Structures of Optimal Contracts in Dynamic Mechanism Design with One Agent

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We consider dynamic mechanism design problems in which a principal procures up to one unit of a product/service in every period from an agent who is privately informed about its marginal production cost in each period, which are i.i.d. random variables. Continued participation by the agent is costly: the agent needs to consume a fixed amount in each period for the partnership to continue. We consider two dynamic models which differ mainly in the principal’s credibility on deferring compensation into the future. In one model, the agent has sufficient trust in the principal’s credibility to enter into a long term contract, and therefore thinks dynamically about how each period’s report affects future payments. In the second model, the agent does not accept promises of future payments from the principal. Therefore, incentives in each period have to be provided through cash payments.

We provide regularities conditions on the distribution of the private information for both models such that the optimal contracts offer at most two different procurement levels depending on the reported production cost in each period. These results potentially facilitate tractable computational procedures. Technically, our results generalize the “ironing” technique introduced in the seminal paper Myerson [12] to dynamic settings, and is based on infinite-dimensional optimization theory. The analysis focuses on what we call the “dynamic virtual valuation,” a generalization of the Myersonian virtual valuation in the static setting, and is potentially relevant to other dynamic mechanism design problems. Furthermore, we establish additional properties of the optimal policy and value functions. In particular, in the second model, where no promise of future payment is allowed, we show that the dynamic approach reduces the efficiency loss from the asymmetry of information, compared with its single period counter-part. That is, the optimal dynamic contract induces more trade than the optimal static contract in each period.

Key words: dynamic mechanism design, optimization, dynamic programming, dynamic virtual valuation

1. Introduction

In his seminal work, Myerson [12] analyzed what is arguably the most fundamental mechanism design model, in which a seller (principal) aims at selling one unit of an item to one of several potential buyers (agents) with private information on the item’s valuation. Under mild regularity conditions concerning the distribution of the item’s private valuation, it can be shown that when there is only one potential buyer, a take-it-or-leave-it offer (a single selling price)
characterizes the optimal mechanism. The selling price is characterized by the so-called “virtual valuation” crosses 0, which summarizes the marginal value the trade brings to the principal for each agent type, adjusted to incorporate the agent’s incentives. The regularity conditions guarantee that the virtual valuation is monotonically increasing in the agent’s private valuation, such that the optimal price is set at the point where the virtual valuation is zero. This seminal work initiated a large literature on mechanism design and auctions [see, for example, 11, 7]. When the regularity conditions do not hold, Myerson further showed that the optimal mechanism is slightly more complex, but can still be characterised by various levels of purchase quantities, determined by the shape of the virtual valuation. The technique is often referred to as “ironing” in the literature [see, for example, 15].

Our paper focuses on analyzing dynamic versions of this fundamental model in procurement (rather than purchasing) settings, and generalize the ironing technique to dynamic mechanism design. In particular, we consider two different dynamic models in which a principal repeatedly procures up to one unit of product/service from an agent whose variable production costs over time are commonly known i.i.d. random variables. In both settings, partnership is costly, i.e., a fixed cost of consumption occurs in each period. In addition, both models allow for endogenous termination of the partnership, which reflects a key trade-off for the principal that only occurs in dynamic settings. The main distinction between the two models we study is whether the principal has the credibility of issuing long term contracts, in the form of promised payments in the future.

In the first setting, the principal has the credibility of issuing long term contract with the agent (i.e. commitment power). Correspondingly, the agent thinks dynamically about how each period’s revelation of private information affects future payments. We refer to this model as the “model (P),” standing for promises of future payments, or promised value, as referred to in this literature [9].

In the second setting, the agent perceives the collaboration myopically, and does not accept promises of future payments. Therefore, the principal has to provide incentive in each period by cash payment. The agent has access to a bank account which cumulates surplus beyond a fixed consumption $S$ in each period, or fund this consumption if the net payment from the principal in a period is below $S$. The partnership would terminate if the balance in the cash account is not high enough to cover this fixed cost. We refer to this model as the “model (C),” standing for cash payment in each period.

Similar to the regularity conditions in Myerson [12], we provide conditions on the distributions of uncertain private information for each of the two models, under which the optimal procurement
quantity in each period depends on at most two thresholds on the revealed marginal production cost in the period. That is, the principal procures the full unit in a period if the marginal production cost is lower than a lower threshold, and procures nothing if the production cost is greater than a higher threshold. In between these two thresholds, the optimal procurement level is a constant between zero and one. A take-it-or-leave-it offer is a special case of the two threshold policy, when the two thresholds coincide with each other. In particular, for model (P), we provide a condition under which there is only one threshold, similar to the basic static setting. When the distribution of the marginal production cost is uniform, this condition is not satisfied, but the optimal procurement quantity still follows a two threshold structure, in which the higher threshold is the upper bound of the support of the distribution. The two threshold optimal procurement policy structure generalizes the simple take-it-or-leave-it contract structure in Myerson [12]. Fundamentally, such two threshold policies arise as a consequence of the interplay between ironing (due to the agent’s incentives) and the principal’s dynamic optimization (the latter is not present in the static setting). These features highlight the additional complexities of optimal mechanism design in dynamic settings.

In order to demonstrate aforementioned results, we generalize the concept of virtual valuation in the basic Myerson model to what we call *dynamic virtual valuations*. This generalization constitutes the methodological contribution of this work. For each of the two models, the dynamic virtual valuation consists of two components: one from the current period’s profits, resembling virtual valuation in the basic Myerson model, and the other based on the principal’s value function to account for future profits. In each of the two models, we characterize the dynamic virtual valuation as a functional derivative. Since the value function is defined through a Bellman equation, the structural properties of the dynamic virtual valuation are much more involved than in the corresponding basic single-period setting. In the spirit of Myerson’s approach, our regularity conditions on input parameters still guarantee that the dynamic virtual valuations have certain structures. For example, in model (P), one set of sufficient conditions guarantees that the dynamic virtual valuation monotonically non-increasing, which yields the aforementioned single threshold procurement structure. For uniformly distributed marginal production cost, on the other hand, we show that the dynamic virtual valuation is convex, which yields the two threshold procurement structure, following arguments in infinite dimensional optimization. By contrast, in model (C), the sufficient conditions guarantee that the dynamic virtual valuation is monotonic in the marginal production cost almost anywhere, except for one possible discontinuity, which gives rise to the possibility of the two threshold policy. We believe the concept of dynamic virtual valuation will also play a central role in other dynamic mechanism design settings.
Our models also reveal interesting dynamics associated with the optimal mechanism. For example, according to model (C), it might be optimal for the principal to terminate interactions with the agent despite the fact that the expected gain from each game stage is positive. Furthermore, we show that the principal’s procurement amount is always higher compared with the single period model because of the incentives generated by potential future gains.

One of the two models that we study in this paper, Model (P), is based Krishna et al. [6]. However, our paper differs from Krishna et al. [6] by deriving results on the structure of the optimal contract based on the ironing technique. In contrast, Krishna et al. [6] characterizes interesting dynamics, which can be used to interpret the so-called “sweat equity” phenomenon. Practical examples of their model are drawn from the venture capital market, where founders launch a business based on their personal expertise but do not possess sufficient financial resources, from franchising situations, where an owner wants to expand into a specific market but lacks idiosyncratic knowledge about local factors, and from other areas.

Our paper contributes to the growing literature concerning revenue-maximizing dynamic mechanism design problems initiated by Baron and Besanko [1], in which the authors consider a principal with full commitment power and an agent that privately observes costs over two periods. Dynamic contracting and dynamic mechanism design have attracted much more attention in recent years. Battaglini [2], for example, considers revenue-maximizing long-term contracts from a monopolist in a model with an infinite time horizon when the valuation of the buyer changes in a Markovian fashion over time. There is another recent stream of literature concerning dynamic mechanism design that focuses on socially efficient allocation [see, for example, 3, and references therein].

A recent paper, Pavan et al. [13], proposes a general framework for dynamic mechanism design that allows for multiple agents with serially correlated private information, as long as utilities are quasi-linear and each agent’s private information is unidimensional in each period. The paper provides envelope-theory-type results for the derivative of an agent’s expected equilibrium utility with respect to private information, and describes how to construct optimal payments given an allocation rule. Although our model, along with many other existing models of dynamic adverse selection, is a special case of the modeling framework of [13], our analysis and results are new and have a different focus.

Methodologically, the recursive formulations of our dynamic mechanism design model with a dynamic agent date back to Thomas and Worrall [14]. That paper reveals, in a specific dynamic adverse selection problem, that repeated principal-agent interactions can be modeled as a dynamic optimization that maximizes the principal’s utility subject to the agent’s incentive compatibility
constraint. Interestingly, the agent’s total future utility serves as a state variable in the principal’s dynamic optimization problem. Such a recursive formulation is also valid in our Model (P). Fernandes and Phelan [5] extends the problem studied in [14] to an endowment process where the private information is serially correlated over time, and demonstrates that the incentive compatibility constraints can still be formulated in a recursive manner. This, again, allows for dynamic programming representations of the problem.

Dynamic contracting problems have also received considerable attention recently in the Operations Research and Management Science literature. Li et al. [8], for example, study a long-term moral hazard problem for a firm seeking to induce efforts from competing suppliers, while Zhang et al. [18] and Lobel and Xiao [10] study dynamic adverse selection models that seek to efficiently manage inventory systems. In particular, the modeling framework of Li et al. [8] also contains promised utility, which captures agents’ value functions under the optimal contract. Zhang [16, 17] study theoretical properties and solution approaches to dynamic adverse selection problems with serially correlated private information.

In the remainder of this section, we briefly revisit a basic mechanism design model in a static setting, which serves as a foundation for the two dynamic models to be introduced in Section 2. Sections 3 and 4 provide the analysis and results for these two dynamic models, respectively. In these two sections, we also provide one numerical example for each of the two dynamic models to illustrate the optimal policy structure. Finally, Section 5 offers some concluding remarks.

1.1. Benchmark Static Model The most basic mechanism involves a seller seeking to profit from selling one unit of an item to a buyer, whose valuation on the item is unknown to the seller. See, for example, Krishna [7, Chapter 5]. In order to lay the foundation for the rest of the paper, here we present the equivalent procurement version of the basic mechanism design problem – a buyer trying to procure up to one unit of a product/service (later referred to as “good”) from a seller who has private information concerning the marginal production cost – as the benchmark model. Our paper focuses on analyzing dynamic versions of this basic model. The model and results presented in this section are well known, but, in our view, they serve as important foundations for the rest of the paper.

Specifically, we assume that the principal (buyer) values the good at unit price $p$. The agent (producer) is privately informed about its type (variable production cost) $c$, which is drawn from a commonly known distribution $F$ with support $[c, \bar{c}]$. With no loss of generality we focus on a direct mechanism $(q(c'), m(c'))$, in which $q(c')$ is the purchase quantity and $m(c')$ the monetary
payment from the principal to the agent, both depending on the reported type $c'$. In designing the mechanism, the principal solves the following optimization problem.

\[
J_S = \max_{q,m} \mathbb{E}[pq(c) - m(c)] \\
\text{s.t.} \\
m(c) - cq(c) \geq m(c') - cq(c') & \quad \forall c, c' \in [\underline{c}, \bar{c}], \\
m(c) - cq(c) \geq 0 & \quad \forall c \in [\underline{c}, \bar{c}].
\]

Here the principal tries to maximize the expected profit while ensuring that the agent truthfully reports the type, captured by the incentive compatibility (IC) constraint, and is willing to participate, reflected in the individual rationality (IR) constraint.

The following result is well known and is attributed to Myerson [12].

**Lemma 1** The optimal procurement function $q$ solves the following optimization model

\[
\max_q \mathbb{E}[\rho(c)q(c)] \\
\text{subject to } 0 \leq q \leq 1 \text{ and } q(c) \text{ non-increasing in } c, \text{ where} \\
\rho(c) = p - c - \frac{F(c)}{f(c)}.
\]

Function $\rho$ is commonly referred to as the “virtual valuation” in the literature. This celebrated result implies that under the assumption that this virtual valuation is non-increasing in $c$, which is ensured when $F(c)/f(c)$ is non-decreasing, the optimal mechanism is a simple take-it-or-leave-it offer. Indeed, this follows directly from the optimization formulation in Lemma 1. To be specific, monotonicity of $\rho(c)$ implies that there exists a threshold $\hat{c} \in [\underline{c}, \bar{c}]$, such that $\rho(c) \geq 0$ if and only if $c$ is below the threshold. As a result, the maximization problem is “separable,” and it is optimal to set procurement quantity $q^*(c) = 1$ when $c$ is below the threshold $\hat{c}$, and $q^*(c) = 0$ when $c$ is above the threshold.

In what follows, we extend the above static model to two different dynamic settings. In each setting, we propose a corresponding concept of virtual valuation that we call “dynamic virtual valuation.” Each dynamic virtual valuation has two components, one from the current period’s profits, and the other based on the principal’s value function to account for future profits. Since dynamic virtual valuation depends on the value function, which is defined through the Bellman Equation, the structural properties of dynamic virtual valuations are much more involved than the static virtual valuation. In the spirit of Myerson’s approach, we provide conditions on the
probability distributions of private information such that the dynamic virtual valuations have certain structures. In turn, these structures allow us to establish the optimality of one- or two-threshold-structures of optimal procurement policies.

2. Dynamic Models

In this section, we consider two extensions of the static model introduced in Section 1.1 to two different dynamic settings. That is, we assume that the principal designs a direct mechanism to procure up to one unit of a good from the agent at each (discrete) time period over an infinite time horizon. Similar to the static setting, in each period \( t \) the agent is privately informed about the marginal production cost \( c \), which follows a commonly known distribution \( F \) with support \([c, \bar{c}]\), which are assumed to be i.i.d. over time periods. Our goal is to characterize the impact of the dynamics on the optimal contracts. The next two subsections present the two settings, respectively.

2.1. Promised Value Model (P)

The promised value model, in which the principal has the commitment power and the agent is forward looking, is very similar to the model studied in Krishna et al. [6]. In particular, the principal commits to a dynamic mechanism, which determines the procurement quantity \( q_t \) and payment \( m_t \) in each period \( t \), depending on the entire history of the reported private information. Correspondingly, the agent decides what information to reveal, taking into consideration its impact on current as well as future payments. In each period, in addition to the income \( m_t \) and variable production cost \( c_t q_t \), the agent also bears a cost \( S \), representing, for example, consumption or rent and salary payments. Overall, the agent’s utility is represented by the discounted summation of single period profits \( m_t - c_t q_t - S \) over the entire future, which may be affected by the current period’s report through the mechanism. Similar to [14], the repeated strategic interactions between the principal and the agent can be modeled as a dynamic program in which the agent’s utility (promised value) serves as a summary statistic of the history, and therefore is the state variable.

Formally, denote \( h_t = \{c_0, \ldots, c_t\} \) to represent the production costs from the beginning of the time horizon to period \( t \), that is observable only by the agent. Consider “direct mechanisms” under which the agent reports (truthfully or not) the production cost in each period. Denote \( h'_t = \{c'_0, \ldots, c'_t\} \) to represent the reported history. The procurement quantity \( q_t \) and monetary payment \( m_t \) in each period depend on the entire reported history \( h'_t \). In general, the agent’s utility depends on both the true history \( h_t \) and reported history \( h'_t \), as well as the agent’s reporting
strategy, denoted as $i$, that maps $(h_t, h'_{t-1})$ into a report $c'_t$ in each period $t$. Therefore, the agent’s total discounted utility can be represented as

$$w_0(i) = \mathbb{E} \left[ \sum_{t=0}^{\infty} \beta^t \left( m_t(h'_t) - c_t q_t(h'_t) - S \right) \right],$$

where $\beta$ is the discount factor, in which for any given true history $h_t$ and reported $h'_{t-1}$, the new report is

$$h'_t = (h'_{t-1}, i(h_t, h'_{t-1})).$$

Following the “revelation principle,” without loss of generality, we can focus on direct mechanisms, under which the agent truthfully reveal the marginal production cost $c_t$ in each period. Therefore, the principal can maximize over mechanisms such that any history-dependent reporting strategy must not outperform the one that always reports the truth for the agent. Denote $\bar{i}$ to represent a “truthful reporting strategies,” such that if truth has been reviewed so far, strategy $\bar{i}$ keeps reporting the truth. Formally, $\bar{i}_t((h_{t-1}, c_t), h_{t-1}) = c_t$. The “revelation principle” states in any equilibrium of the aforementioned Bayesian game, there exists a payoff equivalent mechanism under which the truth revelation strategy $\bar{i}$ is the best response by the agent. Therefore, the principal can maximize over mechanisms that satisfy the following condition

$$w_0(\bar{i}) \geq w_0(i), \quad \forall i.$$

Note that the incentive compatibility constraint (2) requires that if an agent has been reporting true costs, he or she is better off by continuing to tell the truth, or,

$$m_t(h_{t-1}, c) - c q_t(h_{t-1}, c) + \beta w_t(h_{t-1}, c) \geq m_t(h_{t-1}, c') - c q_t(h_{t-1}, c') + \beta w_t(h_{t-1}, c'), \quad \forall h_{t-1}, c, c', \text{ and}$$

$$w_t(h_t) = \mathbb{E} \left[ m_{t+1}(h_t, \bar{c}) - \bar{c} q_{t+1}(h_t, \bar{c}) + \beta w_{t+1}(h_t, \bar{c}) \right].$$

The following lemma formally establishes the equivalence between the original incentive compatibility constraint (2) and the recursive formulation (3), similar to Lemma 2.1 in [5].

**Lemma 2** Mechanism $\{m_t, q_t\}$ satisfies incentive compatibility constraints (2) if and only if it satisfies the recursive formulation (3).

Focus on the principal’s decision in period $t$. In order to ensure incentive compatibility, the principal also needs to commit to future payments summarized by the agent’s value function $w_t$. As a result, in the beginning of the subsequent period $t + 1$ (and therefore in each period), the committed future payment $w_t$ has an impact on the principal’s current period decision, and therefore needs to be in the state space of the principal’s optimization problem. The principal, whose time discount factor is
also \( \beta \), faces the following dynamic optimization problem. Here we use \( J_t(h_{t-1}, v) \) to represent the principal’s value function, \( v = w_{t-1}(h_{t-1}) \) to represent the agent’s value function at the beginning of the period, and \( w \) to replace \( w_t \).

\[
J_t(h_{t-1}, v) = \max_{q, m, w} \mathbb{E}[pq(c) - m(c) + \beta J_{t+1}({h_{t-1}, c}, w(c))] 
\]

subject to the following constraints, in light of (3),

\[
m(c) - cq(c) + \beta w(c) \geq m'(c') - cq(c') + \beta w(c') , \quad \forall c, c' , \quad (IC_d) 
\]

and

\[
v = \mathbb{E}[m(c) - cq(c) + \beta w(c)] - S . \quad (PK) 
\]

Constraint (PK) is commonly referred to in the literature as the “promise keeping” constraint. Here we omit the decision variables’ dependence upon history in the expressions because they already are implied from the state variable \( h_{t-1} \). It can also be verified that the optimal value function \( J_t \) is constant over its first argument \( h_{t-1} \) for a given \( v \). That is, the principal’s value function and the optimal dynamic mechanism depends on history only through the agent’s value function \( v \) in the beginning of the period. Furthermore, we still require constraint (IR) as in the static model, such that variable production costs need to be reimbursed. Note that we do not require the fixed cost to be paid in cash in each period under this (IR), so it is assumed that the agent is able to fund the fixed cost. (In an alternative setting, one can replace (IR) with \( m(c) - cq(c) - S \geq 0 \), which is a tighter constraint, and therefore yields less value for the principal. Our analysis demonstrates that the dynamics of the system in our model setting are different, and, arguably, more interesting, than in this alternative setting.)

Because the agent’s future value function \( w(c) \) (often referred to as the “promised value” [9]) is one of the decision variables, it is important to include some additional constraints to ensure economic reality as well as the model being bounded mathematically. First, the agent is only willing to continue the collaboration with the principal if the agent’s value function is non-negative, i.e., \( w(c) \geq 0 \). Furthermore, the principal cannot promise a total utility from future payments that is higher than the maximal potential profit one can generate over time. In particular, the highest value that can be generated from the system is through the so-called “first best” production policy \( \bar{q}(c) \), defined as

\[
\bar{q}(c) = \begin{cases} 
1 , & c \leq p , \\
0 , & c > p .
\end{cases} \quad (4) 
\]

Therefore, the first best profit in each period is \( \mathbb{E}[(p - c)\bar{q}(c)] - S \) for the system. As a result, the agent’s value function \( w(c) \) cannot exceed the present value of first best cash flow,

\[
\bar{v} = \frac{\mathbb{E}[(p - c)\bar{q}(c)] - S}{1 - \beta} . \quad (5) 
\]
We also assume that $\bar{v} \geq 0$ (otherwise the principal would not participate in the game). Equivalently, this implies the following condition on the fixed cost $S$

$$S \leq \mathbb{E}[(p - c)\bar{q}(c)] .$$

To summarize, we have the following box constraint for the promised value $w(c)$,

$$0 \leq w(c) \leq \bar{v} \quad \forall c \in [\underline{c}, \bar{c}].$$

Overall, the dynamic mechanism design problem considered in this section can be modeled as the following dynamic optimization problem, in which $J_\pi$ represents the principal’s value function, whose time discount factor is also $\beta$.

$$J_\pi(v) = \max_{q,m} \mathbb{E}[pq(c) - m(c) + \beta J_\pi(w(c))]$$

s.t. (IC\textsubscript{d}), (IR), (PK), and (BW).

2.2. Cash payment model (C) In this setting, the agent perceives the collaboration with the principal myopically. That is, in each period, the agent only aims at maximizing the profit of the current period. Furthermore, the profit from each period is cumulated in a cash account, which is used to fund a fixed consumption $S$ per period. If the payment in a period exceeds the production cost and the consumption $S$, the cash account balance increases. Otherwise, the cash account is depleted. We assume that the initial balance of the cash account is common knowledge. According to the revelation principle, the principal always knows the account balance following a direct mechanism. We denote $x$ to represent the account balance. In the beginning of each period, depending on $x$, the principal announces a mechanism that includes the quantity-payment pair, $q(c)$ and $m(c)$, that satisfy the (IC) constraint as in the static model. We also assume that the mechanism has to satisfy the same (IR) constraint, so that the agent cannot be forced into using the cash account to fund the variable production cost. In addition, before knowing each period’s marginal production cost $c$, it is natural to assume that the agent is willing to participate only if the expected profit from production exceeds the fixed cost $S$, that is, $\mathbb{E}[m(c) - cq(c)] \geq S$. In certain applications such a constrain may not be required. For example, a retiree may be willing to work on a part-time job that does not pay to fully cover consumption. Therefore, we assume the following constraint, which is mathematically more general, for a generic parameter $S'$ that represents a required compensation

$$\mathbb{E}[m(c) - cq(c)] \geq S'.$$

When $S' = S$, we recover the assumption that the expected profit needs to exceed the fixed cost. When $S' = 0$, on the other hand, the constrain is automatically satisfied with (IR), and therefore...
is removed. For more general $S'$ values, constraint (ES) allows for other potential compensation deals.

Thus, given the current period cash account balance $x$, and the marginal production cost $c$, the account balance in the beginning of the next period becomes

$$x^+(c) = (x + m(c) - cq(c) - S)/\beta , \quad \forall c \in [c, \bar{c}] . \quad (AB)$$

in which $\beta \in (0,1)$ reflects the interest payment to the cash account. The strategic interaction between the principal and agent continues only if $x^+(c) \geq 0$.

The principal’s objective in designing the contract is to maximize the total future payoff with a discount factor $\beta$. In a recursive form, denote $J_C : \mathbb{R} \rightarrow \mathbb{R}$ to represent the value function of the principal that depends on the cash account balance $x$. The mechanism design problem can be expressed as the following dynamic optimization model. For $x < 0$, we have $J_C(x) = 0$. For non-negative $x$, we have

$$J_C(x) := \max \{ \max_{q,m} E[pq(c) - m(c) + \beta J_C(x^+(c))], 0 \} \quad \text{s.t. } \quad (IC), (IR), (ES) \text{ and } (AB) , \quad (C)$$

where the outer maximization reflects that the principal may terminate the strategic interaction whenever it becomes too costly to provide an expected profit exceeding $S$. Alternatively, the principal can also terminate the interaction with the agent by not replenish the cash account so that the fixed consumption $S$ drives the cash account balance to below zero. Therefore, our model studies the optimal way to dynamically adjust the time horizon of the collaboration. Similar to condition (6) here we assume that the required compensation $S'$ does not offset the gains of trade, namely

$$S' \leq E[(p - c)\bar{q}(c)] . \quad (7)$$

**Remark 2.1 (Extreme Cases and Dynamics)** In the special case with fixed cost $S = 0$, the dynamics change considerably. In fact, as will be explained in the next section in the analysis of this model, the problem reduces to repetitions of static mechanism design problems as presented in Section 1.1.

Consider another case, in which the cash account balance is high enough such that $x \geq \bar{x} := S/(1 - \beta)$. Constraints (AB) and (IR) imply that the next period cash balance $x^+ \geq x > S$. This, essentially, removes constraints on participation due to fixed costs. Therefore, similar to model (P), in which promised value $v$ is defined on the interval $[0, \bar{v}]$, interesting dynamics occur in model (C)

1 In Section 4, we show that under this assumption the outer “max” in (C) can be removed.
only when \( x \) falls in the interval \([0, \bar{x})\), and when there is uncertainty whether the partnership will eventually dissolve \((x < 0)\) or become permanent \(x \geq \bar{x}\).\(^2\)

In the following two sections we analyze the two dynamic models, respectively.

3. Optimal policy for Model \((P)\) In this section we analyze model \((P)\) introduced in the last section, where the principal faces a dynamic agent.

We begin with Lemma 3, which removes the decision variables \(m\) in the direct mechanism from the optimization formulation \((P)\), following standard treatment of the \((IC_d)\) and \((IR)\) constraints [see, for example, Chapter 5 in 7].

**Lemma 3** Model \((P)\) is equivalent to

\[
J_P(v) = \max_{(q,w) \in \Pi(v)} E[(p - c)q(c)] - v - S + \beta E[J_P(w(c))] \tag{8}
\]

for \(v \in [0, \bar{v})\), in which \((q, w)\) are subject to the following constraints.

\[
0 \leq q(c) \leq 1, \quad q(c) \text{ is non-increasing, } E[Q(c)] \leq v + S, \tag{9}
\]

and

\[
0 \leq w(c) \leq \min \left\{ (v + S - E[Q(c)] + Q(c))/\beta, \bar{v} \right\}, \tag{10}
\]

where \(Q(c) := \int_{\underline{c}}^c q(t)dt\) is the information rent for an agent with type \(c\).

It is convenient to study the total value function of the system that combines the principal’s and agent’s value functions, \(V(v) = J_P(v) + v\). It follows that solving \((8)\) is equivalent to solving

\[
V(v) = (\Gamma V)(v) = \max_{(q,w)} E[(p - c)q(c)] - S + \beta E[V(w(c))] \tag{11}
\]

subject to \((9)\) and \((10)\).

Next we define the agent’s cumulative expected surplus \(v^*\) associated with the first-best allocation \(\bar{q}\),

\[
v^* = \frac{E\bar{Q}(c) - S}{1 - \beta} = \frac{1}{1 - \beta} \left\{ \int_{\underline{c}}^{\bar{c}} F(c)dc - S \right\}, \tag{12}
\]

where \(\bar{q}\) is defined in \((4)\), and \(\bar{Q}(c)\) is given by \(\bar{Q}(c) = \int_{\underline{c}}^c \bar{q}(t)dt\). The following result slightly generalizes Krishna et al. [6] to cases with \(S \geq 0\) and \(c \in [\underline{c}, \bar{c}]\).

\(^2\)Reaching the region of \(x \geq \bar{x}\) can be perceived as the agent gaining permanent employment status. To avoid confusion about \(x\) approaching infinity after \(x > \bar{x}\), we can revise the model to allow the agent to keep the cash account balance at \(\bar{x}\) and consume the rest in each period. Such a revision is equivalent to the original model.
Proposition 1 (Structure of Value Function, essentially in Krishna et al. [6]) The following properties hold:

(i) $V$ is an increasing and concave function on $[0, \bar{v}]$.

(ii) $V(v) = \bar{v}$ for all $v \geq v^*$, and $(q(c) = \bar{q}(c), w(c) = v^*)$ is an optimal solution to $(IV)(v^*)$.

(iii) For any feasible policy $q$ in (11) at a given $v \in [0, \bar{v}]$, there is an optimal promised utility function $w$ that satisfies

$$w(c) = \min \{ (v + S - \mathbb{E}[Q(c)] + Q(c))/\beta, \; v^* \}.$$  \hspace{1cm} (13)

Furthermore, the optimal net payment $m(c) - cq(c) = \max \{ (v + S - \mathbb{E}[Q(c)] + Q(c))/\beta - v^*, 0 \}$.

Proposition 1(iii) implies that the principal pays a fixed salary to reimburse the agent’s variable production cost only to secure the agent’s participation. Rewards for better types (lower marginal production costs) are in the form of future payments, or promised value, as much as possible.

Proposition 1(ii) further states that $v^*$ is an absorbing state of promised value, following this specific optimal mechanism. At this point, the principal repeatedly procures according to the first-best allocation. It is worth noting that the optimal mechanism is not unique. An alternative optimal mechanism allows the promised value to increase to a different absorbing state, $\bar{v}$, at which point the principal’s value function $J_P(\bar{v})$ remains at 0. This means that the principal essentially operates as if the enterprise belongs to the agent. This is consistent with the “sweat equity” interpretation provided in Krishna et al. [6].

Proposition 1 allows us to remove the promised value decision $w(c)$ from the optimization problem (11) with (13). Furthermore, because function $V(v)$ is a constant on $v \in [v^*, \bar{v}]$, if we extend the domain of $V$ such that $V(v) = \bar{v}$ for any $v > \bar{v}$, we can further simplify (11) by replacing $w(c)$ with $g(c)$, defined as

$$g(c) = \frac{v + S - \mathbb{E}[Q(c)] + Q(c)}{\beta}.$$  \hspace{1cm} (14)

Therefore, the optimization on the right hand side of dynamic program (11) can be expressed as $\max_q G_P(q, v)$ subject to constraint (9), where we define

$$G_P(q, v) = \mathbb{E} [(p - c)q(c)] - S + \beta \mathbb{E} [V(g(c))]$$ . \hspace{1cm} (15)

At this point, we can see some of the impact of including a positive fixed cost $S$ in the model. In the case of $S = 0$, if in a period the optimal promised value $v$ becomes zero, then constraint $\mathbb{E}[Q(c)] \leq v$ implies that the future procurement amount and promised value will remain zero.
Therefore, state \( v = 0 \) is an absorbing state, at which point the collaboration terminates. When \( S > 0 \), on the other hand, even if \( v = 0 \), the (PK) constraint guarantees that \( w(c) > 0 \) for some \( c \in [c; \bar{c}] \). Therefore, there is always a chance that the collaboration will revive, as long as the agent can finance the fixed cost \( S \).

3.1. Dynamic Virtual Valuation and Optimal Procurement Structure  Recall the benchmark static model as shown in Section 1.1. In that model, the optimal procurement policy is a step function with one threshold, when the virtual valuation \( \rho(c) \) is decreasing in \( c \). This monotone virtual valuation condition ensures “single crossing” – \( \rho(c) \) crosses zero once. Under such a condition the optimization problem becomes separable in \( c \). Here, for Model (P), we derive a similar quantity, which we call the “dynamic virtual valuation.” This generalization of the virtual valuation relies on the functional derivative of the objective function \( G_{\xi}(q,v) \) for maximization over \( q \).

Proposition 2  For any measurable function \( h: [c; \bar{c}] \to \mathbb{R} \), we have
\[
\frac{d}{d\delta} G_{\xi}(q + \delta h, v) \bigg|_{\delta=0} = \mathbb{E}[h(c)\xi_{\xi}(c)],
\]
where the dynamic virtual valuation \( \xi_{\xi} \) is defined as
\[
\xi_{\xi}(c) = p - c - \frac{F(c)}{f(c)} \left( \int_{c}^{\bar{c}} V'(g(\tau))f(\tau)d\tau \right) + \frac{1}{f(c)} \left( \int_{c}^{\bar{c}} V'(g(\tau))f(\tau)d\tau \right).
\]

Unlike the (static) virtual valuation \( \rho(c) = p - c - F(c)/f(c) \), the dynamic virtual valuation \( \xi_{\xi}(c) \) contains additional terms that depend on the value function \( V \). The dynamic virtual valuation \( \xi_{\xi}(c) \) therefore captures the value of future partnerships with the agent when the current marginal production cost is \( c \). In the static model, the simple assumption of \( F(c)/f(c) \) non-decreasing in \( c \) implies single crossing as mentioned before. The monotonicity of dynamic virtual valuation \( \xi_{\xi}(c) \), however, requires a different condition and is harder to establish. Even if it is monotonic, there is an additional constraint, \( \mathbb{E}[Q(c)] \leq v + S \), which prevents the optimization problem from being separable in \( c \) as in the static model (see Lemma 1). However, we are still able to characterize the following structures of the dynamic virtual valuation under mild conditions on the distribution of the production cost, which are essential in showing the optimality of the corresponding structures of the optimal procurement levels.

\(^3\)The alternative setting of the (IR) constraint in the paper, where each period’s payment has to cover the fixed cost \( S \), yields the same dynamics as the \( S = 0 \) case. The only difference is that each period’s payment \( m \) is increased by \( S \).
Theorem 1 (Properties of Dynamic Virtual Valuations, Model (P))

1. Assume that the production cost distribution satisfies the following sufficient condition,

\[
\frac{1 - F(c)}{f(c)} \text{ is non-decreasing in } c. \quad \text{(Suff)}
\]

Then the dynamic virtual valuation \( \xi_P(c) \) is decreasing in \([c, \bar{c}]\).

2. Assume that the production cost distribution follows a uniform distribution, which does not satisfy condition (Suff).\(^4\) The dynamic virtual valuation \( \xi_P(c) \) is convex in \(c\) on \([c, \bar{c}]\).

It is worth noting that exponential distributions and Pareto distributions satisfy condition (Suff).

The structure of the dynamic virtual valuation implies the following optimal policy structure, which constitutes the main result for model (P).

Theorem 2 (Structure of Optimal Procurement, Model (P))

1. Under condition (Suff), the optimal procurement policy \( q^* \) follows a single threshold \( \bar{c}(v) \) policy structure, that is, there is a threshold \( \bar{c}(v) \) such that it is optimal to procure a whole unit when \( c < \bar{c}(v) \), and 0 unit otherwise.

2. If the production cost follows a uniform distribution, the optimal procurement policy \( q^* \) follows a \((\bar{c}(v), \gamma(v))\) policy structure, under which it is optimal to procure a whole unit when \( c < \bar{c}(v) \), and a partial \( \gamma(v) \in [0, 1] \) unit otherwise.

The structure of optimal procurement policies revealed in Theorem 2 can be different from the structure in the static model. This highlights the difference between static and dynamic models. On the other hand, the structure of the optimal policy remains simple, which can be directly used to develop computational algorithms.

The proof of Theorem 2(2), which is detailed in the Appendix, corresponds to the “ironing” technique of Myerson [12]. In the next subsection, we present a numerical example for the uniform distribution case to illustrate the two-threshold structure described above.

3.2. A numerical example

Here we present a numerical example for the optimal mechanism and the value function associated with model (P), for which the marginal production cost follows a uniform distribution. According to Theorem 1, the structure is more complex in this case. We use the value iteration algorithm, and exploit the structural results of the optimal policy established in

\(^4\) In the proof we provide a sufficient condition, beyond the uniform distribution, for the result to hold.
the previous section. In particular, we can restrict attention to searching for one or two thresholds candidate solutions for the optimal procurement $q(c)$, which further depends on the state of the dynamic program, the promise value $v$.

For simplicity, we focus on numerical examples where production cost $c$ has a uniform $[0,1]$ distribution. We have tested a large number of examples and present one of them here to clearly illustrate our points. Other model parameters in the examples are $p = 1.5$, $S = 0.1$ and $\beta = 0.85$.

Figure 1 depicts the value function $V(v)$, which is increasing concave and converging to $\bar{v} = 6$ following Proposition 1. The figure also depicts the corresponding principal’s value function $J_P(v)$. Function $J_P(v)$ is maximized at $v = 0.4$. Therefore, at the beginning of the time horizon, the principal proposes promised utility $v = 0.4$ as the starting point of the mechanism. Furthermore, in this setting $v^* = 2.6667$, even though function $V(v)$ appears essentially constant when $v$ is less than $v^*$.

![Figure 1. Optimal value function $V(v)$ and $J_P(v)$, with $p = 1.5$, $S = 0.1$, $\beta = 0.85$, $\underline{c} = 0$, $\bar{c} = 1$, and $c$ uniformly distributed.](image)

Figure 2(a) and (b) depict the optimal $\hat{c}(v)$ and $\gamma(v)$, respectively. In particular, the solid curve in 2(a) depicts the optimal $\hat{c}(v)$, while the dashed curve depicts the optimal threshold if we limit the policy to be within the class of single threshold policies. We observe from the figure that when $v$ takes values between about 0.12 and 0.4, it is indeed optimal for the principal to offer a two-procurement-level policy as described in Theorem 2.

Next, we focus on a particular value of $v = 0.25$ and demonstrates the optimal procurement policy and dynamic virtual valuation in Figure 2. In particular, Figure 2(c) shows the optimal
procurement quantity \( q(c) \), which has a two-level structure as described in Theorem 2, with \( \widehat{c} \) and \( \gamma(v) \) values consistent with Figure 2 at \( v = 0.25 \). Figure 2(d), on the other hand, shows the dynamic virtual valuation \( \xi_P(c) \), which is convex, as described in Theorem 1(2).

### 4. Optimal Policy for Model (C)

In this section, we establish our main results for the cash payment model (C) introduced in Section 2, in which there is an intertemporal cash account that allows the agent to save net cash payments to meet future participation costs. Similar to Lemma 3, we have the following revenue equivalence result for model (C).

**Lemma 4** Under assumption (7), the dynamic optimization problem (C) is equivalent to:

\[
J_C(x) = \max_{\bar{u}, q} E[\rho(c)q(c)] + \beta E \left[ J_C \left( (x + \bar{u} - S + Q(c)) / \beta \right) \right] - \bar{u}
\]

subject to

\[
\bar{u} \geq 0; \quad q \text{ non-increasing; } 0 \leq q(c) \leq 1; \quad \mathbb{E}[Q(c)] \geq S' - \bar{u},
\]

where \( \rho(c) = p - c - F(c)/f(c) \) as defined in (1). Furthermore, the net payment associated with the optimal solution \((q^*, \bar{u}^*)\) can be expressed as

\[
m^*(c) - cq^*(c) = \bar{u}^* + Q^*(c).
\]
Here, the decision variable \( \bar{u} \) represents the net payment when the agent’s marginal production cost is at the highest level \( \bar{c} \).

If \( S = 0 \), the next period cash account balance, \( (x + \bar{u} + Q(c))/\beta \), remains non-negative from any non-negative \( x \). As a result, the boundary condition \( J_C(x) = 0 \) when \( x < 0 \) never plays a role in the model. Furthermore, if \( S' \) is also equal to 0, the (ES) constraint is always satisfied. In this case, one can verify through induction that the optimal value function \( J_C \) is a constant with respect to state variable \( x \). Therefore, the model reduces to a standard one period procurement auction problem applied to each period. That is, it is optimal to set \( q^*(c) = 1 \) when the virtual valuation \( \rho(c) = p - c - F(c)/f(c) \geq 0 \), and \( q^*(c) = 0 \) otherwise. The same is true when \( S > 0 \) and \( x \geq \bar{x} \), as long as \( S \leq \mathbb{E}[m^*(c) - cq^*(c)] \), as mentioned in Remark 2.1.

In contrast, when \( S > 0 \) and \( x < \bar{x} = S/(1 - \beta) \), the model becomes much harder to analyze. Due to the boundary condition, the value function \( J_C(x) \) may not be continuous at \( x = 0 \). As a result, its integral on the right-hand side of the Bellman equation implies that the \( J_C \) function lacks concavity properties that are often handy in maximization problems. In a numerical example that we will show later in Subsection 4.2, it is clear that even for uniformly distributed private information, the \( J_C \) function is indeed discontinuous at \( x = 0 \) and non-concave.

Despite the discontinuity at zero and non-concavity, the value function still enjoys some important properties that we will exploit in our analysis. We begin by showing that the value function \( J_C \) is increasing and also a Lipschitz function with constant 1 when \( x \) is positive, as summarized in the following result.

Proposition 3 (Structure of Value Function for Model (C)) The value function \( J_C \) satisfies the following properties:

(i) \( J_C(x) \) is non-decreasing in \( x \);

(ii) For any \( x \geq \bar{x} \),

\[
J_C(x) = \frac{1}{1 - \beta} \max \mathbb{E}[pq(c) - m(c)] \text{ s.t. (IC), (IR), (ES)};
\]

and

(iii) For any \( x > x' \geq 0 \), we have \( \frac{J_C(x) - J_C(x')}{x - x'} \leq 1 \).

Proposition 3 echoes Remark 2.1, that when the cash balance \( x \) is large enough, the optimal value function \( J_C(x) \) is a constant, which is realized by implementing the optimal static mechanism repeatedly. In particular, if \( S' = 0 \), \( J_C(x) = J_S/(1 - \beta) \) for all \( x \geq \bar{x} \).
In what follows it will be convenient to represent the value function as the sum of a step function and a Lipschitz function: \( J_C(x) = \tilde{J}_C(x) + J_C(0)1_{\{x\geq 0\}} \). When writing derivative \( J'_C \) we refer to \( \tilde{J}'_C \), which is well defined almost everywhere.

4.1. Dynamic Virtual Valuation and Optimal Procurement Structure

Similar to the analysis in the previous section, we focus on deriving the function derivative of the objective function in the maximization problem (16). Specifically, the maximization on the right-hand side of the Bellman equation (16) can be expressed as \( \max_{q, \bar{u}} G_C(q, u) \) subject to corresponding constraints on \( \bar{u} \) and \( q \), where

\[
G_C(q, \bar{u}) = E[\rho(c)q(c)] + \beta E\left[\tilde{J}_C\left(\frac{(x + \bar{u} - S + Q(c))}{\beta}\right)\right] + \beta J_C(0)E\left[1_{\{x + \bar{u} - S + Q(c)\geq 0\}}\right] - \bar{u}.
\]

Here we define the following threshold \( \Psi(q) \) on the marginal production cost \( c \), above which the next period’s cash account would be negative, and therefore the collaboration between the principal and agent terminates.

\[
\Psi(q) = \max\{c : x + \bar{u} - S + Q(c) \geq 0 \mid c \in [\underline{c}, \bar{c}]\}. \tag{17}
\]

It follows that if \( \Psi(q) \in (\underline{c}, \bar{c}) \), we have \( x + \bar{u} - S + Q(\Psi(q)) = 0 \).

**Proposition 4** For any measurable function \( h : [\underline{c}, \bar{c}] \to \mathbb{R} \), we have

\[
\frac{d}{d\delta} G_C(q + \delta h, \bar{u}) \bigg|_{\delta = 0} = E[\xi_c(c)h(c)]
\]

where the dynamic virtual valuation \( \xi_c(c) \) for model (C) is defined, for any feasible \( q \) function, as

\[
\xi_c(c) = \rho(c) + \frac{1}{f(c)} \int_{\min\{c, \Psi(q)\}}^{\min\{c, \Psi(q)\}} J'_C\left(\frac{(x + \bar{u} - S + Q(t))}{\beta}\right)f(t)dt + \frac{\beta J_C(0)f(\Psi(q))}{q(\Psi(q))f(c)}1_{\{c \geq \Psi(q)\}}. \tag{18}
\]

Compared with the (static) virtual valuation \( \rho(c) \), the dynamic virtual valuation \( \xi_c(c) \) contains additional terms related to the value function \( J_C \), which captures the dynamic value of the future partnership with the agent, whose current marginal production cost is \( c \). Moreover, we note that the term \( \int_{\xi}^{c} J'_C\left(\frac{(x + \bar{u} - S + Q(t))}{\beta}\right)f(t)dt \) is a well behaved function since \( J_C \) is monotonic and a Lipschitz function with constant 1, according to Lemma 7. Finally, although the dynamic virtual valuation \( \xi_P \) for model (P) is continuous, here \( \xi_C \) may have a “vertical jump” at \( \Psi(q) \) if \( \Psi(q) \) is in the interior of the support \( [\underline{c}, \bar{c}] \).

**Remark 4.1 (Dynamics Reduce Inefficiencies)** We note that \( J_C \geq 0 \) and its derivative \( J'_C \geq 0 \), for any density function \( f \). Therefore, we have \( \xi_c(c) \geq \rho(c) \) for all \( c \in [\underline{c}, \bar{c}] \), following (18).
In turn, this shows that, under the dynamic model, the optimal procurement quantity is larger than under the static case for any given realized private information. This is a consequence of the incentives from potential future gains. Therefore, the dynamic aspect of the model reduces inefficiencies due to private information compared with the static model.

Parallel to Theorem 1, we provide a sufficient condition on the probability distribution that yields certain structures of the dynamic virtual valuation.

**Theorem 3 (Properties of Dynamic Virtual Valuation, Model (C))** Consider the following condition on the probability distribution

\[
\frac{F(c)}{f(c)} \text{ is non-decreasing in } c. \tag{19}
\]

For any \(x \in [0, S]\), feasible allocation function \(q\), and \(\lambda \in [0, 1]\), we have the following properties:

1. \(\xi C(c) + \lambda F(c)/f(c)\) is non-increasing on \([c, \Psi(q))\) and \([\Psi(q), \bar{c}]\) under condition (19) and \(f(c)\) is non-decreasing in \(c\); \(\tag{20}\)

2. \(\xi C(c) f(c)/F(c) + \lambda\) is non-increasing on \([c, \Psi(q))\) and \([\Psi(q), \bar{c}]\) under condition (19) and \(p \geq \bar{c}\). \(\tag{21}\)

Theorem 3 provides conditions under which the dynamic virtual valuation is well behaved. These conditions allow for a variety of distributions. For example, condition (19) is satisfied by any distribution with a non-increasing density function, e.g. truncated exponential distribution, but also allows distributions with increasing density. Indeed, the class of distribution \(F(c) = (c - \underline{c})^k / (\bar{c} - \underline{c})^k\) for any \(k \geq 1\) satisfies conditions (19) and (20) simultaneously. (Note that for \(k = 1\) we have the uniform distribution over \([\underline{c}, \bar{c}]\).)

The structure of the dynamic virtual valuation revealed in Corollary 1 implies the following two threshold structure of the optimal procurement policy in this model.

**Theorem 4 (Structure of Optimal Procurement with Low Cash Balance)** Suppose that condition (19) holds together with either (20) or (21). For any given \(x \in [0, S]\), there are two thresholds \(c(1) \leq c(2)\) and a level \(\gamma \in [0, 1]\) such that the optimal procured function \(q^*\) that solves (16) satisfies

\[
q^*(c) = \begin{cases} 
1 & , c \in \left[\underline{c}, c(1)\right) \\
\gamma & , c \in \left[c(1), c(2)\right) \\
0 & , c \in \left[c(2), \bar{c}\right] 
\end{cases} \tag{22}
\]
Theorem 4 is parallel to Theorem 2 for model (P). In fact, the optimal procurement structure for the uniform distribution in Theorem 2 may be perceived as a two-threshold structure as well, with the corresponding higher threshold \( c_2 \) being always the upper bound \( \bar{c} \). The difference in structure is due to differences in the dynamic virtual valuation, as well as the constraint \( \mathbb{E}[Q] \leq v + S \) in (9) for model (P) replaced with the (ES) constraint in model (C) (or, \( \mathbb{E}[Q] \geq S' - \bar{u} \) in (16)). The quantity \( \lambda \) in Theorem 4, in fact, corresponds to the “dual variable” for this constraint.

Note that Theorems 3 and 4 continue to hold if we remove the (ES) constraint from model (C). The only change in the results is to set \( \lambda = 0 \).

Furthermore, the following result establishes that two-threshold structures are needed only if the cash balance constraint is potentially binding. That is, in the case of a large cash position, when the current cash position \( x \) is larger than the participation cost \( S \), the optimal mechanism remains a single threshold structure similar to the static case.

**Theorem 5 (Structure of Optimal Procurement with High Cash Balance)** Suppose that condition (19) holds together with either (20) or (21). For any given \( x \in [S, \infty) \), we have \( \Psi(q) = \bar{c} \), and therefore \( \xi_C(c) \) is non-increasing on \([c, \bar{c}]\). Furthermore, the optimal \( q \) that solves optimization (16) has the following single threshold structure: there is a threshold \( \tilde{c} \in [c, \bar{c}] \) such that \( q(c) = 1 \) if \( c \leq \tilde{c} \) and \( q(c) = 0 \) otherwise.

Theorems 4 and 5 imply an indirect mechanism in which the agent does not have to report the exact production cost in each period in the model (C) setting. A single threshold corresponds to a take-it-or-leave it offer. In the two threshold case, the principal offers price \( c_2 \) to procure up to \( \gamma(x) \) units, and price \( c_1 \) for the remaining \( 1 - \gamma(x) \). Note that in the model presented in the previous section, we have to implement the direct mechanism because the current period payment is to reimburse the production cost, which depends on a specific report of the marginal production cost.

Finally, we study optimal choices for \( \bar{u} \) to completely characterize the mechanism. In contrast to the static case where the optimal \( \bar{u} \) is set to zero, in the dynamic setting, \( \bar{u} \), the net payment to the highest cost type, may be positive for certain \( x \) values to avoid termination of the game for future trading opportunities.

**Theorem 6 (Optimal \( \bar{u} \), Model (C))** Assume that condition (20) holds. At state \( x \), the optimal solution \((q^*, \bar{u}^*)\) satisfies

\[
\bar{u}^* = \begin{cases} 
\max \left\{ 0, S' - \mathbb{E}[Q^*(c)] \right\}, & x \geq \min\{S, S - S' + \mathbb{E}[Q^*(c)]\}; \\
S - x \text{ or } \max \left\{ 0, S' - \mathbb{E}[Q^*(c)] \right\}, & x < \min\{S, S - S' + \mathbb{E}[Q^*(c)]\}.
\end{cases}
\]
Furthermore, when $S' = 0$, there exists a threshold $\hat{S} \in [0, S)$ such that the optimal net payment to the highest cost type, $\bar{u}^*$, has the following structure

$$
\bar{u}^* = \begin{cases} 
0, & x \geq S \\
S - x, & \hat{S} \leq x < S \\
0, & x < \hat{S}
\end{cases}
$$

and the two-threshold structure described in Theorem 4 may only occur when $x < \hat{S}$.

The implication of Theorem 6 is that when the cash balance is higher than $S$, the agent with the highest marginal production cost receives non-negative net payment, $m(\bar{c}) - \bar{c}q(\bar{c}) = \bar{u} \geq 0$. Better (lower) cost induces better payment. Therefore, in this case, the game continues regardless of the cost level $c$ because the agent is able to afford the participation cost $S$ funded by the cash account.

When cash balance $x$ becomes less than $S$, the highest marginal production cost induces two possible net payment. In the first case, a net payment of $S - x$ guarantees that the worst type agent has $S$ in the account, and therefore is able to continue working with the principal into the next period. This is an interesting possibility that arise because future collaboration is profitable, which justifies paying extra in the current period. In the other case, the agent with the highest marginal production cost will not be able to afford the consumption $S$ to continue into the next period. In this situation, the interaction continues into the next period only if the agent’s marginal production cost $c$ is low enough such that $\bar{u}^* + Q^*(c) \geq S$.

The optimal structures derived in Theorems 4, 5 and 6 can also substantially simplify computation by restricting the search over incentive compatible mechanisms to those following the corresponding structures. Take, for example, the case of $S' = 0$. When $x \geq S$, we only need to search within a family of mechanisms parameterized by one parameter. We then search through three parameters $(c(1), c(2)\text{ and } \gamma)$ only when $x < S$. At each state $x$, the decision variable $\bar{u}$ can take at most two possible values. These structures will be very helpful in the computational study we report in the next Section.

### 4.2. A numerical example

Similar to the study of model (P), we have tested a large set of model parameters, with production cost $c$ taking uniform $[0, 1]$ distributions. Interestingly, we never observe the two-threshold policy as presented in Theorem 4 despite our extensive numerical study. That is, the optimal procurement policy turns out to always follow single threshold structures for any state, cash account balance $x$, in all our examples. In fact, we suspect that the two-threshold structure may only occur on a measure zero set for the cash model, although we have not been able to formally prove such a result even for the uniform distribution case. In this section we present
two particular examples to illustrate our thinking. In the first example, we take $p = 1.5$, $S = 2$, $S' = 0$ and $\beta = 0.9$.

The value function $J_C(x)$ is, in general, not concave, as illustrated in Figure 3. It is also evident that function $J_C(x) > 0$ for all $x \geq 0$, and therefore is not continuous at $x = 0$. In fact, global concavity is only ensured if gains from trade are such that no type is ever discontinued from the game. On the other hand, Figure 3 also confirms that value function $J_C(x)$ is increasing and Lipschitz continuous with constant 1, as stated in Lemma 7. We also observe that $J_C(x)$ converges to $J_S/(1 - \beta)$ for $x \geq \bar{x} = S/(1 - \beta) = 20$, as stated in Remark 2.1.

![Figure 3. Value function of a cash payment model, with $p = 1.5$, $S = 2$, $\beta = 0.9$, $c = 0$, $\bar{c} = 1$, and $c$ uniformly distributed. (The plot on the right is a zoomed in version of the one on the left.)](image)

Furthermore, Figure 4 illustrates that the net payment $\bar{u}$ when the agent’s cost is highest equals $S - x$ when $x \in [\hat{S}, S)$, and equals 0 when $x$ is outside the interval. (Here $\hat{S} = 1.8736$.) This is consistent with Theorem 6. The bottom plot of Figure 4 demonstrates that the thresholds $c(1)$ and $c(2)$ are, in fact, the same for all $x$ values, despite the fact that they are allowed to be different when $x < \hat{S}$ in the optimization algorithm. As a result, next we investigate the dynamic virtual valuations.

Following Theorem 6, we focus on $x < \hat{S} = 1.8736$. In particular, Figure 5 demonstrates the optimal procurement quantities $q(c)$ (up) and dynamic virtual valuations $\xi_C(c)$ (down) when $x$ equals 1.7 (left) and 1.8 (right), respectively. As we observe, for either $x$ value, the dynamic virtual valuation function contains a vertical upward “jump” at the corresponding $\Psi(q)$’s. For the case $x = 1.7$, the dynamic virtual valuation is always positive. Therefore the corresponding procurement quantities are set at the highest value 1 for all $c$. For the case $x = 1.8$, on the other hand, the
Figure 4. Optimal $\bar{u}$ and thresholds $c(1)$ and $c(2)$ as functions of cash balance $x$, with $p = 1.5$, $S = 2$, $\beta = 0.9$, $\xi = 0$, $\bar{c} = 1$, and $c$ uniformly distributed.

dynamic virtual valuation $\xi_C(c)$ crosses zero once on the left of $\Psi(q)$, at $c(1)$, and once on the right of it at $c(2) = \bar{c} = 1$. It is clear that the optimal procurement quantity $q(c)$ should be set at its upper bound 1 when $c < c(1)$, because the derivative $\xi_C(c) > 0$ in that region. When $c \in (c(1), \Psi(q))$, on the other hand, derivative $\xi_C(c) < 0$. However, since the procurement function $q(c)$ needs to be non-increasing, and the integral of $\xi_C(c)$ between $c(1)$ and $c(2)$ (the total areas beneath $\xi_C$) is clearly positive, the optimal procurement quantities $q(c)$ should still be set as high as possible, and therefore at the upper bound 1.

As we increase the $x$ value to 1.8735, very close to $\tilde{S}$, the vertical jump $\Psi(q)$ moves to the right (see Figure 7, left). As a result, the integral of $\xi_C$ between $c(1)$ and $c(2)$ decreases to a value very close to zero, but still positive. The corresponding optimal procurement policy remains the same. When we increase $x$ to 1.8736, however, $x$ becomes $\tilde{S}$. As a result, the single price structure becomes optimal, following Theorem 6. The corresponding policy and dynamic virtual valuations are presented on the right of Figure 7. Therefore, the two-threshold structure of the optimal procurement policy does not arise in this example.

In fact, we have not been able to construct a convincing numerical example showing that the two-threshold policy structure exists in optimal policies, at least when private information follows a uniform distribution. The aforementioned numerical example is quite representative of the cases where the two-threshold policy structure “almost” emerges, in the sense that the dynamic virtual valuation contains a vertical jump and crosses zero multiple times.
Figure 5. Optimal procurement quantities and dynamic virtual valuations for cash balance $x = 1.7$ and $x = 1.8$, with $p = 1.5$, $S = 2$, $\beta = 0.9$, $\xi = 0$, $\bar{c} = 1$, and $c$ uniformly distributed.

Figure 6. Optimal procurement quantities and dynamic virtual valuations for cash balance $x = 1.8735$ and $x = 1.8736$, with $p = 1.5$, $S = 2$, $\beta = 0.9$, $\xi = 0$, $\bar{c} = 1$, and $c$ uniformly distributed.

When we enforce the constraint (ES) with $S' = S$, on the other hand, the two-threshold policy structure appears even less likely to emerge. Figure 7, for example, illustrates such a case. As we increase the $x$ value to 0.3225, the vertical jump at $\Psi(q)$ moves to the right. A further increase of $x$ to 0.3230 (the smallest increase according to our discretization), however, moves $\Psi(q)$ to $\bar{c}$. As a result, the single-threshold structure is optimal for all $x$. 
Figure 7. Optimal procurement quantities and dynamic virtual valuations for cash balance $x = 0.3225$ and $x = 0.3230$, with $p = 1.1$, $S = S' = 0.48$, $\beta = 0.8$, $\xi = 0$, $\bar{c} = 1$, and $c$ uniformly distributed.

On the other hand, we also cannot prove that the optimal policy is indeed a single price policy in general. We suspect that the two-threshold structure only occurs, if at all, in a measure zero point of $x$, as illustrated in the first example in this section. We leave it as an open question for future research.

5. Conclusion In this paper, we extend the basic mechanism design model from its static setting to dynamic settings. The results, especially for the optimal procurement policy, highlight the impact of the dynamic nature of the problem. Despite the much more complex mathematical analysis, as compared with the analysis for the static counterpart, we show that the optimal procurement policies still exhibit relatively simple structures. These simple structures facilitate computation and suggest the possibility of developing easy-to-compute and easy-to-implement mechanisms that are near optimal in more complex settings.
Appendix. Proofs

Proof of Lemma 3  Denote function $u(c) = m(c) - cq(c) + \beta w(c)$. The (IC$_d$) constraint implies that $u(c) = \max_{\hat{c}} m(\hat{c}) - cq(\hat{c}) + \beta w(\hat{c})$, and therefore it is convex as it is the upper envelope of a linear functions. Moreover, by the envelope theorem, $-q(c)$ is an element of the subgradient of $u$ at $c$. Convexity of $u(c)$ also implies that $-q(c)$ is non-decreasing. Denote $\bar{\bar{u}} = m(\hat{\bar{c}}) - cq(\hat{\bar{c}}) + \beta w(\hat{\bar{c}})$, we have

$$m(c) - cq(c) + \beta w(c) = u(c) = \bar{\bar{u}} + \int_c^{\bar{\bar{e}}} q(\tau)d\tau = \bar{\bar{u}} + Q(c) .$$

(PK) implies $\bar{\bar{u}} = v + S - \mathbb{E}[Q(c)]$. So

$$m(c) - cq(c) = v + S - \mathbb{E}[Q] + Q(c) - \beta w(c).$$

(IR) then implies that

$$w(c) \leq (v + S - \mathbb{E}[Q] + Q(c))/\beta .$$

Also, $w(\hat{c}) \geq 0$ implies that

$$0 \leq v + S - \mathbb{E}[Q] + Q(c) \quad \text{for all } c \in [\hat{c}, \bar{\bar{c}}] \Rightarrow \mathbb{E}[Q] \leq v + S .$$

Finally we replace the $m(c)$ in the objective function with $cq(c) + v + S - \mathbb{E}[Q] + Q(c) - \beta w(c)$.

Proof of Proposition 1  First, argument $v$ does not appear in the objective function in the optimization problem $\Gamma \hat{V}$ for any function $\hat{V} : [0, \bar{\bar{v}}] \rightarrow R$. And the constraint set $\Pi(v)$ is such that $\Pi(v) \subseteq \Pi(v')$ for $v \leq v'$. Therefore $(\Gamma \hat{V})(v) \leq (\Gamma \hat{V})(v')$ for $v \leq v'$, or, $\Gamma \hat{V}$ is increasing for any function $\hat{V}$. In particular, since $V = \Gamma \hat{V}$, we must have $V$ is increasing.

To show the concavity of $V$, we consider the value iteration algorithm $\Gamma^k \hat{V} = \Gamma(\Gamma^{k-1} \hat{V})$, $k = 1, 2, \ldots$ starting from a concave increasing function $\hat{V}$. For any $\lambda \epsilon (0, 1)$ and $v_i \epsilon [0, \bar{\bar{v}}]$ for $i = 1, 2$, denote $(q_i, w_i)$ to be the optimal solution in the optimization $(\Gamma \hat{V})(v_i)$. Obviously, $(\lambda q_1 + (1 - \lambda)q_2, \lambda w_1 + (1 - \lambda)w_2)$ is feasible to the optimization problem $(\Gamma \hat{V})(\lambda v_1 + (1 - \lambda)v_2)$, and generates an objective function that is at least as good as $\lambda(\Gamma \hat{V})(v_1) + (1 - \lambda)(\Gamma \hat{V})(v_2)$ due to concavity of $\hat{V}$. As a result,

$$(\Gamma \hat{V})(\lambda v_1 + (1 - \lambda)v_2) \geq \lambda(\Gamma \hat{V})(v_1) + (1 - \lambda)(\Gamma \hat{V})(v_2) .$$

This implies that $V = \lim_{k \rightarrow \infty} \Gamma^k \hat{V}$ is concave.
For part (ii), when \( v \geq v^* \), the first best production policy \( q(c) = \bar{q}(c) \) is a feasible solution together with \( w(c) = v^* \), which means the profit in each period for the system is exactly the first best profit. That is, \( V(v) = \bar{v} \), and \( (q(c), v^*) \) is an optimal solution to (IV)(v*).

Part (iii) follows from the monotonicity of \( V \).

\[\text{Proof of Proposition 2}\] For a measurable function \( h : [\underline{c}, \bar{c}] \rightarrow \mathbb{R} \), let \( H(c) = \int_{\xi}^{\bar{c}} h(t)dt \). Recall that \( V \) is concave so that \( V' \) exists almost surely. Therefore, we have

\[
\frac{d}{d\delta} G_v(q + \delta h)|_{\delta = 0} = \mathbb{E}[(p - c)h(c)] + \beta \frac{d}{d\delta} \int_{\xi}^{\bar{c}} V\left( (v + S - \mathbb{E}[[Q + \delta H(c)] + Q(c) + \delta H(c)]) / \beta \right) f(c) dc
\]

\[
= \mathbb{E}[(p - c)h(c)] + \int_{\xi}^{\bar{c}} (H(c) - \mathbb{E}[H]) V' \left( (v + S - \mathbb{E}[Q(c)] + Q(c)) / \beta \right) f(c) dc
\]

\[
= \mathbb{E}[(p - c)h(c)] + \int_{\xi}^{\bar{c}} \left( \int_{\xi}^{\bar{c}} h(\tau)d\tau - \int_{\xi}^{\bar{c}} h(t)F(t)dt \right) V'(g(c)) f(c) dc
\]

\[
= \mathbb{E}[(p - c)h(c)] + \int_{\xi}^{\bar{c}} h(c) \left( \int_{\xi}^{\bar{c}} V'(g(\tau)) f(\tau) d\tau \right) dc - \int_{\xi}^{\bar{c}} h(c) F(c) \left( \int_{\xi}^{\bar{c}} V'(g(\tau)) f(\tau) d\tau \right) dc
\]

where \( g(c) \) is defined in (14).

\[\text{Proof of Theorem 1}\] Recall that \( V' \geq 0 \), \( V'' \leq 0 \), \( g' \leq 0 \). (For simplicity of exposition, we assume these functions are twice continuously differentiable. If otherwise, we can resort to standard approximation arguments and subsequently taking limits as \( V \) is concave.) The dynamic virtual valuation is

\[
\xi_v(c) = p - c - \frac{F(c)}{f(c)} \left( \int_{\xi}^{\bar{c}} V'(g(\tau)) f(\tau) d\tau \right) + \frac{1}{f(c)} \left( \int_{\xi}^{\bar{c}} V'(g(\tau)) f(\tau) d\tau \right)
\]  

(23)

Define \( A(c) := \int_{\xi}^{c} V'(g(\tau)) f(\tau) d\tau \geq 0 \), so that \( A'(c) = -V'(g(c)) f(c) \).

\[
\xi_v(c) = p - c + \frac{1 - F(c)}{f(c)} \frac{A(c)}{1 - F(c)} = p - c + \frac{1 - F(c)}{f(c)} \left( A(c) - \frac{A(c) - A(c)}{1 - F(c)} \right)
\]

\[
= p - c - \frac{1 - F(c)}{f(c)} \left( \frac{A(c) - A(c)}{1 - F(c)} \right)_{B(c)}
\]

\[
B'(c) = \frac{f(c) A(c)}{(1 - F(c))^2} - \frac{V'(g(c)) f(c)}{1 - F(c)} = \frac{f(c)}{(1 - F(c))^2} \left[ A(c) - V'(g(c))(1 - F(c)) \right]_{D(c)}
\]
Since $D'(c) = A'(c) - V''(g(c))g'(c) \left(1 - F(c)\right) + V'(g(c))f(c) = -V''(g(c))g'(c) \left(1 - F(c)\right) \leq 0$ and $D(\tilde{c}) = 0$, we have $D(c) \geq 0$, and, therefore, $B'(c) \geq 0$. Further,

$$B(\tilde{c}) = \frac{1}{1 - F(\tilde{c})} A(\tilde{c}) - A(c) = 0.$$ 

Therefore $B(c) \geq 0$ for all $c \in [\bar{\xi}, \tilde{c}]$, and

$$\xi_\tau'(c) = -1 - \frac{d}{dc} \left(\frac{1 - F(c)}{f(c)}\right) B(c) - \frac{1 - F(c)}{f(c)} B'(c) < 0,$$

where the last inequality follows from $\frac{1 - F(c)}{f(c)}$ being non-decreasing, according to (Suff). Therefore $\xi_\tau$ is also decreasing in $[\bar{\xi}, \tilde{c}]$, and hence (1).

Now consider the (2). We show the result under the following sufficient condition

$$f(c) \text{ is non-increasing in } c \text{, and } \frac{f'(c)F(c)}{f^2(c)} \text{ is non-decreasing in } c \quad \text{(Suff2)}$$

which is satisfied by the uniform distribution. We can write

$$\xi_\tau(c) = p - c - \frac{F(c)}{f(c)} \left(\int_\bar{\xi}^c V'(g(\tau))f(\tau)d\tau - \frac{1}{F(c)} \int_\bar{\xi}^c V'(g(\tau))f(\tau)d\tau\right)$$

$$= p - c - \frac{F(c)}{f(c)} \left(A(c) - \frac{1}{F(c)} \int_\bar{\xi}^c V'(g(\tau))f(\tau)d\tau\right).$$

So that

$$\xi_\tau(c) = -1 - \left(\frac{F(c)}{f(c)}\right) \left(A(c) - \frac{1}{F(c)} \int_\bar{\xi}^c V'(g(\tau))f(\tau)d\tau\right) - \frac{F(c)}{f(c)} \left(\frac{1}{F(c)} A(\Phi(q))\right)'$$

$$= -1 - \left(\frac{F(c)}{f(c)}\right) \left(A(c) - \frac{1}{F(c)} \int_\bar{\xi}^c V'(g(\tau))f(\tau)d\tau\right) + \frac{F(c)}{f(c)} A(\phi(q))$$

$$= -1 - A(c) + \frac{F(c)f'(c)}{f^2(c)} \left(A(c) - \frac{1}{F(c)} \int_\bar{\xi}^c V'(g(\tau))f(\tau)d\tau\right) + V'(g(c))$$

Note that because $V' \geq 0$, we have $0 \leq A(c) \leq A(\tilde{c})$.

$$\xi_\tau''(c) = \left(\frac{f'(c)F(c)}{f^2(c)}\right) \left(\int_\bar{\xi}^c V'(g(\tau))f(\tau)d\tau\right)$$

$$- \frac{f'(c)}{f(c)F(c)} \left(V'(g(c))F(c) - \int_\bar{\xi}^c V'(g(\tau))f(\tau)d\tau\right) - \frac{q(c)V''(c)}{\beta}.$$ 

Direct calculations yield $\tilde{B}'(c) = -\frac{f(c)}{F^2(c)} \tilde{D}(c)$ and $\tilde{D}'(c) = -\frac{q(c)V''(c)}{\beta} \geq 0$. Therefore, since $\tilde{D}'(q) = 0$, we have that $\tilde{D}(c) \geq 0$ for all $c \in [\bar{\xi}, \tilde{c}]$. Furthermore, $\tilde{B}'(c) \leq 0$ and $\tilde{B}(\tilde{c}) = 0$ implies $\tilde{B}(\tilde{c}) \geq 0$. Therefore (Suff2) implies that $\xi_\tau(c)$ is convex in $[\bar{\xi}, \tilde{c}]$. 

\[\square\]
Proof of Theorem 2} The (infinite dimensional) optimization problem can be formulated as
\[
\max_q G_p(q, v) \equiv \max_q \min_{\lambda, \eta, \eta_0, \eta_1} \mathcal{L}(\lambda, \eta_m, \eta_0, \eta_1; q) := G_p(q, v) - \lambda(\int F(c) q(c) dc - v - S)
\]

\[
s.t. \int F(c) q(c) dc \leq v + S
\]

\[
\eta(c) \leq 1, q \in \mathcal{M}_+
\]

where $\mathcal{L}$ is the Lagrangian function
\[
\mathcal{L}(\lambda, \eta_m, \eta_0, \eta_1; q) = G_p(q, v) + \int \{\eta_m(c) + \eta_0(c) - \eta_1(c) - \lambda F(c)\} q(c) dc + \lambda(v + S) + \int \eta_1(c) dc,
\]

cone of functions $\mathcal{M}_+ = \{r : [\epsilon, \bar{c}] \rightarrow \mathbb{R} : r \text{ non-increasing and non-negative function}\}$, and $\mathcal{M}^*$ is its dual, $\lambda \geq 0$ is a scalar, and $\eta_1$ is a non-negative function. The first-order conditions requires that at an optimal primal-dual pair ($\lambda, \eta_m, \eta_1; q$), we have that for any (functional) direction $h : [\epsilon, \bar{c}] \rightarrow \mathbb{R}$
\[
0 = \frac{d}{d\delta} \mathcal{L}(\lambda, \eta_m, \eta_1; q + \delta h) \bigg|_{\delta=0} = \int_{\epsilon}^{\bar{c}} \left[ f(c) \xi_p(q, c) - \lambda F(c) + \eta_0(c) - \eta_1(c) \right] h(c) dc
\]

where $\xi_p(q, c)$ is the dynamic virtual valuation. Thus, optimal primal-dual pairs satisfy
\[
\eta_1(c) - \{\eta_m(c) + \eta_0(c)\} = f(c)\{\xi_p(q, c) - \lambda F(c)/f(c)\}.
\]

The dual cone of non-increasing and non-negative functions is characterized by ($\mathcal{M}_+)^* = \{\eta_m : \int_{\epsilon}^{\bar{c}} \eta_m(c) dc \geq 0$, for all $c \in [\epsilon, \bar{c}]\}$, and changes on $q$ occurs at $c$ only if $\int_{\epsilon}^{\bar{c}} \eta_m(c) dc = 0$.

Define also $\check{c}_1 := \sup\{c : q(c) = 1\}$. We can assume that $\check{c}_1 < \bar{c}$ and that $\liminf_{c \rightarrow \check{c}_1} q(c) > 0$, otherwise the result holds. Complementary slackness implies that $\eta_1(c) = 0$ for $c > \check{c}_1$, and therefore
\[
\eta_1(c) - \eta_m(c) = f(c)\{\xi_p(q, c) - \lambda F(c)/f(c)\} \quad \text{for } c < \check{c}_1
\]
\[
\eta_m(c) = -f(c)\{\xi_p(q, c) - \lambda F(c)/f(c)\} \quad \text{for } c > \check{c}_1
\]

First we prove (1). For any choice of $q$, under (Suff) we have that $f(c)$ and $\{\xi_p(q, c) - \lambda F(c)/f(c)\}$ are non-increasing functions. (Indeed, by Theorem 1(1) we have $\xi_p(q, c)$ decreasing and $-\lambda F(c)/f(c)$ is decreasing by (Suff).) Therefore $f(c)\{\xi_p(q, c) - \lambda F(c)/f(c)\}$ is decreasing and it crosses zero at most once say at $\check{c}_1$.

We claim that $\check{c}_1 \geq \check{c}_1$. Suppose $\check{c}_1 < \check{c}_1$. Since neither monotonicity nor nonnegativity is binding at $\check{c}_1$, we have $\int_{\epsilon}^{\check{c}_1} \eta_m(c) dc = 0$. Because $f(c)\{\xi_p(q, c) - \lambda F(c)/f(c)\} > 0$ for any $\check{c}_1 < c < \check{c}_1$, we have $\int_{\epsilon}^{\check{c}_1} \eta_m(t) dt = \int_{\check{c}_1}^{\check{c}_1} \eta_m(t) dt < 0$, violating dual feasibility.

Furthermore, because $f(c)\{\xi_p(q, c) - \lambda F(c)/f(c)\} < 0$ for all $c > \check{c}_1$, the optimal allocation satisfies $q(c) = 0$ for $c > \check{c}_1$. (Otherwise setting $h(c) = -1\{c > \check{c}_1\} q(c)$ we obtain a direction that yields improvement. This can also be seen by complementary slackness condition.) This implies that $\check{c}_1 = \check{c}_1$ and the result follows.
Following similar argument as before, we know that \( \tilde{c}_1 = \bar{c}_1 \).

Note that \( \eta_m(c) < 0 \) for \( c > \bar{c}_2 \) and \( \eta_m(c) < 0 \) for \( \bar{c}_1 < c < \bar{c}_2 \). Since \( \int_{\bar{c}_1}^{\bar{c}} \eta_m(c) dc = 0 \), it follows that \( \int_{\bar{c}}^{\bar{c}} \eta_m(t)dt = \int_{\tilde{c}_1}^{\bar{c}} \eta_m(t)dt > 0 \) for any \( \bar{c}_1 < c < \tilde{c} \). In turn, \( q \) can only change value at \( \tilde{c}_1 \) and we have \( q(c) = \gamma \) for \( c > \tilde{c}_1 \).

\[ \begin{align*}
\text{Proof of Proposition 3 and Auxiliary Technical Lemmas for Model (C)}
\end{align*} \]

Proposition 3(i),(ii) and (iii) follow from Lemma 5, 6 and 7 below respectively.

\[ \begin{align*}
\text{Lemma 5} \quad \text{Value function } J_C(x) \text{ is non-decreasing in } x.
\end{align*} \]

**Proof:** For any non-decreasing function \( J(x) \) such that \( J(x) = 0 \) for \( x < 0 \) and \( \lim_{x \to \infty} J(x) < +\infty \), denote operator \( T \) such that

\[ (TJ)(x) := \left\{ \begin{array}{ll}
\max \left\{ \max_{q,m:(IC), (IR),(ES),(AB)} \mathbb{E} [pq(c) - m(c) + \beta J(x^+(c))] , 0 \right\}, & x \geq 0, \\
0, & x < 0.
\end{array} \right. \]

Following convergence of dynamic programming algorithms [Proposition 9.16 in 4], we have \( J_C = T^\infty J \).

Consider any \( x_1 \) and \( x_2 \) such that \( x_1 \leq x_2 \). If \( x_1 < 0 \), we have \( 0 = (TJ)(x_1) \leq (TJ)(x_2) \). If \( x_1 \geq 0 \), denote \( q_1 \) and \( m_1 \) to represent the optimal solution of the inner maximization problem in \( (TJ)(x_1) \), and \( x^+_1(c) \) and \( x^+_2(c) \) to represent the corresponding account balance under mechanism \( q_1 \) and \( m_1 \), following initial account balances \( x_1 \) and \( x_2 \), respectively. Following (AB), \( x^+_1(c) \leq x^+_2(c) \), which
implies that \( J(x_1^+(c)) \leq J(x_2^+(c)) \). Furthermore, \( q_1 \) and \( m_1 \) are feasible to the inner optimization in \((TJ)(x_2)\). Therefore \( (TJ)(x_1) \leq (TJ)(x_2) \).

The monotonicity of \( TJ \) implies monotonicity of \( J_C \).

**Lemma 6** For any \( x \geq S/(1 - \beta) \),
\[
J_C(x) = \frac{1}{1 - \beta} \max \mathbb{E}[pq(c) - m(c)] \text{ s.t. (IC), (IR), (ES)};
\]

**Proof:** For any \( x \geq 0 \), \( J_C(x) \leq \frac{1}{1 - \beta} \max \mathbb{E}[pq(c) - m(c)] \text{ s.t. (IC), (IR), (ES)} \) since the value function will be no less than itself with one constraint removed. To establish the reverse inequality for any \( x \geq S/(1 - \beta) \), note that for \( m, q \) satisfying (IR),
\[
x^+(c) \geq \frac{1}{\beta} \left( \frac{S}{1 - \beta} - S + m(c) - cq(c) \right) = \frac{S}{1 - \beta} + \frac{1}{\beta}(m(c) - cq(c)) \geq \frac{S}{1 - \beta}.
\]
Therefore,
\[
J_C(x) \geq J_C\left( \frac{S}{1 - \beta} \right) = \max\{ \max_{q,m} \mathbb{E}[pq(c) - m(c) + \beta J_C(x^+(c))] \text{ s.t. (IC), (IR), (ES) and (AB)},
\geq (\max \mathbb{E}[pq(c) - m(c)] \text{ s.t. (IC), (IR), (ES)}) + \beta J_C\left( \frac{S}{1 - \beta} \right),
\]
which implies
\[
J_C(x) \geq J_C\left( \frac{S}{1 - \beta} \right) \geq \frac{1}{1 - \beta} \max \mathbb{E}[pq(c) - m(c)] \text{ s.t. (IC), (IR), (ES)}.
\]

**Lemma 7** For any \( x > x' \geq 0 \), we have \( \frac{J_C(x) - J_C(x')}{x - x'} \leq 1. \)

**Proof:** Monotonicity of \( J_C \) implies the first inequality. Denote \((q^*, \bar{u}^*)\) to be an optimal solutions of \( J_C(x) \). Clearly \((q^*, \bar{u}^* + x - x')\) is a feasible solution to \( J_C(x') \) given that \( x - x' > 0 \).
\[
J_C(x) - J_C(x') \leq E[p^*(c)q(c)] + \beta E\left[ J_C\left( (x + \bar{u}^* - S + Q^*(c)) / \beta \right) \right] - \bar{u}^* - E[p^*(c)q(c)] - \beta E\left[ J_C\left( (x' + \bar{u}^* + x - x' - S + Q^*(c)) \right) \right] - \bar{u}^* + x - x' = x - x'.
\]

**Proof of Lemma 4** Here we show that the outer maximization in model (C) can be removed under assumption (7). The remaining results in the Lemma follows the same derivation as in the proof of Lemma 3.

Consider the first best allocation \( \bar{q} \) and payment \( \bar{m}(c) = pq(c) \). It is easy to verify that \((\bar{q}, \bar{m})\) satisfies constraints (IC), (IR) and (ES). Therefore, the inner maximization in (C) must yield a non-negative value.
Proof of Proposition 4  Express $G_C(q, \bar{u})$ as

$$G_C(q, \bar{u}) = \mathbb{E}[\rho(c)q(c)] + \beta \int_{\xi}^{\Phi(q)} \tilde{J}_C \left( (x + \bar{u} - S + Q(c))/\beta \right) f(c) dc + \beta J_C(0) \int_{\xi}^{\Phi(q)} f(c) dc - \bar{u}. $$

Similar to the definition of $Q(c)$, we define $H(c) = \int_{\xi}^{\tilde{c}} h(t) dt$. Following the Newton-Leibniz rule for continuous function $\tilde{J}_C(x + \bar{u} - S + Q(c)), we have the following.

$$\frac{d}{d\delta} G_C(q + \delta h, \bar{u}) \bigg|_{\delta=0} = \mathbb{E}[p(c)h(c)] + \int_{\xi}^{\Phi(q)} \tilde{J}_C' \left( (x + \bar{u} - S + Q(c))/\beta \right) H(c) f(c) dc + \beta \frac{d}{d\delta} \Psi(q + \delta h) \bigg|_{\delta=0} J_C(0) f(\Psi(q))$$

When $x + \bar{u} - S + Q(\Psi(q)) = 0$ we have $\tilde{J}_C(\bar{u}) = 0$. Further,

$$\int_{\xi}^{\Phi(q)} \tilde{J}_C' \left( (x + \bar{u} - S + Q(c))/\beta \right) H(c) f(c) dc = \int_{\xi}^{\Phi(q)} \tilde{J}_C' \left( (x + \bar{u} - S + Q(c))/\beta \right) \int_{\xi}^{\tilde{c}} h(t) dt f(c) dc + \beta \frac{d}{d\delta} \Psi(q + \delta h) \bigg|_{\delta=0} J_C(0) f(\Psi(q))$$

By definition $x + \bar{u} - S + Q(\Psi(q) + \delta h)) + \delta H(\Psi(q) + \delta H)) = 0$, and we have

$$\frac{d}{d\delta} \Psi(q + \delta h) \bigg|_{\delta=0} = Q'(\Psi(q)) + H(\Psi(q)) = 0.$$ 

Then

$$\frac{d}{d\delta} \Psi(q + \delta h) \bigg|_{\delta=0} = \mathbb{E}\left[ \frac{1}{q(\Psi(q))} \int_{\xi}^{\tilde{c}} h(c) dc \right] = \mathbb{E}\left[ \frac{h(c)}{q(\Psi(q))} \right] 1_{c \geq \Psi(q)}$$

Finally, we replaced $J_C'$ with $J_C$ since $J_C'(c) = J_C(c)$ when $c \geq 0$.

\[\blacksquare\]

Proof of Theorem 3  Denote $g(c; x, J_C) = \int_{\xi}^{c} J_C' \left( (x + \bar{u} - S + Q(t))/\beta \right) f(t) dt \geq 0$ and recall that $\Psi(q) = \max\{c : x + \bar{u} - S + Q(c) \geq 0, c \in [\xi, \bar{c}]\}$.

First assume condition (19) holds together with (20). For $c < \Psi(q)$, we have

$$\frac{d}{dc} \left( \xi_C(c) + \lambda \frac{F(c)}{f(c)} \right) = J_C' \left( (x + \bar{u} - S + Q(c))/\beta \right) - 1 - (1 - \lambda) \left( \frac{f'(c) \int_{c}^{\xi_C(c)} F(c) \frac{1}{f(c)^2} dc}{\int_{c}^{\xi_C(c)} f(c)^2 \geq 0} \right) - \frac{f'(c) g(c; x, J_C)}{\int_{c}^{\xi_C(c)} f(c)^2 \geq 0} \leq 0$$

where we used that $J_C' \left( (x + \bar{u} - S + Q(c))/\beta \right) \leq 1$ by Lemma 7, $f'(c) F(c)/f(c)^2 \leq 1$ by condition (20), and $f'(c) \geq 0$ by (19).
For $c \geq \Psi(q)$, by similar arguments we have

$$\frac{d}{dc} \left( \xi_c(c) + \lambda \frac{F(c)}{f(c)} \right) = -1 - (1 - \lambda) \left( 1 - \frac{f'(c)F(c)}{f(c)^2} \right) - \frac{\beta f(\Psi(q))J_c(0) f'(c)}{q(\Psi(q)) f(c)^2} < 0.$$  

Next assume condition (19) holds together with (21). Since $F(c)/f(c)$ is non-decreasing under condition (19), $f(c)/F(c)$ is non-increasing. That is $\frac{d}{dc} \left( \frac{f(c)}{F(c)} \right) \leq 0$. For $c < \Psi(q)$, we have

$$\frac{d}{dc} \left( \xi_c(c) \frac{f(c)}{F(c)} + \lambda \right) = \left( J_c \left( \frac{x + \bar{u} - S + Q(c)}{\beta} - 1 \right) \frac{f(c)}{F(c)} \right) \left( 1 - \frac{f(c)}{F(c)} \right) \frac{d}{dc} \left( \frac{f(c)}{F(c)} \right) + \frac{f(c) g(c; x, J_c)}{F^2(c)} \leq 0.$$  

For $c \geq \Psi(q)$, on the other hand,

$$\frac{d}{dc} \left( \xi_c(c) \frac{f(c)}{F(c)} + \lambda \right) = -\frac{f(c)}{F(c)} + (p - c) \frac{d}{dc} \left( \frac{f(c)}{F(c)} \right) - \left( \frac{\beta f(\Psi(q)) J_c(0)}{q(\Psi(q))} + g(\Psi; x, J_c) \right) \frac{f(c)}{F^2(c)} \leq 0.$$  

\[\blacksquare\]

Theorem 3 immediately implies the following structure, which is used directly in the proof of Theorem 4.

**Corollary 1** Suppose that condition (19) holds together with either (20) or (21). Then for any $x \in [0, S]$, feasible $q$ and $\lambda \in [0, 1]$, there exist two thresholds $c_{(1)} \in [\underline{c}, \Psi(q)]$ and $c_{(2)} \in [\Psi(q), \bar{c}]$ such that

i. $\xi_c(c) f(c) + \lambda F(c) > 0$ for $c \in (\underline{c}, c_{(1)})$;

ii. $\xi_c(c) f(c) + \lambda F(c) < 0$ for $c \in (c_{(1)}, \Psi(q))$;

iii. $\xi_c(c) f(c) + \lambda F(c) > 0$ for $c \in (\Psi(q), c_{(2)})$;

iv. $\xi_c(c) f(c) + \lambda F(c) < 0$ for $c \in (c_{(2)}, \bar{c})$.

**Proof of Theorem 4** The (infinite dimensional) optimization problem can be formulated as

$$\max_{q, \bar{u}} G_C(q, \bar{u}) \quad \equiv \max_{q, \bar{u}} \min_{\lambda_S, \lambda_u, \eta_m, \eta_1} \mathcal{L}(\lambda_S, \lambda_u, \eta_m, \eta_1; q, \bar{u})$$

subject to

$$\begin{align*}
F(c) q(c) dc & \geq S' - \bar{u} \\
F(c) & \leq 1, q \in \mathcal{M}_+, \bar{u} \geq 0
\end{align*}$$

where $\mathcal{L}$ is the Lagrangian function

$$\mathcal{L}(\lambda_S, \lambda_u, \eta_m, \eta_1; q, \bar{u}) := G_C(q, \bar{u}) + \int \lambda_S F(c) - \eta_1(c) + \eta_m(c) q(c) dc + \int -\eta_1(c) dc + (\lambda_u + \lambda_S) \bar{u} - \lambda_S S',$$

set $\mathcal{M}_+ = \{ r : [\underline{c}; \bar{c}] \to \mathbb{R} : r \text{ non-increasing, non-negative function} \}$ is the cone of non-increasing non-negative functions, and $(\mathcal{M}_+)^*$ is its dual, $\lambda_S, \lambda_u$ are scalars, and $\eta_1$ is a non-negative functions.
We argue that \( \bar{\lambda} \) which implies that otherwise the result holds. Complementary slackness implies that the first-order conditions require that at an optimal primal-dual pair \((\lambda_S, \lambda_u, \eta_m, \eta_1; q, \bar{u})\), we have that for any (functional) direction \( h : [\underline{c}, \bar{c}] \to \mathbb{R} \)

\[
0 = \frac{d}{d\delta} \mathcal{L}(\lambda_S, \lambda_u, \eta_m, \eta_1; q + h, \bar{u}) \bigg|_{\delta=0} = \int_{\underline{c}}^{\bar{c}} \left[ f(c)\xi_C(c) + \lambda_S F(c) - \eta_1(c) + \eta_m(c) \right] h(c) dc
\]

where \( \xi_C(c) \) is the dynamic virtual valuation. Thus, optimal primal-dual pairs satisfy (a.s. in \( c \in [\underline{c}, \bar{c}] \))

\[
\eta_1(c) - \eta_m(c) = f(c)\xi_C(c) + \lambda_S F(c).
\]

The dual cone of non-increasing and non-negative functions is characterized by \((\mathcal{M}_+)^* = \{ \eta_m : \int_{\underline{c}}^{\bar{c}} \eta_m(c) dc \geq 0 \text{ for all } c \in [\underline{c}, \bar{c}] \}\), and changes in \( q \) occurs at \( c \) only if \( \int_{\underline{c}}^{\bar{c}} \eta_m(c) dc = 0 \).

Define also \( \bar{c}_1 := \sup\{ c : q(c) = 0 \} \). We can assume that \( \bar{c}_1 < \bar{c} \) and that \( \liminf_{c \to \bar{c}_1} q(c) > 0 \), otherwise the result holds. Complementary slackness implies that \( \eta_1(c) = 0 \) for \( c > \bar{c}_1 \). Therefore we have

\[
\eta_1(c) - \eta_m(c) = f(c)\xi_C(c) + \lambda_S F(c) \quad \text{for } c < \bar{c}_1,
\]

\[
\eta_m(c) = -\{ f(c)\xi_C(c) + \lambda_S F(c) \} \quad \text{for } c > \bar{c}_1.
\]

Moreover, the optimality condition for \( \bar{u} \) yields

\[
0 = \frac{d}{d\bar{u}} \mathcal{L}(\lambda_S, \lambda_u, \eta_m, \eta_1; q, \bar{u}) = -1 + \lambda_S + \lambda_u + \int_{\underline{c}}^{\bar{c}} f(c) dc + \beta J_c(0) \frac{f(\Psi(q))}{q(\Psi(q))}
\]

which implies that \( \lambda_S \leq 1 \). Following Corollary 1 with \( \lambda = \lambda_S \leq 1 \), there are at most three values \( c_1, \Psi(q), \bar{c}_2 \) with \( \underline{c} \leq c_1 < \Psi(q) \leq c_2 \leq \bar{c} \), such that function \( \{ \xi_C(c)f(c) + \lambda_S F(c) \} \) crosses zero. We argue that \( \bar{c}_1 \geq c_1 \). Suppose otherwise. Since neither monotonicity nor nonnegativity is binding at \( \bar{c}_1 \), we have \( \int_{\underline{c}}^{\bar{c}_1} \eta_m(c) dc = 0 \). Because \( f(c)\xi_C(c) + \lambda_S F(c) > 0 \) for \( \bar{c}_1 < c < c_1 \) by Corollary 1(i), which implies \( \int_{\underline{c}}^{\bar{c}_1} \eta_m(t) dt = \int_{\underline{c}}^{\bar{c}_1} \eta_m(t) dt < 0 \), violating dual feasibility.

Because \( \int_{\bar{c}_1}^{\bar{c}_2} \eta_m(c) dc = 0 \), for any \( \bar{c}_1 < c < \bar{c} \) we have \( \int_{\bar{c}_1}^{\bar{c}} \eta_m(t) dt = \int_{\bar{c}_1}^{\bar{c}} \eta_m(c) dc \geq 0 \). Note that \( \eta_m(c) > 0 \) for \( c_1 \leq \bar{c}_1 < c < \Psi(q) \) by Corollary 1(ii), while \( \eta_m(c) < 0 \) for \( \Psi(q) < c < c_2 \) by Corollary 1(iii), and \( \eta_m(c) > 0 \) for \( c_2 < c < \bar{c} \) by Corollary 1(iv). Therefore, besides \( c = \bar{c}_1, \int_{\bar{c}_1}^{\bar{c}} \eta_m(c) dc = 0 \) can only occur at a single value \( \bar{c}_2 \in (\Psi(q), c_2] \), and at a single value \( \bar{c}_3 \in [c_2, \bar{c}] \). However, dual feasibility requires that \( \int_{\bar{c}_2}^{\bar{c}_3} \eta_m(c) dc \geq 0 \), and \( \eta_m(c) < 0 \) for \( c \in (\bar{c}_2, c_2) \), it follows that \( \bar{c}_2 = c_2 = \bar{c}_3 \).

\[\square\]

**Proof of Theorem 5** The result is self-explanatory given previous results.

\[\square\]
Proof of Theorem 6

Step 1. (Main Step) Define $y = x + \bar{u}$ and $\bar{y}(q, x) = x + \max\{0, S' - \mathbb{E}[Q(c)]\}$ for any feasible $q$. With a slight abuse of notation, we have

$$G_C(q, y, x) = \mathbb{E}[\rho(c)q(c)] + \beta \mathbb{E}\left[\hat{J}_C\left((y - S + Q(c))/\beta\right)\right] + \beta J_C(0)\mathbb{E}\left[1_{\{y - S + Q(c) \geq 0\}}\right] - y + x.$$ 

so that our problem can be expressed as

$$\max_{y; 0 \leq q(c) \leq 1, q \geq 0} \max_{y; y \geq \bar{y}(q, x)} G_C(q, y, x).$$

(24)

Fix an arbitrary $x$ and a feasible allocation function $q$. We first consider the case that $\bar{y}(q, x) > S$ which implies $1_{\{y - S + Q(c) \geq 0\}} = 1$ for all feasible $y$. In this case,

$$\frac{\partial G_C}{\partial y} = \int_{\xi} \dot{J}_C'\left((y - S + Q(c))/\beta\right)f(c)dc - 1.$$ 

Following $0 \leq \dot{J}_C' \leq 1$ from Lemmas 5 and 7, we have

$$-1 \leq \frac{\partial G_C}{\partial y} \leq 0.$$ 

(25)

Therefore it is optimal to have $y^*$ at the lower bound $\bar{y}(q, x)$, so that $\bar{u} = \max\{0, S - \mathbb{E}[Q(c)]\}$.

Next we consider the case that $\bar{y}(q, x) \leq S$, or, $x \leq \bar{x}(q) := \min\{S, S' - \mathbb{E}[Q(c)]\}$. By Step 2 below we have that $G_C$ is convex in $y \in [\bar{y}(q, x), S]$. In turn, maximization of a convex function over $y \in [\bar{y}(q, x), S]$ is achieved at the boundary values. Those corresponds to either $\bar{u} = \max\{0, S' - \mathbb{E}[Q(c)]\}$ or $\bar{u} = S - x$.

For the case of $S' = 0$, denote

$$G^q_C(x; \bar{u} = 0) = \max_{q; 0 \leq q \leq 1, q \geq 0} \mathbb{E}[\rho(c)q(c)] + \beta \mathbb{E}\left[\hat{J}_C\left((x - S + Q(c))/\beta\right)\right],$$

$$G^q_C(x; \bar{u} = x - S) = \max_{q; 0 \leq q \leq 1, q \geq 0} \mathbb{E}[\rho(c)q(c)] + \beta \mathbb{E}\left[J_C(Q(c)/\beta)\right] + x - S.$$ 

It follows that the function $G^q_C(x; \bar{u} = x - S)$ is linear in $x$. As an upper envelope of convex functions, function $G^q_C(x; \bar{u} = 0)$ is still convex in $x \in [0, S]$. As a result, functions $G^q_C(x; \bar{u} = 0)$ and $G^q_C(x; \bar{u} = x - S)$, as functions of $x$, intersect at most in two values of $x$. Obviously one of them is $x = S$. The other potential intersect, less than $S$, is denoted as $\tilde{S}$. When $x \in (\tilde{S}, S)$, clearly, $G^q_C(x; \bar{u} = x - S) > G^q_C(x; \bar{u} = 0)$. Finally, when $\bar{u} = S - x$, $\Psi(q) = \bar{c}$. Therefore there is no “vertical jump” in $\xi_C$, or two-threshold structure in $q$.

Step 2. (Auxiliary Calculations) In this step we show that $G_C(q, y, x)$ is convex in $y \in [\bar{y}(q, x), S]$. We have

$$\frac{\partial G_C}{\partial y} = \int_{\xi} \Psi \dot{J}_C'\left((y - S + Q(c))/\beta\right)f(c)dc + \beta J_C(0)\frac{\partial \Psi}{\partial y}f(\Psi) - 1.$$ 

(26)
and we will show (26) is non-decreasing in $y$.

Consider the second term of the right hand side of (26). By the definition, $\Psi$ satisfies $y - S + Q(\Psi) = 0$. Therefore we have $1 + Q'(\Psi)\frac{\partial \Psi}{\partial y} = 0$ so that
\[
\frac{\partial \Psi}{\partial y} = \frac{1}{q(\Psi)} > 0
\]  
which establishes that $\Psi$ is increasing in $y$. Therefore, since $q$ is non-increasing itself, $\frac{\partial \Psi}{\partial y}$ is non-decreasing in $y$. Furthermore, $f(\Psi)$ is also non-decreasing in $y$ as $\Psi$ is non-decreasing by (27) and $f$ is non-decreasing by C1.

Next we turn to the first term of the right hand side of (26). First note that for any $t \in [0, (y - S + Q(c))/\beta]$, denote $\Phi(t)$ so solve $y - S + Q(\Phi(t)) = \beta t$ which implies that $\frac{\partial \Phi(t)}{\partial y} = \frac{1}{q(\Phi(t))} > 0$. By a change of variables we can write
\[
\int_{\Psi}^{y} \tilde{J}_C'(y - S + Q(c))/\beta f(c)dc = \beta \int_{0}^{y - S + Q(c)/\beta} \tilde{J}_C'(t) \frac{f(\Phi(t))}{q(\Phi(t))} dt.
\]  
Since $\Phi$ is non-decreasing in $y$ and $q$ is non-increasing, the relation above is non-decreasing in $y$ as $\tilde{J}_C' \geq 0$ and $f$ is non-decreasing by C1. Thus the first term of the right hand side of (26) is also increasing in $y$.

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References


