Optimal structural policies for ambiguity and risk averse inventory and pricing models: Online Appendix

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A Review on k-convexity and symmetric k-convexity

In this section, we review some important properties of k-convexity and symmetric k-convexity that are used in this paper; see Chen (2003) and Simchi-Levi et al. (2005) for more details.

The concept of k-convexity was introduced by Scarf (1960) to prove the optimality of an (s, S) for the traditional inventory control problem.

Definition A.1 *A real-valued function f is called k-convex for $k \geq 0$, if for any $x_0 \leq x_1$ and $\lambda \in [0, 1]$,*

$$f((1-\lambda)x_0 + \lambda x_1) \leq (1-\lambda)f(x_0) + \lambda f(x_1) + \lambda k. \tag{18}$$

Below we summarize properties of k-convex functions.

Lemma A.1 *(a) A real-valued convex function is also 0-convex and hence k-convex for all $k \geq 0$. A k_1-convex function is also a k_2-convex function for $k_1 \leq k_2$.*

(b) If $f_1(y)$ and $f_2(y)$ are k_1-convex and k_2-convex respectively, then for $\alpha, \beta \geq 0$, $\alpha f_1(y) + \beta f_2(y)$ is $(\alpha k_1 + \beta k_2)$-convex.

(c) If $f(y)$ is k-convex and w is a random variable, then $E\{f(y-w)\}$ is also k-convex, provided $E\{|f(y-w)|\} < \infty$ for all y.

(d) Assume that f is a continuous k-convex function and $f(y) \to \infty$ as $|y| \to \infty$. Let S be a minimum point of g and s be any element of the set

$$\{x | x \leq S, f(x) = f(S) + k\}.$$

Then the following results hold.

(i) $f(S) + k = f(s) \leq f(y)$, for all $y \leq s$.

(ii) $f(y)$ is a non-increasing function on $(-\infty, s)$.

(iii) $f(y) \leq f(z) + k$ for all y, z with $s \leq y \leq z$.

Proposition A.1 *If $f(x)$ is a K-convex function, then function*

$$g(x) = \min_{y \geq x} Q\delta(y - x) + f(y),$$

is $\max\{K, Q\}$*-convex.*

Recently a weaker concept of symmetric k-convexity was introduced by Chen & Simchi-Levi (2004a) when they analyze the joint inventory and pricing problem with fixed ordering cost.

Definition A.2 *A function $f : \Re \to \Re$ is called symmetric k-convex for $k \geq 0$, if for any $x_0, x_1 \in \Re$ and $\lambda \in [0, 1]$,*

$$f((1 - \lambda)x_0 + \lambda x_1) \leq (1 - \lambda)f(x_0) + \lambda f(x_1) + \max\{\lambda, 1 - \lambda\}k. \tag{19}$$

A function f is called symmetric k-concave if $-f$ is symmetric k-convex.

Observe that k-convexity is a special case of symmetric k-convexity. The following results describe properties of symmetric k-convex functions, properties that are parallel to those summarized in Lemma A.1 and Proposition A.1. Finally, notice that the concept of symmetric k-convexity can be easily extended to *multi-dimensional* space.

Lemma A.2 *(a) A real-valued convex function is also symmetric* 0*-convex and hence symmetric k-convex for all $k \geq 0$. A symmetric k_1-convex function is also a symmetric k_2-convex function for $k_1 \leq k_2$.*

(b) If $g_1(y)$ and $g_2(y)$ are symmetric k_1-convex and symmetric k_2-convex respectively, then for $\alpha, \beta \geq 0$, $\alpha g_1(y) + \beta g_2(y)$ is symmetric $(\alpha k_1 + \beta k_2)$-convex.

(c) If $g(y)$ is symmetric k-convex and w is a random variable, then $E\{g(y - w)\}$ is also symmetric k-convex, provided $E\{|g(y - w)|\} < \infty$ for all y.

(d) Assume that g is a continuous symmetric k-convex function and $g(y) \to \infty$ as $|y| \to \infty$. Let S be a global minimizer of g and s be any element from the set

$$X := \{x | x \leq S, g(x) = g(S) + k \text{ and } g(x') \geq g(x) \text{ for any } x' \leq x\}.$$

Then we have the following results.

(i) $g(s) = g(S) + k$ and $g(y) \geq g(s)$ for all $y \leq s$.

(ii) $g(y) \le g(z) + k$ *for all* y, z *with* $(s+S)/2 \le y \le z$.

Proposition A.2 *If* $f(x)$ *is a symmetric* K*-convex function, then the function*

$$g(x) = \min_{y \le x} Q\delta(x-y) + f(y)$$

is symmetric $\max\{K, Q\}$*-convex. Similarly, the function*

$$h(x) = \min_{y \ge x} Q\delta(x-y) + f(y)$$

is also symmetric $\max\{K, Q\}$*-convex.*

Proposition A.3 *Let* $f(\cdot,\cdot)$ *be a function defined on* $\Re^n \times \Re^m \to \Re$. *Assume that for any* $x \in \Re^n$ *there is a corresponding set* $C(x) \subset \Re^m$ *such that the set* $C \equiv \{(x,y) \mid y \in C(x), x \in \Re^n\}$ *is convex in* $\Re^n \times \Re^m$. *If* f *is symmetric* k*-convex over the set* C*, and the function*

$$g(x) = \inf_{y \in C(x)} f(x,y)$$

is well defined, then g *is symmetric* k*-convex over* $\Re^n$.

Proposition A.4 *(see Chen et al. (2007) and Simchi-Levi et al. (2005)) If* $g(x,\tilde{\xi})$ *is concave,* k*-concave or symmetric* k*-concave for any given* $\tilde{\xi}$*, then* $\mathcal{CE}^R_{\tilde{\xi}}(g(x,\tilde{\xi}))$ *is also concave,* k*-concave or symmetric* k*-concave.*

Proposition A.5 *If* $f(x,t)$ *is concave,* k*-concave or symmetric* k*-concave for any given* t*. Then* $g(x) = \min_t f(x,t)$ *(assumed to be well-defined) is also concave,* k*-concave or symmetric* k*-concave.*

B Proof of Lemma 2.2

(a) It is a well known result (Pratt (1964)) that the certainty equivalent is increasing with the risk tolerance level. That is, $\mathcal{CE}^{R'}_{\tilde{\xi}}\left(g(\tilde{\xi})\right) \le \mathcal{CE}^{R}_{\tilde{\xi}}\left(g(\tilde{\xi})\right)$ for any $f_{\tilde{\xi}} \in \Theta$. Therefore

$$\mathcal{G}^{R'}_{\Theta}\left(g(\tilde{\xi})\right) = \min_{f_{\tilde{\xi}} \in \Theta} \mathcal{CE}^{R'}_{\tilde{\xi}}\left(g(\tilde{\xi})\right) \le \min_{f_{\tilde{\xi}} \in \Theta} \mathcal{CE}^{R}_{\tilde{\xi}}\left(g(\tilde{\xi})\right) = \mathcal{G}^{R}_{\Theta}\left(g(\tilde{\xi})\right).$$

(b) Define $h(x) = \mathcal{G}^1_{f_{\tilde{\xi}}}\left(xg(\tilde{\xi})\right)$ and

$$H(x,R) = \begin{cases} Rh(x/R), & \text{if } R > 0, \\ 0, & \text{if } R = 0, x = 0, \\ +\infty, & \text{if } R < 0. \end{cases}$$

Lemma 2.1 part (d) implies that h is concave, which in turn implies that $H(x, R)$ is concave in (x, R) (see Rockafellar (1970), page 35). Thus, $\mathcal{G}^R_{f_{\tilde{\xi}}}\left(g(\tilde{\xi})\right) = H(1, R)$ is concave in R for $R > 0$.

(c) For a given $f_{\tilde{\xi}} \in \Theta$,

$$\frac{\partial \mathcal{CE}^R_{\tilde{\xi}}(g(\tilde{\xi}))}{\partial R} = -\ln \mathbf{E}\left[\exp\{-g(\tilde{\xi})/R\}\right] - \frac{\mathbf{E}\left[g(\tilde{\xi})\exp\{-g(\tilde{\xi})/R\}\right]}{R\mathbf{E}\left[\exp\{-g(\tilde{\xi})/R\}\right]}.$$

(It is easy to verify that under our assumptions, the integration and differential can be interchanged when taking the derivative of $\mathcal{CE}^R_{\tilde{\xi}}(g(\tilde{\xi}))$ with respect to R.) The above formula immediately implies that for any two constants $\delta > 0$ and $M > 0$, there exists a constant $\kappa > 0$ such that for any continuous function $g(\tilde{\xi})$ with $|g(\tilde{\xi})| \leq M$ for any $\tilde{\xi}$,

$$0 \leq \frac{\partial \mathcal{CE}^R_{\tilde{\xi}}(g(\tilde{\xi}))}{\partial R} \leq \kappa.$$

Since when $f_{\tilde{\xi}} \in \Theta$ attains the minimum in the definition of $\mathcal{G}^R_{\Theta}$, we have that for any $R' \geq R$,

$$\mathcal{G}^{R'}_{\Theta}(g(\tilde{\xi})) - \mathcal{G}^R_{\Theta}(g(\tilde{\xi})) \leq \mathcal{CE}^{R'}_{\tilde{\xi}}(g(\tilde{\xi})) - \mathcal{CE}^R_{\tilde{\xi}}(g(\tilde{\xi})) \leq \left|\frac{\partial \mathcal{CE}^R_{\tilde{\xi}}(g(\tilde{\xi}))}{\partial R}\right|(R' - R) \leq \kappa(R' - R).$$

Thus, part (c) holds.

C Results for the finite horizon models

Theorem C.1 *For the ambiguity and risk averse inventory and pricing model (5),*

(a) an (s, S, A, p) policy is optimal, and

(b) an (s, S) policy is optimal when pricing is not a decision variable.

The result is similar to the one corresponding to the risk averse inventory and pricing model analyzed in Chen et al. (2007). The proof is similar as well by observing that the minimum envelope of (symmetric) k-concave functions is still (symmetric) k-concave (see Proposition A.5).

We now further restrict the demand function $D_t(p_t, \tilde{\varepsilon}_t)$ and the ambiguity set Θ_t and show that the structure of the optimal inventory and pricing policy may be reduced to an (s, S, p) policy.

Theorem C.2 *Consider the risk neutral ambiguity averse inventory and pricing formulation, i.e., Equation (5) with $\rho_t \to \infty$. If*

- *the demand function is additive, i.e., parameter* α_t *in the demand function*

$$D_t(p_t, \tilde{\varepsilon}_t) = \tilde{\beta}_t - \alpha_t p_t$$

is a constant; and

- *every* $\tilde{\beta}_t$ *in set* Θ_t *has the same expectation,*

then an (s, S, p) *policy is optimal.*

The result and proof are similar to Chen & Simchi-Levi (2004a), Theorem 3.1. For the completeness of the paper, here we provide a key step for the proof, which basically shows that a higher order-up-to level y_t leads to a higher expected end-period inventory level.

Proof.

Note that (5) under the conditions specified in the theorem can be written as follows.

$$J_t(x) = \max_{y:y\geq x} -k\delta(y-x) + \max_{p:\underline{p}_t\leq p\leq\bar{p}_t} I_t(y,p) \ ,$$

where

$$I_t(y,p) = R(p) + F_t(y + \alpha_t p),$$

$$R(p) = (p - c_t)(\beta_t - \alpha_t p) \ ,$$

with β_t being the expectation of $\tilde{\beta}_t$, and

$$F_t(y) = \min_{f_{\tilde{\beta}_t}\in\Theta_t} E_{f_{\tilde{\beta}_t}}\left[-\hat{h}(y-\tilde{\beta}_t) + \gamma J_{t-1}(y-\tilde{\beta}_t)\right] \ .$$

Define a new function $K(y,z)$ as follows.

$$K(y,z) = R((z-y)/\alpha_t) + F_t(z).$$

Then, we have that

$$\max_{p:\underline{p}_t\leq p\leq\bar{p}_t} I_t(y,p) = \max_{z:\underline{p}_t\leq(z-y)/\alpha_t\leq\bar{p}_t} K(y,z).$$

Since $R(\cdot)$ is concave and $K(y,z)$ is supermodular, the above optimization problem has an optimal solution $z^*(y)$, which is non-deceasing in y. This implies that the higher the order-up-to level y in period t, the higher the expected inventory level at the end of period t. This result is parallel to Lemma 2 in Chen & Simchi-Levi (2004a). The remaining proof follows from the same steps as the one for Theorem 3.1 in Chen & Simchi-Levi (2004a) and is omitted.

■

From Theorem C.1, we know that an (s, S, A, p) policy is optimal for the following dynamic programming recursion

$$G_t(x) = \max_{y \geq x, p \in \mathcal{P}} -k\delta(y-x) + \mathcal{G}_{\Theta}^{R_t}\left[P(y,p;\tilde{\varepsilon}) + \gamma G_{t-1}(y - D(p,\tilde{\varepsilon}))\right] \ .$$

Next we present a technical result which illustrates that the parameters s_t and S_t are uniformly bounded for $t = 1, 2, \ldots$. For this purpose, let the lower bound $\underline{s}$ be a constant such that $\underline{s} \leq y^*$ and

$$\hat{h}(\underline{s}) = \hat{h}(y^*) + k,$$

where y^* is a minimizer of $\hat{h}$.

Also define the upper bound to be $\bar{S} = S^0 + \Xi$, in which $S^0 \geq y^*$ and $\hat{h}(S^0) = \hat{h}(y^*) + k$, and Ξ is the demand upper bound in Assumption 2.1. Indeed, the demand upper bound assumption is only used here to derive an upper bound for S_t. It is worth noticing that in the risk neutral case, such an assumption is not needed (see Chen & Simchi-Levi (2004b)).

The main result in this subsection is the following theorem.

Theorem C.3 *$\underline{s}$ and $\bar{S}$ are the lower and upper bound for s_t and S_t. That is,*

$$\underline{s} \leq s_t \leq S_t \leq \bar{S} \ .$$

In order to prove Theorem C.3, we first make the following observations.

Lemma C.1 *(a) $G_t(x) \geq G_t(x') - k$ for $x \leq x'$.*

(b) $G_t(x) \leq G_t(x')$ for any $x \leq x' \leq y^*$.

(c) $g_t(y,p) \leq g_t(y',p)$ for any $y \leq y' \leq y^*$ and $p \in \mathcal{P}$, where

$$g_t(y,p) = \mathcal{G}_{\Theta}^{R_t}\left[P(y,p;\tilde{\varepsilon}) + \gamma G_{t-1}(y - D(p,\tilde{\varepsilon}))\right].$$

Proof. For two inventory levels $x \leq x'$, one can always first raise the inventory level from x to x' by paying a fixed cost k and then follow the same optimal strategy for the inventory level x'. Therefore (a) holds.

We now prove parts (b) and (c) by induction. First, we have that part (b) holds for $t = 0$ since $G_0(x) = 0$. Now assume that (b) holds for $G_{t-1}(x)$. Then for any $y \leq y' \leq y^*$ and $p \in \mathcal{P}$,

$$\begin{aligned} P(y,p;\tilde{\varepsilon}) &= (p-c)D(p,\tilde{\varepsilon}) - \hat{h}(y' - D(p,\tilde{\varepsilon})) + \hat{h}(y' - D(p,\tilde{\varepsilon})) - \hat{h}(y - D(p,\tilde{\varepsilon})) \\ &\leq P(y',p;\tilde{\varepsilon}), \end{aligned}$$

where the inequality holds since $\hat{h}(x)$ is convex and $y \leq y' \leq y^*$. Thus, for any $y \leq y' \leq y^*$ and $p \in \mathcal{P}$,

$$\begin{aligned} g_t(y,p) &= \mathcal{G}_{\Theta}^{R_t}\left[P(y,p;\tilde{\varepsilon}) + \gamma G_{t-1}(y - D(p,\tilde{\varepsilon}))\right] \\ &\leq \mathcal{G}_{\Theta}^{R_t}\left[P(y',p;\tilde{\varepsilon}) + \gamma G_{t-1}(y' - D(p,\tilde{\varepsilon}))\right] \\ &= g_t(y',p), \end{aligned}$$

where the inequality follows from the induction assumption. Thus, part (c) holds. Finally, part (b) follows immediately from part (c), and the fact that $G_t(x) = \max_p g_t(x,p)$. ■

Now we are ready to prove Theorem C.3.

Proof. First, note that for any $p \in \mathcal{P}$ and $y \leq \underline{s}$, we have that

$$\begin{aligned} P(y,p;\tilde{\varepsilon}) &= (p-c)D(p,\tilde{\varepsilon}) - \hat{h}(y^* - D(p,\tilde{\varepsilon})) + \hat{h}(y^* - D(p,\tilde{\varepsilon})) - \hat{h}(y - D(p,\tilde{\varepsilon})) \\ &\leq P(y^*,p;\tilde{\varepsilon}) + \hat{h}(y^*) - \hat{h}(y) \\ &\leq P(y^*,p;\tilde{\varepsilon}) - k, \end{aligned}$$

where the first inequality holds since $\hat{h}$ is convex and thus has increasing differences, and the second inequality follows from the definition of $\underline{s}$. In addition, from Lemma C.1 (b), we have $G_{t-1}(y - D(p,\tilde{\varepsilon})) \leq G_{t-1}(y^* - D(p,\tilde{\varepsilon}))$ for $y \leq \underline{s} \leq y^*$. Therefore,

$$\begin{aligned} g_t(y,p) &= \mathcal{G}_{\Theta}^{R_t}\left[P(y,p;\tilde{\varepsilon}) + \gamma G_{t-1}(y - D(p,\tilde{\varepsilon}))\right] \\ &\leq \mathcal{G}_{\Theta}^{R_t}\left[P(y^*,p;\tilde{\varepsilon}) + \gamma G_{t-1}(y^* - D(p,\tilde{\varepsilon}))\right] - k \\ &= g_t(y^*,p) - k. \end{aligned}$$

This implies that it is optimal to place an order for an inventory level no more than $\underline{s}$.

We now show that $S_t \leq \bar{S}$. Note that for any $p \in \mathcal{P}$ and $y \geq \bar{S}$, we have that

$$\begin{aligned} P(y,p;\tilde{\varepsilon}) &= (p-c)D(p,\tilde{\varepsilon}) - \hat{h}(y^* + \Xi - D(p,\tilde{\varepsilon})) + \hat{h}(y^* + \Xi - D(p,\tilde{\varepsilon})) - \hat{h}(y - D(p,\tilde{\varepsilon})) \\ &\leq P(y^* + \Xi,p;\tilde{\varepsilon}) + \hat{h}(y^*) - \hat{h}(y - \Xi) \\ &\leq P(y^* + \Xi,p;\tilde{\varepsilon}) - k, \end{aligned}$$

where again the first inequality holds since $\hat{h}$ is convex and the second inequality follows from the definition of $\bar{S}$. Thus, for any $y \geq \bar{S}$,

$$\begin{aligned} g_t(y,p) &= \mathcal{G}_{\Theta}^{R_t}\left[P(y,p;\tilde{\varepsilon}) + \gamma G_{t-1}(y - D(p,\tilde{\varepsilon}))\right] \\ &\leq \mathcal{G}_{\Theta}^{R_t}\left[P(y^* + \Xi,p;\tilde{\varepsilon}) - k + \gamma G_{t-1}(y - D(p,\tilde{\varepsilon}))\right] \\ &\leq \mathcal{G}_{\Theta}^{R_t}\left[P(y^* + \Xi,p;\tilde{\varepsilon}) - k + \gamma G_{t-1}(y^* + \Xi - D(p,\tilde{\varepsilon})) + \gamma k\right] \\ &\leq g_t(y^* + \Xi,p), \end{aligned}$$

where the second inequality follows from Lemma C.1 (a). Thus, $S_t \leq \bar{S}$. ■

D Proof for Theorem 3.2

The above theorem is built upon the following two lemmas, which resemble Lemmas 4 and 5 in Chen & Simchi-Levi (2004b) respectively.

Lemma D.1 *(a)* $\varphi(x) = 0$ *for any* $x \leq s^*$.

(b) $\varphi(x) \leq \varphi(S^*) = k$ *for any* x.

(c) $Q(x) \geq Q(s^*)$ *for any* $x \in [s^*, S^*]$.

(d) $\varphi(x) \geq 0$ *for any* $x \leq S^*$.

(e) $s^* \leq x^*$.

(f) $S^* \geq y^*$.

Proof. Part (a) follows directly from the construction of function $\varphi(x) = \varphi(x, s^*)$. Part (b) follows from the definition of s^*, and part (e) follows from Lemma 3.1. From proposition 3.1 (f), we have $S^* \geq y^*$ and thus part (f) holds.

We now prove part (c). Note that $\varphi(S^* - D(p, \tilde{\varepsilon}), s^*) \leq \varphi(S^*, s^*) = k$. We have that

$$\begin{aligned} \varphi(S^*, s^*) &= \max_{p \in \mathcal{P}} \mathcal{G}[P(S^*, p; \tilde{\varepsilon}) + \gamma\varphi(S^* - D(p, \tilde{\varepsilon}), s^*)] - Q(s^*) \\ &\leq \max_{p \in \mathcal{P}} \mathcal{G}[P(S^*, p; \tilde{\varepsilon}) + \gamma\varphi(S^*, s^*)] - Q(s^*) \\ &= Q(S^*) - Q(s^*) + \gamma\varphi(S^*, s^*)\ , \end{aligned}$$

in which the last equation follows Lemma 2.1(b). Therefore,

$$Q(S^*) - Q(s^*) \geq (1 - \gamma)\varphi(S^*, s^*) = (1 - \gamma)k \geq 0.$$

Since $Q(x)$ is concave and $s^* \leq x^*$, we have $Q(x) \geq Q(s^*)$ for $x \in [s^*, S^*]$, i.e., part (c) holds.

We now prove part (d). Clearly, $\varphi(x) = 0$ for $x \leq s^*$. We now assume the induction hypothesis that $\varphi(x) \geq 0$ for $x \leq \bar{x}$ for some $\bar{x} \in [s^*, S^*]$. Then for any $x \in [\bar{x}, \min(\bar{x} + \eta, S^*)]$, $\varphi(x - D(p, \tilde{\varepsilon})) \geq 0$ following the induction hypothesis. Therefore we have,

$$\begin{aligned} \varphi(x) &= \max_{p \in \mathcal{P}} \mathcal{G}[P(x, p; \tilde{\varepsilon}) + \gamma\varphi(x - D(p, \tilde{\varepsilon}))] - Q(s^*) \\ &\geq \max_{p \in \mathcal{P}} \mathcal{G}[P(x, p; \tilde{\varepsilon})] - Q(s^*) \\ &= Q(x) - Q(s^*) \\ &\geq 0, \end{aligned}$$

where the last inequality follows from part (c). Thus, part (d) holds. ■

Lemma D.2 $\varphi(x)$ *is a symmetric* k*-concave function.*

Proof. We prove by induction that for any $x_0 \le x_1$, $\lambda \in [0,1]$,

$$\varphi(x_\lambda) \ge (1-\lambda)\varphi(x_0) + \lambda\varphi(x_1) - \max\{1-\lambda, \lambda\}k, \tag{20}$$

where $x_\lambda = (1-\lambda)x_0 + \lambda x_1$.

Since $\varphi(x) = 0$ for $x \le s^*$, the inequality (20) trivially holds for $x_1 \le s^*$. We assume, as the induction hypothesis, that (20) holds for any $x_0 \le x_1 \le \bar{x}$ for some $\bar{x}$. We next show that the inequality continues to hold for any $x_0 \le x_1 \le \bar{x} + \eta$.

Let $p^*(x)$ be an optimal solution for problem (12) with $s = s^*$. We distinguish between three different cases.

(1) $x_0 > s^*$. In this case, let $p_\lambda = (1-\lambda)p^*(x_0) + \lambda p^*(x_1)$. We have

$$\begin{aligned}
\varphi(x_\lambda) &\ge \mathcal{G}[P(x_\lambda, p_\lambda; \tilde{\varepsilon}) + \gamma\varphi(x_\lambda - D(p_\lambda, \tilde{\varepsilon}))] - Q(s^*) \\
&\ge \mathcal{G}[(1-\lambda)(P(x_0, p^*(x_0); \tilde{\varepsilon}) + \gamma\varphi(x_0 - D(p^*(x_0), \tilde{\varepsilon}))) \\
&\qquad +\lambda(P(x_1, p^*(x_1); \tilde{\varepsilon}) + \gamma\varphi(x_1 - D(p^*(x_1), \tilde{\varepsilon}))) - \max\{1-\lambda, \lambda\}k] - Q(s^*) \\
&\ge (1-\lambda)\mathcal{G}[P(x_0, p^*(x_0); \tilde{\varepsilon}) + \gamma\varphi(x_0 - D(p^*(x_0), \tilde{\varepsilon}))] \\
&\qquad +\lambda\mathcal{G}[P(x_1, p^*(x_1); \tilde{\varepsilon}) + \gamma\varphi(x_1 - D(p^*(x_1), \tilde{\varepsilon}))] - Q(s^*) - \max\{1-\lambda, \lambda\}k \\
&= (1-\lambda)\varphi(x_0) + \lambda\varphi(x_1) - \max\{1-\lambda, \lambda\}k,
\end{aligned}$$

where the first inequality holds since p_λ is a feasible price; the second inequality follows from the monotonicity of the operator $\mathcal{G}$, the concavity of the function $P(S^*, p, \tilde{\varepsilon})$ in p, the fact that $x_0 - D(p^*(x_0), \tilde{\varepsilon}) \le \bar{x}$ and $x_1 - D(p^*(x_1), \tilde{\varepsilon}) \le \bar{x}$; and the induction assumption, and the last inequality holds since the operator $\mathcal{G}$ preserves concavity.

(2) $x_0 \le s^*$ and $x_\lambda \le S^*$. In this case, we have from Lemma D.1 that $\varphi(x_0) = 0$, $\varphi(x_\lambda) \ge 0$ and $\varphi(x_1) \le k$. Thus, the inequality (20) holds.

(3) $x_0 \le s^* \le S^* \le x_\lambda$. In this case, x_λ can be expressed as the convex combination of S^* and x_1, i.e., $x_\lambda = (1-\mu)S^* + \mu x_1$ for some $\mu \in [0,1]$. It is straightforward to verify that

$\mu \leq \lambda$. We now have that

$$\begin{aligned}
\varphi(x_\lambda) &\geq (1-\mu)\varphi(S^*) + \mu\varphi(x_1) - \max\{1-\mu, \mu\}k \\
&= (1-\lambda)\varphi(x_0) + \lambda\varphi(x_1) - \max\{1-\lambda, \lambda\}k \\
&\quad +(1-\mu)k + (\mu-\lambda)\varphi(x_1) \\
&\quad + \max\{1-\lambda, \lambda\}k - \max\{1-\mu, \mu\}k \\
&\geq (1-\lambda)\varphi(x_0) + \lambda\varphi(x_1) - \max\{1-\lambda, \lambda\}k \\
&\quad +(1-\mu)k + (\mu-\lambda)k \\
&\quad + \max\{1-\lambda, \lambda\}k - \max\{1-\mu, \mu\}k \\
&\geq (1-\lambda)\varphi(x_0) + \lambda\varphi(x_1) - \max\{1-\lambda, \lambda\}k,
\end{aligned}$$

where the first inequality follows from case (1), the second inequality from the fact that $\varphi(x) \leq k$ for any x and the last inequality from a simple algebraic manipulation.

Thus, the inequality (20) holds for $x_0 \leq x_1 \leq \bar{x} + \eta$ and the proof is complete. ■

Proof of Theorem 3.2

Define for any x,

$$O(x) = \max_{p \in \mathcal{P}} \mathcal{G}[P(x, p; \tilde{\varepsilon}) + \gamma\varphi(x - D(p, \tilde{\varepsilon}))] - Q(s^*).$$

It is clear that $O(x) = \varphi(x)$ for $x \geq s^*$, while for $x \leq s^*$, $\varphi(x) = 0$ and $O(x) = Q(x) - Q(s^*) \leq 0$. Following Lemma D.1, for any x we have

$$O(x) \leq O(S^*) = \varphi(S^*) = k \ .$$

We distinguish between three different cases to show the result.

(1) $x \leq s^*$. In this case, we have

$$\begin{aligned}
\varphi(x) = 0 &= -k + \max_{y: y \geq x} O(y) \\
&= \max_{y: y \geq x} -k\delta(y-x) + \max_{\underline{p} \leq p \leq \bar{p}} \mathcal{G}\left[P(y, p, \tilde{\varepsilon}) - Q(s^*) + \gamma\varphi(y - D(p, \tilde{\varepsilon}))\right] \ ,
\end{aligned}$$

with the maximum achieved at $y^* = S^*$.

(2) $x \in (s^*, S^*]$. In this case, $\varphi(x) \geq 0$, and for any $y > x$, $-k\delta(y-x) + O(y) \leq 0$. Therefore

$$\varphi(x) = O(x) = \max_{y: y \geq x} -k\delta(y-x) + \max_{\underline{p} \leq p \leq \bar{p}} \mathcal{G}\left[P(y, p, \tilde{\varepsilon}) - Q(s^*) + \gamma\varphi(y - D(p, \tilde{\varepsilon}))\right] \ ,$$

with the maximum achieved at $y^* = x$.

(3) $x > S^*$. In this case, for any $y > x$, x can be expressed as the convex combination of S^* and y, i.e., there exists $\lambda \in [0,1]$ such that $x = (1-\lambda)S^* + \lambda y$. Since φ is symmetric k-concave, we have that

$$\begin{aligned}\varphi(x) &\geq (1-\lambda)\varphi(S^*) + \lambda\varphi(y) - \max\{1-\lambda, \lambda\}k \\ &= \varphi(y) + (1-\lambda)(\varphi(S^*) - \varphi(y)) - \max\{1-\lambda, \lambda\}k \\ &\geq \varphi(y) - k \\ &= O(y) - k,\end{aligned}$$

where the first inequality follows from the symmetric k-concavity of φ and the last inequality follows from the fact that $\varphi(y) \leq \varphi(S^*) = k$ for any y. Thus, it is optimal not to place an order, or, $y^* = x$.

The proof of Theorem 3.2 is now complete.

References

X. Chen (2003). 'Coordinating Inventory Control and Pricing Strategies with Random Demand and Fixed Ordering Cost'. Ph.D. thesis, Massachusetts Institute of Technology.

X. Chen, et al. (2007). 'Risk Aversion in Inventory Management'. *Operations Research* **55**(5):828–842.

X. Chen & D. Simchi-Levi (2004a). 'Coordinating Inventory Control and Pricing Strategies with Random Demand and Fixed Ordering Cost: The Finite Horizon Case'. *Operations Research* **52**(6):887–896.

X. Chen & D. Simchi-Levi (2004b). 'Coordinating Inventory Control and Pricing Strategies with Random Demand and Fixed Ordering Cost: The Infinite Horizon Case'. *Mathematics of Operations Research* **29**(3):698–723.

J. Pratt (1964). 'Risk Aversion in the Small and in the Large'. *Econometrica* **32**(1-2):122–136.

R. T. Rockafellar (1970). *Convex Analysis*. Princeton University Press, Princeton.

H. Scarf (1960). 'The optimality of (s, S) policies for the dynamic inventory problem'. In *Proceedings of the 1st Stanford Symposium on Mathematical Methods in the Social Sciences*. Stanford University Press, Stanford, CA.

D. Simchi-Levi, et al. (2005). *The Logic of Logistics: theory, Algorithms, and Applications for Logistics and Supply Chain Management, Second Edition*. Springer-Verlag, New York.