(h)=\nu_h - c_h B, \quad q(l)=p(l)=r(l)=0.$$

This control $\vec{a}_+$ satisfies (IC), (IR) and (FE) for $w=0$ from Assumption 1. Assumption 3(v) is satisfied. Q.E.D.

Lemma 1

We have

$$\frac{\mathrm{d}}{\mathrm{d}w}u_l(w) = -c_l < 0, \ \frac{\mathrm{d}}{\mathrm{d}w}u_h(w) = -c_h < 0, \ \frac{\mathrm{d}}{\mathrm{d}w}u_2(w) = -c_h + \frac{\alpha_l}{\alpha_h}(-c_h+c_l) < 0,$$

because $c_h > c_l > 0$ and $\alpha_l = 1-\alpha_h < 1$. Furthermore, (11), (12) and (13) directly follow the definitions of r^* in Assumption 1, (7), and (8)-(10).

Theorem 1

We first show the existence of thresholds $\bar{w}_{ps}$ and $\bar{w}_{sr}^l$. Because $\Delta(w)$ decreases in w and $\lim_{w\uparrow\infty}\Delta(w)=-\infty$, $u_2(w)$ increases in w and $\lim_{w\uparrow\infty}u_2(w)=\infty$, and $\Delta(w^*)\geq u_l(w^*)=\frac{1}{\alpha_h}u_2^*(w)$ (Lemma 1), there must exists a unique $\bar{w}_{ps}\geq w^*$ such that $\Delta(\bar{w}_{ps})=\frac{1}{\alpha_h}u_2(\bar{w}_{ps})$. The same logic implies that there must exists a unique $\bar{w}_{sr}^l\geq w^*$ such that $\Delta(\bar{w}_{sr}^l)=u_l(\bar{w}_{sr}^l)$. Finally, $\bar{w}_{sr}^l\geq\bar{w}_{ps}$ follows from the fact that $\frac{1}{\alpha_h}u_2(w)\leq u_l(w)$ for $w\geq w^*$.

The dual of linear optimization $\Phi(w)$ defined in (5) is

$$
\begin{aligned}
\min_{\lambda_\theta\leq 0,\mu_\theta\leq 0,\xi_\theta\leq 0,\delta_\theta\geq 0} \quad & \delta_h+\delta_l && (60)\\
\text{s.t.} \quad & -\lambda_h+\lambda_l-\mu_h\geq\alpha_h && (61)\\
& \lambda_h-\lambda_l-\mu_l\geq\alpha_l && (62)\\
& \nu_h(\lambda_h+\mu_h)-\nu_l\lambda_l-w+B\xi_h+\delta_h\geq\alpha_h\Delta(w) && (63)\\
& -\nu_h\lambda_h+\nu_l(\lambda_l+\mu_l)-w+B\xi_l+\delta_l\geq\alpha_l\Delta(w) && (64)\\
& -c_h(\lambda_h+\mu_h)+c_l\lambda_l+\xi_h=0 && (65)\\
& c_h\lambda_h-c_l(\lambda_l+\mu_l)+\xi_l=0 && (66)
\end{aligned}
$$

We analyze the dual problem above and suppose $\Delta(w^*)\geq u_l(w^*)=u_h(w^*)$. Moreover, we separate the analysis into three cases with thresholds w^*, $\bar{w}_{ps}$, and $\bar{w}_{sr}^l$.

Case 1. We consider the wait time $w\leq w^*$ and solve the dual problem. From the definition of w^* and the concavity of $V(w)$ in Proposition 1, we have

$$\Delta(w)-u_h(w^*)=\Delta(w)+\nu_h-c_hr^*=\Delta(w)+\nu_l-c_lr^*=\Delta(w)-u_l(w^*)\geq 0,\ \forall w\leq w^*. \tag{67}$$

Primal feasible solution (16) yields a primal objective value

$$\alpha_h[\nu_h-c_h(w+B)]+\alpha_l\left[\frac{c_l\nu_h-c_h\nu_l}{c_l-c_h}\right]+\Delta(w). \tag{68}$$

It can be verified that the following solution is dual feasible and yields a dual objective value equal to (68),

$$
\begin{aligned}
&\lambda_l=\xi_l=0,\ \lambda_h=\alpha_l\frac{c_l}{c_l-c_h},\ \mu_l=\alpha_l\frac{c_h}{c_l-c_h},\ \xi_h=-c_h\alpha_h,\ \mu_h=\alpha_l\frac{c_l}{c_h-c_l}-\alpha_h,\\
&\delta_h=\alpha_h\left[\Delta(w)+\nu_h-c_h(w+B)\right]=\alpha_h\left[\Delta(w)-u_h(w)\right],\ \delta_l=\alpha_l\left[\Delta(w)+\frac{c_l\nu_h-c_h\nu_l}{c_l-c_h}\right].
\end{aligned}
$$

In particular, $\mu_h\leq 0$ follows from Assumption 1. When $w\leq w^*$, we have $\delta_h\geq 0$ from

$$\Delta(w)-u_h(w)\geq\Delta(w)-u_h(w^*)\geq 0,$$

provided by (67), the concavity of $V(w)$, and Lemma 1.

Similarly, $\delta_l\geq 0$ follows from (67), the concavity of $V(w)$, and Lemma 1 that

$$
\begin{aligned}
&\Delta(w)+\frac{c_l\nu_h-c_h\nu_l}{c_l-c_h}=\Delta(w)+\frac{c_l\nu_h}{c_l-c_h}-\frac{c_h\nu_l}{c_l-c_h}+\frac{c_h\nu_h}{c_l-c_h}-\frac{c_h\nu_h}{c_l-c_h}=\Delta(w)+\nu_h-\frac{c_h\nu_l-c_h\nu_h}{c_l-c_h}\\
&=\Delta(w)+\nu_h-c_h\frac{\nu_l-\nu_h}{c_l-c_h}=\Delta(w)+\nu_h-c_hr^*=\Delta(w)-u_h(w^*)\geq\Delta(w^*)-u_h(w^*)\geq 0.
\end{aligned}
$$

Case 2. (Serving both without delay) We consider $w \in [w^*, \bar{w}_{ps}]$, in which $\bar{w}_{ps}$ is defined as

$$\nu_h - c_h(\bar{w}_{ps} + B) + \Delta(\bar{w}_{ps}) - \alpha_l \left[\nu_l - c_l(\bar{w}_{ps} + B) + \Delta(\bar{w}_{ps})\right] = 0, \tag{69}$$

Which is equivalent to

$$\Delta(\bar{w}_{ps}) = \frac{1}{\alpha_h} u_2(\bar{w}_{ps}).$$

Together with Lemma 1, we have

$$\Delta(w) - u_l(w) = \nu_l - c_l(w+B) + \Delta(w) \geq \Delta(w) - u_h(w) = \nu_h - c_h(w+B) + \Delta(w) \geq 0, \ \forall w \in [w^*, \bar{w}_{ps}] \tag{70}$$

We notice that

$$\begin{aligned} &\nu_h - c_h s(w^* + B) + \Delta(w^*) - \alpha_l \left[\nu_l - c_l s(w^* + B) + \Delta(w^*)\right] \\ &= \alpha_h \left[\nu_h - c_h(w^* + B) + \Delta(w^*)\right] = \alpha_h [\Delta(w^*) - u_h(w^*)] \geq 0. \end{aligned}$$

from the assumption $\Delta(w^*) - u_h(w^*) \geq 0$. Moreover, from the concavity of $V(\cdot)$ and the monotonicity of $u_2(w)$ in Lemma 1, we have

$$u_2(w) + \alpha_h \Delta(w) \geq 0, \ \forall w \in [w^*, \bar{w}_{ps}]. \tag{71}$$

We then propose a primal feasible solution (17) as

$$q^h = q^l = 1, \ p^h = p^l = \nu_h - c_h w + B, \ r^h = r^l = w + B;$$

yields a primal objective value

$$\nu_h - c_h(w + B) + \Delta(w). \tag{72}$$

It can be verified that the following solution is dual-feasible and yields a dual objective value equal to (72),

$$\begin{aligned} &\lambda_h = \mu_l = 0, \ \lambda_l = -\alpha_l, \ \mu_h = -1, \ \xi_h = \alpha_l c_l - c_h, \ \xi_l = -\alpha_l c_l, \\ &\delta_h = \nu_h - c_h(w+B) + \Delta(w) - \alpha_l \left[\nu_l - c_l(w+B) + \Delta(w)\right], \\ &\delta_l = \alpha_l \left[\nu_l - c_l(w+B) + \Delta(w)\right]. \end{aligned}$$

In particular, the feasibility of dual variable $\delta_h \geq 0$ follows from (71), and $\delta_l \geq 0$ follows from (70).

Case 3. (Only serving type l) We solve the dual problem when $w \in (\bar{w}_{ps}, \bar{w}_{sr}^l)$. The threshold $\bar{w}_{sr}^l$ is defined as,

$$\nu_l - c_l(\bar{w}_{sr}^l + B) + \Delta(\bar{w}_{sr}^l) = 0,$$

which is equivalent to $\Delta(\bar{w}_{sr}^l) = u_l(\bar{w}_{sr}^l)$. We have the primal feasible solution (18) as

$$q^h = 0, \ q^l = 1, \ p^h = 0, \ p^l = \nu_l - c_l(w+B), \ r^h = 0, \ r^l = s(w+B).$$

This yields the primal objective value

$$\alpha_l \left[\nu_l - c_l(w+B) + \Delta(w)\right]. \tag{73}$$

It can be verified that the following solution is dual-feasible when $w \in (\bar{w}_{ps}, \bar{w}_{sr}^l)$ and yields a dual objective value equal to (73),

$$\begin{aligned} &\lambda_h = \mu_l = 0, \ \lambda_l = -\alpha_l, \ \mu_h = -1, \ \xi_h = \alpha_l c_l - c_h, \ \xi_l = -\alpha_l c_l, \\ &\delta_h = 0, \ \delta_l = \alpha_l \left[\nu_l - c_l(w+B) + \Delta(w)\right]. \end{aligned}$$

Theorem 2

We first show the existence of thresholds $\bar{w}_{ds}$ and $\bar{w}_{sr}^h$. Because $\Delta(w)$ decreases in w and $\lim_{w\uparrow\infty}\Delta(w)=-\infty$, $u_h(w)$ increases in w and $\lim_{w\uparrow\infty}u_2(w)=\infty$, and $\Delta(w^*)\geq u_h(w^*)$ as the assumption for this proposition, there must exist a unique $\bar{w}_{ds}\leq w^*$ such that $\Delta(\bar{w}_{dp})=u_h(\bar{e}_{fp})$. The same logic implies that there must exists a unique $\bar{w}_{sr}^h\leq w^*$ such that $\Delta(\bar{w}_{sr}^h)=u_l(\bar{w}_{sr}^h)$. Finally, $\bar{w}_{sr}^h\geq\bar{w}_{dp}$ follows from facts that $u_h(w^*)\geq u_h(w)$ for $w\leq w^*$ and that $\Delta(w)$ is decreasing.

The dual of linear optimization $\Phi(w)$ defined in (5) is

$$\min_{\lambda_\theta\leq 0,\mu_\theta\leq 0,\xi_\theta\leq 0,\delta_\theta\geq 0}\delta_h+\delta_l \tag{74}$$

$$\text{s.t.}\quad -\lambda_h+\lambda_l-\mu_h\geq\alpha_h \tag{75}$$

$$\lambda_h-\lambda_l-\mu_l\geq\alpha_l \tag{76}$$

$$\nu_h(\lambda_h+\mu_h)-\nu_l\lambda_l-s(w+B)\xi_h+\delta_h\geq\alpha_h\Delta(w) \tag{77}$$

$$-\nu_h\lambda_h+\nu_l(\lambda_l+\mu_l)-s(w+B)\xi_l+\delta_l\geq\alpha_l\Delta(w) \tag{78}$$

$$-c_h(\lambda_h+\mu_h)+c_l\lambda_l+\xi_h=0 \tag{79}$$

$$c_h\lambda_h-c_l(\lambda_l+\mu_l)+\xi_l=0 \tag{80}$$

Now we solve the dual problem above and suppose $\Delta(w^*)+\nu_h-c_hs(r^*)=\Delta(w^*)-u_h(w^*)=\Delta(w^*)-u_l(w^*)<0$. We separate the analysis in two cases with thresholds $\bar{w}_{dp}$ and $\bar{w}_{sr}^h$.

Case 1. We consider the interval $w\in[0,\bar{w}_{dp}]$ and we have $\bar{w}_{dp}<w^*$, in which $\bar{w}_{dp}$ is defined as

$$\Delta(\bar{w}_{dp})-u_h(\bar{w}_{dp})=0.$$

From the definition of $\bar{w}_{dp}$ and the concavity of $V(w)$, we have

$$\Delta(w)-u_h(w^*)=\Delta(w)-u_l(w^*)\geq 0,\ \forall w\leq\bar{w}_{dp}. \tag{81}$$

Primal feasible solution (16) yields a primal objective value

$$\alpha_h[\nu_h-c_h(w+B)]+\alpha_l\left[\frac{c_l\nu_h-c_h\nu_l}{c_l-c_h}\right]+\Delta(w). \tag{82}$$

It can be verified that the following solution is dual-feasible and yields a dual objective value equal to (82),

$$\lambda_l=\xi_l=0,\ \lambda_h=\alpha_l\frac{c_l}{c_l-c_h},\ \mu_l=\alpha_l\frac{c_h}{c_l-c_h},\ \xi_h=-c_h\alpha_h,\ \mu_h=\alpha_l\frac{c_l}{c_h-c_l}-\alpha_h,$$
$$\delta_h=\alpha_h\left[\Delta(w)+\nu_h-(w+B)c_h\right],\ \delta_l=\alpha_l\left[\Delta(w)+\frac{c_l\nu_h-c_h\nu_l}{c_l-c_h}\right].$$

In particular, $\mu_h\leq 0$ follows from Assumption 1. Given condition $\Delta(w^*)-u_h(w^*)\geq 0$, we have (81) and Lemma 1, which imply $\delta_h\geq 0$ for all $w\leq\bar{w}_{dp}$.

Moreover, $\delta_l\geq 0$ follows from (81) that

$$\Delta(w)+\frac{c_l\nu_h-c_h\nu_l}{c_l-c_h}=\Delta(w)+\frac{c_l\nu_h}{c_l-c_h}-\frac{c_h\nu_l}{c_l-c_h}+\frac{c_h\nu_h}{c_l-c_h}-\frac{c_h\nu_h}{c_l-c_h}$$
$$=\Delta(w)+\nu_h-\frac{c_h\nu_l-c_h\nu_h}{c_l-c_h}=\Delta(w)+\nu_h-c_h\frac{\nu_l-\nu_h}{c_l-c_h}=\Delta(w)+\nu_h-c_h(w^*+B)=\Delta-u_h(w^*)\geq 0.$$

Case 2. We consider the case when $w \in (\bar{w}_{dp}, \bar{w}_{sr}^h]$ in which $\bar{w}_{sr}^h$ is defined as,

$$\nu_h - c_h(\bar{w}_{sr}^h + B) + \Delta(\bar{w}_{sr}^h) = 0,$$

which is equivalent to

$$u_h(\bar{w}_{sr}^h) = \Delta(\bar{w}_{sr}^h).$$

We have the primal feasible solution (22) as

$$q^h = 1,\ q^l = 0,\ p^h = \nu_h - c_h(w+B),\ p^l = 0,\ r^h = w + B,\ r^l = 0;$$

yields a primal objective value

$$\alpha_h\left[\nu_h - c_h(w+B) + \Delta(w)\right]. \tag{83}$$

It can be verified that the following solution is dual-feasible and yields a dual objective value equal to (83),

$$\begin{aligned}&\lambda_l = \xi_l = 0,\ \lambda_h = \alpha_l \frac{c_l}{c_l - c_h},\ \mu_l = \alpha_l \frac{c_h}{c_l - c_h},\ \xi_h = -c_h\alpha_h,\ \mu_h = \alpha_l \frac{c_l}{c_h - c_l} - \alpha_h,\\ &\delta_h = \alpha_h\left[\Delta(w) + \nu_h - (w+B)c_h\right],\ \delta_l = 0.\end{aligned}$$

The term δ_h is non-negative only when $w \le \bar{w}_{sr}^h$ as the optimal solution when $w \in (\bar{w}_{dp}, \bar{w}_{sr}^h]$.

Appendix B: Proofs in Section 4

Proposition 2

Similar to Proposition 1, the proof is to verify Assumption 3 with the continuous type setting in Section (4.1).

We define the control $\vec{a} := \{q(\theta), p(\theta), r(\theta)\}_{\theta\in\Theta=[c_l,c_h]}$, the admission probability $\rho(\theta,\vec{a}) = q(\theta)$, the service time $\tau(\theta,\vec{a},w) = B$, and the reward

$$R(\theta,\vec{a},w) := \begin{cases} p(\theta)/q(\theta), & \text{if } q(\theta) = 1,\\ 0,\ \text{if } q(\theta) = 0. \end{cases}$$

First, the lead time $w + \tau(\theta,\vec{a},w) = w + B$ is increasing in w, which satisfies Assumption 3(i). Second, the feasible set $A(w)$ consists of (26), (27) and (FE), among which only (FE) involves w, in $r \ge q(w+B)$. That is, when $q = 1$ and w increases, the feasible choices of r becomes less, which stasifies Assumption 3(ii).

Assumption 3(iii) holds from a feasible control $\vec{a}_0$ such that $q(\theta) = p(\theta) = 0$ and $r(\theta) = 0$ for all $\theta \in \Theta$ and $w \ge 0$. One can verify that this control satisfies (26), (27) and (FE), and $\rho(\theta,\vec{a}_0) = q(\theta) = 0$.

Next, we verify Assumption 3(iv). The reward function $R(\theta,\vec{a},w)$ is independent of w, so the monotonicity of the function is satisfied. The $R(\theta,\vec{a},w)$ is upper bounded by $\bar{\nu} + r^*c_h$ for all w from (27). Also, define

$$\bar{w} := \frac{\bar{\nu} + r^*c_l}{c_l} - B.$$

Following (FE) and (27), the only feasible control is to set $p(\theta) = q(\theta) = 0$ for all $\theta \in \Theta$, when $w \ge \bar{w}$ which implying that the $R(\theta,\vec{a},w)$ function is non-positive.

Finally, define a control $\vec{a}_+ := \{q(\theta), r(\theta), p(\theta)\}_{\theta\in\Theta}$ such that

$$q(\theta) = 1, \quad r(\theta) = B, \quad p(\theta) = \bar{\nu} + c_h(r^* - B), \theta = c_h$$

and

$$q(\theta) = p(\theta) = r(\theta) = 0,\ \forall \theta \ne c_h.$$

The control satisfies (26), (27) and (FE) for $w = 0$ from Assumption 1. This verifies Assumption 3(v).

Lemma 2

Constraint (IC) implies

$$U(\theta)=\bar{\nu}q(\theta)+\theta[r^*q(\theta)-r(\theta)]-p(\theta)=\max_{\theta'}\bar{\nu}q(\theta')+\theta[r^*q(\theta')-r(\theta')]-p(\theta'),$$

which is the maximum of a collection of linear functions in θ, and hence is convex. Furthermore, the Envelop Theorem implies that (28) holds. Finally, convexity of $U(\theta)$ implies that $U'(\theta)$ is non-decreasing, or (29).

For any functions $q(\theta)$ and $r(\theta)$ that satisfies (29) and $p(\theta)$ as defined in (30), we have

$$U(\theta)=\bar{\nu}q(\theta)+\theta[r^*q(\theta)-r(\theta)]-p(\theta)=\bar{u}+\int_{\hat{\theta}}^{\theta}[r^*q(x)-r(x)]\mathrm{d}x,$$

which implies $U(\hat{\theta})=\bar{u}$, and

$$\mathrm{u}(\theta,q(\theta'),p(\theta'),r(\theta'))=\bar{u}+(\theta-\theta')[r^*q(\theta')-r(\theta')]+\int_{\hat{\theta}}^{\theta'}[r^*q(x)-r(x)]\mathrm{d}x,$$

which implies that

$$U(\theta)-\mathrm{u}(\theta,q(\theta'),p(\theta'),r(\theta'))=\int_{\theta'}^{\theta}\{[r^*q(x)-r(x)]-[r^*q(\theta')-r(\theta')]\}\,\mathrm{d}x\geq 0,$$

where the last inequality holds because $r^*q(\theta)-r(\theta)$ is non-decreasing in θ, which implies (IC).

Lemma 3

If we let $\hat{\theta}$ in Lemma 2 to be either $\check{c}$ or $\hat{c}$, then (27) constraint in (25) implies that $\bar{u}=U(\check{c})=U(\hat{c})\geq 0$. From (25), $\bar{u}\geq 0$, and Lemma 2, we have

$$\begin{aligned}
&\Upsilon(w)=\max_{q(\theta),p(\theta),r(\theta):\Pi}\int_{\theta\in\Theta}[p(\theta)+q(\theta)\Delta(w)]f(\theta)\mathrm{d}\theta\\
&=\max_{q(\theta),p(\theta),r(\theta):\Pi}\int_{c_l}^{\check{c}}[p(\theta)+q(\theta)\Delta(w)]f(\theta)\mathrm{d}\theta\\
&\qquad+\int_{\check{c}}^{\hat{c}}[p(\theta)+q(\theta)\Delta(w)]f(\theta)\mathrm{d}\theta+\int_{\hat{c}}^{c_h}[p(\theta)+q(\theta)\Delta(w)]f(\theta)\mathrm{d}\theta\\
&=\max_{q(\theta),r(\theta),\bar{u}\geq 0,\check{c},\hat{c}:\Pi'}-\bar{u}\\
&\qquad+\int_{c_l}^{\check{c}}\left\{\bar{\nu}q(\theta)+\theta[r^*q(\theta)-r(\theta)]+\int_{\theta}^{\check{c}}[r^*q(x)-r(x)]\mathrm{d}x+\Delta(w)q(\theta)\right\}f(\theta)\mathrm{d}\theta\\
&\qquad+\int_{\check{c}}^{\hat{c}}\big[q(\theta)(\bar{\nu}+\Delta(w))\big]f(\theta)\mathrm{d}\theta\\
&\qquad+\int_{\hat{c}}^{c_h}\left\{\bar{\nu}q(\theta)+\theta(r^*q(\theta)-r(\theta)-\int_{\hat{c}}^{\theta}[r^*q(x)-r(x)]\mathrm{d}x+\Delta(w)q(\theta)\right\}f(\theta)\mathrm{d}\theta\\
&=\max_{q(\theta),r(\theta),\check{c},\hat{c}:\Pi'}\int_{c_l}^{\check{c}}\left\{(\bar{\nu}+\Delta(w))q(\theta)+[r^*q(\theta)-r(\theta)]\left[\theta+\frac{F(\theta)}{f(\theta)}\right]\right\}f(\theta)\mathrm{d}\theta\\
&\qquad+\int_{\check{c}}^{\hat{c}}\big[q(\theta)(\bar{\nu}+\Delta(w))\big]f(\theta)\mathrm{d}\theta+\int_{\hat{c}}^{c_h}\left\{(\bar{\nu}+\Delta(w))q(\theta)+[r^*q(\theta)-r(\theta)]\left[\theta-\frac{1-F(\theta)}{f(\theta)}\right]\right\}f(\theta)\mathrm{d}\theta.
\end{aligned}$$

The third equality follows from Lemma 2. The last equality follows from (31), (32), (33), and (34) and $-\bar{u}=0$ in the maximization.

Theorem 3

To prove Theorem 3, we first analyze (35) as the following proposition.

PROPOSITION 5. *The optimal solution $\left(q^*(\theta), p^*(\theta), r^*(\theta), \check{c}^*, \hat{c}^*\right)$ that solves $\Upsilon(w)$ in (35) satisfy:*

(1) if $w > w^*$, $\check{c}^* = \check{c}(w)$, and $\hat{c}^* = c_h$, in which $\check{c}(w)$ is defined in (38);

—for $\theta \le \check{c}(w)$, $q^*(\theta) = 1$, $r^*(\theta) = w + B$, and $p^*(\theta) = \bar{\nu} + \check{c}(w)(w^* - w)$;

—for $\theta > \check{c}(w)$, on the other hand, $q^*(\theta) = r^*(\theta) = p^*(\theta) = 0$;

(2) if $w \le w^*$ and $\bar{\nu} + \Delta(w) \ge 0$, $q^*(\theta) = 1$, $\forall \theta \in \Theta$ and $\check{c}^* = \hat{c}^* = \bar{c}$;

—for $\theta < \bar{c}$, $r^*(\theta) = r^*$, and $p^*(\theta) = \bar{\nu}$;

—for $\theta \ge \bar{c}$, on the other hand, $r^*(\theta) = r + B$, and $p^*(\theta) = \bar{\nu} + \bar{c}(w^* - w)$;

(3) if $w \le w^*$ and $\bar{\nu} + \Delta(w) < 0$, $\check{c} = c_l$ and $\hat{c}^* = \hat{c}(w)$ in which $\hat{c}(w)$ is defined in (40);

—for $\theta \ge \hat{c}(w)$, $q^*(\theta) = 1$, $r^*(\theta) = w + B$, and $p^*(\theta) = \bar{\nu} + \hat{c}(w)(w^* - w)$;

—for $\theta < \hat{c}(w)$, on the other hand, $q^*(\theta) = r^*(\theta) = p^*(\theta) = 0$.

Proof of Proposition 5 We analyze the equation (35) by separating three situations:

1. The case when $w > w^*$
2. The case when $w \le w^*$ and $\bar{\nu} + \Delta(w) \ge 0$
3. The case when $w \le w^*$ and $\bar{\nu} + \Delta(w) < 0$

Case 1 When $w > w^*$, (34) cannot be satisfied together with (FE), which implies that $\hat{c} = c_h$. Furthermore, (FE) together with (33) can only be satisfied with $q(\theta) = r(\theta) = 0$ for all $\theta \in [\check{c}, \hat{c}]$. Therefore, (35) is reduced to

$$\Upsilon(w) = \max_{q(\theta), r(\theta), \check{c} \in [c_l, c_h] : (\mathrm{FE}), (29), (32)} \int_{c_l}^{\check{c}} \left\{ (\bar{\nu} + \Delta(w)) q(\theta) + [r^* q(\theta) - r(\theta)] \left[\theta + \frac{F(\theta)}{f(\theta)} \right] \right\} f(\theta) \mathrm{d}\theta. \tag{84}$$

By ignoring the constraint (29) for the moment, we solve $r(\theta) = q(\theta)(w + B)$, because $\theta + F(\theta)/f(\theta) > 0$. Therefore, (84) becomes

$$\max_{q(\theta) \in [0,1], \check{c} \in [c_l, c_h]} \int_{c_l}^{\check{c}} q(\theta) \left\{ \bar{\nu} + \Delta(w) + [r^* - (w + B)] \left[\theta + \frac{F(\theta)}{f(\theta)} \right] \right\} f(\theta) \mathrm{d}\theta. \tag{85}$$

For any given $\check{c}$, the optimal $q(\theta)$ takes value 0 or 1 depending on the sign of the term

$$\mathcal{H}(w, \theta) := \bar{\nu} + \Delta(w) + (w^* - w) \left[\theta + \frac{F(\theta)}{f(\theta)} \right]. \tag{86}$$

That is, the optimal solution to (85) for a given $\check{c}$ is

$$q(\theta) = \begin{cases} 1 & \text{if } \mathcal{H}(w, \theta) \ge 0 \\ 0 & \text{if } \mathcal{H}(w, \theta) < 0 \end{cases}. \tag{87}$$

Because $\theta + \frac{F(\theta)}{f(\theta)}$ increases in θ following Assumption 2, (87) implies that we can define a threshold

$$\check{c}(w) := \max \left\{ \max \left\{ \theta \in [c_l, \check{c}] \middle| \mathcal{H}(w, \theta) \ge 0 \right\}, c_l \right\}, \ \forall w > w^*, \tag{88}$$

and have

$$q(\theta) = \begin{cases} 1 & \forall \theta \le \check{c}(w) \\ 0 & \forall \theta > \check{c}(w) \end{cases}, \text{ and } r(\theta) = \begin{cases} w + B & \forall \theta \le \check{c}(w) \\ 0 & \forall \theta > \check{c}(w) \end{cases}. \tag{89}$$

It is easy to verify that the optimal q and r for (85) given $\check{c}$ also satisfies (29) and (32). Therefore, they are the optimal solution to (84) for a given $\check{c}$. Finally, the integrant in (84),

$$\left\{(\bar{\nu}+\Delta(w))q(\theta)+[dq(\theta)-r(\theta)]\left[\theta+\frac{F(\theta)}{f(\theta)}\right]\right\}=(\bar{\nu}+\Delta(w))-(w-w^*)\left[\theta+\frac{F(\theta)}{f(\theta)}\right]\geq 0$$

and is decreasing in θ for $\theta\leq\check{c}(w)$, which implies that the optimal $\check{c}$ must be $\check{c}(w)$. As a result, we have

$$\Upsilon(w)=\int_{c_l}^{\check{c}(w)}\left\{\bar{\nu}+\Delta(w)+(w^*-w)\left[\theta+\frac{F(\theta)}{f(\theta)}\right]\right\}f(\theta)\mathrm{d}\theta,\ \forall w>w^*.$$

Case 2 We analyze (35) when $w\leq w^*$ and $\bar{\nu}+\Delta(w)\geq 0$. Now we ignore the constraint (29) for a moment to solve (35). The first integral in (35) has the integrand

$$(\bar{\nu}+\Delta(w))q(\theta)+[r^*q(\theta)-r(\theta)]\left[\theta+\frac{F(\theta)}{f(\theta}\right].$$

Given $\bar{\nu}+\Delta(w)\geq 0$ and

$$[q(\theta)d-r(\theta)]\left[\theta+\frac{F(\theta)}{f(\theta}\right]\leq 0$$

from (32), we have

$$q(\theta)=1, r(\theta)=r^*,\ \forall\theta\in[c_l,\check{c}], \tag{90}$$

in (35) for this case. Therefore, the first two integrals in (35) can be combined and thus (35) without (29) becomes

$$\begin{aligned}\Upsilon(w)=&\max_{q(\theta),r(\theta),\hat{c}\in[c_l,c_h]:(\mathrm{FE}),(33),(34)}\int_{c_l}^{\hat{c}}\{(\bar{\nu}+\Delta(w))q(\theta)\}f(\theta)\mathrm{d}\theta\\&+\int_{\hat{c}}^{c_h}\left\{(\bar{\nu}+\Delta(w))q(\theta)+[r^*q(\theta)-r(\theta)]\left[\theta-\frac{1-F(\theta)}{f(\theta)}\right]\right\}f(\theta)\mathrm{d}\theta,\end{aligned} \tag{91}$$

Furthermore, from (33), $\bar{\nu}+\Delta(w)\geq 0$, and (90), we have

$$q(\theta)=1, r(\theta)=w^*+B=r^*,\ \forall\theta\in[c_l,\hat{c}].$$

Hence, we can further write (91) as

$$\begin{aligned}&\max_{q(\theta),r(\theta),\hat{c}\in[c_l,c_h]:(\mathrm{FE}),(34)}\int_{c_l}^{\hat{c}}\{(\bar{\nu}+\Delta(w))\}f(\theta)\mathrm{d}\theta\\&\quad+\int_{\hat{c}}^{c_h}\left\{(\bar{\nu}+\Delta(w))q(\theta)+[r^*q(\theta)-r(\theta)]\left[\theta-\frac{1-F(\theta)}{f(\theta)}\right]\right\}f(\theta)\mathrm{d}\theta\\&=\max_{q(\theta),r(\theta),\hat{c}\in[c_l,c_h]:(\mathrm{FE}),(34)}\int_{c_l}^{\hat{c}}\{(\bar{\nu}+\Delta(w))\}f(\theta)\mathrm{d}\theta\\&\quad+\int_{\hat{c}}^{\max\{\hat{c},\bar{c}\}}\left\{(\bar{\nu}+\Delta(w))q(\theta)+[r^*q(\theta)-r(\theta)]\left[\theta-\frac{1-F(\theta)}{f(\theta)}\right]\right\}f(\theta)\mathrm{d}\theta\\&\quad+\int_{\max\{\hat{c},\bar{c}\}}^{c_h}\left\{(\bar{\nu}+\Delta(w))q(\theta)+[r^*q(\theta)-r(\theta)]\left[\theta-\frac{1-F(\theta)}{f(\theta)}\right]\right\}f(\theta)\mathrm{d}\theta,\end{aligned} \tag{92}$$

in which $\bar{c}$ is defined in (36). Following (34) and (36), we have

$$[r^*q(\theta)-r(\theta)]\left[\theta-\frac{1-F(\theta)}{f(\theta)}\right]<0,\forall\theta<\bar{c},\ \text{and}\ [r^*q(\theta)-r(\theta)]\left[\theta-\frac{1-F(\theta)}{f(\theta)}\right]\geq 0,\forall\theta>\bar{c}$$

which implies that the optimal $\hat{c}$ in (92) must be $\bar{c}$ to make the second integral empty. Furthermore, the optimal decisions for the right-hand-side of (92) are $q(\theta)=1$ and $r(\theta)=w+B$ for the interval $\theta\in[\hat{c},c_h]$, because of the constraint (FE), $\bar{\nu}+\Delta(w)>0$, and $\left[\theta-\frac{1-F(\theta)}{f(\theta)}\right]\geq 0$. It can be easily verified that the above solution satisfies (29) ignored at the beginning and, therefore, is indeed optimal to (91). We summarize the optimal solution to (92) as follows

$$q(\theta)=\begin{cases}1\ ,\ \forall\ \theta\leq\bar{c},\\ 1\ ,\ \forall\ \theta>\bar{c},\end{cases}\quad ,\text{ and } r(\theta)=\begin{cases}r^* & ,\forall\theta\leq\bar{c},\\ w+B & ,\forall\theta>\bar{c},\end{cases}$$

where $\hat{c}=\bar{c}$ and $\check{c}=c_l$, and the solution satisfies (29) as the optimal solution to (91). Therefore, we have

$$\Upsilon(w)=\int_{c_l}^{\bar{c}}\{(\bar{\nu}+\Delta(w))\}f(\theta)\mathrm{d}\theta+\int_{\bar{c}}^{c_h}\left\{(\bar{\nu}+\Delta(w))+(w^*-w)\left[\theta-\frac{1-F(\theta)}{f(\theta)}\right]f(\theta)\right\}\mathrm{d}\theta,$$

when $w\leq w^*$ and $\bar{\nu}+\Delta(w)\geq 0$.

Case 3 Now consider (35) when $w\leq w^*$ and $\bar{\nu}+\Delta(w)<0$. From (32), we have $\check{c}=c_l$, because the integrant is negative in the interval $[c_l,\check{c})$. The optimal $q(\theta)=0$ for all $\theta\in[c_l,\hat{c}]$ in (35), because $\bar{\nu}+\Delta(w)<0$. Therefore, the equation (35) becomes

$$\Upsilon(w)=\max_{q(\theta),r(\theta),\hat{c}\in[c_l,c_h]:(\mathrm{FE}),(29),(34)}\int_{\hat{c}}^{c_h}\left\{(\bar{\nu}+\Delta(w))q(\theta)+[r^*q(\theta)-r(\theta)]\left[\theta-\frac{1-F(\theta)}{f(\theta)}\right]\right\}f(\theta)\mathrm{d}\theta,\qquad(93)$$

Following the definition of $\bar{c}$ in (36), we have

$$(\bar{\nu}+\Delta(w))q(\theta)+[r^*q(\theta)-r(\theta)]\left[\theta-\frac{1-F(\theta)}{f(\theta)}\right]\leq 0,\forall\theta\leq\bar{c},$$

which further implies that the optimal $\hat{c}\geq\bar{c}$.

Now we ignore constraint (29). Given $\hat{c}\geq\bar{c}$, we have $\theta-(1-F(\theta))/f(\theta)\geq 0$ for $\theta\in[\hat{c},c_h]$. Together with constraint (FE), we solve $r(\theta)=q(\theta)(w+B)$ for all $\theta\in[\hat{c},c_h]$. Therefore, we rewrite (93) as

$$\max_{q(\theta)\in[0,1]\hat{c}\in[\bar{c},c_h]}\int_{\hat{c}}^{c_h}q(\theta)\left\{(\bar{\nu}+\Delta(w))+(w^*-w)\left[\theta-\frac{1-F(\theta)}{f(\theta)}\right]\right\}f(\theta)\mathrm{d}\theta,\qquad(94)$$

Define

$$\mathcal{G}(w,\theta):=\bar{\nu}+\Delta(w)+(w^*-w)\left[\theta-\frac{1-F(\theta)}{f(\theta)}\right].$$

The optimal solution to (94) for a given $\hat{c}$ is

$$q(\theta)=\begin{cases}1 & \text{if } \mathcal{G}(w,\theta)\geq 0\\ 0 & \text{if } \mathcal{G}(w,\theta)<0.\end{cases}\qquad(95)$$

Because $\theta-\frac{1-F(\theta)}{f(\theta)}$ increases in θ following Assumption 2, (95) implies that we can define a threshold

$$\hat{c}(w):=\min\left\{\min\left\{\theta\in[\bar{c},c_h]\middle|\mathcal{G}(w,\theta)\geq 0\right\},c_h\right\},\ \forall w\leq w^*,\bar{\nu}+\Delta(w)<0,\qquad(96)$$

and have

$$q(\theta)=\begin{cases}1 & \forall\theta\geq\hat{c}(w)\\ 0 & \forall\theta<\hat{c}(w)\end{cases},\text{ and } r(\theta)=\begin{cases}w+B & \forall\theta\geq\hat{c}(w)\\ 0 & \forall\theta<\hat{c}(w)\end{cases}.\qquad(97)$$

It is easy to verify that q and r in (97) also satisfy (29). Therefore, they are the optimal solution to (93) for a given $\hat{c}$. Finally, the integrant in (93),

$$\left\{(\bar{\nu}+\Delta(w))q(\theta)+[r^*q(\theta)-r(\theta)]\left[\theta-\frac{1-F(\theta)}{f(\theta)}\right]\right\}=(\bar{\nu}+\Delta(w))+(w^*-w)\left[\theta-\frac{1-F(\theta)}{f(\theta)}\right]\geq 0$$

and is increasing in θ for $\theta\geq\hat{c}(w)$, which implies that the optimal $\hat{c}$ must be $\hat{c}(w)$. As a result, we have

$$\Upsilon(w)=\int_{\hat{c}(w)}^{c_h}\left\{\bar{\nu}+\Delta(w)+[r^*-(w+B)]\left[\theta-\frac{1-F(\theta)}{f(\theta)}\right]\right\}f(\theta)\mathrm{d}\theta,\ \forall w\leq w^*,\bar{\nu}+\Delta(w)<0.\quad Q.E.D.$$

Complete the Proof of Theorem 3: First consider $\bar{\nu}+\Delta(w^*)\geq 0$. In this case we have

$$\bar{\nu}+\Delta(w)\geq 0, \forall w\in[0,w^*],$$

from the concavity of $V(w)$ in Proposition 2. Case (1)i. then follows Proposition 5's case (2). Case (1)ii. follows from Proposition 5's case (1). Moreover, $\check{c}(w)$ is non-increasing in w because of the monotonicity of $\Delta(w)$, also following the concavity of $V(w)$ in Proposition 2.

Now consider $\bar{\nu}+\Delta(w^*)<0$, which implies that $w_D<w^*$ again due to the monotonicity of $\Delta(w)$ (following concavity of $V(w)$ in Proposition 2). When $w\leq w_D$, we have $\bar{\nu}+\Delta(w)\geq 0$, and case (2)i. follows from Proposition 5's case (2). For $w\in(w_D,w^*]$, we have $\bar{\nu}+\Delta(w)<0$. Case (2)ii. then follows Proposition 5's case (3). When $w>w^*$, the optimal policy follows Proposition 5's case (1), in which $\check{c}(w)=c_l$.

Moreover, concavity of $V(w)$ in Proposition 2 implies that the threshold $\hat{c}(w)$ is non-decreasing in w due to the decreasing $\Delta(w)$.

Proposition 3

Similar to the verifications in Proposition 1 and Proposition 2, the proof of Proposition 3 is established by verifying Assumption 3 given the setting in Section 4.2.

We define the control $\vec{a}:=\{q^\theta,r^\theta,p^\theta\}_{\theta\in\{h,l\}}$, the admission probability $\rho(\theta,\vec{a})=q^\theta$, the service time $\tau(\theta,\vec{a},w)=b^\theta$, and the reward

$$R(\theta,\vec{a},w):=\begin{cases} p^\theta/q^\theta-k(b^\theta), & \text{if } q^\theta=1,\\ 0, \text{ if } q^\theta=0.\end{cases}$$

First, the lead time $w+\tau(\theta,\vec{a},w)=w+b^\theta$ increases in w given any b^θ. Thus, Assumption 3(i) is satisfied. the feasible set $A(w)$ is specified as the constraints in (43). In (43), only $r^\theta\geq q^\theta(w+b^\theta)$ involves w. It is clear that Assumption 3(ii) holds.

Assumption 3(iii) holds as defining a control

$$\vec{a}_0:=\{q^\theta,p^\theta,r^\theta,b^\theta\}_{\theta\in\{h,l\}},$$

where $q^\theta=p^\theta=r^\theta=b^\theta=0$ for all $\theta\in\Theta=\{h,l\}$ and $w\geq 0$. This control $\vec{a}_0$ indeed satisfies the constraints in (43), and $\rho(\theta,\vec{a}_0)=q^\theta=0$.

Next, consider Assumption 3(iv). The reward function $R(\theta,\vec{a},w)$ is independent of w. Similar to Proposition 1, the $R(\theta,\vec{a},w)$ is upper bounded by ν_h for all w from (43), and we define

$$\bar{w}:=\frac{\nu_l}{c_l}.$$

Following (43), the only admissible control is to set $p^\theta=q^\theta=r^\theta=b^\theta=0$ for all $\theta\in\{h,l\}$ when $w\geq\bar{w}$. This implies that the $R(\theta,\vec{a},w)$ function is non-positive when $w\geq\bar{w}$.

Finally, define a control $\vec{a}_+:=\{q^h,r^h,p^h,b^h,q^l,r^l,p^l,b^l\}$ such that

$$q^h=1,\quad r^h=b^h=B,\quad p^h=\nu_h-c_hB,$$

and

$$q^l=p^l=r^l=b^l=0.$$

Given $w=0$, the control satisfies the constraints in (43) when $w=0$ from Assumption (1). This verifies Assumption 3(v).

Proposition 4

We separate the analysis of (43) into three cases: $(q^h,q^l)=(1,1)$, $(q^h,q^l)=(1,0)$, and $(q^h,q^l)=(0,1)$.

Case 1: $q^h=q^l=1$ Given $q^h=q^l=1$, the problem (43) can be decomposed into a bi-level optimization problem. The second layer problem is fixing b^θ in $\Phi_s(w)$ and solves the resulting linear optimization problem with variables q^θ, p^θ, and r^θ. Then, the first layer problem only solves b^θ as solving the original $\Phi(s)(w)$ by utilizing results from the second layer problem. Moreover, the first layer problem can be solved efficiently since two decisions b^h and b^l are separable.

That is, we consider

$$
\begin{aligned}
\max_{b^h,b^l}\quad & \alpha_h(V(w+b^h)-V(w)-k(b^h))+\alpha_l(V(w+b^l)-V(w)-k(b^l))+J(b^h,b^l,w) \\
s.t.\quad & b^h\geq 0 \\
& b^l\geq 0,
\end{aligned}
\tag{98}
$$

where $J(b^h,b^l,w)$ is derived from the problem:

$$
\begin{aligned}
J(b^h,b^l,w):=\max_{p^h,r^h,p^l,r^l}\ & \alpha_h p^h+\alpha_l p^l \\
s.t.\ & p^h+c_h r^h\leq p^l+c_h r^l, \\
& p^l+c_l r^l\leq p^h+c_l r^h, \\
& p^h+c_h r^h\leq \nu_h, \\
& p^l+c_l r^l\leq \nu_l, \\
& w+b^\theta-r^\theta\leq 0,\ \theta\in\{h,l\}, \\
& p^\theta\geq 0,\ \theta\in\{h,l\}.
\end{aligned}
$$

We denote $J(b^h,b^l,w)$ as the primal problem:

$$
\begin{aligned}
\max_{p^h\geq 0,r^h,p^l\geq 0,r^l}\quad & \alpha_h p^h+\alpha_l p^l \\
s.t.\quad & \begin{pmatrix} -1 & 1 & c_h & -c_h \\ 0 & 1 & c_h & 0 \\ 1 & -1 & -c_l & c_l \\ 1 & 0 & 0 & c_l \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} p^l \\ p^h \\ r^h \\ r^l \end{pmatrix}\leq\begin{pmatrix} 0 \\ \nu_h \\ 0 \\ \nu_l \\ -(w+b^h) \\ -(w+b^l) \end{pmatrix}.
\end{aligned}
$$

We then obtain an equivalent dual problem

$$
\begin{aligned}
\min_{\lambda\geq 0}\quad & \lambda_2\nu_h+\lambda_4\nu_l-\lambda_5(w+b^h)-\lambda_6(w+b^l) \\
s.t.\quad & \begin{pmatrix} -1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & -1 & 0 & 0 & 0 \\ c_h & c_h & -c_l & 0 & -1 & 0 \\ -c_h & 0 & c_l & c_l & 0 & -1 \end{pmatrix}\begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \\ \lambda_5 \\ \lambda_6 \end{pmatrix}=\begin{pmatrix} \alpha_h \\ \alpha_l \\ 0 \\ 0 \end{pmatrix}.
\end{aligned}
$$

In the dual problem, we first focus on constraints:

$$-\lambda_1+\lambda_3+\lambda_4=\alpha_l \tag{99}$$

$$\lambda_1+\lambda_2-\lambda_3=-\alpha_h \tag{100}$$

$$c_h\lambda_1+c_h\lambda_2-c_l\lambda_3-\lambda_5=0 \tag{101}$$

$$-c_h\lambda_1+c_l\lambda_3+c_l\lambda_4-\lambda_6=0 \tag{102}$$

We now solve $J(b^h,b^l,w)$ when $w+b^l\le r^*$. We propose a feasible solution to the primal problem:

$$\begin{aligned}&p^h=\nu_l-c_lr^*-c_h((w+b^h)-r^*),\quad p^l=\nu_l-c_lr^*,\\&r^h=w+b^h,\quad r^l=r^*,\end{aligned}$$

and the corresponding objective value is

$$\nu_l-c_lr^*-\alpha_hc_h((w+b^h)-r^*). \tag{103}$$

For the dual problem, we have a feasible solution:

$$\begin{aligned}&\lambda_1=\frac{\alpha_lc_l}{c_h-c_l},\quad \lambda_2=\frac{\alpha_hc_h-c_l}{c_h-c_l},\quad \lambda_3=0,\\&\lambda_4=\frac{\alpha_lc_h}{c_h-c_l},\quad \lambda_5=\alpha_hc_h,\quad \lambda_6=0,\end{aligned}$$

which yields the objective value

$$\frac{\alpha_hc_h-c_l}{c_h-c_l}\nu_h+\frac{\alpha_lc_h}{c_h-c_l}\nu_l-\alpha_hc_h(w+b^h). \tag{104}$$

The feasibility of dual solution requires $\alpha_h\ge\frac{c_l}{c_h}$ and is satisfied by the assumption on α_h. We then rewrite (104):

$$\begin{aligned}&\frac{\alpha_hc_h-c_l}{c_h-c_l}\nu_h+\frac{\alpha_lc_h}{c_h-c_l}\nu_l-\alpha_hc_h(w+b^h)\\=&\frac{1}{c_h-c_l}\left(c_h\alpha_h\nu_h-c_l\nu_h+c_h\alpha_l\nu_l\right)-c_h\alpha_h(w+b^h)\\=&\frac{1}{c_h-c_l}\left(c_h\alpha_h\nu_h-c_l\nu_h+c_h\nu_l-c_h\alpha_h\nu_l\right)-c_h\alpha_h(w+b^h)\\=&\frac{c_h\nu_l-c_l\nu_h}{c_h-c_l}-\alpha_hc_h\left((w+b^h)-\frac{\nu_h-\nu_l}{c_h-c_l}\right)\\=&\frac{c_h\nu_l}{c_h-c_l}-\frac{c_l\nu_h}{c_h-c_l}-\frac{c_l\nu_l}{c_h-c_l}+\frac{c_l\nu_l}{c_h-c_l}-\alpha_hc_h\left((w+b^h)-r^*\right)\\=&\frac{(c_h-c_l)\nu_l}{c_h-c_l}-c_l\frac{\nu_h-\nu_l}{c_h-c_l}-\alpha_hc_h\left((w+b^h)-r^*\right)\\=&\nu_l-c_lr^*-\alpha_hc_h((w+b^h)-r^*).\end{aligned}$$

As a result, the objective value (103) of primal problem is equal to the objective value (104), so the proposed feasible solution is optimal when $w+b^l\le r^*$.

Next, we define

$$\bar{w}_h:=\frac{\nu_h}{c_h},$$

and we solve $J(b^h, b^l, w)$ when $w + b^l > r^*$ and $w + b^h \leq \bar{w}_h$. We propose a feasible solution to the primal problem:

$$\begin{aligned} &p^h = \nu_h - c_h(w+b^h), \quad p^l = \nu_h - c_h(w+b^h) - c_l((w+b^l)-(w+b^h)), \\ &r^h = w+b^h, \quad r^l = w+b^l, \end{aligned}$$

and the objective value is

$$\nu_h - c_h(w+b^h) - \alpha_l c_l((w+b^l)-(w+b^h)). \tag{105}$$

For the dual problem, we have a feasible solution:

$$\begin{aligned} &\lambda_1 = 0, \quad \lambda_2 = 1, \quad \lambda_3 = \alpha_l, \\ &\lambda_4 = 0, \quad \lambda_5 = c_h - \alpha_l c_l, \quad \lambda_6 = \alpha_l c_l, \end{aligned}$$

which yields the objective value

$$\begin{aligned} &\lambda_2 \nu_h + \lambda_4 \nu_l - \lambda_5(w+b^h) - \lambda_6(w+b^l) \\ =&\nu_h - (c_h - \alpha_l c_l)(w+b^h) - \alpha_l c_l(w+b^l) \\ =&\nu_h - c_h(w+b^h) - \alpha_l c_l((w+b^l)-(w+b^h)). \end{aligned} \tag{106}$$

As a result, the objective value (105) of primal problem is equal to the objective value (106), so the proposed feasible solution is optimal when $w + b^h < \bar{w}_h$ and $w + b^l > r^*$.

For $w < \bar{w}_h$, we characterize the function $J(b^h, b^l, w)$:

$$J(b^h, b^l, w) = \begin{cases} \nu_l - c_l s(r^*) - \alpha_h c_h((w+b^h)-r^*), \text{ if } w+b^l \leq r^*, \\ \nu_h - c_h s(w+b^h) - \alpha_l c_l((w+b^l)-(w+b^h)), \text{ if } w+b^l > r^*. \end{cases} \tag{107}$$

The value of $J(b^h, b^l, w)$ changes based on whether $w + b^l$ exceeds r^* or not. We then consider the problem (98) to be

$$\max\{Q_1(w), Q_2(w)\}, \tag{108}$$

where $Q_1(w)$ is defined as

$$\begin{aligned} Q_1(w) := \max_{b^h, b^l} \ & \nu_l - c_l r^* + \alpha_h \left(V(w+b^h) - V(w) - k(b^h) - c_h((w+b^h)-r^*)\right) \\ & + \alpha_l (V(w+b^l) - V(w) - k(b^l)) \\ s.t. \ \ & b^h \geq 0, \\ & b^l \geq 0, \\ & b^l \leq r^* - w, \end{aligned}$$

and $Q_2(w)$ is defined as

$$\begin{aligned} Q_2(w) := \max_{b^h, b^l} \ & \nu_h - c_h(w+b^h) + \alpha_h \left(V(w+b^h) - V(w) - k(b^h))\right) \\ & + \alpha_l \left(V(w+b^l) - V(w) - k(b^l) - c_l((w+b^l)-(w+b^h))\right) \\ s.t. \ \ & b^h \geq 0, \\ & b^l \geq r^* - w, \end{aligned}$$

Both objectives in $Q_1(w)$ and $Q_2(w)$ are concave in b^h and b^l.

Case 2: $q^h = 1$ and $q^l = 0$ Now we derive the case for $q^h = 1$ and $q^l = 0$, the optimization problem (43) becomes

$$\max_{p^h, r^h, b^h} \alpha_h \left(p^h - k(b^h) + V_s(w + b^h) - V_s(w)\right) \tag{109}$$
$$s.t. \quad \nu_h - c_h r^h - p^h \geq 0 \tag{110}$$
$$0 \geq \nu_l - c_l r^h - p^h \tag{111}$$
$$\nu_h q^h - c_h r^h - p^h \leq 0 \tag{112}$$
$$r^h \geq w + b^h, \; p^h \geq 0,$$
$$b^h \in [0, \infty].$$

Constraints (110) and (111) can be combined as

$$\nu_h - c_h r^h \geq \nu_l - c_l r^h,$$

which the constraint

$$r^h \leq \frac{\nu_h - \nu_l}{c_h - c_l} = r^*. \tag{113}$$

Therefore, the problem (109) is only feasible when $r^h \leq w^*$ Moreover, p^h and r^h are linear in (109) and we thus have $p^h = \nu_h - c_h r^h$ from (110) and $r^h = w + b^h$. As a result, (109) becomes

$$Q_3(w) := \max_{b^h} \quad \alpha_h \left(\nu_h - c_h(w + b^h) - k(b^h) + V_s(w + b^h) - V_s(w)\right)$$
$$b^h \leq w^* - w,$$
$$b^h \geq 0.$$

This becomes a single variable optimization where the objective is concave in b^h due to the convexity of $k(b^h)$ and the concavity of $V(w)$.

Case 3: $q^h = 0$ and $q^l = 1$ Lastly, we derive the case for $q^h = 0$ and $q^l = 1$, the optimization problem (43) becomes

$$\Phi_s(w) := \max_{p^l, r^l, b^l} \alpha_l \left(p^l - k(b^l) + V_s(w + b^l) - V_s(w)\right) \tag{114}$$
$$s.t. \quad 0 \geq \nu_h - c_h r^l - p^l \tag{115}$$
$$\nu_l - c_l r^l - p^l \geq 0 \tag{116}$$
$$\nu_l q^l - c_l r^l - p^l \leq 0 \tag{117}$$
$$r^l \geq q^l(w + b^l), \; p^l \geq 0,$$
$$b^l \in [0, \infty].$$

Constraints (115) and (116) can be combined as

$$\nu_h - c_h r^l \leq \nu_l - c_l r^l,$$

which the constraint

$$r^l \geq \frac{\nu_h - \nu_l}{c_h - c_l} = r^*. \tag{118}$$

Therefore, the problem (114) is only feasible when $r^l \geq w^*$ Moreover, p^l and r^l are linear in (114) and we thus have $p^l = \nu_l - c_l r^l$ from (116) and $r^l = w + b^l$. As a result, (114) is equivalent to

$$\begin{aligned} Q_4(w) := \max_{b^l} \quad & \alpha_l \left(\nu_l - c_l(w + b^l) - k(b^l) + V_s(w + b^l) - V_s(w)\right) \\ & b^l \geq w^* - w. \end{aligned}$$

This becomes a single variable optimisation where the objective is concave in b^l due to the convexity of $k(b^h)$ and the concavity of $V(w)$.

As a result, we show that

$$\max_{b^\theta \geq 0} \{\Phi(b^h, b^l, w)\} = \max\{\max\{Q_1(w), Q_2(w)\}, Q_3(w), Q_4(w), 0\}. \tag{119}$$

Due to the feasible region on the state w, we then separate (119) based on w and simplify as (44)-(51) which completes the proof.