

Online Appendix

Appendix

Proof of Proposition 1

In this proof we show that starting from a function \mathcal{J} that satisfies conditions **(C1)**, **(C2)** and **(C3)**, $\Gamma\mathcal{J}$ also satisfies them. First note that function $\mathcal{J}_0 = 0$ for all r (obtained from $J_0 = 0$ for all (r, j)) satisfies all these conditions.

Condition (5) for \mathcal{J} implies

$$(u - c) + \frac{\gamma}{N} \max_{m:m \neq l} \mathcal{J}(r + e_m - e_l) \geq \frac{\gamma}{N} \mathcal{J}(r) ,$$

which further implies

$$(\Gamma\mathcal{J})(r) = N_r(u - c) + \frac{\gamma}{N} \left[(N - N_r)\mathcal{J}(r) + \sum_{l:r_l \geq 1} \max_{m:m \neq l} \mathcal{J}(r + e_m - e_l) \right] . \quad (27)$$

1. Condition **(C1)**

We first show that condition (3) holds for $\Gamma\mathcal{J}$ in (27). That is, $(\Gamma\mathcal{J})(r) = (\Gamma\mathcal{J})(r')$. Note that

$$(\Gamma\mathcal{J})(r') = N_{r'}(u - c) + \frac{\gamma}{N} \left[(N - N_{r'})\mathcal{J}(r') + \sum_{l:r'_l \geq 1} \max_{m:m \neq l} \mathcal{J}(r' + e_m - e_l) \right] .$$

Obviously, $N_r = N_{r'}$, and $\mathcal{J}(r) = \mathcal{J}(r')$ due to (3). We only have the following cases.

- $l \neq i, j$. For $m \neq i, j$, we have $\mathcal{J}(r + e_m - e_l) = \mathcal{J}(r' + e_m - e_l)$ since $(r + e_m - e_l)_i = (r' + e_m - e_l)_j$ and $(r + e_m - e_l)_j = (r' + e_m - e_l)_i$. Furthermore, we have $\mathcal{J}(r + e_i - e_l) = \mathcal{J}(r' + e_j - e_l)$ and $\mathcal{J}(r + e_j - e_l) = \mathcal{J}(r' + e_i - e_l)$. Therefore, $\max_{m:m \neq l} \mathcal{J}(r + e_m - e_l) = \max_{m:m \neq l} \mathcal{J}(r' + e_m - e_l)$.

- $r_i \geq 1$ and $l = i$. For $m \neq j$, we have $\mathcal{J}(r + e_m - e_i) = \mathcal{J}(r' + e_m - e_j)$. Furthermore, we have $\mathcal{J}(r + e_j - e_i) = \mathcal{J}(r' + e_i - e_j)$. Therefore, $\max_{m:m \neq l} \mathcal{J}(r + e_m - e_l) = \max_{m:m \neq l} \mathcal{J}(r' + e_m - e_l)$.

- $r_j \geq 1$ and $l = j$. The same logic as in the previous case reveals $\max_{m:m \neq l} \mathcal{J}(r + e_m - e_l) = \max_{m:m \neq l} \mathcal{J}(r' + e_m - e_l)$.

2. Condition **(C2)**

First of all, when $r_i = r_j + 1$, then symmetry condition **C1** implies $\mathcal{J}(r - e_i + e_j) = \mathcal{J}(r)$ so the condition (4) holds. Therefore we only need to show the result carries through value iteration when $r_i \geq r_j + 2$. Denote $m_l^* = \arg \max_{m:m \neq l} \mathcal{J}(r + e_m - e_l)$, which has to be a minimum scrip holder other than l . That is,

$$(\Gamma\mathcal{J})(r) = N_r(u - c) + \frac{\gamma}{N} \left[(N - N_r)\mathcal{J}(r) + \sum_{l:r_l \geq 1} \mathcal{J}(r + e_{m_l^*} - e_l) \right]$$

Consider the following cases.

1. $r_j > 0$. In this case, since $r_i \geq r_j + 2$, we know that $N_r = N_{r-e_i+e_j}$, and for any l with $r_l > 0$, we also have $(r - e_i + e_j)_l > 0$, and vice versa.

1.1 $r_i \geq r_j + 3$. Note that for any l such that $r_l \geq 1$, we have $(r + e_{m_l^*} - e_l)_i > (r + e_{m_l^*} - e_l)_j$. Therefore

$$(\Gamma J)(r) \leq N_r(u - c) + \frac{\gamma}{N} \left[(N - N_r) \mathcal{J}(r - e_i + e_j) + \sum_{l:r_l \geq 1} \mathcal{J}(r + e_{m_l^*} - e_l - e_i + e_j) \right] \leq (\Gamma J)(r - e_i + e_j).$$

1.2 $r_i = r_j + 2$ and $m_i^* \neq j$. In this case we also have that for any l such that $r_l \geq 1$, $(r + e_{m_l^*} - e_l)_i > (r + e_{m_l^*} - e_l)_j$. Therefore the situation is the same as the previous case.

1.3 $r_i = r_j + 2$ and $m_i^* = j$. In this case we only have that for any $l \neq i$ such that $r_l \geq 1$, $(r + e_{m_l^*} - e_l)_i > (r + e_{m_l^*} - e_l)_j$. Therefore

$$\begin{aligned} (\Gamma J)(r) &= N_r(u - c) + \frac{\gamma}{N} \left[(N - N_r) \mathcal{J}(r) + \mathcal{J}(r - e_i + e_j) + \sum_{l \neq i: r_l \geq 1} \mathcal{J}(r + e_{m_l^*} - e_l) \right] \\ &\leq N_r(u - c) + \frac{\gamma}{N} \left[\mathcal{J}(r) + (N - N_r) \mathcal{J}(r - e_i + e_j) + \sum_{l \neq i: r_l \geq 1} \mathcal{J}(r + e_{m_l^*} - e_l - e_i + e_j) \right] \end{aligned}$$

Note that if $m_i^* = j$, then j is a minimum scrip holder in r other than i . Following symmetry, we have either $\max_{m:m \neq i} \mathcal{J}(r - 2e_i + e_j + e_m) = \mathcal{J}(r)$ (when j is the unique minimum scrip holder in r other than i) or $\max_{m:m \neq i} \mathcal{J}(r - 2e_i + e_j + e_m) = \mathcal{J}(r - e_i + e_j)$ (when j is not the unique minimizer in r other than i). In either case,

$$(\Gamma J)(r) \leq (\Gamma J)(r - e_i + e_j).$$

2. $r_j = 0$. In this case $N_r = N_{r-e_i+e_j} - 1$.

$$\begin{aligned} (\Gamma J)(r - e_i + e_j) &\geq (N_r + 1)(u - c) + \frac{\gamma}{N} \left[(N - N_r - 1) \mathcal{J}(r - e_i + e_j) + \max_{m:m \neq j} \mathcal{J}(r - e_i + e_j + e_m - e_j) \right] \\ &+ \sum_{l:r_l \geq 1} \mathcal{J}(r + e_{m_l^*} - e_l - e_i + e_j) \geq N_r(u - c) + \frac{\gamma}{N} \left[(N - N_r) \mathcal{J}(r - e_i + e_j) + \sum_{l:r_l \geq 1} \mathcal{J}(r + e_{m_l^*} - e_l + e_j - e_l) \right] \geq \Gamma \mathcal{J}(r), \end{aligned}$$

where the second inequality follows Condition **(C3)**, that is,

$$\frac{N}{\gamma} (u - c) + \max_{m:m \neq j} \mathcal{J}(r - e_i + e_j + e_m - e_j) \geq \mathcal{J}(r - e_i + e_j);$$

and the last inequality follows the same logic as in the previous cases.

3. Condition **(C3)**

First, we show condition (5). From the previous proof, we know that for \mathcal{J} satisfying (C1) and (C2), when $r_j > r_i$, we have $\mathcal{J}(r) \leq \mathcal{J}(r + e_i - e_j)$, which implies the condition automatically. Therefore we focus on cases when $r_i \geq r_j$. Furthermore, player i must be the minimum scrip holder in r except j .

Denote m_l^* to represent the player that achieves $\max_{m:m \neq l} \mathcal{J}(r + e_m - e_l)$.

- $r_j \geq 2$. In this case $N_r = N_{r+e_i-e_j} = N$. Therefore,

$$\begin{aligned} (\Gamma\mathcal{J})(r) - (\Gamma\mathcal{J})(r+e_i-e_j) &= \frac{\gamma}{N} \sum_l \left(\max_{m:m \neq l} \mathcal{J}(r+e_m-e_l) - \max_{m:m \neq l} \mathcal{J}(r+e_i-e_j+e_m-e_l) \right) \\ &\leq \frac{\gamma}{N} \sum_l \left(\mathcal{J}(r+e_{m_l^*}-e_l) - \mathcal{J}((r+e_{m_l^*}-e_l)+e_i-e_j) \right) \end{aligned}$$

— $|I_{r_i}| = 2$. This means that $I_{r_i} = \{i, j\}$ and i is the unique minimum scrip holder in r except j . Thus for each possible value of l , i is also the minimum scrip holder in $r+e_{m_l^*}-e_l$; note that if $r_i = r_j$ and $l = k \neq i, j$, then by (C1) we have $\mathcal{J}(r+e_i-e_k) = \mathcal{J}(r+e_j-e_k)$ and we can choose $m_k^* = j$ to make this statement true.

Therefore, condition (5) implies the following result:

$$(\Gamma\mathcal{J})(r) - (\Gamma\mathcal{J})(r+e_i-e_j) \leq \frac{\gamma}{N} \left[N \frac{N}{\gamma} (u-c) \right] < \frac{N}{\gamma} (u-c).$$

— $|I_{r_i}| > 2$. This means that player i is not a unique minimum scrip holder in r except j ; in other words, another player(s) has r_i scrips. For $l = j$, we know from (C1) that $\mathcal{J}(r+e_{m_k^*}-e_j) = \mathcal{J}(r+e_k-e_j)$ for any player $k \neq i$ with $r_k = r_i$. Thus, i is a minimum scrip holder in $r+e_k-e_j$ and we can apply condition (5).

For $l = i$, i is the minimum scrip holder in $r+e_j-e_i$ except j , so again we can apply condition (5). For $l = k \neq i, j$ with $r_k > r_i$, i is the minimum scrip holder in $r+e_j-e_k$ except j , so again we can apply condition (5).

Finally, for $l = k \neq i, j$ with $r_k = r_i$, i is no longer a minimum scrip holder in $r+e_j-e_k$. In this case, if $r_i = r_j$, we know from (C1) that $\mathcal{J}(r+e_{m_k^*}-e_k) = \mathcal{J}(r+e_j-e_k)$ and by choosing $m_k^* = j$, we see that $(r+e_j-e_k)_i < (r+e_j-e_k)_j$ and from (C1) and (C2) we have $\mathcal{J}(r+e_j-e_k) \leq \mathcal{J}(r+e_j-e_k+e_i-e_j)$. On the other hand, if $r_i > r_j$, we have $|I_{(r+e_j-e_k)_i}| \geq 3$ and includes i, j , and k ; since $(r+e_j-e_k)_j \geq 2$ we know condition (6) holds so we can apply condition (7). Putting all these cases together, we again have

$$(\Gamma\mathcal{J})(r) - (\Gamma\mathcal{J})(r+e_i-e_j) \leq \frac{\gamma}{N} \left[N \frac{N}{\gamma} (u-c) \right] < \frac{N}{\gamma} (u-c).$$

- $r_j = 1, r_i > 1$. In this case $N = N_r = N_{r+e_i-e_j} + 1$.

$$(\Gamma\mathcal{J})(r) - (\Gamma\mathcal{J})(r+e_i-e_j) = (u-c) + \frac{\gamma}{N} \sum_{l:l \neq j} \left(\max_{m:m \neq l} \mathcal{J}(r+e_m-e_l) - \max_{m:m \neq l} \mathcal{J}(r+e_i-e_j+e_m-e_l) \right)$$

Following the exact same line of argument as above, we have

$$(\Gamma\mathcal{J})(r) - (\Gamma\mathcal{J})(r+e_i-e_j) \leq (u-c) + \frac{\gamma}{N} \left((N-1) \frac{N}{\gamma} (u-c) \right) < \frac{N}{\gamma} (u-c).$$

• $r_j = 1, r_i = 1$. In this case we must have $\max_{m:m \neq l} \mathcal{J}(r+e_m-e_l) = \max_{m:m \neq l} \mathcal{J}(r+e_i-e_j+e_m-e_l) = \mathcal{J}(r+e_i-e_l)$, when $l \neq i, j$. And $\max_{m:m \neq i} \mathcal{J}(r+e_m-e_i) = \mathcal{J}(r+e_j-e_i) \leq \mathcal{J}(r) = \max_{m:m \neq i} \mathcal{J}(r+e_i-e_j+e_m-e_i)$. As a result, $(\Gamma\mathcal{J})(r) - (\Gamma\mathcal{J})(r+e_i-e_j) \leq (u-c)$.

Now we focus on showing condition (7). Again, from the previous proof, we know that for \mathcal{J} satisfying (C1) and (C2), when $r_j > r_i$, we have $\mathcal{J}(r) \leq \mathcal{J}(r + e_i - e_j)$, which implies the condition automatically. Therefore we focus on cases when $r_i \geq r_j$. Also, note that in order for condition (6) to hold, we know that no players can have 0 coupons.

- $r_j \geq 2$. In this case $N_r = N_{r+e_i-e_j} = N$.

$$\begin{aligned} (\Gamma\mathcal{J})(r) - (\Gamma\mathcal{J})(r + e_i - e_j) &= \frac{\gamma}{N} \sum_l \left(\max_{m:m \neq l} \mathcal{J}(r + e_m - e_l) - \max_{m:m \neq l} \mathcal{J}(r + e_i - e_j + e_m - e_l) \right) \\ &\leq \frac{\gamma}{N} \sum_l \left(\mathcal{J}(r + e_{m_l^*} - e_l) - \mathcal{J}((r + e_{m_l^*} - e_l) + e_i - e_j) \right) \end{aligned}$$

(1) There exists $r_k = 1$. According to condition (6), all other players in I_{r_i} must have exactly r_i scrip. For any player $l \neq k$, $r + e_{m_l^*} - e_l$ satisfies (6). Note that $\mathcal{J}(r + e_j - e_k) = \mathcal{J}(r + e_i - e_k)$ since $r_j = r_i$. Thus by choosing $m_k^* = j$, we have

$$(\Gamma\mathcal{J})(r) - (\Gamma\mathcal{J})(r + e_i - e_j) \leq \frac{\gamma}{N} \left((N-1) \frac{N}{\gamma} (u-c) \right) < \frac{N}{\gamma} (u-c).$$

(2) $r_l \geq 2$ for all $l \in I_{r_i}$. By choosing $m_l^* \neq i$ when possible, $r + e_{m_l^*} - e_l$ always satisfies condition (6), which implies our result.

- $r_j = 1$. Condition (6) implies that for all players $l \in I_{r_i} \setminus j$, $r_l = r_i$. Therefore, i achieves $\max_{i:i \neq j} \mathcal{J}(r + e_i - e_j)$, which has been shown earlier in the proof.

We have shown that starting from a function \mathcal{J} that satisfies conditions (C1), (C2), and (C3), $\Gamma\mathcal{J}$ also satisfies them. Therefore, the conditions hold for any function \mathcal{J} through value iteration. The set of functions that satisfy the three conditions is closed (and convex), which implies that the limit \mathcal{J}^* also satisfy (C1)-(C3). See, for example, Proposition 1 of Smith and McCardle (2002). (Strictly speaking, the proposition in Smith and McCardle (2002) directly implies that (C1) and (C2) are closed, convex cone properties. Its proof, on the other hand, clearly indicates that (C3) is a closed, convex property.)

Q.E.D.

Steady state probabilities for random provider selection rule: Here we provide closed form expressions on the steady state probabilities and the probability of no trade under the random provider selection rule.

PROPOSITION 7. *In a system with N players and R scrips following the random provider selection rule, the steady state probability of any possible scrip distribution among players is $\rho = 1 / \binom{R+N-1}{N-1}$. Furthermore,*

the probability that trade would not occur due to service requester's lack of scrip is $\rho \sum_{k=1}^{N-1} \binom{N-1}{k-1} \binom{R-1}{N-k-1}$.

Proof: Under the random provider selection rule, it is obvious that the transition probability matrix is doubly stochastic, and therefore the steady state probability is the same for all states. The number of states for a system with N players and R scrips is the number of combinations of distributing R scrips to N players, which is $\binom{R+N-1}{N-1}$. This implies the expression for ρ .

The event of no trade occurs when the service requester has zero scrips. Divide the event into sub-events of exactly $k \leq N-1$ players having zero scrips. The possible ways of having exactly k zero scrip players is $\binom{N}{k} \binom{R-1}{N-k-1}$, where the second term is the number of ways of distributing the R scrips among the remaining $N-k$ players, which is the same as distributing $R-N+k$ scrips among $N-k$ players without no-zero-scrip restrictions. As a result, the probability of no trade is

$$\sum_{k=1}^{N-1} \rho \frac{k}{N} \binom{N}{k} \binom{R-1}{N-k-1} = \rho \sum_{k=1}^{N-1} \binom{N-1}{k-1} \binom{R-1}{N-k-1}.$$

Q.E.D.

Proof of Lemma 3: This result follows directly from the Folk Theorem for stochastic games (Theorem 9, Dutta (1995)). In particular, we need to verify the following three conditions to invoke Theorem 9 in Dutta (1995):

1. Unichain selection rule guarantees that the set of feasible long-run average payoffs is independent of the starting state.
2. Each player's long-run average min-max payoff needs to be independent of the starting state. Instead of considering the min-max payoff, we consider the "always refuse providing service" equilibrium strategy, which generates 0 long run average payoff, and therefore independent of the starting state.
3. The dimension of the set of feasible payoffs should be N . This condition holds because any single player i can be ruled out from service, which creates a payoff vector that has a constant positive value for all players except player i , who has value zero. The resulting N vectors, each corresponding to one player being ruled out, span \mathfrak{R}^N .

Q.E.D.

Proof of Theorem 3: Start the value iteration algorithm (10)-(12) from $\bar{V}(r, j)$ defined as,

$$\bar{V}(r, 1) = u \sum_{k=0}^{r_1-1} \left(\frac{\gamma}{N}\right)^k, \text{ if } r_1 \geq 1; \text{ and } \bar{V}(r, j) = 0, \text{ otherwise.}$$

Note that $\bar{V} \geq 0$ in general, and, in particular, $\bar{V}(r, 1) \geq u$ if $r_1 \geq 1$.

Next we verify that $T\bar{V} \geq \bar{V} \geq 0$.

1. When $j = 1$ and $r_1 = 0$ we have,

$$(T\bar{V})(r, 1) = \frac{\gamma}{N} \sum_{j'} \bar{V}(r, j') = 0 = \bar{V}(r, 1) .$$

2. When $j = 1$ and $r_1 > 0$ we have,

$$(T\bar{V})(r, 1) = u + \frac{\gamma}{N|\Upsilon(r, 1)|} \sum_{j'} \sum_{i \in \Upsilon(r, 1)} \bar{V}(r - e_1 + e_i, j') = u \sum_{k=0}^{r_1-1} \left(\frac{\gamma}{N}\right)^k = \bar{V}(r, 1) . \quad (28)$$

3. When $j \neq 1$ and $r_j = 0$ we have,

$$(T\bar{V})(r, j) = \frac{\gamma}{N} \sum_{j'} \bar{V}(r, j') \geq 0 = \bar{V}(r, j) .$$

4. When $j \neq 1$, $r_j > 0$, and $1 \notin \Upsilon(r, j)$ we have,

$$(T\bar{V})(r, j) = \frac{\gamma}{N|\Upsilon(r, j)|} \sum_{j'} \sum_{i \in \Upsilon(r, j)} \bar{V}(r - e_j + e_i, j') \geq 0 = \bar{V}(r, j) .$$

5. When $j \neq 1$, $r_j > 0$, and $1 \in \Upsilon(r, j)$, following (28) we have, $(T\bar{V})(r - e_j + e_1, 1) \geq u$. Therefore,

$$\begin{aligned} (T\bar{V})(r, j) &= \left[-c + \gamma/N \sum_{i \in \Upsilon(r, j)} \left(\bar{V}(r - e_j + e_i, 1) + \sum_{j' \neq 1} \bar{V}(r - e_j + e_i, j') \right) \right] / |\Upsilon(r, j)| \\ &= \left[-c + \gamma/N \sum_{i \in \Upsilon(r, j)} \bar{V}(r - e_j + e_i, 1) \right] / |\Upsilon(r, j)| \\ &\geq \left[-c + (\gamma/N) \bar{V}(r - e_j + e_1, 1) \right] / |\Upsilon(r, j)| \\ &\geq \left[-c + u\gamma/N \right] / |\Upsilon(r, j)| \geq 0 = \bar{V}(r, j) , \end{aligned}$$

in which the first inequality is due to $\bar{V} \geq 0$, the second inequality due to $(r - e_j + e_1)_1 \geq 1$ and therefore $\bar{V}(r - e_j + e_1, 1) \geq u$, and the last inequality from $u/c \geq N/\gamma$.

Monotonicity and convergence of operator T imply that $V^\gamma = \lim_{t \rightarrow \infty} T^t \bar{V} \geq \bar{V} \geq 0$. Q.E.D.

Proof of Lemma 4: Consider a scrip distribution vector $r_{n_1 n_2}^{n_0}$, where exactly n_0 players have zero scrip, n_1 players have 1 scrip, and n_2 players have more than 1 scrip; therefore $n_0 + n_1 + n_2 = N$. For $n_0 \geq 1$, under the minimum scrip selection rule, from state $(r_{n_1 n_2}^{n_0}, j)$, the probability of transitioning into another state also with n_0 zero scrip players (including itself) is $(n_0 + n_1)/N$ (this happens when either a zero scrip player or a one scrip player becomes the service requester, j). Otherwise, with probability n_2/N , the system transitions into a state with exactly $n_0 - 1$ zero scrip players. For $n_0 = 0$, the probability of transitioning into a state with one zero scrip player is n_1/N , and the probability of transitioning into a state where there are no zero

scrip players is n_2/N . This implies that when $n_0 \geq 2$ (and $N \geq 3$), the system never transitions into state $(r_{n_1 n_2}^{n_0}, j)$ from another state with fewer number of zero scrip players. In addition, since $R \geq N$, we must have $n_2 > 0$ when $n_0 \geq 1$. That is, there is a positive probability when $n_0 \geq 1$ that the system transitions out of state $(r_{n_1 n_2}^{n_0}, j)$ and into another state with one fewer zero scrip players. Starting from (r_{01}^{N-1}, j) , where the R scrips concentrate in one player, leaving the other $N - 1$ players with 0 scrips, the above analysis implies that state (r_{01}^{N-1}, j) must be transient. Decrease n_0 one by one from $N - 2$ to 2, and we have that state $(r_{n_1 n_2}^{n_0}, j)$ is transient as long as $n_0 \geq 2$.

Note that for the special case when $N = 2$, it is impossible to have a state where more than one player has 0 scrips since the $R \geq 2$ scrips must be held by at least one player. Q.E.D.

Proof of Lemma 5: We need to show that starting from any function V that satisfies (14), the condition still holds after one step value iteration defined by the right hand side of recursive equations (10)-(12), denoted as TV . For simplification of notation, we denote $\nu(r) = \sum_j V(r, j)$. We will show that every term $(TV)(r, j)$ corresponds to an equivalent term $(TV)(r', \hat{j})$ for some \hat{j} and vice versa, thus implying the result.

(1) $j = 1$. We show the equivalence of $(TV)(r, 1)$ with $(TV)(r', 1)$ under the following two possibilities. Note that if $r_1 = 0$, then $r'_1 = 0$.

(a) $r_1 = 0$. By directly applying the induction hypothesis (14),

$$(TV)(r, 1) = \frac{\gamma}{N} \nu(r) = \frac{\gamma}{N} \nu(r') = (TV)(r', 1)$$

(b) $r_1 > 0$.

$$(TV)(r, 1) = u + \frac{\gamma}{N|\Upsilon(r, 1)|} \sum_{i \in \Upsilon(r, 1)} \nu(r - e_1 + e_i)$$

$$(TV)(r', 1) = u + \frac{\gamma}{N|\Upsilon(r', 1)|} \sum_{i \in \Upsilon(r', 1)} \nu(r' - e_1 + e_i)$$

For $i \in \Upsilon(r, 1) \neq l, m$, note that $i \in \Upsilon(r', 1)$ and $\nu(r - e_1 + e_i) = \nu(r' - e_1 + e_i)$ by the induction hypothesis. If $l \in \Upsilon(r, 1)$, then $m \in \Upsilon(r', 1)$ and $\nu(r - e_1 + e_l) = \nu(r' - e_1 + e_m)$ by the induction hypothesis. Similarly, if $m \in \Upsilon(r, 1)$, then $l \in \Upsilon(r', 1)$ and $\nu(r - e_1 + e_m) = \nu(r' - e_1 + e_l)$. Therefore, $(TV)(r, 1) = (TV)(r', 1)$ in this case.

(2) $j \neq l, m$. We show the equivalence of $(TV)(r, j)$ with $(TV)(r', j)$ under the following possibilities. Note that if $r_j = 0$, then $r'_j = 0$. Also, if $1 \notin \Upsilon(r, j)$, then $1 \notin \Upsilon(r', j)$.

(a) $r_j = 0$. By directly applying the induction hypothesis (14),

$$(TV)(r, j) = \frac{\gamma}{N} \nu(r) = \frac{\gamma}{N} \nu(r') = (TV)(r', j)$$

(b) $r_j > 0$ and $1 \notin \Upsilon(r, j)$.

$$(TV)(r, j) = \frac{\gamma}{N|\Upsilon(r, j)|} \sum_{i \in \Upsilon(r, j)} \nu(r - e_j + e_i)$$

$$(TV)(r', j) = \frac{\gamma}{N|\Upsilon(r', j)|} \sum_{i \in \Upsilon(r', j)} \nu(r' - e_j + e_i)$$

For $i \in \Upsilon(r, j) \neq l, m$, note that $i \in \Upsilon(r', j)$ and $\nu(r - e_j + e_i) = \nu(r' - e_j + e_i)$ by the induction hypothesis. If $l \in \Upsilon(r, j)$, then $m \in \Upsilon(r', j)$ and $\nu(r - e_j + e_l) = \nu(r' - e_j + e_m)$ by the induction hypothesis. Similarly, if $m \in \Upsilon(r, j)$, then $l \in \Upsilon(r', j)$ and $\nu(r - e_j + e_m) = \nu(r' - e_j + e_l)$. Therefore, $(TV)(r, j) = (TV)(r', j)$ in this case.

(c) $r_j > 0$ and $1 \in \Upsilon(r, j)$.

$$(TV)(r, j) = \frac{-c}{|\Upsilon(r, j)|} + \frac{\gamma}{N|\Upsilon(r, j)|} \sum_{i \in \Upsilon(r, j)} \nu(r - e_j + e_i)$$

$$(TV)(r', j) = \frac{-c}{|\Upsilon(r', j)|} + \frac{\gamma}{N|\Upsilon(r', j)|} \sum_{i \in \Upsilon(r', j)} \nu(r' - e_j + e_i)$$

Similarly to the previous case, for $i \in \Upsilon(r, j) \neq l, m$, note that $i \in \Upsilon(r', j)$ and $\nu(r - e_j + e_i) = \nu(r' - e_j + e_i)$ by the induction hypothesis. If $l \in \Upsilon(r, j)$, then $m \in \Upsilon(r', j)$ and $\nu(r - e_j + e_l) = \nu(r' - e_j + e_m)$ by the induction hypothesis. Similarly, if $m \in \Upsilon(r, j)$, then $l \in \Upsilon(r', j)$ and $\nu(r - e_j + e_m) = \nu(r' - e_j + e_l)$. Therefore, $(TV)(r, j) = (TV)(r', j)$ in this case.

(3) $j = l$. We show the equivalence of $(TV)(r, l)$ with $(TV)(r', m)$ under the following possibilities. Note that if $r_l = 0$, then $r'_m = 0$. Also, if $1 \notin \Upsilon(r, l)$, then $1 \notin \Upsilon(r', m)$.

(a) $r_l = 0$. By directly applying the induction hypothesis (14),

$$(TV)(r, l) = \frac{\gamma}{N} \nu(r) = \frac{\gamma}{N} \nu(r') = (TV)(r', m)$$

(b) $r_l > 0$ and $1 \notin \Upsilon(r, l)$.

$$(TV)(r, l) = \frac{\gamma}{N|\Upsilon(r, l)|} \sum_{i \in \Upsilon(r, l)} \nu(r - e_l + e_i)$$

$$(TV)(r', m) = \frac{\gamma}{N|\Upsilon(r', m)|} \sum_{i \in \Upsilon(r', m)} \nu(r' - e_m + e_i)$$

For $i \in \Upsilon(r, l) \neq m$, note that $i \in \Upsilon(r', m)$ and $\nu(r - e_l + e_i) = \nu(r' - e_m + e_i)$ by the induction hypothesis. If $m \in \Upsilon(r, l)$, then $l \in \Upsilon(r', m)$ and $\nu(r - e_l + e_m) = \nu(r' - e_m + e_l)$ by the induction hypothesis. Therefore, $(TV)(r, l) = (TV)(r', m)$ in this case.

(c) $r_l > 0$ and $1 \in \Upsilon(r, l)$.

$$(TV)(r, l) = \frac{-c}{|\Upsilon(r, l)|} + \frac{\gamma}{N|\Upsilon(r, l)|} \sum_{i \in \Upsilon(r, l)} \nu(r - e_l + e_i)$$

$$(TV)(r', m) = \frac{-c}{|\Upsilon(r', m)|} + \frac{\gamma}{N|\Upsilon(r', m)|} \sum_{i \in \Upsilon(r', m)} \nu(r' - e_m + e_i)$$

Similarly to the previous case, for $i \in \Upsilon(r, l) \neq m$, note that $i \in \Upsilon(r', m)$ and $\nu(r - e_l + e_i) = \nu(r' - e_m + e_i)$ by the induction hypothesis. If $m \in \Upsilon(r, l)$, then $l \in \Upsilon(r', m)$ and $\nu(r - e_l + e_m) = \nu(r' - e_m + e_l)$ by the induction hypothesis. Therefore, $(TV)(r, l) = (TV)(r', m)$ in this case.

(4) $j = m$. This case is identical to case 3 above. Following the same logic with l and m interchanged, one can show that $(TV)(r, m) = (TV)(r', l)$.

Q.E.D.

Proof of Proposition 4

Obviously, constant 0 functions of the corresponding dimensions satisfy conditions (16)-(18). We just need to show that starting from any functions \bar{V} and V that satisfy (16)-(18), the condition still holds after one step value iteration defined by the right hand side of equations (10)-(12), denoted as $T\bar{V}$ and TV , respectively.

First consider condition (16). That is, we show that if conditions (16), (17), and (18) hold for some functions V and \bar{V} , condition (16) also holds for TV and $T\bar{V}$. For simplification of notation, we denote $\nu(r) = \sum_j V(r, j)$, and $\bar{\nu}$ similarly, which will be used in the proof.

$$(1.1) \quad j = 1, r_1 = 0.$$

$$(TV)(r, 1) = (\gamma/N)\nu(r) \geq (\gamma/N)\bar{\nu}(r + e_k) = (T\bar{V})(r + e_k, 1) .$$

$$(1.2) \quad j = 1, r_1 > 0,$$

$$(TV)(r, 1) = u + \gamma/(N|\Upsilon(r, 1)|) \sum_{i \in \Upsilon(r, 1)} \nu(r - e_1 + e_i)$$

$$(T\bar{V})(r + e_k, 1) = u + \gamma/(N|\Upsilon(r + e_k, 1)|) \sum_{i \in \Upsilon(r + e_k, 1)} \bar{\nu}(r - e_1 + e_i + e_k) .$$

Consider the following 3 possibilities.

(i) $\Upsilon(r + e_k, 1) = \Upsilon(r, 1)$. In this case, $\bar{V}(r - e_1 + e_i + e_k, j') \leq V(r - e_1 + e_i, j')$ from induction hypothesis.

Therefore we have $(T\bar{V})(r + e_k, 1) \leq (TV)(r, 1)$.

(ii) $\Upsilon(r + e_k, 1) \cup \{k\} = \Upsilon(r, 1)$. In this case, $\bar{\nu}(r - e_1 + e_i + e_k)$ is the same for all $i \in \Upsilon(r + e_k, 1)$. Therefore $\sum_{i \in \Upsilon(r + e_k, 1)} \bar{\nu}(r - e_1 + e_i + e_k) / |\Upsilon(r + e_k, 1)|$ equals $\bar{\nu}(r - e_1 + e_i + e_k)$ for any particular i . The same can be said for $\nu(r - e_1 + e_i)$. The result then follows the comparison between $\nu(r - e_1 + e_i)$ and $\bar{\nu}(r - e_1 + e_i + e_k)$ according to the induction hypothesis.

(iii) $\Upsilon(r, 1) = \{k\}$. We have $V(r - e_1 + e_k, j') \geq \bar{V}(r - e_1 + e_i + e_k, j')$ for any $i \neq 1$. Therefore,

$$(TV)(r, 1) = u + \gamma/N\nu(r - e_1 + e_k)$$

$$\geq u + \gamma/(N|\Upsilon(r + e_k, 1)|) \sum_{i \in \Upsilon(r + e_k, 1)} \bar{\nu}(r - e_1 + e_i + e_k) = (T\bar{V})(r + e_k, 1) .$$

(2.1.1) $j \neq 1, r_j = 0, k \neq j,$

$$(TV)(r, j) = (\gamma/N)\nu(r) \geq (\gamma/N)\bar{\nu}(r + e_k) = (T\bar{V})(r + e_k, j) .$$

(2.1.2.1) $j \neq 1, r_j = 0, k = j, 1 \notin \Upsilon(r + e_j, j)$

$$(T\bar{V})(r + e_j, j) = \gamma/(N|\Upsilon(r + e_j, j)|) \sum_{i \in \Upsilon(r + e_j, j)} \bar{\nu}(r + e_i)$$

$$\leq \gamma/(N|\Upsilon(r + e_j, j)|) \sum_{i \in \Upsilon(r + e_j, j)} \nu(r) = (\gamma/N)\nu(r) = (TV)(r, j) .$$

(2.1.2.2) $j \neq 1, r_j = 0, k = j, 1 \in \Upsilon(r + e_j, j)$. Lemma 4 implies that no more than one player has 0 scrips in any state in the recurrent class. Therefore, we must have $r_1 > 0$ and hence,

$$(T\bar{V})(r + e_j, j) = \left(-c + \gamma/N \sum_{i \in \Upsilon(r + e_j, j)} \bar{\nu}(r + e_i) \right) / |\Upsilon(r + e_j, j)|$$

$$= \frac{-c}{|\Upsilon(r + e_j, j)|} + \frac{\gamma}{N|\Upsilon(r + e_j, j)|} \left(\sum_{i \in \Upsilon(r + e_j, j) \setminus 1} \bar{\nu}(r + e_i) + \bar{\nu}(r + e_1) \right)$$

$$\leq \frac{-c}{|\Upsilon(r + e_j, j)|} + \frac{\gamma}{N|\Upsilon(r + e_j, j)|} \left(\sum_{i \in \Upsilon(r + e_j, j) \setminus 1} \bar{\nu}(r + e_i) + \nu(r) + \frac{cN}{\gamma} \right)$$

$$\leq \frac{-c}{|\Upsilon(r + e_j, j)|} + \frac{\gamma}{N|\Upsilon(r + e_j, j)|} \left(\sum_{i \in \Upsilon(r + e_j, j) \setminus 1} \nu(r) + \nu(r) + \frac{cN}{\gamma} \right)$$

$$= \frac{-c}{|\Upsilon(r + e_j, j)|} + \frac{\gamma}{N|\Upsilon(r + e_j, j)|} \left(|\Upsilon(r + e_j, j)|\nu(r) + \frac{cN}{\gamma} \right)$$

$$= \frac{\gamma}{N}\nu(r) = (TV)(r, j)$$

where the first inequality follows from (17), and the second inequality follows from (16).

$$(2.2.1) \quad j \neq 1, r_j > 0, 1 \notin \Upsilon(r, j), 1 \notin \Upsilon(r + e_k, j).$$

$$(TV)(r, j) = \gamma / (N |\Upsilon(r, j)|) \sum_{i \in \Upsilon(r, j)} \nu(r - e_j + e_i)$$

$$(T\bar{V})(r + e_k, j) = \gamma / (N |\Upsilon(r + e_k, j)|) \sum_{i \in \Upsilon(r + e_k, j)} \bar{\nu}(r - e_j + e_k + e_i).$$

Following the 3 cases as in (1.2), we have the result.

(2.2.2) $j \neq 1, r_j > 0, 1 \notin \Upsilon(r, j), 1 \in \Upsilon(r + e_k, j)$. This means that we must have $\Upsilon(r, j) = \{k\}$, and $r_1 = r_k + 1$ (therefore $r_1 > 0$).

$$\begin{aligned} (T\bar{V})(r + e_k, j) &= \left(-c + \gamma/N \sum_{i \in \Upsilon(r + e_k, j)} \bar{\nu}(r - e_j + e_k + e_i) \right) / |\Upsilon(r + e_k, j)| \\ &= \frac{-c}{|\Upsilon(r + e_k, j)|} \\ &+ \frac{\gamma}{N |\Upsilon(r + e_k, j)|} \left(\sum_{i \in \Upsilon(r + e_k, j) \setminus 1} \bar{\nu}(r - e_j + e_k + e_i) + \bar{\nu}(r - e_j + e_k + e_1) \right) \\ &\leq \frac{-c}{|\Upsilon(r + e_k, j)|} \\ &+ \frac{\gamma}{N |\Upsilon(r + e_k, j)|} \left(\sum_{i \in \Upsilon(r + e_k, j) \setminus 1} \bar{\nu}(r - e_j + e_k + e_i) + \nu(r - e_j + e_k) + \frac{cN}{\gamma} \right) \\ &\leq \frac{-c}{|\Upsilon(r + e_k, j)|} \\ &+ \frac{\gamma}{N |\Upsilon(r + e_k, j)|} \left(\sum_{i \in \Upsilon(r + e_k, j) \setminus 1} \nu(r - e_j + e_k) + \nu(r - e_j + e_k) + \frac{cN}{\gamma} \right) \\ &= \frac{-c}{|\Upsilon(r + e_k, j)|} + \frac{\gamma}{N |\Upsilon(r + e_k, j)|} \left(|\Upsilon(r + e_k, j)| \nu(r - e_j + e_k) + \frac{cN}{\gamma} \right) \\ &= \frac{\gamma}{N} \nu(r - e_j + e_k) = (TV)(r, j). \end{aligned}$$

Here, the first inequality follows from (17), the second inequality follows from (16), and the last equation follows from the fact that $\Upsilon(r, j)$ is a singleton with only one element, k .

$$(2.3.1) \quad j \neq 1, r_j > 0, 1 \in \Upsilon(r, j), 1 \in \Upsilon(r + e_k, j).$$

$$(TV)(r, j) = \left(-c + \gamma/N \sum_{i \in \Upsilon(r, j)} \nu(r - e_j + e_i) \right) / |\Upsilon(r, j)|$$

$$(T\bar{V})(r + e_k, j) = \left(-c + \gamma/N \sum_{i \in \Upsilon(r + e_k, j)} \bar{\nu}(r - e_j + e_k + e_i) \right) / |\Upsilon(r + e_k, j)|$$

Similar to case (1.2), we consider the three cases. When $\Upsilon(r + e_k, j) = \Upsilon(r, j)$, we have the result directly from condition (16). Since here we assume that $1 \in \Upsilon(r, j)$, it is impossible that $\Upsilon(r, j) = \{k\}$.

Now we focus on the case where $|\Upsilon(r + e_k, j)| + 1 = |\Upsilon(r, j)| = L > 1$, that is, $\Upsilon(r + e_k, j) \cup \{k\} = \Upsilon(r, j)$. Since, by Lemma 4, no more than one player can have 0 scrips in any state in the recurrent class, we must have $r_1 > 0$.

$$\begin{aligned}
& L(L-1) (T\bar{V}(r + e_k, j) - T\bar{V}(r, j)) \\
&= -c + \frac{\gamma}{N} \left[L \left(\sum_{i \in \Upsilon(r+e_k, j) \setminus \{1\}} \bar{\nu}(r - e_j + e_k + e_i) + \bar{\nu}(r - e_j + e_k + e_1) \right) \right. \\
&\quad \left. - (L-1) \left(\sum_{i \in \Upsilon(r+e_k, j) \setminus \{1\}} \nu(r - e_j + e_i) + \nu(r - e_j + e_k) + \nu(r - e_j + e_1) \right) \right] \\
&= -c + \frac{\gamma}{N} \left[(L-1) \sum_{i \in \Upsilon(r+e_k, j) \setminus \{1\}} \left(\bar{\nu}(r - e_j + e_i + e_k) - \nu(r - e_j + e_i) \right) \right. \\
&\quad \left. + \sum_{i \in \Upsilon(r+e_k, j) \setminus \{1\}} \left(\bar{\nu}(r - e_j + e_i + e_k) - \nu(r - e_j + e_k) \right) - \nu(r - e_j + e_k) \right. \\
&\quad \left. + (L-1) \left(\bar{\nu}(r - e_j + e_k + e_1) - \nu(r - e_j + e_1) \right) + \bar{\nu}(r - e_j + e_k + e_1) \right] \\
&\leq -c + \frac{\gamma}{N} \left[\bar{\nu}(r - e_j + e_k + e_1) - \nu(r - e_j + e_k) \right] \leq -c + \frac{\gamma}{N} \frac{cN}{\gamma} = 0,
\end{aligned}$$

where the first inequality follows (16), and the second inequality follows (17).

(2.3.2) $j \neq 1$, $r_j > 0$, $1 \in \Upsilon(r, j)$, $1 \notin \Upsilon(r + e_k, j)$. This is not possible when we assume $k \neq 1$.

Next, we consider condition (18) when $r_1 = 0$. That is, we show that if (16) and (17) hold for some V and \bar{V} , condition (18) also holds for TV and $T\bar{V}$. Since, by Lemma 4, no more than one player can have 0 scrips in any state in the recurrent class, we know that player 1 is the only one with 0 scrips and hence,

$$\sum_j TV(r, j) = -c(N-1) + \frac{\gamma}{N} \left(\sum_{j \neq 1} \nu(r + e_1 - e_j) + \nu(r) \right)$$

In this case $\sum_j T\bar{V}(r + e_1, j)$ has the following two possibilities.

R0.1 Player 1 remains the only minimum scrip holder in $r + e_1$. There is a set L with $|L| = l$ of other players having the second least number of scrips.

$$\sum_j T\bar{V}(r + e_1, j) = u - c(N-1) + \frac{\gamma}{N} \left(\sum_{j \neq 1} \bar{\nu}(r + 2e_1 - e_j) + \frac{1}{l} \sum_{i \in L} \bar{\nu}(r + e_i) \right)$$

Therefore,

$$\begin{aligned}
\sum_j T\bar{V}(r + e_1, j) - \sum_j TV(r, j) &= u + \frac{\gamma}{N} \left(\frac{1}{l} \sum_{i \in L} \left(\bar{\nu}(r + e_i) - \nu(r) \right) \right. \\
&\quad \left. + \sum_{j \neq 1} \left(\bar{\nu}(r + 2e_1 - e_j) - \nu(r + e_1 - e_j) \right) \right)
\end{aligned}$$

$$\begin{aligned} &\leq u + \frac{\gamma}{N} \left(\sum_{j \neq 1} \left(\bar{\nu}(r + 2e_1 - e_j) - \nu(r + e_1 - e_j) \right) \right) \\ &\leq u + \frac{\gamma}{N} \frac{cN}{\gamma} (N-1) = u + c(N-1) \leq \frac{N}{\gamma} \left[\frac{N}{\gamma} - (N-1) \right] c, \end{aligned}$$

where the first inequality follows induction hypothesis (16), the second inequality follows (17), and the last inequality follows condition (15).

R0.2 Player 1 becomes a member of the set L , $|L| = l$, of minimum scrip holders in $r + e_1$.

$$\begin{aligned} \sum_j T\bar{V}(r + e_1, j) &= u - c - \frac{N-l}{l}c \\ &+ \frac{\gamma}{N} \left(\sum_{j \notin L} \frac{1}{l} \left(\sum_{i \in L \setminus 1} \bar{\nu}(r + e_1 + e_i - e_j) + \bar{\nu}(r + 2e_1 - e_j) \right) \right) \\ &+ \sum_{j \in L \setminus 1} \frac{1}{l-1} \left(\sum_{i \in L \setminus j, 1} \bar{\nu}(r + e_1 + e_i - e_j) + \bar{\nu}(r + 2e_1 - e_j) \right) + \frac{1}{l-1} \sum_{i \in L \setminus 1} \bar{\nu}(r + e_i) \end{aligned}$$

Therefore

$$\begin{aligned} \sum_j T\bar{V}(r + e_1, j) - \sum_j TV(r, j) &\leq u + c \left((N-2) - \frac{N-l}{l} \right) \\ &+ \frac{\gamma}{N} \left(\frac{1}{l} \sum_{j \notin L} \left(\bar{\nu}(r + 2e_1 - e_j) - \nu(r + e_1 - e_j) \right) \right. \\ &\left. + \frac{1}{l-1} \sum_{j \in L \setminus 1} \left(\bar{\nu}(r + 2e_1 - e_j) - \nu(r + e_1 - e_j) \right) \right) \\ &\leq u + c \left((N-2) - \frac{N-l}{l} \right) + \frac{\gamma}{N} \left(\frac{N-l}{l} + 1 \right) \frac{cN}{\gamma} \leq \frac{N}{\gamma} \left[\frac{N}{\gamma} - (N-1) \right] c, \end{aligned}$$

where the first inequality follows induction hypothesis (16), the second inequality follows (17), and the last inequality follows condition (15).

Now we consider condition (17) when $r_1 > 0$. Following the definition of ν , we have the following cases for $\sum_j TV(r, j)$.

1. Player 1 is the only minimum scrip holder in both r and $r + e_1$. Let L , $|L| = l$, be the set of the second least number of scrip holders.

$$\begin{aligned} \sum_j T\bar{V}(r + e_1, j) - \sum_j TV(r, j) &= \frac{\gamma}{N} \left(\frac{1}{l} \sum_{i \in L} \left(\bar{\nu}(r + e_i) - \nu(r - e_1 + e_i) \right) \right. \\ &\left. + \sum_{j \neq 1} \left(\bar{\nu}(r + 2e_1 - e_j) - \nu(r + e_1 - e_j) \right) \right) \\ &\leq \frac{\gamma}{N} \left\{ \frac{N}{\gamma} \left[\frac{N}{\gamma} - (N-1) \right] + (N-1) \frac{N}{\gamma} \right\} c = \frac{cN}{\gamma}, \end{aligned}$$

where the inequality follows both (17) and (18). Note that if $r_1 > 1$, using only (17) leads to the same inequality, although not tight.

2. Player 1 is the only minimum scrip holder in r . Let L , $|L| = l$, be the set that includes player one plus the $l - 1$ others with exactly one more scrip ($l > 1$).

$$\sum_j TV(r, j) = u - c(N - 1) + \frac{\gamma}{N} \left(\sum_{j \neq 1} \nu(r + e_1 - e_j) + \frac{1}{l-1} \sum_{i \in L \setminus 1} \nu(r + e_i - e_1) \right)$$

As a result, player 1 is one of l minimum scrip holders in $r + e_1$.

$$\begin{aligned} \sum_j T\bar{V}(r + e_1, j) &= u - c - \frac{N-l}{l}c \\ &+ \frac{\gamma}{N} \left(\sum_{j \notin L} \frac{1}{l} \left(\sum_{i \in L \setminus 1} \bar{\nu}(r + e_1 + e_i - e_j) + \bar{\nu}(r + 2e_1 - e_j) \right) \right) \\ &+ \sum_{j \in L \setminus 1} \frac{1}{l-1} \sum_{i \in L \setminus j} \bar{\nu}(r + e_1 + e_i - e_j) + \frac{1}{l-1} \sum_{i \in L \setminus 1} \bar{\nu}(r + e_i) \end{aligned}$$

Therefore

$$\begin{aligned} \sum_j T\bar{V}(r + e_1, j) - \sum_j TV(r, j) &\leq c \left((N-2) - \frac{N-l}{l} \right) \\ &+ \frac{\gamma}{N} \left(\sum_{j \notin L} \frac{1}{l} \left(\bar{\nu}(r + 2e_1 - e_j) - \nu(r + e_1 - e_j) \right) + \frac{1}{l-1} \sum_{i \in L \setminus 1} \left(\bar{\nu}(r + e_i) - \nu(r + e_i - e_1) \right) \right) \\ &+ \frac{1}{l-1} \sum_{j \in L \setminus 1} \left(\bar{\nu}(r + 2e_1 - e_j) - \nu(r + e_1 - e_j) \right) \\ &\leq c \left((N-2) - \frac{N-l}{l} \right) + \frac{\gamma}{N} \left(\frac{N-l}{l} + 1 \right) \frac{cN}{\gamma} + \frac{\gamma}{N} \frac{N}{\gamma} \left[\frac{N}{\gamma} - (N-1) \right] c = \frac{cN}{\gamma}. \end{aligned}$$

where the first inequality follows from (16), and the second inequality follows both (17) and (18). Similar to before, if $r_1 > 1$, we only need (17), which leads to the same inequality, although not tight.

3. Player 1 is in the set L , $|L| = l \geq 2$ of players with minimum scrips.

$$\begin{aligned} \sum_j TV(r, j) &= u - c - \frac{N-l}{l}c + \frac{\gamma}{N} \left(\sum_{j \notin L} \frac{1}{l} \left(\sum_{i \in L \setminus 1} \nu(r + e_i - e_j) + \nu(r + e_1 - e_j) \right) \right) \\ &+ \sum_{j \in L \setminus 1} \frac{1}{l-1} \left(\sum_{i \in L \setminus \{j, 1\}} \nu(r + e_i - e_j) + \nu(r + e_1 - e_j) \right) + \frac{1}{l-1} \sum_{i \in L \setminus 1} \nu(r + e_i - e_1) \end{aligned}$$

(1) $l = 2$. Therefore in $r + e_1$ there is a unique minimum scrip holder, say player k . That is, $L = \{1, k\}$

and player 1 is among the set M , $|M| = m$, of second minimum scrip holders in $r + e_1$.

$$\begin{aligned} \sum_j T\bar{V}(r + e_1, j) &= u - \frac{c}{m} + \frac{\gamma}{N} \left(\sum_{j \notin L} \bar{\nu}(r + e_1 + e_k - e_j) + \bar{\nu}(r + e_k) \right) \\ &+ \frac{1}{m} \left(\sum_{i \in M \setminus 1} \bar{\nu}(r + e_1 + e_i - e_k) + \bar{\nu}(r + 2e_1 - e_k) \right) \end{aligned}$$

$$\begin{aligned} \sum_j T\bar{V}(r + e_1, j) - \sum_j TV(r, j) &\leq \left(1 + \frac{N-2}{2} - \frac{1}{m}\right) c + \frac{\gamma}{N} \frac{N}{\gamma} \left[\frac{N}{\gamma} - (N-1)\right] c \\ &\quad + \frac{\gamma}{N} \frac{cN}{\gamma} \left(\frac{N-2}{2} + \frac{1}{m}\right) = \frac{cN}{\gamma}, \end{aligned}$$

where the inequality follows both (17) and (18). Again, if $r_1 > 1$, it is fine using (17) only, which leads to a more relaxed version of the same inequality.

(2) $l > 2$. Therefore there are $l - 1$ minimum scrip holders in $r + e_1$.

$$\begin{aligned} \sum_j T\bar{V}(r + e_1, j) &= u + \frac{\gamma}{N} \left(\sum_{j \notin L} \frac{1}{l-1} \sum_{i \in L \setminus 1} \bar{\nu}(r + e_1 + e_i - e_j) \right. \\ &\quad \left. + \frac{1}{l-1} \sum_{i \in L \setminus 1} \bar{\nu}(r + e_i) + \sum_{j \in L \setminus 1} \frac{1}{l-2} \sum_{i \in L \setminus \{1, j\}} \bar{\nu}(r + e_i + e_1 - e_j) \right) \\ \sum_j T\bar{V}(r + e_1, j) - \sum_j TV(r, j) &\leq \left(1 + \frac{N-l}{l}\right) c \\ &\quad + \frac{\gamma}{N} \frac{cN}{\gamma} \left(\frac{N-l}{l}(l-1) + l-2\right) + \frac{\gamma}{N} \frac{N}{\gamma} \left[\frac{N}{\gamma} - (N-1)\right] c = \frac{cN}{\gamma} \end{aligned}$$

4. There is a unique minimum scrip holder $k \neq 1$ with $r_k > 0$, and player 1 is in the set $L, |L| = l$, of players who have the second minimum scrips.

$$\begin{aligned} \sum_j TV(r, j) &= u - \frac{c}{l} + \frac{\gamma}{N} \left(\sum_{j \neq k, 1} \nu(r + e_k - e_j) + \nu(r + e_k - e_1) \right. \\ &\quad \left. + \frac{1}{l} \left(\sum_{i \in L \setminus 1} \nu(r + e_i - e_k) + \nu(r + e_1 - e_k) \right) \right) \\ \sum_j T\bar{V}(r + e_1, j) &= u + \frac{\gamma}{N} \left(\sum_{j \neq k, 1} \bar{\nu}(r + e_1 + e_k - e_j) + \bar{\nu}(r + e_k) \right. \\ &\quad \left. + \frac{1}{l-1} \sum_{i \in L \setminus 1} \bar{\nu}(r + e_1 + e_i - e_k) \right) \\ \sum_j T\bar{V}(r + e_1, j) - \sum_j TV(r, j) &\leq \left(\frac{1}{l} + (N-2) + \frac{l-1}{l}\right) c + \frac{\gamma}{N} \frac{N}{\gamma} \left[\frac{N}{\gamma} - (N-1)\right] c = \frac{cN}{\gamma} \end{aligned}$$

5. There is a unique minimum scrip holder $k \neq 1$ with $r_k > 0$, and there are l players who have the second minimum scrips, not including player 1. Following the same logic as before, we obtain

$$\sum_j T\bar{V}(r + e_1, j) - \sum_j TV(r, j) \leq \frac{cN}{\gamma}$$

6. There is a unique minimum scrip holder $k \neq 1$ with $r_k = 0$.

$$\sum_j T\bar{V}(r + e_1, j) - \sum_j TV(r, j) \leq \frac{cN}{\gamma}$$

7. There are $l \geq 2$ minimum scrip holders, not including Player 1.

$$\sum_j TV\bar{V}(r + e_1, j) - \sum_j TV(r, j) \leq \frac{cN}{\gamma}$$

Q.E.D.

Proof of Proposition 5

First we show the result for the general case where $N \geq 3$. Denote $\nu_1 = \sum_j V(e, j)$, where e is the vector of all ones; $\nu_2 = \sum_j V(e + e_1 - e_2, j)$, that is, player 1 has two scrips while another player has zero; $\nu_3 = \sum_j V(e - e_1 + e_2, j)$, that is, player 1 has zero scrips while another player has two; $\nu_4 = \sum_j V(e + e_2 - e_3, j)$. Due to symmetry (see Lemma 5), (e, j) , $(e + e_1 - e_2, j)$, $(e - e_1 + e_2, j)$, and $(e + e_2 - e_3, j)$ for any j are the only types of recurrent states in the system with $R = N$ scrips. The corresponding recursive equations (10) - (12) give us

$$\begin{aligned} \nu_1 &= (u - c) + (\gamma/N)(\nu_2 + \nu_3 + (N - 2)\nu_4) , & \nu_2 &= u + (\gamma/N)(\nu_1 + (N - 1)\nu_2) , \\ \nu_3 &= -(N - 1)c + (\gamma/N)(\nu_1 + \nu_3 + (N - 2)\nu_4) , & \nu_4 &= u + (\gamma/N)(\nu_1 + \nu_3 + (N - 2)\nu_4) \end{aligned}$$

The solution is

$$\begin{aligned} \nu_1 &= \frac{(u - c)N}{(1 - \gamma)(N + \gamma)} \geq 0 , & \nu_2 &= \left(1 - \frac{\gamma}{N}(N - 1)\right)^{-1} \left(\frac{\gamma}{N}\nu_1 + u\right) \geq 0 , \\ \nu_3 &= \left(1 - \frac{\gamma}{N}(N - 1)\right)^{-1} \left[\frac{\gamma}{N}\nu_1 + (N - 1)c \left(\frac{\gamma(N - 2)}{N} - 1\right) + \frac{\gamma(N - 2)}{N}u\right] , \\ \nu_4 &= \left(1 - \frac{\gamma}{N}(N - 1)\right)^{-1} \left[\frac{\gamma}{N}\nu_1 + \frac{1}{N}((N - \gamma)u - (N - 1)\gamma c)\right] \geq 0 . \end{aligned}$$

$\nu_3 \geq 0$ iff

$$\begin{aligned} \frac{u}{c} &\geq \frac{(\gamma N + (N - 1)(N - \gamma(N - 2))(1 - \gamma)(N + \gamma))}{\gamma(N + (N - 2)(1 - \gamma)(N + \gamma))} = 1 + \frac{N((N - 1) - (N - 2)\gamma)(1 - \gamma)(N + \gamma)}{\gamma(N + (N - 2)(1 - \gamma)(N + \gamma))} \\ &= \frac{N}{\gamma} \left(\frac{((N - 1)(1 - \gamma)(N + \gamma) + N\gamma)}{(N + (N - 2)(1 - \gamma)(N + \gamma))} \right) - (N - 1) = \frac{N}{\gamma} \left(1 + \frac{(1 - \gamma)\gamma}{N + (N - 2)(1 - \gamma)(N + \gamma)} \right) - (N - 1) \end{aligned}$$

Under condition (20), the above inequality holds.

For the special case where $N = 2$ and following the same notation as above, the only types of recurrent states are (e, j) , $(e + e_1 - e_2, j)$, and $(e - e_1 + e_2, j)$ for $j = 1, 2$. Thus we keep our definitions $\nu_1 = \sum_j V(e, j)$, $\nu_2 = \sum_j V(e + e_1 - e_2, j)$, and $\nu_3 = \sum_j V(e - e_1 + e_2, j)$, and we no longer have $\nu_4 = \sum_j V(e + e_2 - e_3, j)$ since there are only two players. The corresponding recursive equations (10) - (12) give us

$$\nu_1 = u - c + \frac{\gamma}{2}(\nu_2 + \nu_3) ; \quad \nu_2 = u + \frac{\gamma}{2}(\nu_1 + \nu_2) ; \quad \nu_3 = -c + \frac{\gamma}{2}(\nu_1 + \nu_3)$$

The solution is

$$\nu_1 = \frac{-2(u-c)}{(\gamma+2)(\gamma-1)} \geq 0; \quad \nu_2 = \frac{-2(u\gamma^2 + c\gamma - 2u)}{(\gamma+2)(\gamma-1)(\gamma-2)}; \quad \nu_3 = \frac{2(c\gamma^2 + u\gamma - 2c)}{(\gamma+2)(\gamma-1)(\gamma-2)}$$

$\nu_2 \geq 0$ iff $\frac{u}{c} \geq \frac{-\gamma}{\gamma^2-2}$, which is true under condition (20); $\nu_3 \geq 0$ iff $\frac{u}{c} \geq \frac{2-\gamma^2}{\gamma}$, which is also true under condition (20).

Q.E.D.

Proof of Proposition 6

Similar to the proof of Proposition 4, we first show that for any functions \bar{V} and V that satisfy (25)-(26), condition (25) holds with $\Xi_\kappa \bar{V}$ and $\Xi_\kappa V$, with any given κ .

First consider condition (25). Still denote $\nu(r) = \sum_j V(r, j)$, and $\bar{\nu}$ similarly.

$$(1.1) \quad j = 1, r_1 = 0.$$

$$(\Xi_\kappa V)(r, 1) = (\gamma/N)\nu(r) \geq (\gamma/N)\bar{\nu}(r + e_k) = (\Xi_\kappa \bar{V})(r + e_k, 1).$$

$$(1.2) \quad j = 1, r_1 > 0,$$

$$\begin{aligned} (\Xi_\kappa V)(r, 1) &= u + \gamma/(N|\Upsilon_\kappa(r, 1)|) \sum_{i \in \Upsilon_\kappa(r, 1)} \nu(r - e_1 + e_i) \\ (\Xi_\kappa \bar{V})(r + e_k, 1) &= u + \gamma/(N|\Upsilon_\kappa(r + e_k, 1)|) \sum_{i \in \Upsilon_\kappa(r + e_k, 1)} \bar{\nu}(r - e_1 + e_i + e_k). \end{aligned}$$

Consider the following 3 possibilities.

(i) $\Upsilon_\kappa(r + e_k, 1) = \Upsilon_\kappa(r, 1)$. In this case, $\bar{V}(r - e_1 + e_i + e_k, j') \leq V(r - e_1 + e_i, j')$ from induction hypothesis. Therefore we have $(\Xi_\kappa \bar{V})(r + e_k, 1) \leq (\Xi_\kappa V)(r, 1)$.

(ii) $\Upsilon_\kappa(r + e_k, 1) \cup \{k\} = \Upsilon_\kappa(r, 1)$. In this case, $\bar{\nu}(r - e_1 + e_i + e_k)$ is the same for all $i \in \Upsilon_\kappa(r + e_k, 1)$. Therefore $\sum_{i \in \Upsilon_\kappa(r + e_k, 1)} \bar{\nu}(r - e_1 + e_i + e_k)/|\Upsilon_\kappa(r + e_k, 1)|$ equals $\bar{\nu}(r - e_1 + e_i + e_k)$ for any particular i . The same can be said for $\nu(r - e_1 + e_i)$. The result then follows the comparison between $\nu(r - e_1 + e_i)$ and $\bar{\nu}(r - e_1 + e_i + e_k)$ according to the induction hypothesis.

(iii) $\Upsilon_\kappa(r, 1) = \{k\}$. We have $V(r - e_1 + e_k, j') \geq \bar{V}(r - e_1 + e_i + e_k, j')$ for any $i \neq 1$. Therefore,

$$\begin{aligned} (\Xi_\kappa V)(r, 1) &= u + (\gamma/N)\nu(r - e_1 + e_k) \\ &\geq u + \gamma/(N|\Upsilon_\kappa(r + e_k, 1)|) \sum_{i \in \Upsilon_\kappa(r + e_k, 1)} \bar{\nu}(r - e_1 + e_i + e_k) = (\Xi_\kappa \bar{V})(r + e_k, 1). \end{aligned}$$

$$(2.1.1) \quad j \neq 1, r_j = 0, k \neq j,$$

$$(\Xi_\kappa V)(r, j) = (\gamma/N)\nu(r) \geq (\gamma/N)\bar{\nu}(r + e_k) = (\Xi_\kappa \bar{V})(r + e_k, j).$$

$$(2.1.2.1) \quad j \neq 1, r_j = 0, k = j, 1 \notin \Upsilon_\kappa(r + e_j, j)$$

$$\begin{aligned} (\Xi_\kappa \bar{V})(r + e_j, j) &= \gamma / (N |\Upsilon_\kappa(r + e_j, j)|) \sum_{i \in \Upsilon_\kappa(r + e_j, j)} \bar{\nu}(r + e_i) \\ &\leq \gamma / (N |\Upsilon_\kappa(r + e_j, j)|) \sum_{i \in \Upsilon_\kappa(r + e_j, j)} \nu(r) = (\gamma/N) \nu(r) = (\Xi_\kappa V)(r, j). \end{aligned}$$

$$(2.1.2.2) \quad j \neq 1, r_j = 0, k = j, 1 \in \Upsilon_\kappa(r + e_j, j).$$

$$\begin{aligned} (\Xi_\kappa \bar{V})(r + e_j, j) &= \left(-c + \gamma/N \sum_{i \in \Upsilon_\kappa(r + e_j, j)} \bar{\nu}(r + e_i) \right) / |\Upsilon_\kappa(r + e_j, j)| \\ &= \frac{-c}{|\Upsilon_\kappa(r + e_j, j)|} + \frac{\gamma}{N |\Upsilon_\kappa(r + e_j, j)|} \left(\sum_{i \in \Upsilon_\kappa(r + e_j, j) \setminus 1} \bar{\nu}(r + e_i) + \bar{\nu}(r + e_1) \right) \\ &\leq \frac{-c}{|\Upsilon_\kappa(r + e_j, j)|} + \frac{\gamma}{N |\Upsilon_\kappa(r + e_j, j)|} \left(\sum_{i \in \Upsilon_\kappa(r + e_j, j) \setminus 1} \bar{\nu}(r + e_i) + \nu(r) + \frac{cN}{\gamma} \right) \\ &\leq \frac{-c}{|\Upsilon_\kappa(r + e_j, j)|} + \frac{\gamma}{N |\Upsilon_\kappa(r + e_j, j)|} \left(\sum_{i \in \Upsilon_\kappa(r + e_j, j) \setminus 1} \nu(r) + \nu(r) + \frac{cN}{\gamma} \right) \\ &= \frac{-c}{|\Upsilon_\kappa(r + e_j, j)|} + \frac{\gamma}{N |\Upsilon_\kappa(r + e_j, j)|} \left(|\Upsilon_\kappa(r + e_j, j)| \nu(r) + \frac{cN}{\gamma} \right) \\ &= \frac{\gamma}{N} \nu(r) = (\Xi_\kappa V)(r, j) \end{aligned}$$

where the first inequality follows from (26), and the second inequality follows from (25).

$$(2.2.1) \quad j \neq 1, r_j > 0, 1 \notin \Upsilon_\kappa(r, j), 1 \notin \Upsilon_\kappa(r + e_k, j).$$

$$\begin{aligned} (\Xi_\kappa V)(r, j) &= \gamma / (N |\Upsilon_\kappa(r, j)|) \sum_{i \in \Upsilon_\kappa(r, j)} \nu(r - e_j + e_i) \\ (\Xi_\kappa \bar{V})(r + e_k, j) &= \gamma / (N |\Upsilon_\kappa(r + e_k, j)|) \sum_{i \in \Upsilon_\kappa(r + e_k, j)} \bar{\nu}(r - e_j + e_k + e_i). \end{aligned}$$

Following the 3 cases as in (1.2), we have the result.

(2.2.2) $j \neq 1, r_j > 0, 1 \notin \Upsilon_\kappa(r, j), 1 \in \Upsilon_\kappa(r + e_k, j)$. This means that we must have $\Upsilon_\kappa(r, j) = \{k\}$, and $r_1 = r_k + 1$.

$$\begin{aligned} (\Xi_\kappa \bar{V})(r + e_k, j) &= \left(-c + \gamma/N \sum_{i \in \Upsilon_\kappa(r + e_k, j)} \bar{\nu}(r - e_j + e_k + e_i) \right) / |\Upsilon_\kappa(r + e_k, j)| \\ &= \frac{-c}{|\Upsilon_\kappa(r + e_k, j)|} + \frac{\gamma}{N |\Upsilon_\kappa(r + e_k, j)|} \left(\sum_{i \in \Upsilon_\kappa(r + e_k, j) \setminus 1} \bar{\nu}(r - e_j + e_k + e_i) + \bar{\nu}(r - e_j + e_k + e_1) \right) \\ &\leq \frac{-c}{|\Upsilon_\kappa(r + e_k, j)|} + \frac{\gamma}{N |\Upsilon_\kappa(r + e_k, j)|} \left(\sum_{i \in \Upsilon_\kappa(r + e_k, j) \setminus 1} \bar{\nu}(r - e_j + e_k + e_i) + \nu(r - e_j + e_k) + \frac{cN}{\gamma} \right) \end{aligned}$$

$$\begin{aligned}
 &\leq \frac{-c}{|\Upsilon_\kappa(r+e_k, j)|} + \frac{\gamma}{N|\Upsilon_\kappa(r+e_k, j)|} \left(\sum_{i \in \Upsilon_\kappa(r+e_k, j) \setminus 1} \nu(r-e_j+e_k) + \nu(r-e_j+e_k) + \frac{cN}{\gamma} \right) \\
 &= \frac{-c}{|\Upsilon_\kappa(r+e_k, j)|} + \frac{\gamma}{N|\Upsilon_\kappa(r+e_k, j)|} \left(|\Upsilon_\kappa(r+e_k, j)|\nu(r-e_j+e_k) + \frac{cN}{\gamma} \right) \\
 &= \frac{\gamma}{N}\nu(r-e_j+e_k) = (\Xi_\kappa V)(r, j) .
 \end{aligned}$$

Here, the first inequality follows from (26), the second inequality follows from (25), and the last equation follows from the fact that $\Upsilon_\kappa(r, j)$ is a singleton with only one element, k .

$$(2.3.1) \quad j \neq 1, r_j > 0, 1 \in \Upsilon_\kappa(r, j), 1 \in \Upsilon_\kappa(r+e_k, j).$$

$$\begin{aligned}
 (\Xi_\kappa V)(r, j) &= \left(-c + \gamma/N \sum_{i \in \Upsilon_\kappa(r, j)} \nu(r-e_j+e_i) \right) / |\Upsilon_\kappa(r, j)| \\
 (\Xi_\kappa \bar{V})(r+e_k, j) &= \left(-c + \gamma/N \sum_{i \in \Upsilon_\kappa(r+e_k, j)} \bar{\nu}(r-e_j+e_k+e_i) \right) / |\Upsilon_\kappa(r+e_k, j)|
 \end{aligned}$$

Similar to case (1.2), we consider the three cases. When $\Upsilon_\kappa(r+e_k, j) = \Upsilon_\kappa(r, j)$, we have the result directly from condition (25). Since here we assume that $1 \in \Upsilon_\kappa(r, j)$, it is impossible that $\Upsilon_\kappa(r, j) = \{k\}$.

Now we focus on the case where $|\Upsilon_\kappa(r+e_k, j)| + 1 = |\Upsilon_\kappa(r, j)| = L > 1$, that is, $\Upsilon_\kappa(r+e_k, j) \cup \{k\} = \Upsilon_\kappa(r, j)$.

$$\begin{aligned}
 &L(L-1) (\Xi_\kappa \bar{V}(r+e_k, j) - \Xi_\kappa \bar{V}(r, j)) \\
 &= -c + \frac{\gamma}{N} \left[L \left(\sum_{i \in \Upsilon_\kappa(r+e_k, j) \setminus 1} \bar{\nu}(r-e_j+e_k+e_i) + \bar{\nu}(r-e_j+e_k+e_1) \right) \right. \\
 &\quad \left. - (L-1) \left(\sum_{i \in \Upsilon_\kappa(r+e_k, j) \setminus 1} \nu(r-e_j+e_i) + \nu(r-e_j+e_k) + \nu(r-e_j+e_1) \right) \right] \\
 &= -c + \frac{\gamma}{N} \left[(L-1) \sum_{i \in \Upsilon_\kappa(r+e_k, j) \setminus 1} \left(\bar{\nu}(r-e_j+e_i+e_k) - \nu(r-e_j+e_i) \right) \right. \\
 &\quad \left. + \sum_{i \in \Upsilon_\kappa(r+e_k, j) \setminus 1} \left(\bar{\nu}(r-e_j+e_i+e_k) - \nu(r-e_j+e_k) \right) - \nu(r-e_j+e_k) \right. \\
 &\quad \left. + (L-1) \left(\bar{\nu}(r-e_j+e_k+e_1) - \nu(r-e_j+e_1) \right) + \bar{\nu}(r-e_j+e_k+e_1) \right] \\
 &\leq -c + \frac{\gamma}{N} \left[\bar{\nu}(r-e_j+e_k+e_1) - \nu(r-e_j+e_k) \right] \leq -c + \frac{\gamma}{N} \frac{cN}{\gamma} = 0 ,
 \end{aligned}$$

where the first inequality follows (25), and the second inequality follows (26).

$$(2.3.2) \quad j \neq 1, r_j > 0, 1 \in \Upsilon_\kappa(r, j), 1 \notin \Upsilon_\kappa(r+e_k, j). \text{ This is not possible when we assume } k \neq 1.$$

Next, we consider condition (26). First, consider the case when $r_1 = 0$, and. Denote C_n^k to be n choose k .

We have the following possibilities.

R0.1 Player 1 is the only one with zero scrip in r and remains the only minimum scrip holder in $r + e_1$. Denote M to be the product of the probability that player 1 provides service when another (random) player is the service requester times $N - 1$.

$$\begin{aligned} \sum_j (TV)(r, j) &= \sum_j \sum_{\kappa_j} (\Xi_{\kappa_j} V)(r, j) / C_{N-1}^K = -cM + \frac{\gamma}{N} \left[\nu(r) + \sum_{j \neq 1} \left(\sum_{\kappa_j: 1 \in \kappa_j} \nu(r + e_1 - e_j) + \sum_{\kappa_j: 1 \notin \kappa_j} \nu(r'_{\kappa_j}) \right) / C_{N-1}^K \right] \\ \sum_j (T\bar{V})(r + e_1, j) &= \sum_j \sum_{\kappa_j} (\Xi_{\kappa_j} \bar{V})(r + e_1, j) / C_{N-1}^K \\ &= u - cM + \frac{\gamma}{N} \left[\sum_{\kappa_1} \bar{\nu}(r + e_{\kappa_1}) + \sum_{j \neq 1} \left(\sum_{\kappa_j: 1 \in \kappa_j} \bar{\nu}(r + 2e_1 - e_j) + \sum_{\kappa_j: 1 \notin \kappa_j} \bar{\nu}(r'_{\kappa_j} + e_1) \right) \right] / C_{N-1}^K \end{aligned}$$

Therefore,

$$\sum_j (T\bar{V})(r + e_1, j) - (TV)(r, j) \leq u + \frac{\gamma}{N} \frac{cN}{\gamma} (N - 1) \leq \frac{N}{\gamma} c - c(N - 1) + c(N - 1) = \frac{cN}{\gamma},$$

where the first inequality follows induction hypothesis (25) as well as (25), and the second inequality follows condition (24).

R0.2 Player 1 is NOT the only one with zero scrip in r . Similar to the previous case, the difference between $\sum_j (T\bar{V})(r + e_1, j)$ and $\sum_j (TV)(r, j)$ is upper bounded by u coming from the case $j = 1$, as well as a collection of c terms. In the summations of all the cases over j , κ_j and service providers, either player 1 is not the service provider, in which case $\bar{\nu}$ and ν differ by e_1 , contributing a term cN/γ in the difference following (25), which needs to be multiplied by the γ/N factor; or player 1 is selected as the service provider, resulting in direct contribution of c in the difference, while the $\bar{\nu}$ and ν terms differ with e_i with $i \neq 1$. Collecting the c terms in all cases, the total difference is, therefore, exactly $c(N - 1)$. Following the logic exactly as case R0.1, we have, again,

$$\sum_j (T\bar{V})(r + e_1, j) - (TV)(r, j) \leq u + c(N - 1) \leq \frac{cN}{\gamma}.$$

Now consider $r_1 > 0$. There is no difference of u between $\sum_j (T\bar{V})(r + e_1, j)$ and $\sum_j (TV)(r, j)$ anymore. Instead, when $j = 1$, the difference between the $\bar{\nu}$ and ν terms are always e_1 . Otherwise all other arguments in R0.2 follows. Consequently, the overall difference is bounded by $cN < cN/\gamma$.

Q.E.D.