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Contextual Areas

Optimal Monitoring Schedule in Dynamic Contracts

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to compute and implement, and intuitive to explain.

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Abstract. Consider a setting in which a principal induces effort from an agent to reduce the

arrival rate of a Poisson process of adverse events. The effort is costly to the agent and

unobservable to the principal unless the principal is monitoring the agent. Monitoring

ensures effort but is costly to the principal. The optimal contract involves monetary

payments and monitoring sessions that depend on past arrival times. We formulate the problem as a stochastic optimal control model and solve the problem analytically. The

optimal schedules of payment and monitoring demonstrate different structures depending

on model parameters. Overall, the optimal dynamic contracts are simple to describe, easy

Keywords: dynamic contract • moral hazard • principal-agent model • optimal control • continuous time • costly state verification

1. Introduction

Adverse events often bring significant damages to an organization or the society. In many situations, better efforts in maintaining and safeguarding a system can reduce the chance of such adverse events. The challenge is that these events may still occur, albeit less frequently, despite the best effort. And efforts are often hard to verify. Furthermore, people in charge of the effort (an agent) often cannot bear the full consequence of an adverse event because of limited liability. In practice, an agent is often a hired employee or subcontractor, who can be paid one way or another but cannot compensate damages. In order to ensure efforts, a principal, be it a firm or a government, may decide to "keep an eye" on the agent, which ensures that adverse events occur at a lower frequency and are not the result of a lack of effort should they happen. For example, Accenture served as an outside vendor to support the IT systems for Kbank of Thailand. According to conversations with Accenture, once in a while, the in-house IT team at Kbank would show up to work together with the Accenture team. Such monitoring activities are often too costly to conduct at all times. The principal can also schedule payments that are contingent on arrivals to motivate effort. How do we induce effort from the agent with minimum payments and monitoring costs? In a dynamic setting in which adverse events occur stochastically over time, what is the optimal schedule to pay and to monitor the agent?

To answer these questions, we study an optimal contract design problem in a dynamic setting in which a risk-neutral principal faces a Poisson process of costly adverse events. (Think of adverse events as system breakdowns or production defects.) The instantaneous rate of the Poisson process can be reduced by a risk-neutral agent if the agent exerts effort at that moment. Effort is costly to the agent and observable to the principal only when the principal monitors the agent. The principal, who can commit to a long-term contract over an infinite horizon in continuous time, needs to trade off direct payments to the agent, versus costly monitoring, in order to induce effort.

We formulate this optimal dynamic contract design problem as a continuous-time stochastic optimal control model and are able to provide complete characterizations of the optimal monitoring and payment schedules, which vary depending on the monitoring cost. As expected, if the monitoring cost is lower than a threshold, the principal should monitor all the time. In this case, the agent's total future utility (commonly referred to as the "promised utility," see, for example, Spear and Srivastava 1987) is always kept at zero. Interesting structures emerge when the monitoring cost is above the threshold. In this case, the promised utility serves as a sufficient statistic of the entire history of arrival times, on which the optimal monitoring and payment schedules critically depend.

Generally speaking, the agent needs to be penalized for each arrival when not being monitored. Because we assume that the agent has limited liability and cannot pay the principal, the penalty takes the form of a downward jump of the promised utility upon each arrival whenever the agent is not being monitored. Between arrivals, the promised utility gradually increases. When downward jumps resulting from arrivals bring the promised utility below a threshold, the principal starts monitoring. Monitoring stops only after the promised utility climbs back to the threshold, during which arrivals do not matter. A flow of payment starts only when the promised utility reaches and stays at an upper bound. As soon as another arrival occurs, the promised utility takes a downward jump from the upper bound, which stops the payment.

The aforementioned movements of the promised utility and payment schedule are not completely new to our model. In fact, both Biais et al. (2010) and Myerson (2015) study similar models as ours without monitoring. Biais et al. (2010) consider the agent as a firm and the principal as an investor, who can change the firm size when the promised utility becomes too low. Myerson (2015), on the other hand, considers a political economy setting, in which the principal can dynamically replace an agent with a new one. A fundamental difference between Biais et al. (2010) and Myerson (2015) is the time discount rate. In Biais et al. (2010), the principal is strictly more patient than the agent although Myerson (2015) assumes the two players' time discount rates are the same. Equal time discount rate in this setting introduces an "infinite back-loading" issue. That is, the principal always prefers to delay the cash payment to the future while promising to pay the corresponding interest. In order to prevent the problem from becoming unbounded, Myerson (2015) introduces an exogenous upper bound on the promised utility. For the different discount rate case, Biais et al. (2010) obtain an endogenous upper bound on the promised utility. We study both the different and equal time discount cases. For the different discount case, although it is also not necessary to introduce an exogenous upper bound, we also describe the optimal contract under such a bound in case the endogenous one is high for the agent to stomach in practice.

Our optimal contract demonstrates subtle and important features that do not arise in Biais et al. (2010) even though the aforementioned movements of the promised utility above the monitoring threshold also appear in Biais et al. (2010) and Myerson (2015). When the principal is more patient than the agent, in particular, the promised utility may always be strictly positive, which means that the individual rationality (IR) constraint may never be binding. This may be

counterintuitive to people familiar with the mechanism design literature, starting from Myerson (1981). As we explain in the paper, raising the promised utility threshold for monitoring allows the principal to shorten the monitoring time, which is preferable when the monitoring cost is high. From a technical point of view, the optimal value function demonstrates the "smooth pasting" phenomenon that often arises in optimal stopping problems (see, for example, Dixit 1994). Smooth pasting does not arise in Biais et al. (2010). Finally, we identify a subtle connection between monitoring and allowing the agent to shirk. Essentially, the optimal contract that allows the agent to shirk can be solved as a special case of the monitoring problem with a specific monitoring cost.

Optimal scheduling of monitoring in a dynamic environment is fundamentally an operations problem. There is a recent stream of papers in the operations research/management science literature that study incentive issues related to auditing/monitoring/ inspecting. Most of these papers compare a few classes of practically useful mechanisms or focus on static settings. Babich and Tang (2012), for example, study three mechanisms (deferred payment mechanism, inspection mechanism, and a mechanism that combines the two) for dealing with product adulteration issues when manufacturers cannot control the suppliers' actions. Rui and Lai (2015) study similar sets of mechanisms in a similar problem setting with endogenous procurement decisions and more general arrival discovery processes. Kim (2015) studies environmental disclosure and inspection policies in a dynamic setting and compares deterministic versus random inspection schedules. Plambeck and Taylor (2016, 2019) also study environmental monitoring and disclosure issues, using static models. Wang et al. (2016) study monetary and inspection instruments to induce the agent to report the occurrence of an adverse environmental event. The paper models this dynamic adverse selection problem as an optimal control model in continuous time and identifies optimal policies.

Monitoring is a way to conduct *costly state verification* under asymmetric information in the economics literature, started from Townsend (1979) for adverse selection issues in a static setting. Dye (1986) extends the idea to moral hazard problems, also in a static setting. In continuous-time dynamic settings, Piskorski and Westerfield (2016) study a model in which the underlying uncertainty is a Brownian motion, and the principal checks the agent following a Poisson process, the rate of which is the design issue. If the agent is found shirking, the principal may terminate the contract. Varas et al. (2017) study a two-state hidden Markov model, whose instantaneous transition rates are affected by the agent's effort. The principal decides the schedule of inspecting the true state of the Markov chain in order to induce effort.

Our model and analysis are rooted in the continuoustime optimal contracting literature. Sannikov (2008) provides the analytical foundation for these types of models. In Sannikov (2008), the agent's effort affects the drift of a Brownian motion. And the optimal contract is solved as the solution of a stochastic optimal control problem. The Brownian motion setup is natural for corporate finance applications (see, for example, DeMarzo and Sannikov 2006, Biais et al. 2007, and Fu 2017 to name a few).

Biais et al. (2010) build upon this framework and study continuous-time optimal contracting based on Poisson processes instead of Brownian motions. One important advantage of Poisson process-based models is that the optimal control policies are often easier to describe and implement. More recently, Sun and Tian (2018) study how to induce an agent to increase the arrival rate of a Poisson process in a continuous-time infinite horizon setting. More broadly, a sequence of recent papers also study optimal contracting problems related to Poisson arrivals (see, for example, Mason and Välimäki 2015, Green and Taylor 2016, Hidir 2017, Shan 2017, Varas 2018). Our paper differs from the aforementioned continuous-time dynamic contracting literature in the monitoring component.

More generally, dynamic moral hazard problem has also been the focus of some recent papers published in operations research. Plambeck and Zenios (2003) study continuous-time control of incentive issues in a make-to-stock production system. Li et al. (2012) investigate how to motivate multiple agents (suppliers) in a discrete time dynamic setting. Their model is also based on the promised utility framework originated from Spear and Srivastava (1987).

The rest of this paper is organized as follows. We introduce the model and supporting concepts in Section 2 and focus on the equal discount rate case in Section 3, in which we introduce general structures of optimal contracts and value functions. Built upon these concepts, Section 4 further investigates the case in which the principal is more patient than the agent. We then present theories that support a computational algorithm for the optimal value function and contract in Section 5, study a simple cyclic monitoring schedule in Section 6, and conclude the paper with a discussion of allowing shirking in Section 7. All the proofs are presented in the appendix. In addition, Appendix E contains a number of extensions and some potential directions for future research.

2. The Model

We consider a principal–agent model in a continuoustime setting. The principal faces a Poisson process with arrival rate λ of adverse events (arrivals), each costing the principal a value K.¹ The principal hires an agent, who can bring down the instantaneous arrival rate to $\lambda = \lambda - \Delta \lambda$ if the agent exerts effort at this point in time. The principal does not observe the effort unless *monitoring* the agent at a cost rate *m* per unit of time. Denote a left-continuous counting process $\{N_t\}_{t>0}$ to represent the total number of arrivals up to time *t*, the rate of which depends on the agent's effort process. Further denote $\Lambda = {\lambda_t}_{t>0}$ to represent the agent's effort process. That is, $\lambda_t \in {\lambda, \lambda}$ at each time epoch t. The principal and the agent are both riskneutral and discount future cash flows with discount rates *r* and ρ , respectively. As is often assumed in the literature (see, for example, Biais et al. 2010), $\rho \ge r > 0$; that is, the principal is no less patient than the agent. We start with $\rho = r$ in Section 3 before moving to consider $\rho > r$ in Section 4.

We assume that the principal has commitment power to issue a long-term contract with the agent. This assumption allows us to formulate the strategic interaction between the principal and the agent as a dynamic optimization/optimal control problem in which the contract is a contingency plan that both players understand that the principal would follow through.² The contract specifies a payment and monitoring schedule over time, which depends on past arrival times. More generally, the contract should also specify when to exert effort and when to shirk. For the most part of the paper, we restrict attention to contracts that always induce effort from the agent until the very end of the paper when we show that the shirking problem is a just a special case.

The agent has limited liability, which means that monetary transfer is from the principal to the agent at any point in time. Therefore, the agent cannot buy out the principal to mitigate misalignment of incentives. Under this assumption, the principal needs to compensate the agent's effort, which costs a constant rate b per unit of time. Therefore, the agent *benefits* from shirking at the rate b. Beyond this flow payment b in the background, denote L_t to represent the principal's cumulative payment to the agent up to time t, such that $dL_t = dI_t + \ell_t dt$, in which the pure jump process I_t represents the cumulative instantaneous payment by time t and ℓ_t is a flow payment at time t with $dI_t \ge 0$ and $\ell_t \ge 0$. (Again, ℓ_t does not include the payment b.)

Denote process $\mathcal{M} = \{m_t\}_{t\geq 0}$ to represent the monitoring schedule under the contract in which $m_t \in \{m, 0\}$ captures the monitoring cost at time t. Monitoring during a time interval $(t, t + \delta]$ guarantees the agent's effort during this time interval. Monitoring may start in one of two ways. First, at any point in time t when an arrival occurs, the principal may decide to start monitoring with a probability $y_t \in [0, 1]$. Second, monitoring may also start at time t in a

"deterministic" fashion with respect to realizations of past uncertainties. In order to formally characterize past uncertainties, it is convenient to split the counting process $\{N_t\}_{t\geq 0}$ into two processes $\{N_t^s\}_{t\geq 0}$ and $\{N_t^n\}_{t>0}$, which represent the total numbers of arrivals that have and have not triggered monitoring up to time *t*, respectively. Therefore, $N_t^s + N_t^n = N_t$. In our setting, there are two sources of randomness. One is from nature (the Poisson arrival) and the other from control (random start of monitoring upon arrival). The counting processes N_t^s and N_t^n fully capture these two sources of uncertainties in the history. Consequently, we define filtration $\mathcal{F} = \{\mathcal{F}_t\}_{t \ge 0}$ such that \mathcal{F}_t captures the entire historical information up to time t specified by the two-variate counting process $\{N_t^s, N_t^n\}_{t>0}$. Overall, a contract Γ consists of \mathcal{F}_t -predictable payment and monitoring processes, L_t and m_t , respectively.³

Before proceeding, it is worth noting that, in order to start monitoring randomly upon arrival, the outcome of the probability process y_t needs to follow a device that is agreed upon and commonly observable by the principal and the agent. In practice, players may use devices such as the last two digits of a stock market index to generate a commonly observed random outcome. It is also worth explaining at this point why we need this randomness in the control policy space. As we show in Section 4, when the two players' discount rates are different and the monitoring cost is relatively low, in order to show that always monitoring is optimal, we need to specify optimal control policies for all positive promised utility values. Such an optimal control policy critically depends on the random start of monitoring.

2.1. Agent's Utility

Given contract Γ and the agent's effort process Λ , the agent's total utility is defined as

$$u(\Gamma, \Lambda) = \mathbb{E}^{\Gamma, \Lambda} \left[\int_0^\infty e^{-\rho \tau} \left(dL_\tau + b \, \mathbb{I}_{\lambda_\tau = \bar{\lambda}} \, d\tau \right) \right]. \tag{1}$$

It is standard and convenient to work with the agent's continuation utility (also referred to as the *promised utility*; see, for example, Spear and Srivastava 1987). That is, the total discounted utility starting from time *t*, defined as the following left-continuous process:

$$W_t(\Gamma, \Lambda) = \mathbb{E}^{\Gamma, \Lambda} \left[\int_t^\infty e^{-\rho(\tau - t)} \left(dL_\tau + b \mathbb{I}_{\lambda_\tau = \bar{\lambda}} \, d\tau \right) \, \middle| \, \mathcal{F}_t \right]. \tag{2}$$

When there is no confusion, we omit (Γ, Λ) and refer to the agent's continuation utility at time *t* as W_t . Clearly, $W_0 = u(\Gamma, \Lambda)$. As is often assumed in the literature, the agent does not commit to staying in the contract. Therefore, we require the following participation (also referred to as the IR constraint):

$$W_t \ge 0$$
, for all $t \ge 0$. (IR)

Later in the paper, the optimal contract and the principal's value function are all expressed as functions of the agent's promised utility so that the optimal contract design problem is essentially a stochastic optimal control with W_t being the state variable.

Under any contract, the agent's promised utility W_t must satisfy the following dynamics.

Lemma 1. For any contract Γ , there exists an \mathcal{F}_t -predictable process $\{(H_t^s, H_t^n)\}_{t\geq 0}$ such that

$$dW_t = \left\{ \rho W_t - b \mathbb{I}_{\lambda_\tau = \bar{\lambda}} + \lambda_t [y_t H_t^s + (1 - y_t) H_t^n] \right\} dt$$
$$- H_t^s dN_t^s - H_t^n dN_t^n - dL_t.$$
(PK)

In order to satisfy the agent's continued participation (IR), H_t^s and H_t^n are less than or equal to W_t .

The condition (PK) stands for "promise keeping," which ensures that W_t is indeed the agent's continuation utility starting from time t. Lemma 1 follows directly from the Martingale representation theorem (theorem T9, Brémaud 1981), and extends lemma 1 in Biais et al. (2010) and lemma 6 in Sun and Tian (2018) to our setting with a multivariate counting process. Here, we provide a heuristic derivation of (PK) following discrete time approximation, which offers an intuitive illustration.

Consider the promised utility at the beginning of a small time interval $[t, t + \delta)$ to be W_t . With probability $\lambda_t \delta y_t$, there is an arrival that triggers monitoring to start. In this case, the promised utility moves to $W_t - H_t^s$. With probability $\lambda_t \delta(1 - y_t)$, on the other hand, there is an arrival that does not trigger monitoring. In this case, the promised utility moves to $W_t - H_t^n$. Finally, with probability $1 - \lambda_t \delta$, no arrival occurs and the promised utility moves to $W_{t+\delta}$. Taking into consideration the potential benefit from shirking and ignoring payment for simplicity, we have

$$W_t = b\delta \mathbb{I}_{\lambda_\tau = \bar{\lambda}} + e^{-\rho\delta} \{\lambda_t \delta [y_t (W_t - H_t^s) + (1 - y_t) (W_t - H_t^n)] + (1 - \lambda_t \delta) W_{t+\delta} \}$$

Following the standard procedure of subtracting W_t from and dividing δ on both sides and letting δ approach zero, we obtain

$$\lim_{\delta \downarrow 0} \frac{W_{t+\delta} - W_t}{\delta} = \rho W_t - b \mathbb{I}_{\lambda_\tau = \bar{\lambda}} + \lambda_t \big[y_t H_t^s + (1 - y_t) H_t^n \big],$$

which explains the continuously changing part of (PK). The additional terms in (PK), $H_t^s dN_t^s$ and $H_t^n dN_t^n$, further capture the jumps in the promised utility resulting from arrivals as mentioned just now. Finally, payment dL_t at time *t* naturally brings down total future payments.

2.2. Incentive Compatibility

In this paper, we mainly focus on contracts that always induce effort from the agent. We first argue that assuming

$$m \le K\Delta\lambda - b,$$
 (3)

the principal should always motivate the agent to exert effort. For any contract that allows shirking, we can always improve it by replacing a shirking period with monitoring. This is because monitoring costs the principal *m* plus the effort cost *b*, and shirking costs the principal does not reimburse effort when the agent shirks under the contract.) In the end of the paper, we consider the situation in which (3) is violated and shirking is allowed.

Condition (3) allows us to transform our problem into an optimal control model over contracts that must always induce effort. Correspondingly, the counting process $\{N_t\}_{t\geq 0}$ admits intensity $\lambda_t = \lambda$ for all $t \geq 0$. If a contract Γ induces the agent to always exert effort, that is,

$$u(\Gamma, \underline{\Lambda}) \ge u(\Gamma, \Lambda)$$
, for $\underline{\Lambda} := \{\lambda_t = \lambda\}_{t>0}$ and $\forall \Lambda$, (4)

then we call contract Γ *incentive compatible*.

Incentive compatibility critically depends on the following ratio between the agent's private benefit *b* and the difference in arrival rates $\Delta\lambda$,

$$\beta := \frac{b}{\Delta \lambda}.$$
 (5)

Intuitively, should the principal be able to charge the agent an amount β for each arrival, the agent would be indifferent between exerting effort or not. Shirking in a small time interval δ brings the agent a benefit $b\delta$, which is offset by the higher penalty cost $\Delta\lambda\beta\delta$. Because charging the agent is not allowed in our setting, the principal instead reduces the agent's promised utility by at least β for each arrival in order to induce effort.⁴ Therefore, the value β is the minimum penalty in the promised utility that induces effort from the agent as mentioned in the introduction. This is formalized in the following result:

Lemma 2. Contract Γ is incentive compatible, that is, it satisfies (4), if and only if the following condition holds:

$$y_t H_t^s + (1 - y_t) H_t^n \ge \beta, \quad \text{if } m_t = 0.$$
 (IC)

The term $y_t H_t^s + (1 - y_t) H_t^n$ is the expected downward jump when there is an arrival at time *t*. Constraint (IC) implies that this expected downward jump is at least β when the principal does not monitor.

Constraint (IC) further implies that either H_t^s or H_t^n has to be at least β . Therefore, we can simultaneously satisfy constraints (IC) and (IR) after the potential

downward jump only if the promised utility W_t is at least β . This implies that, in order to induce effort, the principal has to monitor the agent instead of relying on (IC) whenever the promised utility $W_t < \beta$. We summarize this in the following corollary.

Corollary 1. Under any incentive compatible contract, the agent is monitored ($m_t = m$) when $W_t < \beta$.

When $W_t \ge \beta$, the principal trades off monitoring and enforcing (IC) and should monitor the agent if and only if the shadow cost of the (IC) constraint is higher than the monitoring cost *m*.

2.3. Principal's Utility

Assume that the principal receives a constant revenue flow of *R*. Under an incentive compatible contract, the agent always exerts full effort. The corresponding principal's total discounted utility under an incentive compatible contract Γ is

$$\mathbb{E}^{\Gamma,\underline{\Delta}}\left[\int_{0}^{\infty} e^{-rt} \left((R-m_t)dt - KdN_t - dL_t \right) \right]$$
$$= \frac{R-K\lambda}{r} - \mathbb{E}^{\Gamma,\underline{\Delta}}\left[\int_{0}^{\infty} e^{-rt} \left(m_t dt + dL_t \right)\right].$$

Because the term R only shifts the principal's total utility by a constant and is independent of the contract design, without loss of generality, we set

$$=K\lambda.$$
 (6)

Therefore, the principal's total discounted utility under an incentive compatible contract Γ is

R

$$U(\Gamma) = -\mathbb{E}^{\Gamma,\underline{\Delta}} \left[\int_0^\infty e^{-rt} (m_t dt + dL_t) \right].$$
(7)

Before finishing this section, we present the following verification result, which provides an upper bound of the principal's utility $U(\Gamma)$ over all incentive compatible contracts. This result forms the foundation of proving optimality under various parameter regimes.

Lemma 3. Suppose F(w) is a continuous, concave, and upper-bounded function with $F'(w) \ge -1$. (If F(w) is not differentiable at a point w, we denote F'(w) to be the average between its left and right derivatives.) Consider any incentive compatible contract Γ , which yields the agent's expected utility $u(\Gamma, \underline{\Lambda}) = w = W_0$, followed by the promised utility process $\{W_t\}_{t\ge 0}$ according to (PK). Define a stochastic process $\{\Psi_t\}_{t\ge 0}$, where

$$\Psi_{t} := F'(W_{t})\rho W_{t} - rF(W_{t}) - m_{t} + \lambda y_{t} [F'(W_{t})H_{t}^{s} + F(W_{t} - H_{t}^{s}) - F(W_{t})] + \lambda (1 - y_{t}) [F'(W_{t})H_{t}^{n} + F(W_{t} - H_{t}^{n}) - F(W_{t})].$$
(8)

If the process $\{\Psi_t\}_{t\geq 0}$ is nonpositive almost surely, then we have $F(w) \geq U(\Gamma)$.

3. Equal Discount Rate

In this section, we consider the case in which $r = \rho$. That is, the principal shares the same discount rate with the agent. In this setting, we need to introduce an upper bound for the agent's continuation utility, \bar{w} , as an additional model parameter. This is due to the infinite back-loading problem identified in Myerson (2015) for the equal discount moral hazard problem without monitoring. Without such an upper bound, the principal would keep increasing the agent's promised utility to infinity without payment. We provide a comprehensive discussion on the intuitive reason for such an upper bound toward the end of this section. To avoid triviality, the upper bound \bar{w} is set to be above β .⁵ It is worth noting that we only need such an upper bound when $r = \rho$. When $r < \rho$, the principal is more patient than the agent, it is no longer necessary to introduce \bar{w} , as we discuss in the next section.

When $r = \rho$, the optimal contract structure is not unique. Interestingly, when $r < \rho$ to be discussed in the next section, either contract structure may be optimal, depending on how high the monitoring cost is.

3.1. Optimal Contract Structure: Deterministic and Randomized

For the equal discount case, the optimal contract may take two different forms with the same performance. We first describe the evolution of the promised utility under the deterministic optimal contract (that is, probability y_t in (PK) is always zero). As discussed in the previous section, the (IC) condition implies that the principal has to monitor the agent *if* the promised utility W_t is lower than β . In this section, we establish that, when $r = \rho$, it is optimal to monitor *if and only if* $W_t < \beta$. Moreover, the principal does not penalize the agent for any arrivals (i.e., $H_t^n = H_t^s = 0$) and does not pay the agent while monitoring (i.e., $dL_t = 0$). Therefore, when $W_t < \beta$, (PK) implies that the promised utility evolves according to

$$dW_t = \rho W_t dt. \tag{9}$$

Intuitively, the term $\rho W_t dt$ is the "interest" accrued from the promised utility because of time discounting. Furthermore, whenever $W_t < \beta$ at time *t*, condition (9) implies that the promised utility increases deterministically following the simple exponential curve

$$W_{t+\tau} = W_t e^{\rho \tau}, \tag{10}$$

until $W_{t+\tau}$ reaches the threshold β .

When the promised utility $W_t \in [\beta, \bar{w})$, the principal no longer monitors the agent, and the promised utility takes a downward jump of β upon each arrival. That is, following the optimal contract, $H_t^s = H_t^n = \beta$ in (PK). In this case, the principal still does not pay the agent (i.e., $dL_t = 0$), and (PK) reduces to

$$dW_t = (\rho W_t + \beta \lambda)dt - \beta dN_t.$$
(11)

Compared with (9), the rate of increase in (11) is higher. Besides the interest term $\rho W_t dt$, the term $\beta \lambda dt$ is the "information rent" that the agent receives in the form of faster increase in the promised utility when there is no arrival. To see this intuitively, remember that, in order to motivate effort, the principal charges the agent utility β for each arrival, which occurs with probability λdt . Because the agent cannot actually pay the principal money upon arrivals, when there is no arrival during a period dt, the principal increases the agent's promised utility by $\beta \lambda dt$. This exactly equals the expected decrease of the agent's utility for an arrival during this time period, reflected in the downward jump term βdN_t .

Therefore, if there is no arrival during the time interval $[t, t + \tau]$, then, again, the promised utility increases following the curve

$$W_{t+\tau} = W_t e^{\rho\tau} + \frac{\beta\lambda}{\rho} (e^{\rho\tau} - 1), \qquad (12)$$

as long as $W_{t+\tau} < \bar{w}$. Compared with (10), the additional term involving $\beta \lambda$ is the information rent as discussed earlier.

Without an arrival, the agent's promised utility keeps increasing according to a rate $\rho W_t + \beta \lambda$ until W_t reaches the upper bound \bar{w} . At this point, the promised utility cannot increase any more and stays at \bar{w} until the next arrival. That is, when $W_t = \bar{w}$, the promised utility evolves according to

$$dW_t = -\beta dN_t. \tag{13}$$

In order to keep W_t at \bar{w} , the principal has to pay the agent a flow rate

$$\ell_t = \rho \bar{w} + \beta \lambda, \tag{14}$$

to release the upward pressure that would otherwise occur to the promised utility.

Now we are ready to present a formal definition for a class of contracts that includes the optimal one. Note that, in the following definition, we allow the monitoring threshold to be a more general level *s* although it is just β as described before. This is because the optimal threshold can be higher than β in the next section when $r < \rho$.

Definition 1. Contract $\Gamma_d(w; s, \bar{w})$ is defined as

i. The dynamics of the agent's promised utility, W_t , follows (9) for $W_t \in [0, s)$, (11) for $W_t \in [s, \bar{w})$, and (13) for $W_t = \bar{w}$, starting from $W_0 = w$.

ii. In terms of payments, the agent is not paid when $W_t < \bar{w}$ and is paid at a flow rate (14) when $W_t = \bar{w}$. iii. Regarding monitoring, $m_t = m$ if and only if

In Regarding monormal, $m_t = m$ if and only if $W_t < s$.

In Definition 1, the subscript "*d*" in $\Gamma_d(w; s, \bar{w})$ stands for *deterministic* because monitoring starts in a deterministic fashion with respect to the arrival times. That is, the promised utility W_t is adapted to the filtration generated from the arrival process. The parameter *s* represents the monitoring threshold. Clearly, in this section, we are only interested in contract $\Gamma_d(w; \beta, \bar{w})$. In the following section when we discuss $\rho > r$, however, the threshold *s* could be strictly higher than β , and the upper bound \bar{w} may be replaced with an endogenous one.

Figure 1(a) provides a sample trajectory under contract $\Gamma_d(w^*;\beta,\bar{w})$. On this sample trajectory, there are a total of five arrivals, labeled as time epochs t_1 , t_2, \ldots, t_5 . In the beginning, the promised utility keeps increasing with slope $\rho w_t + \beta \lambda$ until reaching the upper bound \bar{w} at time \hat{t} . Then three arrivals cause three downward jumps of magnitude β that eventually bring the promised utility below the threshold β . Monitoring starts from time t_3 and lasts until time t', during which the arrival at t_4 has no impact. The kink at t' reflects the difference between the slopes ρw_t on the left and $\rho w_t + \lambda \beta$ on the right.

This description shows that it is fairly easy for the principal to implement this contract over time, keeping track of a single number in a simple way. Furthermore, the contract guarantees incentive compatibility on an intuitive level. A monitoring session starts when arrivals occur rather frequently over a period of time. Under monitoring, the promised utility grows rather slowly, which means that payment can only happen far into the future. Therefore, monitoring not only ensures effort at the moment, but also serves as a threat to the agent before its use.

Furthermore, the contract motivates the agent to exert effort besides using the threat of monitoring. When the promised utility is at a level w above β and below \bar{w} , payment starts if there is no arrivals for the next $\frac{1}{\rho} \ln \frac{\rho \bar{w} + \beta \lambda}{\rho w + \beta \lambda}$ period of time (following (12) with $W_t = w$ and $W_{t+\tau} = \bar{w}$). Exerting effort increases the chance of the promised utility reaching \bar{w} and payment. Once payment has started, an arrival brings the promised utility down from \bar{w} to $\bar{w} - \beta$ and pauses the flow payment for at least a period of time of $\frac{1}{\rho} \ln \frac{\rho \bar{w} + \beta \lambda}{\rho \bar{w} + \beta (\lambda - \rho)}$ (again, following (12)). Therefore, once being paid, the agent is willing to exert effort in order to prolong the payment period before it pauses.

Figure 1(b) depicts an alternative contract with the same performance that involves randomization upon arrivals, facing the same sample trajectory as in Figure 1(a). In particular, whenever the promised utility drops to below β , it either jumps to zero or β . For example, at time t_3 , the randomization brings the promised utility to β , and at t_4 , the promised utility lands at zero, which triggers monitoring ever after.

More formally, if an arrival occurs when W_{t-} is in $[\beta, 2\beta)$, the next moment's promised utility lands on zero with probability $2 - W_{t-}/\beta$ and on β with probability $W_{t-}/\beta - 1$. Technically, this means that, in (PK), the jumps are $H_t^s = W_{t-}$ and $H_t^n = W_{t-} - \beta$, and the randomization probability is $y_t = 2 - W_{t-}/\beta$. Therefore, we define the following class of contracts $\Gamma_r(w; \bar{w})$, in which the subscript "r" stands for *randomized*.





Definition 2. Define contract $\Gamma_r(w; \bar{w})$ the same as contract $\Gamma_d(w; \beta, \bar{w})$ in Definition 1 except that the dynamics of the agent's promised utility W_t follows

$$dW_{t} = \begin{cases} 0, & \text{if } W_{t} = 0, \\ (\rho W_{t} + \beta \lambda) dt - W_{t} dN_{t}^{s} - (W_{t} - \beta) dN_{t}^{n}, \\ & \text{if } W_{t} \in [\beta, \min\{\bar{w}, 2\beta\}), \\ (\rho W_{t} + \beta \lambda) dt - \beta dN_{t}, \\ & \text{if } W_{t} \in [\min\{\bar{w}, 2\beta\}, \bar{w}), \\ -\beta dN_{t}, & \text{if } W_{t} = \bar{w}, \end{cases}$$
(15)

starting from $W_0 = w$.

The reason that both deterministic and randomized contracts are optimal is revealed in the next subsection, in which we show that the principal's value function is linear in the interval $[0, \beta]$. For the different discount rate case of Section 4, however, whether the value function is linear or not depends on how high the monitoring cost is. In particular, the deterministic contract is optimal when the monitoring cost is high, and the randomized contract is optimal when the monitoring cost is low.

3.2. Principal's Value Function

Following the evolution of the promised utility described, next we heuristically derive the dynamics of the principal's utility as a function of the agent's promised utility using discrete time approximation. Specifically, denote F(w) to represent the principal's total discounted utility when the agent's promised utility is w.

First, for any $w \in [0, \beta)$, over a small time interval with length δ , the principal incurs a monitoring cost $m\delta$, and following the deterministic contract dynamics (9), the agent's promised utility increases to $we^{\rho\delta}$. Therefore, we have the following expression for the principal's utility function,

$$F(w) = -m\delta + e^{-r\delta}F(we^{\rho\delta}) + o(\delta).$$
(16)

Assuming F(w) is differentiable on $[0, \beta)$, following standard procedures of subtracting F(w) from and dividing δ on both sides, and by letting δ approach zero, we obtain

$$rF(w) = \rho w F'(w) - m. \tag{17}$$

We keep both *r* and ρ in (17) and all the equations in this section because the value function takes the same expressions when *r* < ρ in the next section.

Differential Equation (17) has the following standard solution:

$$F_{\theta}(w) = \theta w^{\frac{r}{\rho}} - \frac{m}{r}, \qquad (L)$$

parameterized with a scalar θ . The tag (L) indicates that the promised utility is *lower* than β . Later in this section, we specify the choice of θ to complete the description of the value function $F_{\theta}(w)$. When $r = \rho$, (L) reduces to

$$F_{\theta}(w) = \theta w - \frac{m}{r}, \qquad (\mathbf{L}_l)$$

which is a linear function of w and, hence, the subscript l in the tag.

Next, for any $w \in [\beta, \bar{w})$, the principal no longer monitors the agent. Following similar heuristic derivations for (16), we reach a delay differential equation (DDE),

$$(\lambda + r)F(w) = \lambda F(w - \beta) + (\rho w + \lambda \beta)F'(w), \qquad (H)$$

where the tag (H) indicates that the promised utility is *higher* than β . We denote function $F_{\theta}(w)$ to be the solution to the DDE (H) on $w \in [\beta, \bar{w})$ with boundary condition (L_l) on $w \in [0, \beta)$.

Finally, for $w = \bar{w}$, the principal pays the agent a flow payment according to (14) and keeps the promised utility at \bar{w} until the next arrival. Standard arguments imply that the principal's value function takes the form of

$$(\lambda + r)F(\bar{w}) = \lambda F(\bar{w} - \beta) - (\rho\bar{w} + \beta\lambda), \qquad (U)$$

where the tag (U) stands for *upper* bound. The combination of (H) and (U) implies that $F'(\bar{w}) = -1$. Intuitively, when the slope of the principal's value function is -1, increasing the promised utility further by an amount costs the principal the same amount. This is consistent with the fact that at this point delaying payment while letting the promised utility increase does not yield further benefit to the principal any more.

In order to specify the optimal value function, we need to determine θ in (L_l). To that end, we introduce function J(w) to be the solution of DDE (H) for $w \ge \beta$ with boundary condition J(w) = 1 instead of (L_l) on $w \in [0, \beta)$. Therefore, function J(w) is independent of θ and the monitoring cost *m*. It is easy to verify that, when $\rho = r$, function $F_{\theta}(w)$, which is the solution to (H) with boundary condition (L_l), can be expressed in terms of J(w) as

$$F_{\theta}(w) = \theta w - \frac{m}{r} J(w), \text{ for } w \le \bar{w}.$$
 (18)

We further extend the function to $w \ge \bar{w}$ with slope -1, that is,

$$F_{\theta}(w) = F_{\theta}(\bar{w}) + \bar{w} - w, \text{ for } w > \bar{w}.$$
(19)

Furthermore, if we define

$$\theta(\bar{w}) = \frac{m}{r} J'(\bar{w}) - 1, \qquad (20)$$

it is easy to verify that $F'_{\theta(\bar{w})}(\bar{w}) = -1$, so function $F'_{\theta(\bar{w})}(\bar{w})$ is differentiable at \bar{w} with slope -1.

Proposition 1. We have the following properties regarding the value function $F_{\theta(\bar{w})}(w)$ defined according to (18)–(20): i. The value $\theta(\bar{w})$ is bounded. Specifically,

$$-1 \le \theta(\bar{w}) < \frac{m}{\beta r}.$$
 (21)

ii. Function $F_{\theta(\bar{w})}(w)$ is linear for $w \in [0, \beta)$ and strictly concave on $[0, \bar{w}]$ with $F'_{\theta(\bar{w})}(\bar{w}) = -1$.

iii. Moreover, for any \bar{w} and \tilde{w} such that $\beta \leq \tilde{w} < \bar{w}$, we have $F_{\theta(\tilde{w})}(w) < F_{\theta(\tilde{w})}(w)$ for any $w \ge 0$.

Figure 2 depicts the value function with different model parameters. As we can see, all the functions plotted in the two subfigures are concave as described in Proposition 1(ii). Therefore, it is easy to find its maximizer

$$w^* = \operatorname*{arg\,max}_{w \ge 0} F_{\theta(\bar{w})}(w)$$

as depicted in Figure 2(a) for the case of m = 4. In order to maximize the total future expected utility, the principal should use contract $\Gamma_d(w^*;\beta,\bar{w})$, starting the contract from promised utility w^* .

Figure 2(a) also demonstrates that the value function decreases with the monitoring cost *m*, which is intuitive and consistent with (18). According to (18) and (20), the slope $\theta(\bar{w})$ increases in *m* and is positive if and only if $m > r/J'(\bar{w})$. In Figure 2(a), $\theta(\bar{w})$ is positive when m = 4, zero when m = 1.9, and negative when m = 1. If $m < r/J'(\bar{w})$, the slope $\theta(\bar{w}) < 0$ and the maximizer of $F_{\theta(\bar{w})}(w)$ is $w^* = 0$. In this case, the monitoring cost is low enough such that it is optimal for the principal to always monitor while keeping the agent's promised utility at zero. This is depicted in the curve with m = 1 in Figure 2(a).

Figure 2(b) further depicts the value function under different exogenous upper bounds on the promised

utility. Consistent with Proposition 1(iii), the value function increases with the upper bound. This is also intuitive. From an optimization point of view, the upper bound puts a constraint on the optimal control problem. Relaxing it improves the objective function. From an economic point of view, a higher upper bound allows the principal to delay payments further into the future and, therefore, improves the principal's utility. This explains the infinite back-loading problem: if allowed, the principal would choose the upper bound to approach infinity.

It is worth discussing the underlying reason why infinite back-loading arises in the equal discount setting. First of all, let us consider the first best/ efficient outcome, one that maximizes the societal utility without private information. In the equal discount setting, the efficient outcome corresponds to the low arrival rate λ with no monitoring because payments have no impact on the total utility of the two players. It is important to realize that, in our setting with arrivals of adverse events, any incentive compatible contract (including the optimal one) cannot induce the first best outcome. This is because the (IC) constraint implies that no matter how high the promised utility is, there is always a positive probability with which the promised utility drops below the threshold β following a number of frequent arrivals, triggering monitoring, and losing efficiency. On the other hand, the higher the promised utility, the longer it takes to start monitoring (i.e., lose efficiency). This explains why the higher the upper bound \bar{w} , the better the objective function. If \bar{w} is infinity, however, the principal essentially does not pay the agent in any finite time, which is no longer a meaningful contract. In fact, a contract becomes meaningless if \bar{w} is too high (say higher than the total wealth of the world) because the principal does not have the credibility of delivering such a promised utility to the agent.





Different Monitoring Costs



Different Upper Bounds ($\tilde{w} < \bar{w}$)

With different discount rates as are discussed in the next section, this infinite back-loading problem does not appear. This is because the cost of early payments to the principal is lower than its benefit to the agent. As a result, it is no longer beneficial for the principal to always delay payment. The corresponding upper bound of promised utility at which payment starts becomes finite.

3.3. Proof of Optimality

So far we have not formally established the connection between the value function $F_{\theta(\bar{w})}(w)$ and either contract structure. Now we establish that $F_{\theta(\bar{w})}(w)$ is indeed the optimal value function.

First, the following proposition states that $F_{\theta(\bar{w})}(w)$ is indeed the value function of both contracts $\Gamma_d(w;\beta,\bar{w})$ and $\Gamma_r(w;\bar{w})$.

Proposition 2. For $F_{\theta(\bar{w})}(w)$ defined according to (18)–(20), we have

$$F_{\theta(\bar{w})}(w) = U(\Gamma_d(w;\beta,\bar{w})) = U(\Gamma_r(w;\bar{w})).$$

Therefore, starting from w^* , contracts $\Gamma_d(w^*;\beta,\bar{w})$ and $\Gamma_r(w^*;\bar{w})$ both yield the maximum $F_{\theta(\bar{w})}(w^*)$ for the principal.

According to both (9) and (15), the promise utility stays at zero forever whenever it falls to zero, according to both contracts $\Gamma_d(w;\beta,\bar{w})$ and $\Gamma_r(w;\bar{w})$. That is, once $W_t = 0$, the principal needs to monitor the agent (and endure the monitoring cost) forever. Following the optimal contract $\Gamma_d(w^*;\beta,\bar{w})$, however, the promised utility hitting zero is a zero measure event. And, starting from a promised utility $w \in (0,\beta)$, dynamics (10) implies that the length of a monitoring episode under the optimal contract $\Gamma_d(w^*;\beta,\bar{w})$ is

$$T_m(w) := \frac{1}{\rho} \left(\ln \beta - \ln w \right), \tag{22}$$

which is finite. Superficially, the randomized contract $\Gamma_r(w^*; \bar{w})$ may appear worse because of the possibility of monitoring forever. In fact, when the promised utility before an arrival is in the interval $(\beta, 2\beta)$, under the deterministic contract $\Gamma_d(w^*; \beta, \bar{w})$, the principal has to pay the monitoring cost for a period of time after each arrival. Under the randomized contract $\Gamma_r(w^*; \bar{w})$, however, there is a chance that monitoring does not happen at all, which balances the chance of monitoring ever after. This explains, intuitively, why these two contracts are equivalent to the risk-neutral principal.

The following theorem, together with Proposition 2, establishes the optimality of value function $F_{\theta(\bar{w})}(w)$ and both contracts $\Gamma_d(w^*;\beta,\bar{w})$ and $\Gamma_r(w^*;\bar{w})$.

Theorem 1. For any incentive compatible contract Γ , which yields an agent's utility $w \leq \overline{w}$, we have

$$U(\Gamma) \le F_{\theta(\bar{w})}(w) \le F_{\theta(\bar{w})}(w^*).$$
(23)

Therefore, contracts $\Gamma_d(w^*;\beta,\bar{w})$ and $\Gamma_r(w^*;\bar{w})$ are both optimal, which yield expected utility $F_{\theta(\bar{w})}(w^*)$ to the principal.

The first inequality in (23) follows from Lemma 3. The second inequality simply follows from w^* being the maximizer of $F_{\theta(\bar{w})}(w)$. Therefore, Theorem 1 and Proposition 2 imply that the principal's expected utility generated from contracts $\Gamma_d(w^*; \beta, \bar{w})$ and $\Gamma_r(w^*; \bar{w})$ is higher than those generated from any other incentive compatible contracts. Hence, these two contracts are both optimal. Finally, it is worth pointing out that various combinations of the contracts $\Gamma_d(w^*; \beta, \bar{w})$ and $\Gamma_r(w^*; \bar{w})$ are also optimal. That is, whenever in the interval $(0, \beta)$, the promised utility wcan either continuously increase following (9) or randomly jump between zero and β , following (15), no matter how it behaved in this interval previously.

4. Different Discount Rates

In this section, we consider the case of $\rho > r$. That is, the principal is more patient than the agent. One important distinction compared with the case of $\rho = r$ is that there exists a finite upper bound \bar{w}^* on the promised utility, which is implied endogenously under the optimal contract. Therefore, we no longer need to introduce the exogenous upper bound as in the previous section. Nevertheless, when $\rho > r$, we can still include an exogenous upper bound \bar{w} for the promised utility in the model, which is discussed in Section 4.3.

For the main parts of this section (Sections 4.1 and 4.2), we show that the structure of the optimal contract changes with model parameters, especially the monitoring cost, as illustrated in Figure 3. In this figure, we vary the principal's discount rate r (as the *x*-axis) and the monitoring cost *m* (as the *y*-axis) while keeping other model parameters fixed. In the following two sections, we show that, if the monitoring cost *m* is above a threshold \overline{m} (the solid curve), the optimal contract takes a structure similar to the deterministic contract defined in the previous section. If *m* is below \bar{m} , on the other hand, it is optimal for the principal to always monitor the agent. Figure 3 depicts two additional dotted curves \hat{m} and \underline{m} in this region. We defer the detailed discussion on them to Section 4.2.

Here, we first define the threshold \bar{m} as

$$\bar{m} := \inf_{w > \beta} \frac{r}{J'(w)}.$$
(24)

Figure 3. (Color online) Split of the Low and High Monitoring Cost



For the case of $\rho = r$, Equation (20) implies that $\theta(\bar{w}) < 0$ for any $\bar{w} > \beta$ when $m < \bar{m}$, which implies that the value function is decreasing, and therefore, it is optimal to always monitor the agent. Later in Section 4.2, we show that, for the case $\rho > r$, it is still optimal to always monitor the agent. Next, we first study the case of $m \ge \bar{m}$.

4.1. High Monitoring Cost

In this section, we investigate the case in which the monitoring cost is above the threshold \bar{m} .

Recall Corollary 1; the principal needs to monitor when the promised utility w is lower than β . When $m \ge \overline{m}$, the principal may still need to monitor the agent even if w is higher than β . That is, the optimal contract is similar to the deterministic contract Γ_d in the previous section, in which monitoring occurs whenever the promised utility is below a threshold $\alpha \ge \beta$.

Following the same heuristic derivation in Section 3.2, define function $F_{\theta,\alpha}(w)$ to be the solution of DDE (H) for $w \in [\alpha, \infty)$ with boundary condition (L) for $w \in [0, \alpha)$. Next, we identify the optimal value function in a two-step procedure. In the first step, we specify α for a given parameter θ . In the second step, we establish θ and the endogenous upper bound \overline{w}^* .

First, we identify the threshold α for a given parameter θ . A key property of the value function $F_{\theta,\alpha}(w)$ is smooth pasting (Dixit and Pindyck 1994) at $w = \alpha$. That is, the left and right derivatives, $F'_{\theta,\alpha}(\alpha_{-})$ and $F'_{\theta,\alpha}(\alpha_{+})$, respectively, are set to be equal to each other if possible.⁶ To this end, it is convenient to define the following function,

$$f(\alpha) := \left(F'_{\theta,\alpha}(\alpha_{-}) - F'_{\theta,\alpha}(\alpha_{+})\right)\left(\rho\alpha + \beta\lambda\right)$$
$$= m - \lambda\theta\alpha^{\frac{r}{\rho}}\left[\left(1 - \frac{r\beta}{\rho\alpha}\right) - \left(1 - \frac{\beta}{\alpha}\right)^{\frac{r}{\rho}}\right], \quad (25)$$

in which $F'_{\theta,\alpha}(\alpha_{-})$ and $F'_{\theta,\alpha}(\alpha_{+})$ are obtained from (L) and (H) with switching point α , respectively. Therefore, we may set $f(\alpha) = 0$ to achieve $F'_{\theta,\alpha}(\alpha_{-}) = F'_{\theta,\alpha}(\alpha_{+})$.

Lemma 4. Function $f(\alpha)$ is increasing in α on $[\beta, \infty)$, and $\lim_{\alpha \to \infty} f(\alpha) = m$.

In order to find the threshold α by solving the equation $f(\alpha) = 0$, denote f^{-1} to represent the inverse function of the monotone function f, and for any θ , define

$$\alpha_{\theta} := \begin{cases} \beta, & \text{if } f(\beta) \ge 0, \\ f^{-1}(0), & \text{if } f(\beta) < 0. \end{cases}$$
(26)

Proposition 3. (i) We have

$$f(\alpha_{\theta}) \ge 0 \text{ and } (\alpha_{\theta} - \beta) f(\alpha_{\theta}) = 0.$$
 (27)

Therefore, if $\alpha_{\theta} = \beta$, we have $F'_{\theta,\alpha_{\theta}}(\alpha_{\theta-}) \ge F'_{\theta,\alpha_{\theta}}(\alpha_{\theta+})$, and if $\alpha_{\theta} > \beta$, we have $F'_{\theta,\alpha_{\theta}}(\alpha_{\theta-}) = F'_{\theta,\alpha_{\theta}}(\alpha_{\theta+})$. (ii) Furthermore, if $\alpha_{\theta} > \beta$, we have $F'_{\theta,\alpha_{\theta}}(\alpha_{\theta+}) < F'_{\theta,\alpha_{\theta}}(\alpha_{\theta-}) < 0$. (iii) Finally, for any $\alpha \in [\beta, \alpha_{\theta}), F'_{\theta,\alpha}(\alpha_{-}) < F'_{\theta,\alpha}(\alpha_{+})$; for $\alpha \in (\alpha_{\theta}, \infty), F'_{\theta,\alpha}(\alpha_{-}) > F'_{\theta,\alpha}(\alpha_{+})$.

Proposition 3(i) and (ii) imply that function $F_{\theta,\alpha_{\theta}}(w)$ is locally concave at α . This is important for us to later show (global) concavity and optimality of this function. Proposition 3(iii) further helps us to establish Proposition 4(ii) to be presented later.

After characterizing α , we now describe the optimal θ and the endogenous upper bound \bar{w}^* .

Lemma 5. (i) Function $F_{\theta,\alpha_{\theta}}(w)$ is supermodular in $(\theta, w) \in \mathbb{R} \times \mathbb{R}^+$. Therefore, derivative $F'_{\theta,\alpha_{\theta}}(w)$ increases in θ for any w. (ii) For any given parameter $m \ge \overline{m}$, there exist positive quantities $\overline{\theta}$ and \overline{w}^* such that

$$\inf_{w>\alpha_{\bar{\theta}}} F'_{\bar{\theta},\alpha_{\bar{\theta}}}(w) = -1 \text{ and } \bar{w}^* := \inf\left\{ \arg\inf_{w>\alpha_{\bar{\theta}}} F'_{\bar{\theta},\alpha_{\bar{\theta}}}(w) \right\}, \quad (28)$$

respectively.

Furthermore, we have

$$0 \le \bar{\theta} < \frac{m}{r} \beta^{-\frac{r}{\rho}} \quad and \quad \bar{w}^* \in [\alpha_{\bar{\theta}}, \infty).$$
(29)

Lemma 5(i) implies that the value θ as defined in (28) is unique and (ii) further indicates that it is upper and lower bounded. Therefore, the value of $\bar{\theta}$ can be identified using binary search. The lower bound zero implies that the value function is nondecreasing on $[0, \alpha_{\bar{\theta}}]$. Therefore, the maximizer of the value function is nonnegative. This further implies that, if the monitoring cost is higher than \bar{m} , it would be too costly for the principal to always monitor the agent. The upper bound can be used in a binary search

algorithm to find $\bar{\theta}$. Finally, (29) also indicates that the endogenous upper bound \bar{w}^* is indeed finite.

Now we are ready to define the following value function F(w) based on $F_{\bar{\theta},\alpha_{\bar{h}}}(w)$,

$$F(w) := \begin{cases} F_{\bar{\theta}, \alpha_{\bar{\theta}}}(w), & \text{if } w \leq \bar{w}^*, \\ F_{\bar{\theta}, \alpha_{\bar{\theta}}}(\bar{w}^*) - (w - \bar{w}^*), & \text{otherwise.} \end{cases}$$
(30)

Here are some key properties of the value function that are essential for proving its optimality.

Proposition 4. For $m \ge \overline{m}$, we have

i. Function F(w) is strictly concave on $w \in [0, \bar{w}^*]$ with F'(w) = -1 for $w \ge \bar{w}^*$.

ii. For any $w \in (\alpha_{\bar{\theta}}, \bar{w}^*]$, we have $rF(w) > \rho w F'(w) - m$.

Proposition 4(i) is a standard property that often arises in the dynamic contracting literature; it is the foundation for proving that F(w) is the optimal value function. Proposition 4(ii), however, appears unique to our setting and requires a novel proof based on Proposition 3. Comparing this differential inequality with the differential Equation (17), it is clear that, for any promised utility w above the threshold $\alpha_{\bar{\theta}}$, the principal is better off not to monitor. This condition is critical in proving optimality of the threshold structure in our contract.

Based on the calculation of threshold $\alpha_{\bar{\theta}}$ and upper bound \bar{w}^* in Lemma 5, we can establish that the optimal contract is $\Gamma_d(w^*; \alpha_{\bar{\theta}}, \bar{w}^*)$ following Definition 1, in which w^* is a maximizer of function F(w) defined in (30).

Figure 4(a) provides a sample sketch of the principal's value function. Under this particular parameter setting, we have $\alpha_{\bar{\theta}} > \beta$, and therefore, according to Proposition 3, the value function demonstrates the smooth pasting property at $\alpha_{\bar{\theta}}$.

Figure 4(b) presents a sample trajectory of the promised utility according to the optimal contract, using the same parameter values as in Figure 4(a). As

we can see, in this particular example, we have $\alpha_{\bar{\theta}} > \beta$ and $w^* > \alpha_{\bar{\theta}}$. Therefore, the principal starts the contract with the initial promised utility w^{*} without monitoring the agent. As long as W_t is above $\alpha_{\bar{\theta}}$, the promised utility W_t takes a downward jump of β for each arrival (at time t_1 , t_2 , t_3 , and t_6 in the figure). In this sample trajectory, the promised utility drops below $\alpha_{\bar{\theta}}$ at time t_3 . The principal starts monitoring the agent at this point while the promised utility W_t cumulates interest and increases along the exponential curve (10) until it reaches $\alpha_{\bar{\theta}}$ regardless of arrivals (t_4 and t_5) in the interval [t_3 , t'). When the promised utility climbs back to $\alpha_{\bar{\theta}}$ (at time t'), it keeps increasing along the other exponential curve (12) as long as there is no arrival. A flow payment starts when W_t reaches \bar{w} at time \bar{t} and stops when another arrival (at t_6 in this figure) drops W_t to below \overline{w}^* again.

Similar to Proposition 2 and Theorem 1 for the equal discount case, the following result establishes the optimality for the case of $\rho > r$.

Theorem 2. For $m \ge \overline{m}$ and any incentive compatible contract Γ that yields an agent's utility w, we have $U(\Gamma) \le F(w) \le F(w^*) = U(\Gamma_d(w^*; \alpha_{\overline{0}}, \overline{w}^*))$. Therefore, contract $\Gamma_d(w^*; \alpha_{\overline{0}}, \overline{w}^*)$ is the optimal contract, which yields utilities $F(w^*)$ for the principal and w^* for the agent.

It is worth noting that, under the optimal contract $\Gamma_d(w^*; \alpha_{\bar{\theta}}, \bar{w}^*)$, it is possible that the agent's promised utility never reaches zero. That is, constraint (IR) is never binding. In fact, as long as $\alpha_{\bar{\theta}} > \beta$, a downward jump induced by an arrival at most brings the promised utility W_t down to $\alpha_{\bar{\theta}} - \beta$ and never lower. (Figure 4 depicts such a case.) This phenomenon contrasts sharply with long-held insights in the optimal mechanism/contract design literature, in which the individual rationality constraint is generally binding. Therefore, a curious reader may wonder if the agent is overpaid under our definition of contract $\Gamma_d(w^*; \alpha_{\bar{\theta}}, \bar{w}^*)$ when the monitoring threshold $\alpha_{\bar{\theta}} > \beta$.



Figure 4. (Color online) High Monitoring Cost (i.e., $m \ge \bar{m}$)

In fact, similar to (22), for any monitoring threshold $\alpha > \beta$, starting from the lowest possible promised utility level $\alpha - \beta$, it takes time $\frac{1}{\rho} \ln \frac{\alpha}{\alpha - \beta}$ for monitoring to stop. This time increases as α decreases and approaches infinity as α decreases to β . Therefore, the higher the value of α , the shorter the monitoring time period during which the principal has to endure the monitoring cost at rate *m*. This explains why, for high monitoring cost *m* (as in the current case), the principal is willing to set a threshold α higher than β in order to avoid long episodes of monitoring the agent. Even though the agent's promised utility is maintained at strictly positive levels, this strategy yields lower monitoring costs than keeping the threshold at β . This phenomenon highlights the trade-off that the principal faces between payments to the agent and monitoring costs.

4.2. Low Monitoring Cost

We now consider the case when the monitoring $\cot m$ is lower than \overline{m} . First, it is helpful to consider the case of $m = \overline{m}$. Following the previous section, it is easy to verify that $\overline{\theta} = 0$ and $\alpha_{\overline{\theta}} = \beta$. In this case, the function F(w) is linear (in fact, constant) for $w \in [0, \beta)$. If we decrease m further and still follow contract Γ_d , then Lemma 5 yields a negative $\overline{\theta}$. Loosely speaking, the corresponding value function is decreasing, and therefore, the optimal contract is to always monitor the agent and keep the promised utility at zero. Indeed, this is the structure of the optimal contract.

More rigorously, contract Γ_d with a starting promised utility zero and monitoring threshold zero is, in fact, not optimal. A value function following (L) with a negative $\bar{\theta}$ is convex instead of concave. A nonconcave value function cannot be optimal because it can be improved through concavification with randomization.

In fact, the exact form of the optimal value function varies with model parameters when $m < \overline{m}$. In Figure 3, two dotted curves, \hat{m} and \underline{m} , further divide the $m < \overline{m}$ area into three regions, each corresponding to a distinct value function form. Here we provide the expressions for \hat{m} and \underline{m} as

$$\underline{m} := (\rho - r)\beta \text{ and}$$

$$\hat{m} := \begin{cases} \beta(\rho - r)(2\lambda + r)/\lambda, & \text{if } r > \rho - \lambda, \\ (\rho + \lambda)\beta, & \text{if } r \le \rho - \lambda. \end{cases}$$
(31)

The simplest value function is a linear function,

$$F(w) = -\frac{m}{r} - w, \tag{32}$$

which is optimal when $m < \underline{m}$. In this case, the monitoring cost is so low that the principal should simply pay off any positive promised utility immediately and start monitoring the agent forever.

If $m \in [\underline{m}, \hat{m})$, the optimal value function is a slightly more complex piecewise linear function of the following form,

$$F(w) = \begin{cases} -\frac{m}{r} - \left[1 - \frac{(\rho - r)\beta - m}{(\lambda + r)\beta}\right]w, & \text{if } w \le \beta, \\ F(\beta) - (w - \beta), & \text{if } w > \beta. \end{cases}$$
(33)

The randomized contract $\Gamma_r(w;\beta)$ following Definition 2 achieves the value function. That is, similar to Proposition 2, we can show that $F(w) = U(\Gamma_r(w;\beta))$. In other words, if, for whatever reason, the initial promised utility is $w > \beta$, then the principal pays the agent $w - \beta$ to bring the promised utility down to β and then keeps it there while paying a flow of interest and information rent, $(\rho + \lambda)\beta$, to the agent until the first arrival. Upon the arrival, the principal start monitoring the agent forever and stops the payment. Because the value function is decreasing, its maximizer is zero. Therefore, the optimal contract $\Gamma_r(0;\beta)$ effectively starts monitoring from the very beginning.

If $m \in [\hat{m}, \bar{m})$, however, the optimal value function is more complex. It is the solution to DDE (H) for $w \in [\beta, \bar{w}^*]$ with boundary condition (L_l) for $w \in [0, \beta)$, where θ in (L_l) and \bar{w}^* are defined as the following,

$$\inf_{w>\beta} F'_{\bar{\theta}}(w) = -1 \quad \text{and} \quad \bar{w}^* = \inf \left\{ \arg \inf_{w>\beta} F'_{\bar{\theta}}(w) \right\}.$$
(34)

Therefore, function F(w), defined as

$$F(w) = \begin{cases} F_{\bar{\theta}}(w), & \text{if } w \le \bar{w}^*, \\ F_{\bar{\theta}}(\bar{w}^*) + \bar{w}^* - w, & \text{if } w > \bar{w}^*, \end{cases}$$
(35)

is linear on $[0, \beta)$ and nonlinear on $[\beta, \bar{w}^*)$ and takes a slope of -1 on $[\bar{w}^*, \infty)$. Furthermore, the next result characterizes \bar{w}^* and $\bar{\theta}$.

Proposition 5. For $m \in [\hat{m}, \bar{m})$ and $\bar{\theta}$ and \bar{w}^* defined in (34), we have

$$\bar{w}^* \ge 2\beta \quad and \quad -1 + \frac{\rho - r}{\lambda} < \bar{\theta} \le 0.$$
 (36)

The randomized contract $\Gamma_r(w; \bar{w}^*)$ achieves the value function. That is, the proof of Proposition 2 already establishes that $F(w) = U(\Gamma_r(w; \bar{w}^*))$. Again, the value function is decreasing. Therefore, following contract $\Gamma_r(0; \bar{w}^*)$, the principal monitors the agent from the beginning forever.

Figure 5 depicts the optimal value functions for different monitoring costs. As we can see from the figure, for the monitoring cost $m_3 \in [0, \underline{m})$, the value function is a straight line with slope -1. If we further increase the monitoring cost to $m_2 \in [\underline{m}, \hat{m})$, the value function becomes a piecewise linear function. If the monitoring cost further increases to $m_1 \in [\hat{m}, \overline{m})$, the value function is nonlinear in the interval $[\beta, \overline{w}^*)$ with



Figure 5. (Color online) Value Functions with Low Monitoring Costs

an endogenous $\bar{w}^* > 2\beta$. Finally, the value function decreases with the monitoring cost.

Now we are ready to show our main result of this section.

Theorem 3. For $m < \overline{m}$ and any incentive compatible contract Γ that yields an agent's utility w, we have $U(\Gamma) \le F(w) \le F(0) = -m/r$, in which concave function F(w) is defined as (32) for $m \in [0, \underline{m})$, (33) for $m \in [\underline{m}, \widehat{m})$, and (35) for $m \in [\widehat{m}, \overline{m})$, where $\overline{\theta}$ and \overline{w}^* are defined in (34). Therefore, it is optimal for the principal to always monitor the agent.

4.3. Exogenous Upper Bound

In certain practical settings, the principal may not be able to allow the promised utility to grow too high before paying the agent. This is especially true if the agent's discount rate is close to the principal's, in which case the endogenous upper bound \bar{w}^* , although finite, tends to be very large. Therefore, in this section, we allow the model to include an exogenous upper bound \bar{w} on the promised utility. That is, our optimization problem has an additional constraint $w \leq \bar{w}$, similar to the case of $\rho = r$.

It is clear that, if the exogenous upper bound \bar{w} is higher than the endogenous \bar{w}^* , then the constraint $w \leq \bar{w}$ is not binding, and it has no effect on the optimal contract. Therefore, we focus on the situation in which, after computing \bar{w}^* without considering \bar{w} , the principal realizes that $\bar{w} < \bar{w}^*$.

An immediate observation is that the threshold \bar{m} , which separates the high and low monitoring cost regions, needs to change from (24) to the following:

$$\bar{m}(\bar{w}) := \inf_{w \in (\beta, \bar{w}]} \frac{r}{J'(w)}.$$
(37)

Obviously, this new threshold $\bar{m}(\bar{w})$ increases in the upper bound \bar{w} and, therefore, is greater than or equal

to \bar{m} defined in (24). Therefore, the principal may choose to always monitor the agent for higher monitoring costs comparing with the base model without \bar{w} . This is intuitive not only mathematically, but also practically. The upper bound pushes the principal to start payments "prematurely." Given the trade-off between payments and monitoring costs, such a pressure makes monitoring more favorable.

Finally, thresholds \hat{m} and \underline{m} do not change with the upper bound \overline{w} . The main results of this section only require slight changes to accommodate the upper bound \overline{w} . For example, in specifying the monitoring threshold $\alpha_{\overline{\theta}}$ and optimal value functions, (28) and (34) are changed to $F'_{\overline{\theta},\alpha_{\overline{\theta}}}(\overline{w}) = -1$ and $F'_{\overline{\theta}}(\overline{w}) = -1$, respectively. The optimal contract for high monitoring cost is $\Gamma_d(w^*; \alpha_{\overline{\theta}}, \overline{w})$ in which w^* is the maximizer of the corresponding updated value function. For the low monitoring cost case, contract $\Gamma_r(0; \overline{w})$ achieves the optimal value function in place of $\Gamma_r(0; \overline{w}^*)$.

5. Computation

It is worth pointing out that the optimal value functions and contracts presented in Sections 3 and 4 are very easy to compute. For the value function $F_{\theta(\bar{w})}(w)$ of Section 3, we only need to first solve the function J(w) following DDE (H) using, for example, the standard shooting method starting from J(w) = 1 for $w \in [0, \beta)$. After obtaining the function J(w), we obtain the slope $\theta(\bar{w})$ using (20). Then the value function $F_{\theta(\bar{w})}(w)$ is readily available following (18). The exact definition of the optimal contract follows the initial promised utility w^* , which is a maximizer of $F_{\theta(\bar{w})}(w)$. After obtaining the optimal contract, implementing it over time becomes very easy as we have already discussed in Section 3.

When the principal is more patient than the agent $(\rho > r)$, threshold \bar{m} defined in (24) is easy to compute. In fact, it has closed form expressions if r and p do not differ too much, as shown in the following result.

Proposition 6. (a) For $r \in (0, \rho - \lambda]$, we have $\overline{m} = (\rho + \lambda)\beta$; (b) for $r \in (\rho - \lambda, \overline{r}]$, we have

$$\bar{m} = (\rho + \lambda)\beta \left[1 - \frac{\beta\rho}{\beta(2\rho + \lambda)}\right]^{\frac{\lambda + r}{\rho} - 1}$$

in which \bar{r} *is the unique solution to the following equation on* $[\rho - \lambda, \rho]$ *,*

$$\left[1 - \frac{\beta\rho}{\beta(2\rho + \lambda)}\right]^{\frac{\lambda + \bar{r}}{\rho} - 1} = 1 - \frac{\rho - \bar{r}}{\lambda}.$$
 (38)

If the monitoring cost is higher than the threshold \bar{m} , the computation is slightly more complex than the case with equal discount. In order to specify the

optimal value function $F_{\theta,\alpha_{\bar{\theta}}}(w)$, we also need to search for the slope θ through a binary search. In Algorithm 1, we provide a pseudo code for the arguably more complex case of $\rho > r$ and $m > \overline{m}(\overline{w})$ with an exogenous upper bound \bar{w} .

Algorithm 1

- 1: Let *Stopping* $\leftarrow 0$, $\theta_l \leftarrow 0$, and $\theta_h \leftarrow m \frac{\beta^{-r/\rho}}{r}$ following (20) lowing (29)
- 2: while *Stopping* = 0, do
- 3: Let $\theta \leftarrow (\theta_l + \theta_h)/2$
- 4: Compute α_{θ} according to (26) in which the function f is defined in (25)
- 5: Use the shooting method to compute function $F_{\theta}(w)$ following DDE (H) for $w \ge \alpha_{\theta}$ with boundary condition (L) on $w \in [0, \alpha_{\theta})$ until a point $\hat{w} \in [\alpha_{\theta}, \bar{w}]$ that must satisfy one of the following cases:
- if $F'_{\theta,\alpha_{\theta}}(\hat{w}) < -1$, then 6:
- 7: Let $\theta_l \leftarrow \theta$
- else, if $(\hat{w} < \bar{w} \text{ and } F_{\theta,\alpha_{\theta}}^{\prime\prime}(\hat{w}) \ge 0)$ or $\hat{w} = \bar{w}$, then 8:

9: **if**
$$(F'_{\theta,\alpha_{\theta}}(\hat{w}) > -1 \text{ and } \hat{w} < \bar{w})$$
, then

- 10: Let $\theta_h \leftarrow \theta$
- else, if $(F'_{\theta,\alpha_{\theta}}(\hat{w}) = -1 \text{ or } \hat{w} = \bar{w})$, then 11:
- 12: Let $\bar{w}^* \leftarrow \hat{w}, \theta \leftarrow \theta$, and *Stopping* $\leftarrow 1$
- 13: end if
- 14: end if
- 15: end while

The logic behind steps 6 and 7 of Algorithm 1 follows from Lemma 5 and Proposition 7.

Proposition 7. Function $F_{\theta,\alpha_{\theta}}(w)$ is strictly concave on $w \in [0, \hat{w}]$ for \hat{w} defined as the following:

$$\hat{w} := \begin{cases} \inf\left\{ \underset{w > \alpha_{\theta}}{\arg\inf} F'_{\theta, \alpha_{\theta}}(w) \right\}, & \text{if } \theta \ge \bar{\theta}, \\ \inf\left\{ w : w \ge \alpha_{\theta} \text{ and } F'_{\theta, \alpha_{\theta}}(w) < -1 \right\}, & \text{if } \theta < \bar{\theta}. \end{cases}$$

$$(39)$$

Following the definition of θ in (28), Lemma 5(i) implies that, if $\theta \ge \theta$, we must have $F_{\theta,\alpha_{\theta}}(w) \ge -1$ for all $w \ge \alpha_{\theta}$. Therefore, the existence of a point \hat{w} such that $F_{\theta,\alpha_{\theta}}(\hat{w}) < -1$ must imply that $\theta < \theta$. Consequently, value θ serves as a lower bound θ_l for θ . Furthermore, Proposition 7 guarantees that, if $\theta < \theta$, for any $w \leq \hat{w}$, we must have $F''_{\theta,\alpha_{\theta}}(w) < 0$. Therefore, the search does not stop prematurely at a point following steps 8 and 9.

The logic behind steps 8 and 9 also follows from Proposition 7 together with Lemma 5(i). In particular, Proposition 7 implies that, for $\theta > \theta$, as soon as we observe a point \bar{w} with $F'_{\theta,\alpha_{\theta}}(\hat{w}) = 0$ for the first time, the point \hat{w} must be the minimum of derivative $F'_{\theta,\alpha_{\theta}}(w)$ over the entire interval $[\alpha_{\theta}, \infty)$. Hence, if $F'_{\theta,\alpha_{\theta}}(\hat{w}) > -1$, we must have $\theta > \theta$.

Overall, the algorithm involves a binary search for θ and solving for the value function given any current choice of θ . This computation, again, is very easy to implement. Overall, simple computation and contract structures make our results easily implementable in practice.

6. A Simple Cyclic Monitoring Schedule

The optimal monitoring and payment schedules are dynamically adjusted following changes of the promised utility. In certain situations, the principal may prefer an even simpler, more "regular" schedule. For example, one may think of a "periodic review" contract, which is determined by a set of parameters (T, N_d, π) .⁷ Under this contract, the principal reviews the performance of the agent every *T* time units. If the number of arrivals during this period is less than or equal to N_d , the agent collects an amount of payment π ; otherwise, the agent is not compensated for this cycle. Although payment to the agent is based on the actual performance, such a contract may not be incentive compatible. Imagine that the number of arrivals is already N_d in the middle of the cycle. In this case, the agent has no incentive to continue the effort for the remaining time in the cycle. Similarly, the agent may not want to bother exerting effort toward the end of a cycle when the number of arrivals is still far below N_d . Lack of full effort also implies that it may be hard to determine the optimal values for the contract parameters because the contract design is no longer an optimization problem but involves a differential game.

Here we propose a different, incentive compatible, cyclic contract that is very easy to compute and manage. Each cycle starts with a flow payment until an arrival, which starts a monitoring episode of a fixed period of time without payment. After the monitoring episode a new cycle starts. This schedule is not only very easy to implement, its optimal parameters (payment level and length of monitoring episode) are also very easy to compute. For the equal discount case, they are even in closed forms.

Now we derive the optimal parameters for this contract. Denote ℓ to be the flow payment, T the length of each monitoring period, and \hat{w} the agent's continuation utility while being paid. An arrival brings down this continuation utility by β as long as $\hat{w} > \beta$. Therefore, we have

$$\hat{w} = \int_0^\infty \lambda e^{-\lambda t} \left[\int_0^t e^{-\rho \tau} \ell d\tau + e^{-\rho t} (\hat{w} - \beta) \right] dt \text{ and}$$
$$\hat{w} - \beta = e^{-\rho T} \hat{w},$$

which imply that

$$\hat{w} = \frac{1}{\rho} \left(\ell - \lambda \beta \right) \text{ and } e^{-\rho T} = \frac{\ell - (\lambda + \rho)\beta}{\ell - \lambda \beta}.$$
 (40)

Clearly, to make the cyclic monitoring schedule meaningful, the flow payment rate ℓ must be no less than $(\lambda + \rho)\beta$. Equation (40) reveals the one-to-one correspondence between the payment ℓ , maximal utility \hat{w} , and monitoring period length *T* from binding incentive constraints. Next, denote $C(\ell)$ to be the principal's cost of this simple contract as a function of the payment ℓ . It is easy to verify that function $C(\ell)$ follows a recursive formulation:

$$\begin{split} C(\ell) &= \int_0^\infty \lambda e^{-\lambda t} \bigg[\int_0^t e^{-r\tau} \ell d\tau + e^{-rt} \int_0^T e^{-r\tau} m d\tau \\ &+ e^{-r(T+t)} C(\ell) \bigg] dt. \end{split}$$

Therefore, following (40),

$$C(\ell) = \frac{m}{r} - \frac{m-\ell}{\lambda + r - \lambda \left(\frac{\ell - (\lambda + \rho)\beta}{\ell - \lambda\beta}\right)^{\frac{r}{\rho}}}$$

from which we can compute the optimal ℓ^* that minimizes $C(\ell)$ following a simple one-dimensional search and then obtain the optimal value following (40).

For the equal discount case (i.e., $r = \rho$), we may express the principal's cost as a simple convex function of payment rate ℓ :

$$C(\ell) = \frac{\ell - \lambda \beta}{r} + \frac{\lambda m \beta}{r \ell},$$

which is minimized at $\ell^* = \sqrt{\lambda m \beta}$ when $m > (\lambda + \rho)^2 \beta / \lambda$. The corresponding monitoring time T^* is

$$T^* = \frac{1}{r} \ln \frac{\sqrt{\lambda m \beta} - \lambda \beta}{\sqrt{\lambda m \beta} - \beta (\lambda + r)}$$

The principal's optimal cost under this contract is $C(\ell^*) = (2\sqrt{\lambda\beta m} - \lambda\beta)/r$. When $m \le (\lambda + \rho)^2 \beta/\lambda$, on the other hand, the optimal flow payment is $\ell^* = (\rho + \lambda)\beta < (\lambda + \rho)\beta$, and the first arrival triggers monitoring forever $(T^* = \infty)$. The corresponding principal's value is $C(\ell^*) = \beta + \lambda m/[r(r + \lambda)]$.

Figure 6 provides numerical comparisons between the optimal policy and the cyclic monitoring schedule proposed. As we can see, the suboptimality of the cyclic monitoring schedule is substantial when m is high and/or the discount rate of the agent is close to that of the principal. The intuition is that, comparing with the cyclic monitoring schedule, under which each arrival triggers a costly monitoring episode, the optimal policy leads to relatively infrequent monitoring in the two scenarios. To be more specific, when *m* is high, the principal reduces monitoring frequency in the optimal policy to avoid high monitoring cost. When the discount rate of the agent is close to that of the principal, the principal sets a high threshold in the promised utility for payment (backloading). As a result, in the optimal policy, it may take many arrivals before a monitoring episode is triggered.

7. Concluding Remarks and Further Discussion

This paper studies the optimal monitoring and payment mechanism to induce an agent's effort in order to reduce the arrival rate of adverse events. Under condition (3), the contract design problem can be formulated as a continuous time optimal control model. The structures of its optimal solution depend on model parameters, in particular, the monitoring cost. The variations of the contract structures highlight the trade-off between the monitoring cost and direct payment to the agent. The optimal contract structures are simple to describe, easy to compute and implement, and intuitive to explain. In particular, the key for computing the optimal contract only involves numerically solving a delay differential equation combined with a single dimensional search.

Our results easily extend to discrete time settings. The only issue worth mentioning is that toward the very end of a monitoring episode, it may be optimal to randomly stop monitoring either in the current period or in the next period. This is because, in discrete time settings, the



Figure 6. (Color online) Performance of the Cyclic Monitoring Schedule with Parameters $\lambda = 10$, $\beta = 1$, r = 0.99, and $\rho = 1.5$ in (a) and m = 5 in (b)

switch between monitoring and nonmonitoring may not be a single threshold any more. Instead, there could be an interval in which the optimal value function is linear. The corresponding control policy when the promised utility w falls in this interval is to randomize it between either end of the interval. The upper bound of the interval corresponds to stop monitoring, and the lower bound of the interval corresponds to continue monitoring for one more period after which the promised utility would increase to the upper bound.

Recall that our paper has focused on the case in which the monitor cost is below $K\Delta\lambda - b$ such that the principal should always induce full effort from the agent. When condition (3) does not hold, we argue that the principal should never monitor but allow the agent to shirk instead. In fact, consider any contract with monitoring and construct another contract during which the monitoring periods are replaced with periods that the agent shirks. (When the agent is shirking, the principal does not pay the effort cost *b*.) Path-wise, the dynamics of the agent's promised utility remain the same although the principal's value improves because $K\Delta\lambda - b < m$. Therefore, in a problem in which the principal allows the agent to shirk, the optimal shirking and payment schedule can be solved exactly the same as we have done in the paper, using a monitoring cost of $m = K\Delta\lambda - b$. Although the shirking problem appears quite challenging for the Brownian motion (Zhu 2013) and the good arrival Poisson (end of section 4 of Sun and Tian 2018) settings, it is readily solved in the bad arrival Poisson setting as a special case of our paper.

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Appendix A. Proofs in Section 2

Proof of Lemma 1. For a generic contract Γ and effort process Λ , following Equations (1) and (2), we define the agent's total expected utility conditioned on the information available at time *t* as

$$u_t(\Gamma, \Lambda) := \mathbb{E}^{\Gamma, \Lambda} \left[\int_0^\infty e^{-\rho \tau} \left(dL_\tau + b \, \mathbb{I}_{\lambda_\tau = \bar{\lambda}} \, d\tau \right) \, \middle| \, \mathcal{F}_t \right]$$
$$= \int_0^t e^{-\rho \tau} \left(dL_\tau + b \, \mathbb{I}_{\lambda_\tau = \bar{\lambda}} \, d\tau \right) + e^{-\rho t} W_t(\Gamma, \Lambda). \tag{A.1}$$

Therefore, $u_0(\Gamma, \Lambda) = u(\Gamma, \Lambda)$. Moreover, it is easy to verify that process $\{u_t\}_{t\geq 0}$ is an \mathcal{F}_t -martingale by conditional expectation's tower property. Define processes

$$M_t^{s,\Lambda} := \int_0^t y_\tau \lambda_\tau \, d\tau - N_t^s$$

and $M_t^{n,\Lambda} := \int_0^t (1 - y_\tau) \lambda_\tau \, d\tau - N_t^n.$ (A.2)

Following the Martingale representation theorem (see, for example, theorem T9 of Brémaud 1981), there exist \mathcal{F}_t -predictable processes $\{H^s_t(\Gamma, \Lambda)\}_{t\geq 0}$ and $\{H^n_t(\Gamma, \Lambda)\}_{t\geq 0}$ such that

$$u_t(\Gamma, \Lambda) = u_0(\Gamma, \Lambda) + \int_0^t e^{-\rho\tau} [H^s_\tau(\Gamma, \Lambda) \, dM^{s,\Lambda}_\tau + H^n_\tau(\Gamma, \Lambda) \, dM^{n,\Lambda}_\tau], \quad \forall t \ge 0.$$
(A.3)

On the one hand, (A.1) implies

$$du_t = e^{-\rho t} [dL_t + b \mathbb{I}_{\lambda_t = \bar{\lambda}} dt - \rho W_t(\Gamma, \Lambda) dt + dW_t(\Gamma, \Lambda)].$$
(A.4)

On the other hand, (A.3) implies

$$du_{t} = e^{-\rho t} \Big[H_{t}^{s}(\Gamma, \Lambda) dM_{t}^{s,\Lambda} + H_{t}^{n}(\Gamma, \Lambda) dM_{t}^{n,\Lambda} \Big]$$

$$= e^{-\rho t} \Big[H_{t}^{s}(\Gamma, \Lambda) \big(y_{t} \lambda_{t} dt - dN_{t}^{s} \big) + H_{t}^{n}(\Gamma, \Lambda) \big((1 - y_{t}) \lambda_{t} dt - dN_{t}^{n} \big) \Big], \qquad (A.5)$$

where the second equality follows from the definitions in (A.2). Combining (A.4) and (A.5) yields (PK). \Box

Proof of Lemma 2. This result corresponds to proposition 1 in Biais et al. (2010). Denote \mathcal{F}_t -measurable random variable $\tilde{u}_t(\Gamma, \Lambda, \underline{\Lambda})$ to represent the agent's utility under effort process Λ before time *t* and effort process $\underline{\Lambda}$ afterward. We have

$$\begin{split} \tilde{u}_t(\Gamma,\Lambda,\underline{\Lambda}) \\ &= \int_0^t e^{-\rho\tau} \left(dL_\tau + b \, \mathbb{I}_{\lambda_\tau = \bar{\lambda}} \, d\tau \right) + e^{-\rho t} W_t(\Gamma,\underline{\Lambda}). \end{split} \tag{A.6}$$

Consider any sample trajectory of $\{N_t^s, N_t^n\}_{t\geq 0}$ and effort process Λ and $\underline{\Lambda}$,

$$\begin{split} \tilde{u}_t(\Gamma,\Lambda,\underline{\Lambda}) &= u_t(\Gamma,\underline{\Lambda}) + \int_0^t e^{-\rho\tau} b \, \mathbb{I}_{\lambda_\tau = \bar{\lambda}} \, d\tau \\ &= u_0(\Gamma,\underline{\Lambda}) + \int_0^t e^{-\rho\tau} \Big[H^s_\tau(\Gamma,\underline{\Lambda}) \, dM^{s,\underline{\Lambda}}_\tau \\ &+ H^n_\tau(\Gamma,\underline{\Lambda}) \, dM^{n,\underline{\Lambda}}_\tau + b \, \mathbb{I}_{\lambda_\tau = \bar{\lambda}} \, d\tau \Big] \\ &= u_0(\Gamma,\underline{\Lambda}) + \int_0^t e^{-\rho\tau} \Big[H^s_\tau(\Gamma,\underline{\Lambda}) \, dM^{s,\Lambda}_\tau \\ &+ H^n_\tau(\Gamma,\underline{\Lambda}) \, dM^{n,\Lambda}_\tau \Big] \\ &- \int_0^t e^{-\rho\tau} \big[y_\tau H^s_\tau(\Gamma,\underline{\Lambda}) + (1 - y_\tau) \\ &\times H^n_\tau(\Gamma,\underline{\Lambda}) - \beta \big] \Delta\lambda \mathbb{I}_{\lambda_\tau = \bar{\lambda}} \, d\tau, \end{split}$$

where the first equality follows from (A.1) and (A.6), the second equality from (A.3), the third equality from (A.2) and the definition of β in (5).

Consider any two times t' < t,

$$\begin{split} & \mathbb{E}[\tilde{u}_{t}(\Gamma,\Lambda,\underline{\Lambda})|\mathcal{F}_{t'}] \\ &= u_{0}(\Gamma,\underline{\Lambda}) + \int_{0}^{t'} e^{-\rho\tau} \Big[H^{s}_{\tau}(\Gamma,\underline{\Lambda}) dM^{s,\Lambda}_{\tau} + H^{n}_{\tau}(\Gamma,\underline{\Lambda}) dM^{n,\Lambda}_{\tau} \Big] \\ &- \int_{0}^{t'} e^{-\rho\tau} \Big[y_{\tau} H^{s}_{\tau}(\Gamma,\underline{\Lambda}) + (1-y_{\tau}) H^{n}_{\tau}(\Gamma,\underline{\Lambda}) - \beta \Big] \Delta \lambda \mathbb{I}_{\lambda_{\tau}=\bar{\lambda}} d\tau \\ &- \mathbb{E} \bigg[\int_{t'}^{t} e^{-\rho\tau} \Big[y_{\tau} H^{s}_{\tau}(\Gamma,\underline{\Lambda}) + (1-y_{\tau}) \\ &\qquad \times H^{n}_{\tau}(\Gamma,\underline{\Lambda}) - \beta \Big] \Delta \lambda \mathbb{I}_{\lambda_{\tau}=\bar{\lambda}} d\tau \bigg| \mathcal{F}_{t'} \bigg] \end{split}$$

$$= \tilde{u}_{t'}(\Gamma, \Lambda, \underline{\Lambda}) - \Delta \lambda \mathbb{E} \left[\int_{t'}^{t} e^{-\rho \tau} \left[y_{\tau} H^{s}_{\tau}(\Gamma, \underline{\Lambda}) + (1 - y_{\tau}) H^{n}_{\tau}(\Gamma, \underline{\Lambda}) - \beta \right] \mathbb{I}_{\lambda_{\tau} = \bar{\lambda}} \, d\tau \, \Big| \, \mathcal{F}_{t'} \right]. \tag{A.7}$$

i. On the one hand, if (IC) holds under contract $\Gamma,$ (A.7) suggests that

$$\mathbb{E}\left[\tilde{u}_t(\Gamma,\Lambda,\underline{\Lambda})\,|\,\mathcal{F}_{t'}\right] \leq \tilde{u}_{t'}(\Gamma,\Lambda,\underline{\Lambda}),$$

which implies that the process $\{\tilde{u}_t\}_{t\geq 0}$ is a super-martingale. Taking t' = 0 and letting $t \to \infty$, we have

$$u(\Gamma, \Lambda) = \lim_{t \to \infty} \left\{ \mathbb{E}[\tilde{u}_t(\Gamma, \Lambda, \underline{\Lambda}) | \mathcal{F}_0] \right\} \le \tilde{u}_0(\Gamma, \Lambda, \underline{\Lambda})$$
$$= u(\Gamma, \underline{\Lambda}). \tag{A.8}$$

That is, effort process $\underline{\Lambda}$ dominates any other process Λ under contract Γ , or Γ is incentive compatible if (IC) holds.

ii. On the other hand, suppose (IC) does not hold, or $y_{\tau}H_{\tau}^{s}(\Gamma, \underline{\Lambda}) + (1 - y_{\tau})H_{\tau}^{n}(\Gamma, \underline{\Lambda}) < \beta$ for $m_{\tau} = 0$ over a subset of [0, t] with positive measure. Define an effort process $\Lambda = \{\lambda_{\tau}\}_{\tau>0}$ such that $\lambda_{\tau} = \lambda$ for $\forall \tau \in (t, \infty)$ and for $\forall \tau \in [0, t]$:

$$\lambda_{\tau} = \begin{cases} \bar{\lambda} & \text{if } y_{\tau} H^s_{\tau}(\Gamma, \underline{\Lambda}) + (1 - y_{\tau}) H^n_{\tau}(\Gamma, \underline{\Lambda}) \\ & < \beta \text{ and } m_{\tau} = 0, \\ \lambda & \text{otherwise.} \end{cases}$$

Clearly, we must have

$$\begin{split} &- \mathbb{E} \left[\int_{0}^{t} e^{-\rho\tau} \left[y_{\tau} H_{\tau}^{s}(\Gamma,\underline{\Lambda}) + \left(1 - y_{\tau}\right) H_{\tau}^{n}(\Gamma,\underline{\Lambda}) - \beta \right] \right. \\ & \times \Delta \lambda \mathbb{I}_{\Lambda_{\tau} = \bar{\lambda}} \, d\tau \Big| \mathcal{F}_{t'} \Big] > 0. \end{split}$$

As a result, taking t' = 0 and letting $t \to \infty$ in (A.7), we obtain

$$\begin{split} u(\Gamma, \Lambda) &= \lim_{t \to \infty} \left\{ \mathbb{E}[\tilde{u}_t(\Gamma, \Lambda, \underline{\Lambda}) \,|\, \mathcal{F}_0] \right\} > \tilde{u}_0(\Gamma, \Lambda, \underline{\Lambda}) \\ &= u(\Gamma, \underline{\Lambda}). \end{split}$$

This implies that effort process Λ dominates $\underline{\Lambda}$, or contract Γ is *not* incentive compatible when (IC) does not hold. \Box

Proof of Lemma 3. Following Itô's change of variable formula with function *F* (see, for example, theorem 14.3.2 of Cohen and Elliott 2015), for any $\tau \ge 0$, we have

$$e^{-r\tau}F(W_{\tau}) = F(w) + \int_{0}^{\tau} \left[e^{-rt}dF(W_{t}) - re^{-rt}F(W_{t})dt \right]$$

= $F(w) + \int_{0}^{\tau} e^{-rt}(m_{t}dt + dL_{t}) + \int_{0}^{\tau} e^{-rt}d\mathcal{A}_{t},$ (A.9)

where

$$dst_{t} := dF(W_{t}) - rF(W_{t})dt - m_{t}dt - dL_{t}$$

$$= F'(W_{t}) \Big[\big(\rho W_{t} + \lambda \big(y_{t}H_{t}^{s} + (1 - y_{t})H_{t}^{n} \big) \big) dt$$

$$- \ell_{t}dt \Big] - rF(W_{t})dt$$

$$+ F \big(W_{t} - H_{t}^{s}dN_{t}^{s} - H_{t}^{n}dN_{t}^{n} - dI_{t} \big) - F(W_{t})$$

$$- m_{t}dt - dL_{t}.$$

Further define

$$d\mathfrak{B}_t := [F(W_t - H_t^s) - F(W_t)](dN_t^s - \lambda y_t dt) + [F(W_t - H_t^n) - F(W_t)] \times (dN_t^n - \lambda(1 - y_t)dt).$$
(A.10)

Because function F(w) is concave and $F'(w) \ge -1$, we have

$$\begin{split} d\mathcal{A}_{t} &\leq F'(W_{t}) \big(\rho W_{t} + \lambda \big(y_{t} H_{t}^{s} + (1 - y_{t}) H_{t}^{n} \big) \big) dt \\ &+ F \big(W_{t} - H_{t}^{s} dN_{t}^{s} - H_{t}^{n} dN_{t}^{n} \big) \\ &- F'(W_{t}) \ell_{t} dt - F' \big(W_{t} - H_{t}^{s} dN_{t}^{s} - H_{t}^{n} dN_{t}^{n} \big) dI_{t} \\ &- F(W_{t}) - rF(W_{t}) dt - m_{t} dt - dL_{t} \\ &\leq F'(W_{t}) \big(\rho W_{t} + \lambda \big(y_{t} H_{t}^{s} + (1 - y_{t}) H_{t}^{n} \big) \big) dt \\ &- rF(W_{t}) dt - m_{t} dt + F \big(W_{t} - H_{t}^{s} dN_{t}^{s} \\ &- H_{t}^{n} dN_{t}^{n} \big) - F(W_{t}) \\ &= F'(W_{t}) \big(\rho W_{t} + \lambda \big(y_{t} H_{t}^{s} + (1 - y_{t}) H_{t}^{n} \big) \big) dt \\ &- rF(W_{t}) dt - m_{t} dt + \big[F \big(W_{t} - H_{t}^{s} \big) \\ &- F(W_{t}) \big] dN_{t}^{s} \\ &+ \big[F \big(W_{t} - H_{t}^{n} \big) - F(W_{t}) \big] dN_{t}^{n} \\ &= d\mathcal{R}_{t} + \Psi_{t} dt. \end{split}$$

Therefore, if $\Psi_t \leq 0$, we must have $d\mathcal{A}_t \leq d\mathcal{B}_t$ almost surely. Taking the expectation on both sides of (A.9), we immediately have

$$F(w) \geq \mathbb{E}^{\Gamma,\underline{\Lambda}} \left[e^{-r\tau} F(W_{\tau}) - \int_{0}^{\tau} e^{-rt} (m_{t} dt + dL_{t}) \right]$$

where we use the fact that $\int_0^{\tau} e^{-rt} d\mathcal{B}_t$ is a martingale.

Taking
$$\tau \to \infty$$
, the preceding inequality reduces to

$$F(w) \geq -\mathbb{E}^{\Gamma,\underline{\Lambda}}\left[\int_0^\infty e^{-rt}(m_t dt + dL_t)\right] = U(\Gamma).$$

This completes the proof. \Box

Appendix B. Proofs in Section 3

We first present the following technical lemma.

Lemma B.1. For any $\alpha \ge \beta$, starting with any boundary condition F(w) that is continuous for $w \in [0, \alpha]$, DDE (H) uniquely determines a continuous function F(w) on $w \in [0, \infty)$. Furthermore, function F(w) is increasing on $[\alpha, \infty)$ if either of the following two conditions holds.

- i. Function F(w) is positive and nondecreasing on $[0, \alpha]$.
- ii. Function F(w) is increasing within $[0, \alpha]$ and $F(\alpha) \ge 0$.

Proof. Solving DDE (H) with well-defined boundary conditions over $w \in [0, \alpha]$ is equivalent to solving a sequence of initial value problems over interval $[\alpha + k\beta, \alpha + (k + 1)\beta]$, where k = 0, 1, ... This sequence of initial value problems satisfies the Cauchy–Lipschitz theorem (see, for example, theorem 1.1 of Hartman 1982); therefore, a unique and differentiable solution is guaranteed over $w \in (\alpha, \infty)$.

i. If F(w) is positive and nondecreasing for $w \in [0, \alpha]$, suppose it is nonincreasing for some $w \ge \alpha$. Let

$$\hat{w} := \min\left\{w|F'(w_+) \le 0 \text{ and } w \ge \alpha\right\}.$$

DDE (H) implies

$$rF(\hat{w}) + \lambda \left[F(\hat{w}) - F(\hat{w} - \beta)\right] \le 0$$

which is impossible because $F(\hat{w}) \ge F(\alpha) > 0$ as assumed, and $F(\hat{w}) \ge F(\hat{w} - \beta)$ for $w \in [\alpha, \hat{w}]$ from the definition of \hat{w} . Therefore, we must have F'(w) > 0 for $\forall w \in [\alpha, \infty)$. Part (ii) can be proven similarly. \Box

Lemma B.2. Consider the case $r = \rho$. Function J(w), which is the solution of DDE (H) with boundary condition J(w) = 1 on $w \in [0, \beta)$ is increasing and strictly convex on $w \in [\beta, \infty)$.

Proof. Note that following DDE (H), function J(w) is differentiable for $w > \beta$ and is twice-differentiable except at $w = 2\beta$. Taking derivatives on both sides of DDE (H) yields

$$J''(w) = \frac{(\lambda + r - \rho)J'(w) - \lambda J'(w - \beta)}{\rho w + \beta \lambda}, \text{ for}$$
$$w \in (\beta, 2\beta) \cup (2\beta, \infty). \tag{B.1}$$

In particular, there is

$$J(w) = \frac{r}{\lambda + r} \left(\frac{\rho w + \lambda \beta}{\rho \beta + \lambda \beta} \right)^{\frac{\lambda \pm r}{\rho}} + \frac{\lambda}{\lambda + r}, \text{ for}$$
$$w \in (\beta, 2\beta), \tag{B.2}$$

whose first and second derivatives are

$$J'(w) = \frac{r(\rho w + \lambda \beta)^{\frac{\lambda + r - \rho}{\rho}}}{(\rho \beta + \lambda \beta)^{\frac{\lambda + r}{\rho}}}$$
$$J''(w) = \frac{r(\rho w + \lambda \beta)^{\frac{\lambda + r - 2\rho}{\rho}}}{(\rho \beta + \lambda \beta)^{\frac{\lambda + r}{\rho}}} (\lambda + r - \rho).$$
(B.3)

When $r = \rho$, the closed-form expression in (B.2) is clearly convex; that is, J(w) is convex for $w \in [\beta, 2\beta)$. Consider the point $w = 2\beta$; (B.1) and (B.3) yield

$$J''(2\beta_{+}) = \frac{\lambda r \left[(2\beta\rho + \beta\lambda)^{\frac{\lambda}{\rho}} - (\beta\rho + \beta\lambda)^{\frac{\lambda}{\rho}} \right]}{(2\beta\rho + \beta\lambda)(\beta\rho + \beta\lambda)^{\frac{\lambda+r}{\rho}}} > 0$$

Therefore, we only need to show J''(w) > 0 for all $w > 2\beta$. We prove by contradiction. Suppose, on the contrary, there exists some $w > 2\beta$ such that $J''(w) \le 0$. Define

$$\hat{w} := \min\{w|J'(w) \le J'(w-\beta) \text{ and } w > 2\beta\}.$$

By construction, we have J''(w) > 0 for all $w \in (\beta, \hat{w})$. First, \hat{w} cannot be in $(2\beta, 3\beta]$ because otherwise we would have $\hat{w} - \beta \le 2\beta$, and

$$J'(\hat{w}) > J'(2\beta) \ge J'(\hat{w} - \beta),$$

which contradicts the definition of \hat{w} . Second, \hat{w} cannot be greater than 3β either because otherwise we have

$$J'(\hat{w}) = J'(\hat{w} - \beta) + \int_0^\beta J''(\hat{w} - \beta + x)dx$$

> $J'(\hat{w} - \beta),$

which, again, contradicts the definition of \hat{w} . Therefore, we must have J''(w) > 0 for all $w > 2\beta$. Last but not least,

monotonicity of J(w) follows directly from Lemma B.1, which completes the proof. \Box

Proof of Proposition 1. (i) Given that $\theta(\bar{w}) = \frac{m}{r}J'(\bar{w}) - 1$ and $J'(\bar{w}) \ge 0$ (from Lemma B.2), we have $\theta(\bar{w}) \ge -1$. We prove $\theta(\bar{w}) < \frac{m}{\beta r}$ by contradiction. Suppose, on the contrary, we have $\theta(\bar{w}) \ge \frac{m}{\beta r}$. It's easy to verify that function $F_{\theta(\bar{w})}(w)$ can be decomposed as

$$F_{\theta(\bar{w})}(w) = \left(\theta(\bar{w}) - \frac{m}{r\beta}\right)G_1(w) + \frac{m}{r}G_2(w),$$

for $w \ge 0$,

in which functions $G_1(w)$ and $G_2(w)$ are the solution of DDE (H) with boundary conditions being

$$G_1(w) = w$$
 and $G_2(w) = \frac{w}{\beta} - 1$, $\forall w \in [0, \beta]$,

respectively. Because $G_1(w)$ and $G_2(w)$ are both increasing on $[0,\beta]$ and nonnegative at $w = \beta$, we know that they are both increasing on $[\beta,\infty)$ by Lemma B.1. As such, $F_{\theta(\bar{w})}(\bar{w})$ is increasing on $[\beta,\infty)$ as well, which contradicts to $F'_{\theta(\bar{w})}(w) = -1$. Therefore, contradiction is established, and we must have $\theta_{\bar{w}} < \frac{m}{\beta_r}$.

(ii) We only need to show that $F'_{\theta(\bar{w})}(\beta_+) \leq F'_{\theta(\bar{w})}(\beta_-)$ because, for $w > \beta$, concavity of $F_{\theta(\bar{w})}(w)$ follows immediately from the strict convexity of function J(w) in Lemma B.2 and the decomposition in (18). If $r = \rho$, according to (H), we have

$$F'_{\theta(\bar{w})}(\beta_{+}) = \frac{(\lambda+r)F_{\theta(\bar{w})}(\beta) - \lambda F_{\theta(\bar{w})}(0)}{(r+\lambda)\beta}$$
$$= \frac{\lambda}{\lambda+r}\theta(\bar{w}) \le \theta(\bar{w}) = F'_{\theta(\bar{w})}(\beta_{-}). \tag{B.4}$$

(iii) By the definition of $\theta(\bar{w})$, we know that $\theta(\bar{w})$ is strictly increasing in \bar{w} (recall the convexity of J(w)). Therefore, for any $\bar{w} \in [\beta, \bar{w})$, we have $\theta(\bar{w}) > \theta_{(\bar{w})}$. For any $w \ge 0$, decomposition (18) implies that

$$F_{\theta(\bar{w})}(w) - F_{\theta(\bar{w})}(w) = \left[\theta(\bar{w}) - \theta(\bar{w})\right]w > 0.$$

This completes the proof. \Box

Lemma B.3. Consider a concave function F(w) that satisfies Equations (H), (U), and (19). For any $w \ge \beta$, the following function $\Phi(w, x)$ is increasing in $x \in (-\infty, 0]$ and decreasing in $x \in [0, \infty)$,

$$\Phi(w, x) := F'(w)x + F(w - x).$$
(B.5)

Proof. Taking the first derivative of $\Phi(w, x)$ with respect to x yields

$$\frac{\partial \Phi(w, x)}{\partial x} = F'(w) - F'(w - x).$$

Because F(w) is concave, we know that $\frac{\partial \Phi(w,x)}{\partial x} \ge 0$ when $x \le 0$ and $\frac{\partial \Phi(w,x)}{\partial x} \le 0$ when $x \ge 0$. That is, for any $w \ge \beta$, $\Phi(w, x)$ is increasing in $x \in (-\infty, 0]$ and decreasing in $x \in [0, \infty)$. \Box

Proof of Proposition 2. Starting with any promised utility $W_0 = w \in [0, \bar{w}]$, consider the process $\{W_t\}_{t\geq 0}$ according to (PK) in which the counting processes $\{(N_t^s, N_t^n)\}_{t\geq 0}$ are generated from the effort process $\underline{\Lambda}$ under contracts $\Gamma_d(w; \beta, \bar{w})$ or $\Gamma_r(w; \bar{w})$. Clearly, we must have $0 \le W_t \le \bar{w}$ for $\forall t$.

(i) First, we consider contract $\Gamma_d(w;\beta,\bar{w})$ defined in Definition 1. Following Itô's formula for jump processes, we have

$$dF_{\theta(\bar{w})}(W_t)$$

$$= \left[F_{\theta(\bar{w})}(W_t - dI_t) - F_{\theta(\bar{w})}(W_t)\right] + F'_{\theta(\bar{w})}$$

$$\times (W_t) \left\{\rho W_t + \lambda_t \left[y_t H_t^s + (1 - y_t) H_t^n\right] - \ell_t\right\} dt$$

$$+ \left[F_{\theta(\bar{w})}(W_t - H_t^s) - F_{\theta(\bar{w})}(W_t)\right] dN_t^s$$

$$+ \left[F_{\theta(\bar{w})}(W_t - H_t^n) - F_{\theta(\bar{w})}(W_t)\right] dN_t^n.$$
(B.6)

Following the dynamics of W_t according to Definition 1, we have

$$dF_{\theta(\bar{w})}(W_t) = F'_{\theta(\bar{w})}(W_t) (\rho W_t \mathbb{I}_{W_t < \beta} + (\rho W_t + \lambda \beta) \mathbb{I}_{\beta \le W_t < \bar{w}}) dt + [F_{\theta(\bar{w})}(W_t - \beta) - F_{\theta(\bar{w})}(W_t)] \mathbb{I}_{\beta \le W_t \le \bar{w}} dN_t.$$
(B.7)

Note that dN_t^s and dN_t^n take values zero or one in the preceding expressions.

For any $\tau \ge 0$, we have

$$e^{-r\tau}F_{\theta(\bar{w})}(W_{\tau})$$

$$= F_{\theta(\bar{w})}(w) + \int_{0}^{\tau}F_{\theta(\bar{w})}(W_{t}) de^{-rt}$$

$$+ \int_{0}^{\tau}e^{-rt}dF_{\theta(\bar{w})}(W_{t})$$

$$= F_{\theta(\bar{w})}(w) + \int_{0}^{\tau}e^{-rt}\Big[F'_{\theta(\bar{w})}(W_{t})\big(\rho W_{t}\mathbb{I}_{W_{t}<\beta}$$

$$+ \big(\rho W_{t} + \beta\lambda\big)\mathbb{I}_{\beta\leq W_{t}<\bar{w}}\big) - rF_{\theta(\bar{w})}(W_{t})\Big]dt$$

$$+ \int_{0}^{\tau}e^{-rt}\Big[F_{\theta(\bar{w})}\big(W_{t} - \beta\big) - F_{\theta(\bar{w})}(W_{t})\Big]\mathbb{I}_{\beta\leq W_{t}\leq\bar{w}}dN_{t}. \quad (B.8)$$

From Equations (17) and (H), we know that $F_{\theta(\tilde{w})}(w)$ satisfies

$$\begin{split} F_{\theta(\bar{w})}'(W_t)\rho W_t \mathbb{I}_{W_t < \beta} &= \left[rF_{\theta(\bar{w})}(W_t) + m \right] \mathbb{I}_{W_t < \beta}, \text{ and } \\ F_{\theta(\bar{w})}'(W_t) (\rho W_t + \beta \lambda) \mathbb{I}_{\beta \le W_t < \bar{w}} \\ &= \left[(\lambda + r)F_{\theta(\bar{w})}(W_t) - \lambda F_{\theta(\bar{w})}(W_t - \beta) \right] \mathbb{I}_{\beta \le W_t < \bar{w}}. \end{split}$$

Substituting these equations into (B.8), we obtain

$$e^{-r\tau}F_{\theta(\bar{w})}(W_{\tau})$$

$$= F_{\theta(\bar{w})}(w) + \int_{0}^{\tau} e^{-rt} \left[F_{\theta(\bar{w})}(W_{t} - \beta) - F_{\theta(\bar{w})}(W_{t})\right] \mathbb{I}_{\beta \leq W_{t} \leq \bar{w}} dN_{t} + \int_{0}^{\tau} e^{-rt} \left[(\lambda + r)F_{\theta(\bar{w})}(W_{t}) - \lambda F_{\theta(\bar{w})}(W_{t} - \beta)\right] \mathbb{I}_{\beta \leq W_{t} < \bar{w}} dt$$

$$+ \int_{0}^{\tau} e^{-rt} \left[(rF_{\theta(\bar{w})}(W_{t}) + m)\mathbb{I}_{W_{t} < \beta} - rF_{\theta(\bar{w})}(W_{t})\right] dt$$

$$= F_{\theta(\bar{w})}(w) + \int_{0}^{\tau} e^{-rt} (m\mathbb{I}_{W_{t} < \beta} + (\rho\bar{w} + \beta\lambda)\mathbb{I}_{W_{t} = \bar{w}}) dt + \Omega_{\tau},$$
(B.9)

where the second equality utilizes Equation (U), and the process $\{\Omega_\tau\}_{\tau\geq 0},$ defined as

$$\begin{aligned} \Omega_{\tau} &:= \int_{0}^{\tau} e^{-rt} \big[F_{\theta(\tilde{w})} \big(W_t - \beta \big) - F_{\theta(\tilde{w})} (W_t) \big] \mathbb{I}_{\beta \leq W_t \leq \tilde{w}} \\ &\times (dN_t - \lambda dt), \end{aligned}$$

is a martingale. Taking expectation on both sides of (B.9) and letting $\tau \rightarrow \infty$, we have

$$\begin{split} F_{\theta(\bar{w})}(w) &= -\mathbb{E}^{\Gamma_d(w;\beta,\bar{w}),\Delta} \bigg[\int_0^\infty e^{-rt} \big(m \mathbb{I}_{W_t < \beta} \\ &+ \big(\rho \bar{w} + \beta \lambda \big) \mathbb{I}_{W_t = \bar{w}} \big) dt \bigg] \\ &= -\mathbb{E}^{\Gamma_d(w;\beta,\bar{w}),\Delta} \bigg[\int_0^\infty e^{-rt} \big(m_t \, dt + dL_t \big) \\ &= U \big(\Gamma_d(w;\beta,\bar{w}) \big), \end{split}$$

where the second equality follows from Definition 1.

(ii) Next, we consider contract $\Gamma_r(w; \bar{w})$ defined in Definition 2. Following Itô's formula for jump processes, we have

$$\begin{aligned} dF_{\theta(\bar{w})}(W_t) \\ &= F'_{\theta(\bar{w})}(W_t) (\rho W_t + \lambda \beta) \mathbb{I}_{\beta \le W_t < \bar{w}} dt \\ &+ \{ [F_{\theta(\bar{w})}(\beta) - F(W_t)] dN_t^s \} \mathbb{I}_{\beta \le W_t \le \min\{\bar{w}, 2\beta\}} \\ &+ [F_{\theta(\bar{w})}(0) - F_{\theta(\bar{w})}(W_t)] dN_t^s \} \mathbb{I}_{\beta \le W_t \le \min\{\bar{w}, 2\beta\}} \\ &+ [F_{\theta(\bar{w})}(W_t - \beta) - F_{\theta(\bar{w})}(W_t)] \mathbb{I}_{\min\{\bar{w}, 2\beta\} < W_t \le \bar{w}} dN_t^n. \end{aligned}$$
(B.10)

For any time $\tau \ge 0$, we have

$$\begin{split} e^{-r\tau}F_{\theta(\bar{w})}(W_{\tau}) \\ &= F_{\theta(\bar{w})}(w) + \int_{0}^{\tau}F_{\theta(\bar{w})}(W_{t})de^{-rt} + \int_{0}^{\tau}e^{-rt}dF_{\theta(\bar{w})}(W_{t}) \\ &= F_{\theta(\bar{w})}(w) + \int_{0}^{\tau}e^{-rt}\Big(F'_{\theta(\bar{w})}(W_{t}) \\ &\times (\rho W_{t} + \lambda\beta)\mathbb{I}_{\beta \leq W_{t} < \bar{w}} - rF_{\theta(\bar{w})}(W_{t})\Big)dt \\ &+ \int_{0}^{\tau}e^{-rt}\Big\{ [F_{\theta(\bar{w})}(\beta) - F_{\theta(\bar{w})}(W_{t})]\mathbb{I}_{\beta \leq W_{t} \leq \min\{\bar{w}, 2\beta\}}dN_{t}^{n} \\ &+ [F_{\theta(\bar{w})}(0) - F_{\theta(\bar{w})}(W_{t})]\mathbb{I}_{\beta \leq W_{t} \leq \min\{\bar{w}, 2\beta\}}dN_{t}^{s} \\ &+ [F_{\theta(\bar{w})}(W_{t} - \beta) - F_{\theta(\bar{w})}(W_{t})]\mathbb{I}_{\min\{\bar{w}, 2\beta\} < W_{t} \leq \bar{w}}dN_{t}\Big\} \quad (B.11) \\ &= F_{\theta(\bar{w})}(w) + \int_{0}^{\tau}e^{-rt}(m\mathbb{I}_{W_{t}=0}dt \\ &+ (\rho W_{t} + \beta\lambda)\mathbb{I}_{W_{t}=\bar{w}}dt) + \Omega_{\tau}, \end{split}$$

where the last equality follows from Equation (L), which implies that $F_{\theta(\bar{w})}(0)(2 - \frac{W_t}{\beta}) + F_{\theta(\bar{w})}(\beta)(\frac{W_t}{\beta} - 1) = F_{\theta(\bar{w})}(W_t - \beta)$, and the process $\{\Omega_{\tau}\}_{\tau \geq 0}$, defined as

$$\begin{aligned} \Omega_{\tau} &:= \int_{0}^{\tau} e^{-rt} \Biggl\{ \Biggl(F_{\theta(\bar{w})}(0) - F_{\theta(\bar{w})}(W_{t}) \Biggr) . \\ &\times \Biggl(dN_{t}^{s} - \Biggl(2 - \frac{W_{t}}{\beta} \Biggr) \lambda dt \Biggr) \mathbb{I}_{\beta \leq W_{t} \leq \min\{\bar{w}, 2\beta\}} \\ &+ (F_{\theta(\bar{w})}(\beta) - F_{\theta(\bar{w})}(W_{t})) \\ &\times \Biggl(dN_{t}^{n} - \Biggl(\frac{W_{t}}{\beta} - 1 \Biggr) \lambda dt \Biggr) \mathbb{I}_{\beta \leq W_{t} \leq \min\{\bar{w}, 2\beta\}} \\ &+ (F_{\theta(\bar{w})}(W_{t} - \beta) - F_{\theta(\bar{w})}(W_{t})) \\ &\times (dN_{t} - \lambda dt) \mathbb{I}_{\min\{\bar{w}, 2\beta\} < W_{t} \leq \bar{w}} \Biggr\}, \end{aligned}$$

is a martingale.

Taking expectation on both sides of (B.11) and letting $\tau \rightarrow \infty$, we have

$$\begin{split} F_{\theta(\bar{w})}(w) \\ &= -\mathbb{E}^{\Gamma_r(w;\bar{w}),\Lambda} \bigg[\int_0^\infty e^{-rt} \bigg(m \mathbb{I}_{W_t=0} \, dt + \big(\rho \bar{w} + \beta \lambda\big) \mathbb{I}_{W_t=\bar{w}} \, dt \big) \bigg] \\ &= U(\Gamma_r(w;\bar{w})). \end{split}$$

This completes the proof. \Box

Proof of Theorem 1. By Proposition 1, we know that $F_{\theta(\bar{w})}(w)$ is concave and $F'_{\theta(\bar{w})}(w) \ge -1$. Therefore, we only need to show $\Psi_t \le 0$ holds almost surely (recall Lemma 3).

From Equation (8), we have

.

$$\begin{split} \Psi_t &\leq \lambda \Big[F'_{\theta(\bar{w})}(W_t) \big(y_t H^s_t + \big(1 - y_t \big) H^n_t \big) + y_t F_{\theta(\bar{w})} \\ &\times \big(W_t - H^s_t \big) + \big(1 - y_t \big) F_{\theta(\bar{w})} \big(W_t - H^n_t \big) - F_{\theta(\bar{w})}(W_t) \Big] \\ &+ F'_{\theta(\bar{w})}(W_t) \rho W_t - r F_{\theta(\bar{w})}(W_t) - m_t \\ &\leq \lambda \Phi \big(W_t, y_t H^s_t + \big(1 - y_t \big) H^n_t \big) + F'_{\theta(\bar{w})}(W_t) \rho W_t \\ &- (\lambda + r) F_{\theta(\bar{w})}(W_t) - m_t, \end{split}$$
(B.12)

in which function Φ is defined in (B.5).

i. When $W_t < \beta$, we know that the principal monitors the agent (i.e., $m_t = m$). From Equation (17), we have

$$\Psi_t \le \rho W_t F'_{\theta(\bar{w})}(W_t) - r F_{\theta(\bar{w})}(W_t) - m = 0.$$

ii. When $\beta \le W_t \le \bar{w}$, substituting (H) into inequality (B.12) yields

$$\begin{split} \Psi_t &\leq \lambda \Phi \big(W_t, y_t H^s_t + \big(1 - y_t \big) H^n_t \big) - \lambda \beta F'_{\theta(\bar{w})}(W_t) \\ &- \lambda F_{\theta(\bar{w})} \big(W_t - \beta \big) - m_t. \end{split}$$

• If the principal does not monitor at time *t* (i.e., $m_t = 0$), we must have $y_t H_t^s + (1 - y_t) H_t^n \ge \beta$. Lemma B.3 implies

$$\Psi_t \le \lambda \Phi(W_t, \beta) - \lambda \beta F'_{\theta(\bar{w})}(W_t) - \lambda F_{\theta(\bar{w})}(W_t - \beta)$$

= 0.

• If the principal monitors at time *t* (i.e., $m_t = m$), Lemma B.3 implies

$$\begin{split} \Psi_t &\leq \lambda \Phi(W_t, 0) - \lambda \beta F'_{\theta(\bar{w})}(W_t) \\ &- \lambda F_{\theta(\bar{w})}(W_t - \beta) - m \\ &= -rF_{\theta(\bar{w})}(W_t) + rW_t F'_{\theta(\bar{w})}(W_t) - m \\ &\leq -rF_{\theta(\bar{w})}(\beta) + r\beta F'_{\theta(\bar{w})}(\beta) - m \\ &= -r\Big(\theta(\bar{w})\beta - \frac{m}{r}\Big) + r\beta \theta \bar{w} \frac{\lambda}{\lambda + r} - m \\ &= r\beta \theta(\bar{w}) \Big(\frac{\lambda}{\lambda + r} - 1\Big) \leq 0, \end{split}$$

where the second inequality follows the fact that $-rF(W_t) + rW_tF'(W_t)$ is decreasing in W_t .

To sum up, we must have $\Psi_t \leq 0$. This completes the proof. \Box

Appendix C. Proofs in Section 4

To prove Proposition 6 and Lemma 5, we first characterize the structural properties of function J(w) for the case $r < \rho$.

Lemma C.1. Consider the case $r < \rho$. For any $\alpha \ge \beta$, define function J(w) to be the solution of DDE (H) with boundary condition J(w) = 1 on $w \in [0, \alpha)$. J(w) exhibits the following properties.

i. When $r \leq \rho - \lambda$, J(w) is concave on $[\alpha, \infty)$.

ii. When $\rho - \lambda < r < \overline{r}$, J(w) is convex on $[\alpha, \alpha + \beta]$ and concave on $[\alpha + \beta, \infty)$.

iii. We have $\limsup_{w\to\infty} J'(w) = 0$.

Proof. (i) Note that following DDE (H), function J(w) is differentiable for $w > \alpha$ and is twice differentiable except at $w = \alpha + \beta$. Taking derivatives on both sides of DDE (H) yields

$$J''(w) = \frac{(\lambda + r - \rho)J'(w) - \lambda J'(w - \beta)}{\rho w + \beta \lambda}, \text{ for}$$
$$w \in (\alpha, \alpha + \beta) \cup (\alpha + \beta, \infty). \tag{C.1}$$

Because J(w) is nondecreasing (recall Lemma B.1), we have $J'(w) \ge 0$ and $J'(w - \beta) \ge 0$ for $w \in (\alpha, \infty)$. As such, we must have $J''(w) \le 0$ if $r \le \rho - \lambda$ except $w = \alpha + \beta$, at which point J(w) is differentiable. Therefore, J(w) is concave on (α, ∞) and continuity of function J(w) extends concavity to $w \in [\alpha, \infty)$.

(ii) Starting from boundary conditions J(w) = 1 for $w \in [0, \alpha)$, DDE (H) yields the following:

$$J(w) = \frac{r}{\lambda + r} \left(\frac{\rho w + \lambda \beta}{\rho \beta + \lambda \beta} \right)^{\frac{\lambda + r}{\rho}} + \frac{\lambda}{\lambda + r}, \text{ for}$$
$$w \in (\alpha, \alpha + \beta),$$
(C.2)

whose first and second derivatives are

$$J'(w) = \frac{r(\rho w + \lambda \beta)^{\frac{\lambda + r - \rho}{\rho}}}{(\rho \beta + \lambda \beta)^{\frac{\lambda + r}{\rho}}}$$
$$J''(w) = \frac{r(\rho w + \lambda \beta)^{\frac{\lambda + r - 2\rho}{\rho}}}{(\rho \beta + \lambda \beta)^{\frac{\lambda + r}{\rho}}} (\lambda + r - \rho).$$
(C.3)

Therefore, if $r > \rho - \lambda$, we must have J''(w) > 0, that is, J(w) is convex on $[\alpha, \alpha + \beta]$ because function J(w) is continuous.

Given that the right-hand-side of (38) is increasing, whereas the left-hand side is decreasing in \bar{r} , it is readily shown that Equation (38) has a unique solution within $(\rho - \lambda, \rho)$. Moreover, for $r \in (\rho - \lambda, \bar{r})$, we have

$$\left(\frac{2\rho+\lambda}{\rho+\lambda}\right)^{\frac{\lambda+r}{\rho}-1} < \frac{\lambda}{\lambda+r-\rho}$$

which implies that $J''((\alpha + \beta)_+) < 0$, that is, we have $(\lambda + r - \rho)J'(\alpha + \beta) < \lambda J'(\alpha_+)$.

To prove that J(w) is concave on $[\alpha + \beta, \infty)$, we only need to show $J''(w) \le 0$ for all $w > \alpha + \beta$. Suppose, on the contrary, there exists a $w > \alpha + \beta$ such that J''(w) > 0. Define

$$\hat{w} := \min\{w | (\lambda + r - \rho)J'(w) \ge \lambda J'(w - \beta) \text{ and } w > \alpha + \beta\}.$$

• If $\hat{w} \in (\alpha + \beta, \alpha + 2\beta]$, we have $J'(\alpha + \beta) \ge J'(\hat{w})$ because J(w) is concave within $[\alpha + \beta, \hat{w}]$. We also have $J'(\beta_+) < J'(\hat{w} - \beta)$ because J(w) is convex on $[\alpha, \alpha + \beta]$. Therefore,

$$\begin{aligned} & (\lambda + r - \rho)J'(\hat{w}) \leq (\lambda + r - \rho)J'(\alpha + \beta) \\ & < \lambda J'(\beta_+) < \lambda J'(\hat{w} - \beta). \end{aligned}$$

• If $\hat{w} > \alpha + 2\beta$, then both $J(\hat{w})$ and $J(\hat{w} - \beta)$ are twice continuously differentiable. As such,

$$J'(\hat{w}) = J'(\hat{w} - \beta) + \int_0^\beta J''(\hat{w} - \beta + x)dx < J'(\hat{w} - \beta),$$

which implies $(\lambda + r - \rho)J'(\hat{w}) < \lambda J'(\hat{w} - \beta)$.

That is, $(\lambda + r - \rho)J'(\hat{w}) \ge \lambda J'(\hat{w} - \beta)$ cannot be true for either case. Therefore, J(w) is concave on $[\alpha + \beta, \infty)$.

(iii) For notational convenience, we let $\ell := \limsup_{w \to \infty} J'(w)$. First, we show that ℓ cannot be positive infinity. Suppose, on the contrary, $\ell = \limsup_{w \to \infty} J'(w) = \infty$. Then there exists an increasing divergent sequence $\{w_n\}_{n \ge 1}$ in $(\alpha + \beta, \infty)$ such that $\lim_{w \to \infty} J'(w_n) = \infty$ and

$$w_n = \arg\max_{w \in [0, w_n]} \{J'(w)\}.$$

Then, for each $n \ge 1$ by mean value theorem, there exists $\hat{w}_n \in (w_n - \beta, w_n)$ such that

$$\begin{aligned} (\rho w_n + \beta \lambda) J'(w_n) \\ &= \lambda [J(w_n) - J(w_n - \beta)] + r J(w_n) \\ &= \lambda \beta J'(\hat{w}_n) + r J(w_n). \end{aligned}$$

Rearranging the equation, one gets

$$J'(\hat{w}_n) = \frac{w_n}{\lambda\beta} \left[\rho J'(w_n) - \frac{r}{w_n} J(w_n) \right] + J'(w_n).$$
(C.4)

Because J(0) = 1 and $\{J'(w_n)\}_n$ is an increasing sequence, there is $J(w_n) - 1 \le w_n J'(w_n)$ by construction. For such *n*, from (C.4) and the fact $J(w_n) \le w_n J'(w_n)$, there is

$$J'(\hat{w}_n) \geq \frac{(\rho - r)w_n J'(w_n)}{\lambda \beta}.$$

Because $J'(w_n) > 0$, an immediate result that follows the previous inequality is that

$$\frac{J'(\hat{w}_n)}{J'(w_n)} \ge \frac{(\rho - r)w_n}{\lambda\beta},$$

which goes to infinity as *n* goes to infinity. Therefore, we can obtain $J'(\hat{w}_n) > J'(w_n)$ eventually, which contradicts the definition of w_n because $\hat{w}_n < w_n$. Thus, we have that ℓ must be finite.

Consider a new increasing and divergent sequence $\{w_n\}_{n\geq 1}$ in $(\alpha + \beta, \infty)$ such that $\lim_{n\to\infty} J'(w_n) = \ell$. Then, for all $n \geq 1$, we can find a constant D such that $J(w_n) \leq \ell w_n + D$. Let $\hat{w}_n \in (w_n - \beta, w_n)$, by substituting $J(w_n) \leq \ell w_n + D$ into the differential equations of J(w) for all $n \geq 1$, we have

$$\rho J'(w_n) - rl \le \frac{\lambda \beta [J'(\hat{w}_n) - J'(w_n)] + rD}{w_n}$$

By letting *n* go to infinity, there is

$$(\rho - r)l \le \lambda\beta \liminf_{n \to \infty} \frac{J'(\hat{w}_n)}{w_n}$$

If l > 0, the preceding inequality implies that ℓ must go to infinity; this contradicts to the fact that l is finite. Therefore, we have $l \le 0$. Given that $J'(w) \ge 0$ for all w (recall Lemma B.1), we must have $\ell = 0$. \Box

Proof of Lemma 4. Taking the first derivative with respect to α , we have

$$f'(\alpha) = -\frac{r\lambda\theta}{\rho}\alpha^{\frac{r}{\rho}-1}\left[1 + \frac{\rho-r}{\rho\alpha}\beta - \left(1 - \frac{\beta}{\alpha}\right)^{\frac{r}{\rho}-1}\right].$$

Consider a function

$$h(x) := 1 + (\rho - r)\beta x - (1 - \rho\beta x)^{\frac{r}{\rho} - 1},$$

$$x \in \left[0, \frac{1}{\rho\beta}\right].$$
(C.5)

We know that h(0) = 0 and

$$h'(x) = (\rho - r)\beta \left[1 - (1 - \rho\beta x)^{\frac{r}{\rho}-2}\right] < 0,$$

where the inequality holds because $r < \rho$. Therefore, we must have h(x) < 0, $\forall x \in (0, 1/(\rho\beta)]$. Consequently,

$$f'(\alpha) = -\frac{r\lambda\theta}{\rho}\alpha^{\frac{r}{\rho}-1}h\left(\frac{1}{\rho\alpha}\right) > 0$$

implying that function $f(\alpha)$ is strictly increasing in $\alpha \in [\beta, \infty)$. Moreover, we have

$$\lim_{\alpha \to \infty} f(\alpha) = m - \lambda \theta \lim_{\alpha \to \infty} \frac{1 - \frac{r\beta}{\rho\alpha} - \left(1 - \frac{\beta}{\alpha}\right)^{\frac{r}{\rho}}}{\alpha^{-\frac{r}{\rho}}}$$
$$= m - \lambda \theta \lim_{\alpha \to \infty} \frac{\frac{r\beta}{\rho\alpha^2} - \frac{r\beta}{\rho\alpha^2} \left(1 - \frac{\beta}{\alpha}\right)^{\frac{r}{\rho}-1}}{-\frac{r}{\rho}\alpha^{-\frac{r}{\rho}-1}}$$
$$= m + \lambda \theta \beta \lim_{\alpha \to \infty} \frac{1 - \left(1 - \frac{\beta}{\alpha}\right)^{\frac{r}{\rho}-1}}{\alpha^{1-\frac{r}{\rho}}} = m.$$

This completes the proof. \Box

Proof of Proposition 3. (i) The definition of α_{θ} directly implies that $f(\alpha_{\theta}) \ge 0$. From the definition of $F_{\theta,\alpha}(w)$, for any $\alpha \ge \beta$, we have

$$F'_{\theta,\alpha}(\alpha_{-}) = \frac{r\theta}{\rho} \alpha^{\frac{r}{\rho}-1} \text{ and}$$
$$F''_{\theta,\alpha}(\alpha_{-}) = \frac{r(r-\rho)}{\rho^{2}} \theta \alpha^{\frac{r}{\rho}-2} < 0.$$

From DDE (H), we have

$$\begin{split} & = \frac{1}{\rho \alpha + \lambda \beta} \Big[(\lambda + r) F_{\theta, \alpha}(\alpha) - \lambda F_{\theta, \alpha}(\alpha - \beta) \Big], \\ & = \frac{1}{\rho \alpha + \lambda \beta} \Big[(\lambda + r) \theta \alpha^{\frac{r}{p}} - \lambda \theta (\alpha - \beta)^{\frac{r}{p}} - m \Big]. \end{split}$$

As such,

$$F'_{\theta,\alpha}(\alpha_{-}) - F'_{\theta,\alpha}(\alpha_{+}) = \frac{1}{\rho\alpha + \beta\lambda} f(\alpha).$$
(C.6)

Consider the case $\alpha = \alpha_{\theta}$. If $f(\beta) \ge 0$, the definition of α_{θ} implies that $f(\alpha) = f(\beta) \ge 0$, that is, $F'_{\theta,\alpha_{\theta}}(\alpha_{\theta-}) \ge F'_{\theta,\alpha_{\theta}}(\alpha_{\theta+})$. Otherwise, we must have $f(\alpha_{\theta}) = 0$ and $F'_{\theta,\alpha_{\theta}}(\alpha_{\theta-}) = F'_{\theta,\alpha_{\theta}}(\alpha_{\theta+})$. Therefore, we have (27).

(ii) When $\alpha_{\theta} > \beta$, we must have $f(\alpha_{\theta}) = 0$. From DDE (H), we have

$$\begin{aligned} F_{\theta,\alpha_{\theta}}^{''}(\alpha_{\theta+}) &= \frac{1}{\rho\alpha_{\theta} + \lambda\beta} \Big[(\lambda + r - \rho) F_{\theta,\alpha_{\theta}}^{'}(\alpha_{\theta+}) \\ &- \lambda F_{\theta,\alpha_{\theta}}^{'}(\alpha_{\theta} - \beta) \Big], \\ &= \frac{\theta r}{\rho(\rho\alpha_{\theta} + \lambda\beta)} \Big[(\lambda + r - \rho) \alpha_{\theta}^{\frac{r}{\rho} - 1} - \lambda (\alpha_{\theta} - \beta)_{\rho}^{\frac{r}{\rho} - 1} \Big]. \end{aligned}$$

As such,

$$\begin{split} F''_{\theta,\alpha_{\theta}}(\alpha_{\theta-}) &- F''_{\theta,\alpha_{\theta}}(\alpha_{\theta+}) \\ &= \frac{\theta r \lambda \alpha_{\theta}^{\frac{r}{\rho}-1}}{\rho(\rho\alpha_{\theta}+\beta\lambda)} \bigg[\bigg(1 - \frac{\beta}{\alpha_{\theta}} \bigg)^{\frac{r}{\rho}-1} - 1 - \frac{\beta(\rho-r)}{\rho\alpha_{\theta}} \bigg] \\ &= -\frac{\theta r \lambda \alpha_{\theta}^{\frac{r}{\rho}-1}}{\rho(\rho\alpha_{\theta}+\beta\lambda)} h\bigg(\frac{1}{\rho\alpha_{\theta}} \bigg) > 0, \end{split}$$

where function h(x) is defined in (C.5). This inequality holds because h(x) < 0 for $\forall x \in (0, 1/(\rho\beta)]$.

(iii) Consider an arbitrary $\alpha \ge \beta$. If $\alpha < \alpha_{\theta}$, we must have $f(\alpha) < f(\alpha_{\theta}) = 0$; which, together with (C.6), implies $F'_{\theta,\alpha}(\alpha_{-}) < F'_{\theta,\alpha}(\alpha_{+})$. Otherwise, if $\alpha > \alpha_{\theta}$, we must have $f(\alpha) > f(\alpha_{\theta}) \ge 0$, that is, $F'_{\theta,\alpha}(\alpha_{-}) > F'_{\theta,\alpha}(\alpha_{+})$. \Box

Proof of Lemma 5. We first show that, for any $w \ge 0$, derivative $F'_{\theta,\alpha_{\theta}}(w_{+})$ is increasing in θ . To do so, we define $g(w, \theta) = F_{\theta,\alpha_{\theta}}(w)$, and show $\frac{\partial g(w,\theta)}{\partial \theta}$ is well defined and strictly increasing in w.

• For $\forall w \in [0, \alpha_{\theta})$, we have $\frac{\partial g(w, \theta)}{\partial \theta} = w^{\frac{r}{\theta}}$, which is strictly increasing in w.

• For $w = \alpha_{\theta}$,

$$\frac{\partial g(w, \theta_{+})}{\partial \theta} \Big|_{w=\alpha_{\theta}} = \lim_{\varepsilon \downarrow 0} \frac{F_{\theta+\varepsilon,\alpha_{\theta}}(\alpha_{\theta}) - F_{\theta,\alpha_{\theta}}(\alpha_{\theta})}{\varepsilon} + \frac{F_{\theta,\alpha_{(\theta+\varepsilon)}}(\alpha_{\theta}) - F_{\theta,\alpha_{\theta}}(\alpha_{\theta})}{\varepsilon} \times \frac{d\alpha_{\theta}}{d\theta} = \lim_{\varepsilon \downarrow 0} \frac{(\theta+\varepsilon)\alpha_{\theta}^{\frac{r}{\rho}} - \theta\alpha_{\theta}^{\frac{r}{\rho}}}{\varepsilon} = \alpha_{\theta}^{\frac{r}{\rho}} = \frac{\partial g(w, \theta_{-})}{\partial \theta} \Big|_{w=\alpha_{\theta}},$$

where the second equality follows from $F_{\theta,\alpha_{\{\theta+\varepsilon\}}}(\alpha_{\theta}) = F_{\theta,\alpha_{\theta}}(\alpha_{\theta})$ because $\alpha_{\theta+\varepsilon} \ge \alpha_{\theta}$ for any $\varepsilon > 0$.

• For $w > \alpha_{\theta}$, note that $g(w, \theta) = F_{\theta, \alpha_{\theta}}(w)$ is a solution to DDE (H), parameterized by θ . That is, the following equality also holds:

$$\begin{split} &(\lambda+r)g(w,\theta)\\ &=\lambda g(w-\beta,\theta)+\left(\rho w+\lambda\beta\right)\frac{\partial g(w,\theta)}{\partial w}. \end{split}$$

Taking derivatives w.r.t. $\boldsymbol{\theta}$ on both sides of the preceding equation, we have

$$(\lambda + r)\frac{\partial g(w, \theta)}{\partial \theta}$$
$$= \lambda \frac{\partial g(w - \beta, \theta)}{\partial \theta} + (\rho w + \lambda \beta) \frac{\partial g(w, \theta)}{\partial w \partial \theta}$$

which implies that $\frac{\partial g(w,\theta)}{\partial \theta}$ satisfies DDE (H) as well. Because $\frac{\partial g(0,\theta)}{\partial \theta} > 0$ and $\frac{\partial g(w,\theta)}{\partial \theta}$ is increasing on $w \in [0, \alpha_{\theta}]$, we immediately know that $\frac{\partial g(w,\theta)}{\partial \theta}$ is increasing on $w \in [\alpha_{\theta}, \infty)$ from Lemma B.1.

Summarizing these three cases, we conclude that $\frac{\partial_{\mathcal{C}}(w,\theta)}{\partial \theta}$ is well-defined and strictly increasing in w. Therefore, derivative $F'_{\theta,\alpha_{\theta}}(w_{+})$ is increasing in θ . As a result, we know that $\inf_{w} F'_{\theta,\alpha_{\theta}}(w_{+})$ is increasing in θ . On the one hand, when θ is sufficiently large (i.e., $\theta \to \infty$), we know that α_{θ} approaches ∞ as well. As such,

$$\begin{split} &\lim_{\theta \to \infty} \left\{ \inf_{w > \beta} F'_{\theta, \alpha_{\theta}}(w) \right\} \\ &= \lim_{\theta \to \infty} \left\{ \inf_{w > \beta} \left[\frac{r\theta}{\rho} w^{\frac{r}{\rho} - 1}_{\rho} \right] \right\} > 0 \end{split}$$

On the other hand, when $\theta = 0$, we have $\alpha_{\theta} = \beta$. It is clear that

$$F_{0,\beta}(w) = -\frac{m}{r}J(w)$$
 for $\forall w \ge 0$.

As such,

$$\inf_{w>\beta} F'_{0,\beta}(w) = -\frac{m}{r} \sup_{w>\beta} J'(w) \le -1,$$

where the last inequality holds because J(w) is nondecreasing and $m \ge \overline{m}$.

Therefore, there exists a unique $\bar{\theta} \ge 0$ such that $\inf_{w} F'_{\bar{\theta}, \alpha_{\bar{\alpha}}}(w_{+}) = -1$.

Using similar techniques as in the proof of Proposition 1(i), we can show that $\bar{\theta} < \frac{m}{r}\beta^{-\frac{r}{p}}$. We only need to modify the definition of functions $G_1(w)$ and $G_2(w)$ as

$$G_1(w) = w^{\frac{r}{p}} \text{ and } G_2(w) = \left(\frac{w}{\beta}\right)^{\frac{p}{p}} -1, \text{ for}$$
$$w \in [0, \alpha_{\bar{\theta}}].$$

We omit the detailed proof to avoid redundancy.

Finally, we show that \bar{w}^* , which is determined by (28), is finite. Notice that

$$\bar{\theta} = -\frac{1 - mJ'(\bar{w}^*)/r}{G_1'(\bar{w}^*)}.$$

If, on the contrary, $\bar{w}^* \to \infty$, the preceding equation implies that $\bar{\theta} < 0$ because $\limsup_{w\to\infty} J'(w) = 0$ (recall Lemma C.1(iii)) and $G_1(w)$ is increasing (recall Lemma B.1). This contradicts the fact that $\bar{\theta} > 0$. Therefore, \bar{w}^* must be finite. \Box

To prove Proposition 4, we first show a more general result, as presented in the following lemma.

Lemma C.2. For any $\theta \ge \overline{\theta}$, define $\hat{w} := \inf\{\arg\inf_{w>\beta} F'_{\theta}, \alpha_{\theta}(w)\}$. Then function $F_{\theta,\alpha_{\theta}}(w)$ is strictly concave on $w \in [0, \hat{w}]$.

Proof. First, $F_{\theta,\alpha_{\theta}}(w)$ is nondecreasing and concave in $[0, \alpha_{\theta})$ because $\theta \ge 0$ (recall Lemma 5). Next, from Proposition 3(i)

we know that $F'_{\theta,\alpha_{\theta}}(\alpha_{\theta-}) \ge F'_{\theta,\alpha_{\theta}}(\alpha_{\theta+})$. Therefore, in order to show that F(w) is concave, we only need to show that $F_{\theta,\alpha_{\theta}}(w)$ is concave on $[\alpha_{\theta}, \hat{w}]$.

Recall that $F_{\theta,\alpha_{\theta}}(w)$ is continuous on $[0, \hat{w}]$, and $F'_{\theta,\alpha_{\theta}}(w)$ is continuous on $(\alpha_{\theta}, \hat{w}]$. By the definition of $\bar{\theta}, \bar{w}$ in (28) and given the fact that $\theta \geq \bar{\theta}$, we know that $F'_{\theta,\alpha_{\theta}}(\hat{w}) \geq -1$. Therefore, $F_{\theta,\alpha_{\theta}}(w) + w$ is increasing on $[0, \hat{w}]$.

We prove by contradiction. Suppose, on the contrary, there exists some $w \in (\alpha_{\theta}, \hat{w})$, such that $F'_{\theta,\alpha_{\theta}}(w) \ge 0$ (i.e., $F'_{\theta,\alpha_{\theta}}(w)$ is increasing at w). Define

$$w_1 := \min \left\{ w | F_{\theta, \alpha_{\theta}}^{''}(w) \ge 0 \text{ and } \alpha_{\theta} \le w < \hat{w} \right\}.$$

Furthermore, $F'_{\theta,\alpha_{\theta}}(w)$ must be decreasing on some interval within (w_1, \hat{w}) in order for $F'_{\theta,\alpha_{\theta}}(w)$ to drop to -1 at \hat{w} . As such, define

$$w_2 := \inf \Big\{ w | F_{\theta, \alpha_{\theta}}^{''}(w) < 0 \text{ and } w_1 < w < \hat{w} \Big\}.$$

We claim that $F'_{\theta,\alpha_{\theta}}(w)$ must be continuous at w_2 . To show this, recall that, only when $\alpha_{\theta} = \beta$ and $F'_{\theta,\alpha_{\theta}}(\beta_{-}) > F'_{\theta,\alpha_{\theta}}(\beta_{+})$, would $F''_{\theta,\alpha_{\theta}}(w)$ be discontinuous at point $w = \beta$ or $w = 2\beta$. Suppose, on the contrary, $F''_{\theta,\alpha_{\theta}}(w)$ is not continuous at w_2 , then we must have $\alpha_{\theta} = \beta$ and $w_2 = 2\beta$. As such, differentiating (H) would yield

$$F_{\theta,\alpha_{\theta}}^{''}(\{2\beta\}_{-}) = \frac{(\lambda + r - \rho)F_{\theta,\alpha_{\theta}}^{'}(2\beta) - \lambda F_{\theta,\alpha_{\theta}}^{'}(\beta_{-})}{2\rho\beta + \beta\lambda}$$
$$< \frac{(\lambda + r - \rho)F_{\theta,\alpha_{\theta}}^{'}(2\beta) - \lambda F_{\theta,\alpha_{\theta}}^{'}(\beta_{+})}{2\rho\beta + \beta\lambda}$$
$$= F_{\theta,\alpha_{\theta}}^{''}(\{2\beta\}_{+}),$$

where the inequality follows from Proposition 3(i), and the assumption that $F'_{\theta,\alpha_{\theta}}(w)$ is not continuous at β .

Because $F'_{\theta,\alpha_{\theta}}(w)$ is increasing on $w \in [w_1, w_2]$, we must have $F'_{\theta,\alpha_{\theta}}(\{2\beta\}_{-}) > 0$. Therefore,

$$F_{\theta,\alpha_{\theta}}^{''}(2\beta_{+}) > F_{\theta,\alpha_{\theta}}^{''}(2\beta_{-}) > 0,$$

which is in contradiction to the definition of w_2 . Therefore, $F_{\theta,\alpha_{\theta}}^{''}(w)$ must be continuous at w_2 , and as a result, we must have $F_{\theta,\alpha_{\theta}}^{''}(w_2) = 0$. Consequently, DDE (H) yields

$$(\rho w_2 + \beta \lambda) F_{\theta, \alpha_{\theta}}^{\prime \prime \prime}(w_2) = -\lambda F_{\theta, \alpha_{\theta}}^{\prime \prime}(w_2 - \beta) \le 0,$$

which implies that $F'_{\theta,\alpha_{\theta}}(w_2 - \beta) \ge 0$. Therefore, we must have $w_2 - \beta \ge w_1$ because $F'_{\theta,\alpha_{\theta}}(w)$ is decreasing for $w < w_1$. As such, $F'_{\theta,\alpha_{\theta}}(w)$ is increasing on $[w_2 - \beta, w_2]$. Differentiating (H) at w_2 yields

$$\lambda \Big[F'_{\theta,\alpha_{\theta}}(w_2) - F'_{\theta,\alpha_{\theta}}(w_2 - \beta) \Big]$$

= $(\rho - r)F'_{\theta,\alpha_{\theta}}(w_2) \ge 0,$

which implies that $F'_{\theta,\alpha_{\theta}}(w_2) \ge 0$.

On the one hand, we have

$$\rho w_{2} + \lambda \beta \Big[F'_{\theta,\alpha_{\theta}}(w_{2}) + 1 \Big]$$

$$\leq (\rho w_{2} + \lambda \beta) \Big[F'_{\theta,\alpha_{\theta}}(w_{2}) + 1 \Big]$$

$$= \lambda \Big[F_{\theta,\alpha_{\theta}}(w_{2}) - F_{\theta,\alpha_{\theta}}(w_{2} - \beta) \Big] + r F_{\theta,\alpha_{\theta}}(w_{2})$$

$$+ \rho w_{2} + \beta \lambda$$

$$\leq \lambda \beta F'_{\theta,\alpha_{\theta}}(w_2) + r F_{\theta,\alpha_{\theta}}(w_2) + \rho w_2 + \beta \lambda, \tag{C.7}$$

where the second inequality holds because $F_{\theta,\alpha_{\theta}}(w)$ is convex within $w \in [w_2 - \beta, w_2]$. Rearranging inequality (C.7) yields $F_{\theta,\alpha_{\theta}}(w_2) \ge 0$, which leads to

$$F_{\theta,\alpha_{\theta}}(\hat{w}) + \hat{w} > F_{\theta,\alpha_{\theta}}(w_2) + w_2 \ge w_2 > 0,$$
 (C.8)

because $F_{\theta,\alpha_{\theta}}(w) + w$ is increasing on $[0, \hat{w}]$.

On the other hand, Equation (U), in which \bar{w} is set as \hat{w} , is equivalent to

$$\lambda \left[F_{\theta,\alpha_{\theta}}(\hat{w}) - F_{\theta,\alpha_{\theta}}(\hat{w} - \beta) \right] + r F_{\theta,\alpha_{\theta}}(\hat{w}) + \rho \hat{w} + \beta \lambda = 0.$$

Because $F_{\theta,\alpha_{\theta}}(\hat{w}) > F_{\theta,\alpha_{\theta}}(\hat{w} - \beta) - \beta$, we must have $rF_{\theta,\alpha_{\theta}}(\hat{w}) + \rho\hat{w} < 0$. Consequently,

$$F_{\theta,\alpha_\theta}(\hat{w}) + \hat{w} < -\frac{\rho-r}{r}\hat{w} < 0,$$

which is in contradiction to (C.8). Therefore, $F_{\theta,\alpha_{\theta}}^{"}(w) < 0$ for $\forall w \in (\alpha_{\bar{\theta}}, \hat{w})$. To summarize, $F_{\theta,\alpha_{\theta}}(w)$ is strictly concave within $[0, \hat{w}]$. \Box

Proof of Proposition 4. (i) First note that function $F_{\bar{\theta},\alpha_{\bar{\theta}}}(w)$ is strictly concave on $w \in [0, \alpha_{\bar{\theta}})$. The concavity proof on $w \in [\alpha_{\bar{\theta}}, \bar{w}^*]$ is a special case of the concavity proof in Lemma C.2 with $\theta = \bar{\theta}$ and $\hat{w} = \bar{w}^*$. Therefore, we omit the proof of this part to avoid redundancy.

(ii) We prove by contradiction. Suppose, on the contrary, there exists a $w \in (\alpha_{\bar{\theta}}, \bar{w}^*]$ such that

$$rF_{\bar{\theta},\alpha_{\bar{\theta}}}(w) \le \rho w F'_{\bar{\theta},\alpha_{\bar{\theta}}}(w) - m.$$

Let

$$\hat{w} := \min \left\{ w | rF_{\bar{\theta}, \alpha_{\bar{\theta}}}(w) \le \rho w F'_{\bar{\theta}, \alpha_{\bar{\theta}}}(w) - m \text{ and} \right.$$

$$\alpha_{\bar{\theta}} < w \le \bar{w}^* \right\}.$$
(C.9)

Therefore, we must have the following relationship:

$$rF_{\bar{\theta},\alpha_{\bar{\theta}}}(w) > \rho w F'_{\bar{\theta},\alpha_{\bar{\theta}}}(w) - m, \quad \text{for } w \in (\alpha_{\bar{\theta}}, \hat{w}). \tag{C.10}$$

For notational convenience, we let $\hat{z} := F_{\bar{\theta}, \alpha_{\bar{\theta}}}(\hat{w})$. Figure C.1 illustrates various notations in this proof. We first demonstrate that $F_{\bar{\theta}, \infty}(\hat{w}) > \hat{z}$ by showing $F'_{\bar{\theta}, \alpha_{\bar{\theta}}}(w) < F'_{\bar{\theta}, \infty}(w)$ for all $w \in (\alpha_{\bar{\theta}}, \hat{w}]$ using contradiction. Suppose there exists \tilde{w} such that

$$\begin{split} \tilde{w} &:= \min \Big\{ w | F_{\bar{\theta}, \alpha_{\bar{\theta}}}'(w) \ge F_{\bar{\theta}, \infty}'(w) \text{ and } \\ \alpha_{\bar{\theta}} < w \le \hat{w} \Big\}. \end{split}$$

On one hand, when $\alpha_{\bar{\theta}} = \beta$, from Proposition 3(i), we have $F'_{\bar{\theta},\alpha_{\bar{\theta}}}(\alpha_{\bar{\theta}+}) \leq F'_{\bar{\theta},\alpha_{\bar{\theta}}}(\alpha_{\bar{\theta}-})$. On the other hand, when $\alpha_{\bar{\theta}} > \beta$, from Proposition 3(ii), we have $F'_{\bar{\theta},\alpha_{\bar{\theta}}}(\alpha_{\bar{\theta}+}) < F'_{\bar{\theta},\alpha_{\bar{\theta}}}(\alpha_{\bar{\theta}-}) < 0$. Either of these cases leads to the result that $F'_{\bar{\theta},\alpha_{\bar{\theta}}}(w) < F'_{\bar{\theta},\infty}(w)$ for all $w \in (\alpha_{\bar{\theta}}, \tilde{w})$, according to the definition of \tilde{w} . Thus, the following relationship holds as well:

$$F_{\bar{\theta},\alpha_{\bar{\theta}}}(w) < F_{\bar{\theta},\infty}(w), \ w \in \left(\alpha_{\bar{\theta}},\tilde{w}\right].$$
(C.11)

Then considering (C.10) at $w = \tilde{w}$, we have

$$rF_{\bar{\theta},\alpha_{\bar{\sigma}}}(\tilde{w}) > \rho \tilde{w}F'_{\bar{\theta},\infty}(\tilde{w}) - m = rF_{\bar{\theta},\infty}(\tilde{w}),$$

where the first inequality follows from $F'_{\partial,\alpha_{\bar{\partial}}}(\tilde{w}) \ge F'_{\partial,\infty}(\tilde{w})$ and the next equality follows from (17). Clearly, this is a contradiction to the relationship in (C.11). Therefore, we must have $F'_{\partial,\alpha_{\bar{\partial}}}(w) < F'_{\partial,\infty}(w)$ for all $w \in (\alpha_{\bar{\partial}}, \hat{w})$. As a result, we have $F_{\bar{\partial},\hat{w}}(\hat{w}) = F_{\bar{\partial},\infty}(\hat{w}) > \hat{z}$.

Let

$$\hat{\theta} := \left(\hat{z} + \frac{m}{r}\right)\hat{w}^{-\frac{r}{\rho}}.$$

Then, we must have

$$\hat{\theta} < \left(F_{\bar{\theta},\hat{w}}(\hat{w}) + \frac{m}{r}\right)w^{-\frac{r}{\rho}} = \bar{\theta}.$$

On the one hand, from Equation (C.9), we have

$$F'_{\bar{\theta},\alpha_{\bar{\theta}}}(\hat{w}) \ge \frac{r\hat{z}+m}{\rho\hat{w}} = F'_{\hat{\theta},\hat{w}}(\hat{w}_{-}).$$
(C.12)

On the other hand, given that function $f(\alpha)$ is strictly increasing, we have $\alpha_{\hat{\theta}} < \alpha_{\bar{\theta}}$. Therefore, $\hat{w} > \alpha_{\bar{\theta}} > \alpha_{\hat{\theta}}$. By Proposition 3(i), we have

$$F'_{\hat{\theta},\hat{w}}(\hat{w}_{+}) < F'_{\hat{\theta},\hat{w}}(\hat{w}_{-}).$$
 (C.13)

From DDE (H), we know that

$$\begin{split} F'_{\hat{\theta},\hat{w}}(\hat{w}_{+}) &= \frac{1}{\rho\hat{w} + \beta\lambda} \Big[(\lambda + r) F_{\hat{\theta},\hat{w}}(\hat{w}) - \lambda F_{\hat{\theta},\hat{w}}(\hat{w} - \beta) \Big], \\ F'_{\bar{\theta},\alpha_{\hat{\theta}}}(\hat{w}) &= \frac{1}{\rho\hat{w} + \beta\lambda} \Big[(\lambda + r) F_{\bar{\theta},\alpha_{\hat{\theta}}}(\hat{w}) - \lambda F_{\bar{\theta},\alpha_{\hat{\theta}}}(\hat{w} - \beta) \Big]. \end{split}$$

Because $F_{\hat{\theta},\hat{w}}(\hat{w}) = F_{\bar{\theta},\alpha_{\bar{\theta}}}(\hat{w})$ and $F_{\hat{\theta},\hat{w}}(\hat{w}-\beta) < F_{\bar{\theta},\alpha_{\bar{\theta}}}(\hat{w}-\beta)$, we must have

$$F'_{\hat{\theta},\hat{w}}(\hat{w}_{+}) > F'_{\hat{\theta},\alpha_{\bar{\alpha}}}(\hat{w}). \tag{C.14}$$

Combining (C.13) and (C.14) yields

$$F'_{\hat{\theta},\hat{w}}(\hat{w}_{-}) > F'_{\hat{\theta},\hat{w}}(\hat{w}_{+}) > F'_{\bar{\theta},\alpha_{\bar{\theta}}}(\hat{w})$$

Figure C.1. (Color online) Illustration of $F_{\bar{\theta},\alpha_{\bar{\theta}}}(w)$, $F_{\bar{\theta},\infty}(w)$, and $F_{\bar{\theta},\infty}(w)$



which contradicts (C.12). Therefore, there does not exist a $w > a_{\bar{\theta}}$ such that (C.9) holds, which completes the proof. \Box

To prove Theorem 2, we first present the following lemma.

Lemma C.3. Consider a concave function F(w) that satisfies (H) with $\alpha \ge \beta$ and (U), in which \bar{w} is set as \bar{w}^* , at $w = \bar{w}^* \ge \alpha$, and is linear with slope -1 on $[\bar{w}^*, \infty)$. For any $w \ge \bar{w}^*$,

$$\psi(w) := \lambda F(w - \beta) - (\lambda + r)F(w) - (\rho w + \lambda \beta) \le 0.$$
(C.15)

Proof. Taking the first derivative of $\psi(w)$ yields

$$\psi'(w) = \lambda F'(w - \beta) - (\lambda + r)F'(w) - \rho$$

= $\lambda F'(w - \beta) + \lambda + r - \rho$
 $\leq \lambda F'(\bar{w} - \beta) + \lambda + r - \rho,$

where the second equality follows from F'(w) = -1 for $w \ge \overline{w}^*$ and the last inequality follows from concavity of F(w). Condition (H) implies that

$$\begin{aligned} \lambda F'(\bar{w}^* - \beta) \\ &= (\lambda + r - \rho)F'(\bar{w}^*) - (\rho\bar{w}^* + \beta\lambda)F''(\bar{w}^*_+). \end{aligned}$$

Note that, from the definition of \bar{w}^* in (28), we have

$$\begin{cases} F''(\bar{w}_{+}^{*}) > 0, & \text{if } F''(\bar{w}_{-}^{*}) < F''(\bar{w}_{+}^{*}), \\ F''(\bar{w}_{+}^{*}) = 0, & \text{if } F''(\bar{w}_{-}^{*}) = F''(\bar{w}_{+}^{*}). \end{cases}$$
(C.16)

Therefore,

$$\psi'(w) \le (\lambda + r - \rho)[1 + F'(\bar{w}^*)] - (\rho \bar{w}^* + \beta \lambda) F''(\bar{w}^*_+) \le 0,$$

where the last inequality follows (C.16). Therefore, $\psi(w)$ is decreasing in $w \in [\bar{w}^*, \infty)$. As such, for any $w \ge \bar{w}^*$,

$$\psi(w) \le \psi(\bar{w}^*) = \lambda F(\bar{w}^* - \beta) - (\lambda + r)F(\bar{w}^*) - (\rho\bar{w}^* + \lambda\beta) = 0,$$

where we have used equality (U). This completes the proof. $\hfill\square$

Proof of Theorem 2. The proof is parallel to those of Proposition 2 and Theorem 1. Note that the proof for $F(w^*) = U(\Gamma_d(w^*; \alpha_{\bar{\theta}}, \bar{w}^*))$ is exactly the same as that of Proposition 2 under contract $\Gamma_d(w^*; \beta, \bar{w})$ except that the switching point is $\alpha_{\bar{\theta}}$ instead of β . We omit the proof of this part to avoid redundancy. In the following, we only show that $U(\Gamma) \leq F(w)$ for any incentive compatible contract Γ .

From Proposition 4, we know that F(w) is concave and $F'(w) \ge -1$. Therefore, we only need to show $\Psi_t \le 0$ holds almost surely (recall Lemma 3).

Recall the inequality (B.12). We consider the following four cases. (i) When $W_t < \beta$, we know that the principal monitors the agent (i.e., $m_t = m$). Following (17), we have

$$\Psi_t \le \rho W_t F'(W_t) - rF(W_t) - m = 0.$$

(ii) When $\beta \le W_t \le \alpha_{\bar{\theta}}$, substituting (17) into inequality (B.12) yields

$$\Psi_t \le m - m_t + \lambda \Phi (W_t, y_t H_t^s + (1 - y_t) H_t^n) - \lambda F(W_t).$$

• If the principal does not monitor at time *t* (i.e., $m_t = 0$), condition (IC) indicates that $y_t H_t^s + (1 - y_t) H_t^n \ge \beta$. Following Lemma B.3, we have,

$$\begin{split} \Psi_t &\leq m - m_t + \lambda \Phi(W_t, \beta) - \lambda F(W_t) \\ &= m + \lambda \bar{\theta} W_t^{\frac{r}{\rho}} \left[\left(\frac{\beta r}{\rho W_t} - 1 \right) + \left(1 - \frac{\beta}{W_t} \right)^{\frac{r}{\rho}} \right] \\ &= f(W_t) < 0, \end{split}$$

in which we have used Equation (L) and the fact that $f(W_t)$ is increasing with $f(\alpha_{\bar{\theta}}) = 0$.

• If the principal conducts monitoring at time t (i.e., $m_t = m$), considering Lemma B.3, inequality (B.12) yields

$$\Psi_t \le \lambda \Phi(W_t, 0) - \lambda F(W_t) = \lambda F(W_t) - \lambda F(W_t) = 0.$$

(iii) When $\alpha_{\bar{\theta}} < W_t < \bar{w}^*$, substituting (H) into inequality (B.12) yields

$$\Psi_t \le \lambda \Phi (W_t, y_t H_t^s + (1 - y_t) H_t^n) - \lambda \beta F'(W_t) - \lambda F(W_t - \beta) - m_t.$$

• If the principal does not monitor at time *t* (i.e., $m_t = 0$), we must have $y_t H_t^s + (1 - y_t) H_t^n \ge \beta$. Lemma B.3 implies

$$\Psi_t \leq \lambda \Phi(W_t, \beta) - \lambda \beta F'(W_t) - \lambda F(W_t - \beta) = 0.$$

• If the principal conducts monitoring at time t (i.e., $m_t = m$), Lemma B.3 implies

$$\begin{split} \Psi_t &\leq \lambda \Phi(W_t, 0) - \lambda \beta F'(W_t) - \lambda F(W_t - \beta) - m \\ &= -rF(W_t) + \rho W_t F'(W_t) - m < 0, \end{split}$$

where the last inequality follows from Proposition 4(ii). (iv) When $W_t \ge \bar{w}^*$, we must have $F'(W_t) = -1$, and in-

equality (B.12) reduces to

$$\Psi_t \le \lambda \Phi (W_t, y_t H_t^s + (1 - y_t) H_t^n) - \rho W_t$$
$$- (\lambda + r) F(W_t) - m_t.$$

• If the principal does not monitor at time t (i.e., $m_t = 0$), condition (IC) and Lemma B.3 imply

$$\Psi_t \le \lambda \Phi(W_t, \beta) - \rho W_t - (\lambda + r)F(W_t) = \psi(W_t) \le 0.$$

• If the principal conducts monitoring at time t (i.e., $m_t = m$), Lemma B.3 implies

$$\begin{split} \Psi_t &\leq \lambda \Phi(W_t, 0) - \rho W_t - (\lambda + r) F(W_t) - m \\ &= -rF(W_t) - \rho W_t - m \\ &\leq -rF(\alpha_{\bar{\theta}}) - \rho \alpha_{\bar{\theta}} - m, \end{split}$$

where the second inequality holds because $-rF(w) - \rho w$ is decreasing in *w*. As such, by considering Equation (17), we have

$$\Psi_t \le -\rho \alpha_{\bar{\theta}} \left[1 + F'(\alpha_{\bar{\theta}-}) \right] \le 0.$$

To summarize, we know that $\Psi_t \leq 0$ holds for all the possible cases. This completes the proof. \Box

Proof of Proposition 5. First, following DDE (H), for any θ , we have the following closed-form solution for $F_{\theta}(w)$ for $w \in [\beta, 2\beta]$:

$$F_{\theta}(w) = \mathcal{K}_{\theta} \left(\rho w + \beta \lambda \right)^{\frac{\lambda + r}{\rho}} - \frac{\lambda m}{r(\lambda + r)} + \frac{\theta \lambda \left[\beta \left(\rho - r \right) + (\lambda + r) w \right]}{(\lambda + r)(\lambda + r - \rho)}, \quad (C.17)$$

where the θ -dependent parameter \mathscr{K}_{θ} is defined as

$$\mathcal{H}_{\theta} = -\frac{1}{\lambda + r} \left[m + \frac{\theta \beta (\rho - r)(2\lambda + r)}{\lambda + r - \rho} \right] \\ \times \left(\beta (\rho + \lambda) \right)^{\frac{-\lambda + r}{\rho}}.$$
(C.18)

As such, when

$$\theta \geq \underline{\theta} := -\frac{\lambda + r - \rho}{\lambda},$$

we have

$$\begin{aligned} \mathscr{K}_{\theta} (\beta(\rho+\lambda))^{\frac{\lambda+r}{\rho}} \\ \leq & -\frac{1}{\lambda+r} \left[m - \frac{(\rho-r)\beta(2\lambda+r)}{\lambda} \right] < 0, \end{aligned} \tag{C.19}$$

where the second inequality holds because $m > \hat{m}$.

Second, we show that, for any $w \ge 0$, derivative $F'_{\theta}(w)$ is increasing in θ . To do so, consider the following decomposition of $F_{\theta}(w)$,

$$F_{\theta}(w) = \theta G(w) - \frac{m}{r} J(w),$$

where function G(w) satisfies DDE (H) with boundary condition G(w) = w for all $w \in [0, \beta)$, and function J(w) is defined in Lemma 5. Note that, because G'(w) > 0 for $w \in (0, \beta)$, Lemma B.1 implies that G'(w) > 0 for all $w \in (\beta, \infty)$, which further implies that

$$\frac{\partial F_{\theta}'(w)}{\partial \theta} = G'(w) > 0.$$

On the one hand, when $\theta = 0$, it is clear that

$$F_0(w) = -\frac{m}{r}J(w)$$
, for $w \ge 0$

and therefore,

$$\inf_{w \ge \beta} F'_0(w) = -\frac{m}{r} \sup_{w \ge \beta} J'(w_+) \ge -\frac{\bar{m}}{r} \sup_{w \ge \beta} J'(w_+)$$
$$= -1,$$

where the inequality holds because J(w) is nondecreasing and $m \leq \overline{m}$.

On the other hand, when $\theta = \underline{\theta}$, (C.17) implies that

$$F_{\theta}'(\beta_{+}) = \mathscr{K}_{\underline{\theta}}(\lambda + r)(\rho\beta + \lambda\beta)^{\frac{\lambda+r}{\rho}-1} - 1 < -1,$$

where the inequality holds because $\Re_{\underline{\theta}} < 0$. Therefore,

$$\inf_{w \in \theta} F'_{\theta}(w) \le F'_{\theta}(\beta_{+}) < -1.$$

As such, there exists a unique $\bar{\theta} \in (\underline{\theta}, 0]$ such that $\inf_{w > \beta} F'_{\bar{\theta}}(w) = -1$.

From (C.19), we know that $\mathscr{X}_{\bar{\theta}} < 0$. Given that $r > \rho - \lambda$ when $m \in [\hat{m}, \bar{m}), F_{\bar{\theta}}(w)$ is strictly concave within $[\beta, 2\beta]$, that is, $F'_{\bar{\theta}}(w)$ is strictly decreasing in $(\beta, 2\beta]$. By the definitions of $\bar{\theta}$ and \bar{w} , we must have $\bar{w} \ge 2\beta$. This completes the proof. \Box

Proof of Theorem 3. We first remark that function F(w) is concave. This is obviously true if $m \in [0, \underline{m})$ or $m \in [\underline{m}, \hat{m})$. For $m \in [\hat{m}, \overline{m})$, the concavity proof of F(w) is similar to those of Proposition 1 and Proposition 4 except the slight difference in showing that $F'(\beta_-) \ge F'(\beta_+)$. We omit the detailed proof to avoid redundancy.

Next, we show that $U(\Gamma) \leq F(w)$ for any incentive compatible contract Γ and $\forall w \geq 0$. To do so, we only need to show $\Psi_t \leq 0$ holds almost surely (recall Lemma 3). Given that F(w) takes different forms, depending on the value of *m*, we consider three cases in the following.

Case (a). If $m \in [0, \underline{m})$, given that F(w) is linear and F'(w) = -1 for all $w \ge 0$ (recall Equation (32)), we have

$$\Psi_t = -\rho W_t - rF(W_t) - m_t = -(\rho - r)W_t + (m - m_t).$$

Therefore,

a.1. If $m_t = m$, we have $\Psi_t \le 0$ because $\rho \ge r$. a.2. If $m_t = 0$, Corollary 1 implies $W_t \ge \beta$; consequently,

$$\Psi_t = -(\rho - r)W_t + m \le -(\rho - r)\beta + m \le 0,$$

where the inequality holds because $m \leq \underline{m}$.

Therefore, $\Psi_t \leq 0$ holds almost surely. This completes the proof for $m \in [0, \underline{m}]$.

Case (b). If $m \in [\underline{m}, \hat{m})$, recall that F(w) takes the piecewise linear form of (33). We consider the following two cases.

b.1. When $W_t < \beta$, the principal monitors the agent (i.e., $m_t = m$). Following inequality (B.12), we have

$$\begin{split} \Psi_t &\leq \rho W_t F'(W_t) - rF(W_t) - m \\ &= -\rho W_t \bigg[1 - \frac{(\rho - r)\beta - m}{(\lambda + r)\beta} \bigg] \\ &+ r \bigg[1 - \frac{(\rho - r)\beta - m}{(\lambda + r)\beta} \bigg] W_t \\ &= -(\rho - r) W_t \bigg[1 - \frac{(\rho - r)\beta - m}{(\lambda + r)\beta} \bigg] \leq 0, \end{split}$$

where the second inequality holds because $m \ge \underline{m}$.

b.2. When $W_t \ge \beta$, we have $F'(W_t) = -1$. If the principal does not monitor at time *t* (i.e., $m_t = 0$), considering the (IC) condition and Lemma C.3, we have

$$\begin{split} \Psi_t &\leq \lambda \Phi \big(W_t, \beta \big) + F'(W_t) \rho W_t - (\lambda + r) F(W_t) \\ &= \lambda F \big(W_t - \beta \big) - (\lambda + r) F(W_t) - \big(\rho W_t + \beta \lambda \big) \\ &= \psi(W_t), \end{split}$$

where function $\psi(w)$ is decreasing in $w \in [\beta, \infty)$. As such, we have

$$\Psi_t \le \psi(\beta) = \lambda F(0) - (\lambda + r)F(\beta) - (\rho\beta + \beta\lambda) = 0.$$

If the principal conducts monitoring at time t (i.e., $m_t = m$), Lemma B.3 implies that

$$\begin{aligned} \Psi_t &\leq \rho W_t F'(W_t) - rF(W_t) - m \\ &= -\rho W_t - rF(W_t) - m \\ &\leq -\rho\beta - rF(\beta) - m < 0. \end{aligned}$$

In summary, we always have $\Psi_t \leq 0$, which completes the proof for $m \in [\underline{m}, \hat{m})$.

Case (c). If $m \in [\hat{m}, \bar{m})$, we consider the following three cases.

c.1. When $W_t < \beta$, the principal must monitor the agent (i.e., $m_t = m$). From inequality (B.12) we have

$$\begin{split} \Psi_t &\leq \rho W_t F'(W_t) - r F(W_t) - m \\ &= \rho W_t \bar{\theta} - r \Big(\bar{\theta} W_t - \frac{m}{r} \Big) - m = \bar{\theta} \big(\rho - r \big) W_t \leq 0, \end{split}$$

where we have used Equation (L_l) and the fact that $\bar{\theta} \leq 0$.

c.2. When $\beta \le W_t < \bar{w}^*$, substituting Equation (H) into inequality (B.12) yields

$$\Psi_t \le \lambda \Phi(W_t, y_t H_t^s + (1 - y_t) H_t^n) - \lambda \beta F'(W_t) - \lambda F(W_t - \beta) - m_t.$$

If the principal does not monitor at time *t* (i.e., $m_t = 0$), we must have $y_t H_t^s + (1 - y_t) H_t^n \ge \beta$. By Lemma B.3, we have

$$\Psi_t \le \lambda \Phi(W_t, \beta) - \lambda \beta F'(W_t) - \lambda F(W_t - \beta) = 0.$$

If the principal conducts monitoring at time *t* (i.e., $m_t = m$), by Lemma B.3, we have

$$\begin{split} \Psi_t &\leq \lambda \Phi(W_t, 0) - \lambda \beta F'(W_t) - \lambda F(W_t - \beta) - m \\ &= -rF(W_t) + \rho W_t F'(W_t) - m \\ &\leq -rF(\beta) + \rho \beta F'(\beta_+) - m \\ &\leq -rF(\beta) + \rho \beta F'(\beta_-) - m \leq 0. \end{split}$$

c.3. When $W_t \ge \bar{w}^*$, we must have $F'(W_t) = -1$, and inequality (B.12) reduces to

$$\Psi_t \le \lambda \Phi (W_t, y_t H_t^s + (1 - y_t) H_t^n) - \rho W_t - (\lambda + r) F(W_t) - m_t.$$

If the principal does not monitor at time *t* (i.e., $m_t = 0$), by (IC) condition and Lemma B.3, we have

$$\begin{aligned} \Psi_t &\leq \lambda \Phi(W_t, \beta) - \rho W_t - (\lambda + r) F(W_t) \\ &= \lambda F(W_t - \beta) - (\lambda + r) F(W_t) - (\rho W_t + \beta \lambda) \\ &\leq 0, \end{aligned}$$

where the last inequality follows from Lemma B.3.

If the principal conducts monitoring at time t (i.e., $m_t = m$), by Lemma B.3, we have

$$\begin{split} \Psi_t &\leq \lambda \Phi(W_t, 0) - \rho W_t - (\lambda + r) F(W_t) - m \\ &= -r F(W_t) - \rho W_t - m \\ &\leq -r F(\beta) - \rho \beta - m, \end{split}$$

where the second inequality holds because $-rF(w) - \rho w$ is decreasing in *w*. As such, considering (L_l), we have

$$\begin{split} \Psi_t &\leq -r \Big(\bar{\theta} \beta - \frac{m}{r} \Big) - \rho \beta - m \\ &\leq -r \beta \frac{\lambda + r - \rho}{\lambda} - \rho \beta \\ &= -(\rho - r) \frac{(r + \lambda)\beta}{\lambda} < 0, \end{split}$$

in which the second inequality holds because $\bar{\theta} \ge -1 + (\rho - r)/\lambda$.

In summary, for any contract Γ , we have $\Psi_t \leq 0$ almost surely. This completes the proof. \Box

Appendix D. Proofs in Section 5

Proof of Proposition 6. (i) From Lemma C.1(i), we know that, if $r \le \rho - \lambda$, J'(w) is decreasing on $[\beta, \infty)$. Therefore,

$$\bar{m} = \frac{r}{J'(\beta_+)} = (\rho + \lambda)\beta$$

From Lemma C.1(ii), we know that, if $\rho - \lambda < r < \overline{r}$, J'(w) is increasing on $[\beta, 2\beta]$ and decreasing on $[2\beta, \infty)$. Therefore,

$$\bar{m} = \frac{r}{J'(2\beta)} = \frac{\left[\left(\rho + \lambda\right)\beta\right]^{\frac{\lambda+r}{\rho}}}{\left[\left(2\rho + \lambda\right)\beta\right]^{\frac{\lambda+r}{\rho}-1}}$$

This completes the proof. \Box

Proof of Proposition 7. This proposition follows the same logic as Lemma C.2. Therefore, the proof is omitted.

Appendix E. Further Discussions E.1. Fixed Cost to Start Monitoring

In our paper, we did not consider a fixed cost of start monitoring. If such a fixed cost exists, we expect that there are two different thresholds. When the promised utility falls below the lower threshold, monitoring starts. And monitoring stops after the promised utility increases to be above the higher threshold. The intuition for such a "control band" structure is similar to the (s, S) policy in inventory control with fixed ordering cost.

E.2. General Cost of Arrival

When we introduce the model, we claim our results extend naturally to the case of random cost for each arrival as long as the random cost is not associated with the effort process. In this case, we just need to use K to represent the mean cost per arrival. More generally, however, the effort process may affect the random cost. For example, the agent's effort may affect not only the rate of arrival, but also the distribution of the cost. In this case, the optimal contract needs to take advantage of the information contained in the magnitude of the cost of an arrival. Such a setting is much more complex than the one studied in this paper. Even without monitoring, the dynamic contracting problem with multiple signal types has not been well understood. We suspect that the general optimal contract could be so complex such that a fruitful way to proceed is to explore approximations of the optimal contracts that are easy to compute and implement.

Another way of thinking about arrivals is not to consider them simply as arrivals, as we do in this paper, but as breakdowns of a production process (machine). That is, the agent is a maintenance team, whose effort reduces the arrival rate of breakdowns. The cost, therefore, corresponds to the lost revenue when the machine is down. The breakdown time can be random. Even without monitoring, such a model has not been studied in the literature, and one of the authors of this paper has been working on a related problem in an ongoing research project. It would be interesting to consider combining such a model with monitoring as a potential future research direction.

E.3. Agent More Patient than Principal

Following long traditions of the dynamic contracting literature, we assume that the principal is no less patient than the agent. This is true in most practical settings, in which the principal, as the contract designer, often possesses more resources than the agent. One may wonder what happens if the agent is more patient than the principal or $\rho < r$. In this case, delaying payments is even more beneficial to the principal because the interest the principal collects during the delay is higher than what is demanded by the agent. As a result, we believe that one still needs to introduce the exogenous upper bound on the promised utility to make sure that an optimal solution exists. In order to prove





optimality, we still need to establish concavity of the value function. This appears to be quite challenging when $\rho < r$. Neither the proof techniques in this paper nor the ones in Biais et al. (2010) work when $\rho < r$. We suspect one needs to carefully go through a discrete time model and follow a proof logic of Biais et al. (2007) and Sun and Tian (2018) to show that uniform convergence between the discrete time value function and the continuous-time one. Given the length and complexity of this paper and the relative lack of practical motivation for this technically interesting setting, we consider it outside the scope of this paper.

E.4. Imperfect Monitoring

Monitoring in our setting can be perceived as another signal on the agent's effort level that the principal can pay to obtain besides the arrivals. In this paper, we assume that this signal perfectly reflects the effort level. More generally, one can consider imperfect monitoring. That is, the agent's effort changes the statistic of another stochastic process, which is observable to the principal only if the principal pays for it. For example, in the quality control setting mentioned throughout the paper, the arrivals represent customer complaints. The principal may choose to monitor by conducting costly customer surveys and collect praises from customers. The arrivals of praise constitute the second Poisson process that is observable to the principal only if the principal pays for this information. Our model sheds light on tackling more complex and general incentive systems such as these.

E.5. Opportunity of Replacing Agents

In certain practical situations, the principal has the opportunity of replacing an agent with a $\cot K_r$. Intuitively, if such a cost is relatively low, the principal may prefer such an option over monitoring the agent for a long period of time (when the promised utility drops too low). The general idea of replacing the agent in other dynamic contracting settings has been discussed in detail in Myerson (2015) and section 5.2 of Sun and Tian (2018). Here we provide a description for the case with different discount rates and a high monitoring cost using an example and leave the complete and detailed results for interested readers to work out.

Figure E.1 provides an example of the value function and a corresponding trajectory considering agent replacement. In Figure E.1(a), there is an additional threshold γ compared with Figure 4(a). Smooth pasting works again at this threshold. That is, the value function's left and right derivatives are the same at γ . Furthermore, the value function is linear on $[0, \gamma]$, which implies that, upon an arrival that decreases an agent's promised utility below γ , the principal immediately randomly reset the promised utility to either zero (replacing the agent with a new one) or γ (continue monitoring the current agent). Recall that a new contract starts at promised utility w^* , which maximizes the value function *F*. Therefore, in Figure E.1(b), we have $F(w^*) - F(0) = K_r$.

Figure E.1(b) provides a partial sample trajectory of the agent's promised utility. Arrivals occurring at time t_1 and t_2 bring the promised utility below the threshold γ . At time t_1 , randomization takes the promised utility to γ so the

contract continues with a monitoring session. Randomization at time t_2 , on the other hand, brings the promised utility to zero for the current agent and triggers replacement. The new agent's promised utility starts at w^* and follows the trajectory of the dashed curve in the end.

Endnotes

¹Assuming a constant cost K for each arrival is for simplicity of exposition. Our results naturally extend to the case in which the cost of each arrival is a random variable and K is its mean as long as this random cost is independent to the effort process.

²The commitment power assumption follows a long tradition of dynamic contracting literature, (see, for example, Sannikov 2008, Biais et al. 2010, Myerson 2015, among many other references). Without this assumption, one may have to use the subgame perfect equilibrium concept, which is very hard to describe. In many practical circumstances in which the principal has much more power and resources than the agent, this is a reasonable assumption.

³Note that we have considered a very general class of feasible contracts in our models. Mathematically speaking, one may generalize the contract space even more by introducing a switch control at a nonarrival time with a certain probability. Such a control is essentially "adding points" following the terminology of Brémaud (1981), which involves mathematical tools that are beyond Brémaud (1981). In particular, it would invoke control of "piecewise-deterministic-Markov-processes," a much more sophisticated mathematical framework introduced in Davis (1984). Even if we generalize the class of contract this way, the current optimal contracts remain optimal. Therefore, it is not necessary to introduce additional mathematical complexities with no practical benefits.

⁴Without the limited liability constraint, even if the agent cannot buy out the entire enterprise, the principal can simply charge the agent a cash amount of β to induce effort. Therefore, the limited liability constraint prevents the model from becoming trivial.

⁵ If $\bar{w} < \beta$, the only (IC) contract is to always monitor the agent.

⁶Note that smooth pasting does not arise in Section 3 for the equal discount case because the optimal monitoring threshold is always β although in the different discount case, the threshold can be higher than β .

⁷We appreciate the review team for suggesting this simple contract structure.

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