Chapter 8

WAVEGUIDES AND CAVITIES

This chapter discusses the typical rectangular and cylindrical waveguide and cavity problems. We start with the basic observation that the TEM modes cannot exist in a single-metal waveguide. Then we develop formulations for simple TE and TM modes. Instead of using the vector potentials **A** and **F**, we will solve the field variables directly by expressing all other field components in terms of the z components of the electromagnetic fields. In fact, for general solutions, it appears that the formulation using the field quantities is more convenient than that using the vector potentials.

8.1 General Theory

In this section we discuss time-harmonic waves propagating in z direction parallel to the direction of uniform waveguides. For a uniform waveguide with an arbitrary cross section, we can write the instantaneous expression for the **E** field as

$$\mathbf{E}(x, y, z; t) = \Re e\{\mathbf{e}(x, y)e^{j\omega t - jk_z z}\}, \tag{8.1}$$

where $\mathbf{e}(x, y)$ is a two-dimensional vector phasor that depends only on the transverse coordinates (x, y). With this dependence in z, we can express the transverse field components in terms of z components E_z and H_z. To achieve this goal, we can write the fields and ∇ operator in terms of the transverse components and the z components:

$$\mathbf{e}(x, y) = \mathbf{e}_t + \hat{z}e_z, \quad \mathbf{h}(x, y) = \mathbf{h}_t + \hat{z}h_z, \quad \nabla = \nabla_t + \hat{z}\frac{\partial}{\partial z} = \nabla_t - jk_z \tag{8.2}$$

Substituting equation (8.2) into Maxwell's equations and equating the transverse and z components we arrive at

$$\nabla_t \times \mathbf{E}_t = -j\omega\mu\hat{z}H_z, \tag{8.3}$$

$$-\hat{z} \times \nabla_t E_z - jk_z \hat{z} \times \mathbf{E}_t = -j\omega\mu \mathbf{H}_t \tag{8.4}$$

$$\nabla_t \times \mathbf{H}_t = j\omega\epsilon \hat{z} E_z, \tag{8.5}$$

$$-\hat{z} \times \nabla_t H_z - jk_z \hat{z} \times \mathbf{H}_t = j\omega\epsilon \mathbf{E}_t \tag{8.6}$$

From (8.3)–(8.6) we can thus obtain

$$\mathbf{E}_t = -\frac{j}{k_t^2}[k_z \nabla_t E_z - \omega\mu \hat{z} \times \nabla_t H_z], \tag{8.7}$$

$$\mathbf{H}_t = -\frac{j}{k_t^2}[k_z \nabla_t H_z + \omega\epsilon \hat{z} \times \nabla_t E_z], \tag{8.8}$$

where

$$k_t^2 = k^2 - k_z^2. \tag{8.9}$$

Note (8.7) and (8.8) express the transverse components in terms of the z (longitudinal) components of the fields, and are independent of coordinates. The longitudinal components of the fields satisfy the Helmholtz equations which lead to

$$\nabla_t^2 e_z + k_t^2 e_z = 0, \tag{8.10}$$

$$\nabla_t^2 h_z + k_t^2 h_z = 0, \tag{8.11}$$

as well as the appropriate boundary conditions. For any uniform waveguides, we can use (8.10) and (8.11) together with the boundary conditions to determine $e_z(x, y)$ and $h_z(x, y)$, and then use (8.7)–(8.8) to find other components.

In particular, for Cartesian coordinates $\mathbf{e}_t = \hat{x} e_x + \hat{y} \mathbf{e}_y$, $\nabla_t = \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y}$. Therefore (8.7) and (8.8) become

$$e_x = -\frac{j}{k_t^2}\Big(k_z \frac{\partial e_z}{\partial x} + \omega\mu \frac{\partial h_z}{\partial y}\Big), \tag{8.12}$$

$$e_y = -\frac{j}{k_t^2}\Big(k_z \frac{\partial e_z}{\partial y} - \omega\mu \frac{\partial h_z}{\partial x}\Big), \tag{8.13}$$

$$h_x = -\frac{j}{k_t^2}\Big(k_z \frac{\partial h_z}{\partial x} - \omega\epsilon \frac{\partial e_z}{\partial y}\Big), \tag{8.14}$$

$$h_y = -\frac{j}{k_t^2}\Big(k_z \frac{\partial h_z}{\partial y} + \omega\epsilon \frac{\partial e_z}{\partial x}\Big), \tag{8.15}$$

For cylindrical coordinates, $\mathbf{e}_t = \hat{\rho} e_\rho + \hat{\phi} \mathbf{E}_\phi^0$, $\nabla_t = \hat{\rho}\frac{\partial}{\partial \rho} + \frac{1}{\rho}\hat{\phi}\frac{\partial}{\partial \phi}$. Therefore (8.7) and (8.8) become

$$e_\rho = -\frac{j}{k_t^2}\Big(k_z \frac{\partial e_z}{\partial \rho} + \frac{\omega\mu}{\rho} \frac{\partial h_z}{\partial \phi}\Big), \tag{8.16}$$

$$e_\phi = -\frac{j}{k_t^2}\Big(\frac{k_z}{\rho} \frac{\partial e_z}{\partial \phi} - \omega\mu \frac{\partial h_z}{\partial \rho}\Big), \tag{8.17}$$

$$h_\rho = -\frac{j}{k_t^2}\Big(k_z\frac{\partial h_z}{\partial \rho} - \frac{\omega\epsilon}{\rho}\frac{\partial e_z}{\partial \phi}\Big), \tag{8.18}$$

$$h_\phi = -\frac{j}{k_t^2}\Big(\frac{k_z}{\rho}\frac{\partial h_z}{\partial \phi} + \omega\epsilon\frac{\partial e_z}{\partial \rho}\Big), \tag{8.19}$$

According to the longitudinal components, there are three types of waves:

1. **TEM (transverse electromagnetic) waves:** $E_z = 0$ and $H_z = 0$.
2. **TM (transverse magnetic) waves:** $E_z \neq 0$ and $H_z = 0$.
3. **TE (transverse electric) waves:** $E_z = 0$ and $H_z \neq 0$.

8.1.1 TEM (Transverse Electromagnetic) Waves

For TEM waves, $H_z = 0$ and $E_z = 0$. Therefore, from (8.10) and (8.11) a nontrivial solution of **E** and **H** requires

$$k_t^2 = k^2 - k_z^2 = 0, \tag{8.20}$$

or

$$k_z = k = \omega\sqrt{\mu\epsilon}, \tag{8.21}$$

exactly the same as for plane waves in an unbounded homogeneous medium. The **phase velocity** is

$$u_p = \frac{\omega}{k_z} = \frac{1}{\sqrt{\mu\epsilon}}. \tag{8.22}$$

And the **wave impedance** for TEM waves can be obtained via, for example, (8.4)

$$Z_{TEM} = \frac{e_x}{h_y} = -\frac{e_y}{h_x} = \sqrt{\frac{\mu}{\epsilon}} = \eta. \tag{8.23}$$

Note that both the **phase velocity** and the **wave impedance** for TEM waves are independent of frequency.

Similar to plane waves in unbounded media, the relation between **E** and **H** is

$$\mathbf{H} = \frac{1}{Z_{TEM}}\hat{z}\times\mathbf{E}, \tag{8.24}$$

which can be obtained by (8.4) or (8.6) by letting $E_z = 0$ and $H_z = 0$.

Single-conductor waveguides cannot support TEM waves. To arrive at this conclusion, we note that for TEM waves $E_z = 0$ and $H_z = 0$. However, we know that **H** forms a closed loop. In order for $\mathbf{H}_t$ to form a closed loop, there must be a longitudinal components of electric current (either displacement or conduction current). But by definition, $\mathbf{J}_{dz} = j\omega\epsilon E_z = 0$ for TEM waves; furthermore, and for a single conductor $J_{cz} = 0$ since there is no conductor in the center. Therefore, a single-conductor waveguide cannot support TEM waves. Only waveguides with more than one conductor, e.g. the parallel-plate waveguide, can support pure TEM waves.

8.1.2 TM (Transverse Magnetic) Waves

For TM waves, $H_z = 0$ but $E_z \neq 0$. Hence, equations to be solved are reduced from (8.10), (8.7) and (8.8) to

$$(\nabla_t^2 + k_t^2)e_z(x,y) = 0, \tag{8.25}$$

$$\mathbf{E}_t = -\frac{jk_z}{k_t^2}\nabla_t E_z, \tag{8.26}$$

$$\mathbf{H}_t = -\frac{j\omega\epsilon}{k_t^2}\hat{z}\times\nabla_t E_z, \tag{8.27}$$

The wave impedance is

$$Z_{TM} = \frac{e_x}{h_y} = -\frac{e_y}{h_x} = \frac{\gamma}{j\omega\epsilon}. \tag{8.28}$$

With this wave impedance, the magnetic field is related to the electric field by

$$\mathbf{H} = \frac{1}{Z_{TM}}(\hat{z}\times\mathbf{E}). \tag{8.29}$$

• **Eigenvalues of k_t^2:** Solutions to (8.25) are possible only at some discrete values of $k_t^2 = k_{tn}^2, n = 1, 2, \cdots$. These discrete values are called the eigenvalues.
• **Propagation constant for TM waves:**

$$\gamma_z = jk_z = \sqrt{k_{tn}^2 - k^2} = j\sqrt{\omega^2\mu\epsilon - k_{tn}^2}. \tag{8.30}$$

• **Cutoff frequency:** The value of f such that $k_z = 0$ is called the cutoff frequency and is given by

$$f_{cn} = \frac{k_{tn}}{2\pi\sqrt{\mu\epsilon}}. \tag{8.31}$$

Written in terms of the cutoff frequency, the propagation constant is

$$k_{zn} = k\sqrt{1 - \left(\frac{f_{cn}}{f}\right)^2}. \tag{8.32}$$

In the above, the subscript n is specified to emphasize the fact that k_t and other related constants take discrete values.
• **Above Cutoff Frequency:** If $f > f_{cn}$, then k_{zn} in (8.32) is purely real for a lossless medium. Thus the wave is a propagating wave with a phase constant

$$\beta_z = k\sqrt{1 - \left(\frac{f_{cn}}{f}\right)^2}. \tag{8.33}$$

The wavelength in the waveguide is

$$\lambda_g = \frac{2\pi}{\beta_z} = \frac{\lambda}{\sqrt{1 - \left(\frac{f_{cn}}{f}\right)^2}} \tag{8.34}$$

which is larger than the wavelength $\lambda = 2\pi/k = 1/f\sqrt{\mu\epsilon}$ in an unbounded medium. The phase velocity is

$$u_p = \frac{\omega}{\beta_z} = \frac{c}{\sqrt{1-\left(\frac{f_{cn}}{f}\right)^2}}, \tag{8.35}$$

and the group velocity is

$$u_g = \frac{1}{d\beta_z/d\omega} = c\sqrt{1-\left(\frac{f_{cn}}{f}\right)^2}, \tag{8.36}$$

where $c = 1/\sqrt{\mu\epsilon}$ is the velocity of light in the dielectric medium. From (8.35) it is seen that are dispersive transmission systems for TM waves. The wave impedance is purely resistive and is given by

$$Z_{TM} = \eta\sqrt{1-\left(\frac{f_{cn}}{f}\right)^2} < \eta. \tag{8.37}$$

• **Below Cutoff Frequency:** If $f < f_{cn}$, then k_{zn} in (8.32) is purely imaginary

$$k_{zn} = -j\alpha_z = -jk\sqrt{\left(\frac{f_{cn}}{f}\right)^2 - 1}. \tag{8.38}$$

This makes the field diminish rapidly with z. Therefore, below the cutoff frequency, the wave cannot propagate in the waveguide. The wave impedance becomes purely reactive

$$Z_{TM} = -j\eta\sqrt{\left(\frac{f_{cn}}{f}\right)^2 - 1}. \tag{8.39}$$

8.1.3 TE (Transverse Electric) Waves

For TE waves, $E_z = 0$ but $H_z \neq 0$. Hence, equations to be solved are reduced from (8.2)–(8.5) and (8.8) to

$$(\nabla_t^2 + k_t^2)h_z(x,y) = 0, \tag{8.40}$$

$$\mathbf{E}_t = \frac{j\omega\mu}{k_t^2}\hat{z}\times\nabla_t H_z, \qquad \mathbf{H}_t = -\frac{jk_z}{k_t^2}\nabla_t H_z, \tag{8.41}$$

The wave impedance is

$$Z_{TE} = \frac{e_x}{h_y} = -\frac{e_y}{h_x} = \frac{\omega\mu}{k_z}. \tag{8.42}$$

With this wave impedance, the electric field is related to the magnetic field by

$$\mathbf{E} = -Z_{TE}(\hat{z}\times\mathbf{H}). \tag{8.43}$$

• **Cutoff frequency and propagation constant:**

$$f_{cn} = \frac{k_{tn}}{2\pi\sqrt{\mu\epsilon}}, \tag{8.44}$$

$$k_{zn} = k\sqrt{1 - \left(\frac{f_{cn}}{f}\right)^2}. \tag{8.45}$$

• **Above Cutoff Frequency:** If $f > f_{cn}$, we have $k_{zn} = \beta_{zn}$ with

$$\beta_{zn} = k\sqrt{1 - \left(\frac{f_{cn}}{f}\right)^2}, \tag{8.46}$$

$$\lambda_g = \frac{2\pi}{\beta_{zn}} = \frac{\lambda}{\sqrt{1 - \left(\frac{f_{cn}}{f}\right)^2}} \tag{8.47}$$

$$u_p = \frac{\omega}{\beta_{zn}} = \frac{c}{\sqrt{1 - \left(\frac{f_{cn}}{f}\right)^2}}, \tag{8.48}$$

$$u_g = \frac{1}{d\beta_{zn}/d\omega} = c\sqrt{1 - \left(\frac{f_{cn}}{f}\right)^2}. \tag{8.49}$$

The wave impedance is purely resistive and is given by

$$Z_{TE} = \frac{\eta}{\sqrt{1 - \left(f_{cn}/f\right)^2}} > \eta. \tag{8.50}$$

• **Below Cutoff Frequency:** If $f < f_{cn}$, then k_{zn} in (8.32) is purely imaginary

$$k_{zn} = -j\alpha = -jk\sqrt{\left(\frac{f_{cn}}{f}\right)^2 - 1}. \tag{8.51}$$

This makes the field diminish rapidly with z. Therefore, below the cutoff frequency, the wave cannot propagate in the waveguide. The wave impedance becomes purely reactive

$$Z_{TE} = j\frac{\eta}{\sqrt{\left(f_{cn}/f\right)^2 - 1}}. \tag{8.52}$$

8.2 Parallel-Plate Waveguide

8.2.1 TEM Waves along a Parallel-Plate Waveguide

A y-polarized transverse electromagnetic (TEM) wave propagating in z direction inside two wide parallel plates (width w) at $y = 0$ and $y = b$. If the fringe effect is neglected, the phasor solution of EM field is

$$\mathbf{E} = \hat{y}E_y = \hat{y}E_0 e^{-jkz}, \tag{8.53}$$

$$\mathbf{H} = \hat{x}H_x = -\hat{x}\frac{E_0}{\eta}e^{-jkz}, \tag{8.54}$$

where $k = \omega\sqrt{\mu\epsilon}$, $\eta = \sqrt{\mu/\epsilon}$. Boundary conditions at $y = 0$ and $y = b$ are satisfied since at these plates $E_x = E_z = 0$ and $H_y = 0$.

The surface current density on the upper plate at $y = b$ is

$$\mathbf{J}_{su} = -\hat{y} \times \mathbf{H} = -\hat{z}H_x = \hat{z}\frac{E_0}{\eta}e^{-j\beta z}. \tag{8.55}$$

Similarly, the surface current density on the lower plate is $\mathbf{J}_{sl} = -\mathbf{J}_{su}$.

8.2.2 TM Waves in Parallel-Plate Waveguides

Consider two parallel plates located at $y = 0$ and $y = b$, respectively. The medium between the plates has the constitutive parameters ϵ and μ. The plates are assume infinite in extent in the x direction, and hence the edge effects are negligible. The TM waves propagate in $+z$ direction, and is in-varying in the x direction.

All components of TM waves ($H_z = 0$) can be determined by the $E_z(y, z)$ component alone. The phasor $E_z(y, z) = e_z(y)e^{-jk_z z}$ has this common factor where

$$\frac{d^2 e_z(y)}{dy^2} + k_t^2 e_z(y) = 0. \tag{8.56}$$

The boundary conditions are $e_z(y = 0) = 0$ and $e_z(y = b) = 0$. The solution satisfying (8.47) and the boundary conditions is of the form

$$e_z(y) = A_n \sin(k_{tn} y) \tag{8.57}$$

where the eigenvalues

$$k_{tn} = \frac{n\pi}{b}, \qquad n = 1, 2, 3, \cdots \tag{8.58}$$

and A_n is a is the amplitude dependent on the excitation of the particular TM wave.

Other field component can be found from (8.12)–(8.15). The only other nonzero field components are

$$e_y(y) = -\frac{\gamma}{k_{tn}}A_n \cos(k_{tn} y), \qquad h_x(y) = \frac{j\omega\epsilon}{k_{tn}}A_n \cos(k_{tn} y). \tag{8.59}$$

• **Propagation constant:**

$$\gamma = jk_z = j\sqrt{\left(\omega^2\mu\epsilon - \frac{n\pi}{b}\right)^2}. \qquad (8.60)$$

• **Cutoff frequency:**

$$f_{cn} = \frac{n}{2b\sqrt{\mu\epsilon}}. \qquad (8.61)$$

For $f > f_{cn}$, waves propagate; and waves with $f \le f_{cn}$ are evanescent.
• **Mode order:** TM_n mode for k_{tn}, $n = 1, 2, 3, \cdots$. The TM_0 mode has zero E_z and H_z components, and hence is the same as the TEM mode.
• **Dominant mode: The mode having the lowest cutoff frequency.** For parallel-plate guide, the dominant mode is the TEM mode.

8.2.3 TE Waves in Parallel-Plate Waveguides

For TE waves, $E_z = 0$ and $H_z \neq 0$. The component H_z satisfies

$$\frac{d^2 h_z(y)}{dy^2} + k_t^2 h_z(y) = 0, \qquad (8.62)$$

and the boundary conditions $e_x(y = 0) = 0$ and $e_x(y = b) = 0$ (or $dh_z(y)/dy = 0$ at $y = 0$ and $y = b$). The solutions satisfying (8.63) and the boundary conditions are

$$\begin{aligned} h_z(y) &= B_n \cos(k_{tn} y), \\ h_y(y) &= \frac{jk_z}{k_{tn}} B_n \sin(k_{tn} y), \\ e_x(y) &= \frac{j\omega\mu}{k_{tn}} B_n \sin(k_{tn} y), \end{aligned}$$

where the eigenvalues k_{tn} are the same as (8.49) for TM waves.
• **Cutoff frequencies:** Same as those for TM waves (8.62).
• **TE_n modes:** $n = 1, 2, 3, \cdots$. Note that TE_0 mode does not exist.

8.3 Energy-Transport Velocity

For frequencies above the cutoff frequency, the phase velocity u_p and group velocity u_g are used to describe the propagation of *narrow-band* signals. We now introduce a **energy-transport velocity** u_{en} to describe the propagation of wide-band signals.

In a lossless waveguide, the **energy-transport velocity** u_{en} is defined as the the ratio of the time-averaged propagated power to the time-averaged stored energy per unit guide length:

$$u_{en} = \frac{(P_z)_{av}}{W'_{av}} \qquad \text{(m/s)}, \qquad (8.63)$$

where

$$(P_z)_{av} = \int_S \mathcal{P}_{av} \cdot d\mathbf{s}, \tag{8.64}$$

$$W'_{av} = \int_S [(w_e)_{av} + (w_m)_{av}] ds, \tag{8.65}$$

and the integration is over the cross section of the guide.

For a TM_n mode in a lossless parallel-plate waveguide, for a unit width,

$$(P_z)_{av} = \frac{\omega\epsilon\beta b}{4k_t^2} A_n^2, \quad W'_{av} = \frac{\epsilon b}{4k_t^2} k^2 A_n^2. \tag{8.66}$$

Hence,

$$u_{en} = \frac{(P_z)_{av}}{W'_{av}} = c\sqrt{1-(f_c/f)^2} = u_g. \tag{8.67}$$

This relation also applies to the TE_n mode in a lossless parallel-plate waveguide.

8.3.1 Attenuation in Parallel-Plate Waveguides

In Chapter 4 we discussed the attenuation for TEM waves in parallel-plate waveguides. In general, for TEM, TM, and TE waves, the attenuation constant α consist of two parts: α_d due to losses in the dielectric and α_c due to ohmic power loss in the imperfectly conducting plates

$$\alpha = \alpha_d + \alpha_c. \tag{8.68}$$

- **TEM Modes**

$$\alpha_d = \frac{\sigma}{2}\eta = \frac{\omega\epsilon''}{2}\eta, \tag{8.69}$$

$$\alpha_c = \frac{1}{b}\sqrt{\frac{\pi f \epsilon}{\sigma_c}}, \tag{8.70}$$

where σ is the conductivity of the dielectric, ϵ'' is the imaginary part of the permittivity of the dielectric, and σ_c is the conductivity of the plates.

- **TM Modes**: Under the condition that $\omega\sigma << \omega^2\mu\epsilon - (n\pi/b)^2$,

$$\alpha_d = \frac{\sigma\eta}{2\sqrt{1-(f_c/f)^2}}, \tag{8.71}$$

$$\beta = \omega\sqrt{\mu\epsilon}\sqrt{1-(f_c/f)^2}. \tag{8.72}$$

The other part is

$$\alpha_c = \frac{2}{\eta b}\sqrt{\frac{\pi\mu_c f_c}{\sigma_c}} \frac{1}{\sqrt{(f_c/f)[1-(f_c/f)^2]}}. \tag{8.73}$$

- **TE Modes:** For TE modes, α_d is the same as (8.71), while α_c is given by

$$\alpha_c = \frac{2f_c^2\sqrt{\pi\mu_c f_c/\sigma_c}}{\eta b f^2\sqrt{1-(f_c/f)^2}}. \tag{8.74}$$

8.4 Rectangular Waveguides

Compared to parallel-plate waveguides, rectangular guides have the advantage of no fringing fields because of their enclosed nature. Unlike parallel-plate waveguides, though, rectangular waveguides do not support TEM waves since they are single-conductor waveguides. Only TM and TE modes can be supported. In the following, we consider a rectangular guide with four conducting walls at $x = 0$, $x = a$, $y = 0$, and $y = b$.

8.4.1 TM Waves in Rectangular Waveguides

For TM waves, $H_z = 0$, and $E_z = e_z(x, y)e^{-jk_z z}$ satisfies (8.25) as well as the boundary conditions on the conducting walls:

$$E_z = E_y = 0, \quad \text{at} \quad x = 0 \quad \text{and} \quad x = a, \tag{8.75a}$$

$$E_z = E_x = 0, \quad \text{at} \quad y = 0 \quad \text{and} \quad y = b. \tag{8.75b}$$

Using the method of separation of variables, equations (8.25), (8.75a) and (8.75b) can be solved to yield

$$e_z(x, y) = E_0 \sin(k_{xm}x)\sin(k_{yn}y), \tag{8.76}$$

where the eigenvalues are

$$k_{xm} = \frac{m\pi}{a}, \qquad k_{yn} = \frac{n\pi}{b}, \qquad m, n = 1, 2, 3, \cdots \tag{8.77a}$$

and

$$k_{tmn}^2 = k_{xm}^2 + k_{yn}^2 = \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2. \tag{8.77b}$$

Other field components can be obtained from E_z by equations (8.12)–(8.15):

$$e_x(x, y) = -\frac{\gamma}{k_{tmn}^2} k_{xm} E_0 \cos(k_{xm}x)\sin(k_{yn}y), \tag{8.78}$$

$$e_y(x, y) = -\frac{\gamma}{k_{tmn}^2} k_{yn} E_0 \sin(k_{xm}x)\cos(k_{yn}y), \tag{8.79}$$

$$h_x(x, y) = \frac{j\omega\epsilon}{k_{tmn}^2} k_{yn} E_0 \sin(k_{xm}x)\cos(k_{yn}y), \tag{8.80}$$

$$h_y(x, y) = -\frac{j\omega\epsilon}{k_{tmn}^2} k_{xm} E_0 \cos(k_{xm}x)\sin(k_{yn}y), \tag{8.81}$$

where the propagation constant is

$$\gamma = j\beta = j\sqrt{\omega^2\mu\epsilon - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2}. \tag{8.82}$$

• **Designation of Modes:** TM_{mn} mode is designated for every possible combination of integers m and n. m and n denote the numbers of half-cycle variations of the fields in x and y directions, respectively.

• **Cutoff Frequency and Cutoff Wavelength of TM$_{mn}$ Mode:**

$$(f_c)_{mn} = \frac{1}{2\sqrt{\mu\epsilon}}\sqrt{(m/a)^2 + (n/b)^2}, \tag{8.83a}$$

$$(\lambda_c)_{mn} = \frac{2}{\sqrt{(m/a)^2 + (n/b)^2}}. \tag{8.83b}$$

• **Dominant TM Mode:** Since for TM modes, neither m nor n can be zero, the dominant mode for TM waves is TM$_{11}$ mode.

8.4.2 TE Waves in Rectangular Waveguides

For TE waves, $E_z = 0$ and H_z satisfies the following equation

$$\Big(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + k_t^2\Big) h_z(x, y) = 0, \tag{8.84a}$$

as well as the boundary conditions on the conducting walls:

$$\frac{\partial h_z}{\partial x} = 0 \quad (E_y = 0), \quad \text{at} \quad x = 0 \quad \text{and} \quad x = a, \tag{8.84b}$$

$$\frac{\partial h_z}{\partial y} = 0 \quad (E_x = 0), \quad \text{at} \quad y = 0 \quad \text{and} \quad y = b. \tag{8.84c}$$

The field solutions are then obtained as

$$h_z(x, y) = H_0 \cos(k_{xm}x)\cos(k_{yn}y), \tag{8.85}$$

where the eigenvalues are

$$k_{xm} = \frac{m\pi}{a}, \qquad k_{yn} = \frac{n\pi}{b}, \qquad m, n = 0, 1, 2, 3, \cdots \tag{8.86a}$$

and

$$k_{tmn}^2 = k_{xm}^2 + k_{yn}^2 = \Big(\frac{m\pi}{a}\Big)^2 + \Big(\frac{n\pi}{b}\Big)^2. \tag{8.86b}$$

Other field components can be obtained from E_z by equations

$$e_x(x, y) = \frac{j\omega\mu}{k_{tmn}^2} k_{yn} H_0 \cos(k_{xm}x)\sin(k_{yn}y), \tag{8.87}$$

$$e_y(x, y) = -\frac{j\omega\mu}{k_{tmn}^2} k_{xm} H_0 \sin(k_{xm}x)\cos(k_{yn}y), \tag{8.88}$$

$$h_x(x, y) = \frac{\gamma}{k_{tmn}^2} k_{xm} H_0 \sin(k_{xm}x)\cos(k_{yn}y), \tag{8.89}$$

$$h_y(x, y) = \frac{\gamma}{k_{tmn}^2} k_{yn} H_0 \cos(k_{xm}x)\sin(k_{yn}y), \tag{8.90}$$

where γ is the same as for the TM modes in equation (8.80).

• **TE$_{mn}$ Mode:** Note that m or n (but not both) can be zero. The cutoff frequency has the same expression as (8.81).
• **Dominant TE Mode:** If $a > b$, then the dominant TE mode is TE$_{10}$; otherwise TE$_{01}$ mode is the dominant TE mode.
• **Dominant Mode for Rectangular Waveguides:** Since $m \neq 0$, $n \neq 0$ for TM modes, the dominant mode for rectangular waveguides (with lowest cutoff frequency) is TE$_{10}$ mode if $a > b$.

8.4.3 Attenuation in Rectangular Waveguides

For the all TE and TM modes, the attenuation due to dielectric losses is

$$\alpha_d = \frac{\sigma\eta}{2\sqrt{1-(f_c/f)^2}}. \tag{8.91}$$

The calculation for α_c due to the ohmic loss at the conducting wall is more involved. For the most important mode—TE$_{01}$ mode,

$$(\alpha_c)_{TE_{10}} = \frac{1}{\eta b}\sqrt{\frac{\pi f \mu_c}{\sigma_c[1-(f_c/f)^2]}}\Big[1+\frac{2b}{a}\Big(\frac{f_c}{f}\Big)^2\Big]. \tag{8.92}$$

Similarly, for TM$_{11}$ mode,

$$(\alpha_c)_{TM_{11}} = \frac{2(b/a^2+a/b^2)\sqrt{\frac{\pi f \mu_c}{\sigma_c}}}{\eta ab(1/a^2+1/b^2)\sqrt{1-(f_c/f)^2}}. \tag{8.93}$$

8.5 Circular Waveguides

Waveguides are those with uniform circular cross sections. Similar to rectangular ones, circular waveguides do not support TEM waves since they are single-conductor hollow waveguides. However, both TM and TE waves can be supported by circular waveguides.

The partial differential equations for TM and TE waves are essentially the same as those for rectangular guides. The only difference is that they are now expressed in cylindrical coordinates since it is easier to impose boundary conditions on the circular wall with these coordinates.

Instead of sinusoidal functions for rectangular waveguides, wave solutions in circular waveguides are expressed in terms of special functions known as Bessel functions. We now first introduce Bessel's differential equation and Bessel functions before discussing on TM and TE waves in circular guides.

8.5.1 Bessel's Differential Equations and Bessel Functions

In cylindrical coordinates, the homogeneous Helmholtz equation for e_z can be written as

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\Big(\rho\frac{\partial e_z}{\partial\rho}\Big) + \frac{1}{\rho^2}\frac{\partial^2 e_z}{\partial\phi^2} + k_t^2 e_z = 0, \tag{8.94}$$

where (ρ, ϕ, z) are the cylindrical coordinates. The equation for h_z is the same as (8.94).

The solution to equation (8.94) can be obtained by using the method of separation of variables as illustrated in the textbook. Here we will obtain the solution by making the following observation: since the solution $e_z(\rho, \phi)$ should be the same as $e_z(\rho, \phi + 2k\pi)$ where k is an integer (this is the periodicity of the solution), it can be written as

$$e_z(\rho, \phi) = R(\rho) \cos n\phi, \tag{8.95}$$

where n is an integer. We have chosen $\cos n\phi$ instead of $\sin n\phi$ or a combination of them because these other choices change only the location of the reference $\phi = 0$ angle. Substituting equation (8.95) into (8.94) yields the **Bessel's differential equation**

$$\frac{d^2 R(\rho)}{dr^2} + \frac{1}{\rho}\frac{dR(\rho)}{d\rho} + (k_t^2 - \frac{n^2}{\rho^2})R(\rho) = 0. \tag{8.96}$$

The general solution to equation (8.96) is the combination of $J_n(k_t\rho)$ and $Y_n(k_t\rho)$,

$$R(\rho) = C_n J_n(k_t\rho) + D_n Y_n(k_t\rho), \tag{8.97}$$

where $J_n(k_t\rho)$ is **Bessel function of the first kind**, $Y_n(k_t\rho)$ is **Bessel function of the second kind**, and C_n and D_n are arbitrary constants to be determined by boundary conditions. The Bessel functions are given by

$$J_n(k_t\rho) = \sum_{m=0}^{\infty} \frac{(-1)^m (k_t\rho)^{n+2m}}{m!(n+m)!2^{n+2m}}, \tag{8.98}$$

$$Y_n(k_t\rho) = \frac{(\cos n\pi) J_n(k_t\rho) - J_{-n}(k_t\rho)}{\sin n\pi}. \tag{8.99}$$

For hollow circular waveguides, $D_n = 0$ since $Y_n(k_t\rho)$ is singular (goes to infinity) at $r = 0$. Hence, for circular waveguides, the solution is expressed in terms of Bessel function of the first kind.

- **$J_n(k_t\rho)$ at $r = 0$:** At the origin $r = 0$, $J_0(0) = 1$ (for $n = 0$) and $J_n(0) = 0$ for $n \neq 0$.
- **x_{np}—Zeros of $J_n(x)$:** x_{np} are the arguments at which $J_n(x_{np}) = 0$.
- **x'_{np}—Zeros of $J'_n(x)$:** x'_{np} are the arguments at which $J'_n(x'_{np}) = 0$.

8.5.2 TM Waves in Circular Waveguides

For TM waves, $H_z = 0$, and

$$E_z(\rho, \phi, z) = e_z(\rho, \phi) e^{-\gamma z}, \tag{8.100}$$

where $e_z(\rho, \phi)$ satisfies (8.94). The solution for e_z is then given by

$$e_z(\rho, \phi) = C_n J_n(k_t\rho)) \cos n\phi. \tag{8.101}$$

Similar to fields in Cartesian coordinates, the traverse field components in cylindrical coordinates can be obtained from the z components. For TM modes, the transverse electric field $\mathbf{e}_t = \hat{r}e_\rho + \hat{\phi}e_\phi$ is related to e_z by

$$\mathbf{e}_t = -\frac{\gamma}{k_t^2}\left(\hat{r}\frac{\partial}{\partial\rho} + \hat{\phi}\frac{1}{\rho}\frac{\partial}{\partial\phi}\right)e_z, \tag{8.102}$$

or specifically

$$e_\rho = -\frac{j\beta}{k_t}C_n J_n'(k_t\rho)\cos n\phi, \tag{8.103}$$

$$e_\phi = \frac{j\beta n}{k_t^2 r}C_n J_n(k_t\rho)\sin n\phi. \tag{8.104}$$

The magnetic field can be obtained by using equation (8.20) which gives

$$h_\rho = -\frac{j\omega\epsilon n}{k_t^2 r}C_n J_n(k_t\rho)\sin n\phi, \tag{8.105}$$

$$h_\phi = -\frac{j\omega\epsilon}{k_t}C_n J_n'(k_t\rho)\cos n\phi. \tag{8.106}$$

In the above, $\gamma = j\beta$, J_n' is the derivative of J_n with respect to its argument $(k_t\rho)$, and C_n is a coefficient which depends on the strength of the excitation.

The transverse wavenumber k_t takes discrete numbers, as in rectangular waveguides. This can be determined by the boundary conditions

$$e_z(\rho = a, \phi) = e_\phi(\rho = a, \phi) = 0. \tag{8.107}$$

Using (8.107) in (8.101), we have

$$J_n(k_t a) = 0. \tag{8.108}$$

Therefore,

$$k_t = \frac{x_{np}}{a}, \qquad p = 1, 2, 3, \cdots \tag{8.109}$$

where x_{np} is the p-th zero of the Bessel function $J_n(x)$.

• **TM$_{np}$ Mode:** The mode corresponding to the zero x_{np}.

• **Cutoff Frequency of TM$_{np}$ Mode:**

$$(f_c)_{TM_{np}} = \frac{x_{np}}{2\pi a\sqrt{\mu\epsilon}}. \tag{8.110}$$

Some of the roots x_{np} are given in the textbook. For examples, $x_{01} = 2.405$, $x_{02} = 5.520$, $x_{11} = 3.832$, $x_{12} = 7.016$.

• **Significance of Mode Indices** np**:** n represents the number of half-wave field variations in the ϕ direction, and p represents the number of half-wave field variations in the r direction.

• **TM$_{01}$ Mode:** Since $x_{01} = 2.405$, TM$_{01}$ mode has a cutoff frequency of

$$(f_c)_{TM_{01}} = \frac{2.405}{2\pi a\sqrt{\mu\epsilon}}. \tag{8.111}$$

8.5.3 TE Waves in Circular Waveguides

For TE waves, $E_z = 0$, and

$$H_z(\rho, \phi, z) = h_z(\rho, \phi) e^{-\gamma z}, \tag{8.112}$$

where $h_z(\rho, \phi)$ satisfies (8.94) with e_z replaced by h_z. The solution for h_z is then given by

$$h_z(\rho, \phi) = C'_n J_n(k_t \rho) \cos n\phi. \tag{8.113}$$

For TE modes, the transverse magnetic field $\mathbf{h}_t = \hat{r} h_\rho + \hat{\phi} h_\phi$ and electric field components are

$$h_\rho = -\frac{j\beta}{k_t} C'_n J'_n(k_t \rho) \cos n\phi, \tag{8.114}$$

$$h_\phi = \frac{j\beta n}{k_t^2 r} C'_n J_n(k_t \rho) \sin n\phi, \tag{8.115}$$

$$e_\rho = \frac{j\omega\mu n}{k_t^2 r} C'_n J_n(k_t \rho) \sin n\phi, \tag{8.116}$$

$$e_\phi = \frac{j\omega\mu}{k_t} C'_n J'_n(k_t \rho) \cos n\phi, \tag{8.117}$$

where C'_n is a coefficient which depends on the strength of the excitation.

The transverse wavenumber k_t takes discrete numbers, as in rectangular waveguides. This can be determined by the boundary conditions

$$e_\phi(\rho = a, \phi) = 0. \tag{8.118}$$

Using (8.118) in (8.117), we have

$$J'_n(k_t a) = 0. \tag{8.119}$$

Therefore,

$$k_t = \frac{x'_{np}}{a}, \qquad p = 1, 2, 3, \cdots \tag{8.120}$$

where x'_{np} is the p-th zero of the derivative of Bessel function $J'_n(x)$.

• **TE$_{np}$ Mode:** The mode corresponding to the zero x'_{np}.

• **Cutoff Frequency of TE$_{np}$ Mode:**

$$(f_c)_{TE_{np}} = \frac{x'_{np}}{2\pi a\sqrt{\mu\epsilon}}. \tag{8.121}$$

Some of the roots x'_{np} are given in the textbook. For examples, $x'_{01} = 3.832$, $x'_{02} = 7.016$, $x'_{11} = 1.841$, $x'_{12} = 5.331$.

• **Significance of Mode Indices np:** n represents the number of half-wave field variations in the ϕ direction, and p represents the number of half-wave field variations in the r direction.

• **TE_{11} Mode:** Since $x'_{11} = 1.841$, TE_{11} mode has a cutoff frequency of

$$(f_c)_{TE_{11}} = \frac{1.841}{2\pi a\sqrt{\mu\epsilon}}. \tag{8.122}$$

• **Dominant Mode in a Circular Waveguide:** From equations (8.110) and (8.121), it it seen that the smallest x_{np} or x'_{np} corresponds to the dominant mode since it has the lowest cutoff frequency. Since x'_{11} is the smallest, TE_{11} mode is the dominant mode in a circular waveguide.

8.6 Dielectric Waveguides

In addition to metallic waveguides, dielectric waveguides can also support electromagnetic waves. In this section we will discuss the simplest dielectric waveguides, i.e., dielectric slabs. For simplicity, we assume that the dielectric slabs is infinite in size in both x and z directions, and the parameters for 3 layers of dielectric media are (μ_0, ϵ_0), (μ_d, ϵ_d), and (μ_0, ϵ_0) respectively. The interfaces of the layers are located at $y = -d/2$ and $y = d/2$ respectively. This dielectric slab is a special case of the class of dielectric waveguides.

8.6.1 TM Waves Along A Dielectric Slab

For transverse magnetic waves, $H_z = 0$, and $E_z \neq 0$ satisfies the following equation

$$\frac{d^2 e_z(y)}{dy^2} + k_t^2 e_z(y) = 0, \tag{8.123}$$

where

$$k_t^2 = \gamma^2 + \omega^2\mu\epsilon. \tag{8.124}$$

In the above, for lossless dielectric waveguides, $\gamma = j\beta$. Equation (8.123) governs the field in the three layers in the dielectric slab. The general solution to this equation is exponential functions or sinusoidal functions. However, in order for the waves not to radiate away from the waveguide, the field must decay exponentially away from the slab. Therefore, inside the slab,

$$e_z(y) = E_o \sin k_y y + E_e \cos k_y y, \qquad |y| \leq d/2, \tag{8.125}$$

where

$$k_y^2 = \omega^2\mu_d\epsilon_d - \beta^2. \tag{8.126}$$

Outside the slab,

$$e_z(y) = \begin{cases} \left(E_o \sin\frac{k_y d}{2} + E_e \cos\frac{k_y d}{2}\right)e^{-\alpha(y-d/2)}, & y \geq d/2, \\ \left(-E_o \sin\frac{k_y d}{2} + E_e \cos\frac{k_y d}{2}\right)e^{\alpha(y+d/2)}, & y \leq -d/2, \end{cases} \tag{8.127}$$

where

$$\alpha^2 = \beta^2 - \omega^2\mu_0\epsilon_0 = \omega^2(\mu_d\epsilon_d - \mu_0\epsilon_0) - k_y^2. \tag{8.128}$$

In equation (8.127), the coefficients for the exponential functions are chosen so the E_z is continuous at the slab interfaces.

The other field components can be found from the e_z component using equations (8.2)–(8.5). The nonzero transverse components are given by

$$e_y(y) = \begin{cases} -\frac{j\beta}{\alpha}\Big(E_o \sin\frac{k_y d}{2} + E_e \cos\frac{k_y d}{2}\Big)e^{-\alpha(y-d/2)}, & y \geq d/2, \\ -\frac{j\beta}{k_y}(E_o \cos k_y y - E_e \sin k_y y), & |y| \leq d/2, \\ \frac{j\beta}{\alpha}\Big(-E_o \sin\frac{k_y d}{2} + E_e \cos\frac{k_y d}{2}\Big)e^{\alpha(y+d/2)} & y \leq -d/2 \end{cases} \quad (8.129)$$

$$h_x(y) = \begin{cases} \frac{j\omega\epsilon_0}{\alpha}\Big(E_o \sin\frac{k_y d}{2} + E_e \cos\frac{k_y d}{2}\Big)e^{-\alpha(y-d/2)}, & y \geq d/2, \\ \frac{j\omega\epsilon_d}{k_y}(E_o \cos k_y y - E_e \sin k_y y), & |y| \leq d/2, \\ -\frac{j\omega\epsilon_0}{\alpha}\Big(-E_o \sin\frac{k_y d}{2} + E_e \cos\frac{k_y d}{2}\Big)e^{\alpha(y+d/2)}, & y \leq -d/2. \end{cases} \quad (8.130)$$

From (8.130), the continuity condition for the tangential component of magnetic field h_x at $y = d/2$ and at $y = -d/2$ requires

$$\begin{cases} \Big(\frac{\epsilon_d}{k_y} - \frac{\epsilon_0}{\alpha}\tan\frac{k_y d}{2}\Big)E_o - \Big(\frac{\epsilon_d}{k_y}\tan\frac{k_y d}{2} + \frac{\epsilon_0}{\alpha}\Big)E_e & = 0, \\ \Big(\frac{\epsilon_d}{k_y} - \frac{\epsilon_0}{\alpha}\tan\frac{k_y d}{2}\Big)E_o + \Big(\frac{\epsilon_d}{k_y}\tan\frac{k_y d}{2} + \frac{\epsilon_0}{\alpha}\Big)E_e & = 0. \end{cases} \quad (8.131)$$

Only under the following two conditions do this set of equations can be satisfied:

- **(i) Odd TM Modes:**

In this case, $E_e = 0$ but $E_o \neq 0$, and

$$\Big(\frac{\epsilon_d}{k_y} - \frac{\epsilon_0}{\alpha}\tan\frac{k_y d}{2}\Big) = 0,$$

or equivalently

$$[\omega^2(\mu_d\epsilon_d - \mu_0\epsilon_0) - k_y^2]^{1/2} = \frac{\epsilon_0}{\epsilon_d}k_y \tan\frac{k_y d}{2}. \quad (8.132)$$

The roots for this transcendental equation (8.132) give the discrete values of k_y. In contrast to metallic waveguides, there are only a finite number of possible modes. Since $E_o = 0$, the E_z component is an odd function of y.

For odd TM modes, since the tangential electric field components are zero at $y = 0$, a perfectly conducting plane can be introduced at $y = 0$ without affecting the field distribution. Hence, odd TM modes propagating along a dielectric slab of thickness d are the same as those of the corresponding TM modes supported by a dielectric slab of thickness $d/2$ that is backed by a PEC plane.

Cutoff Frequency: The cutoff frequency for dielectric waveguides has to be determined by the wave behavior outside the slab based on the attenuation constant α in (8.128). When this attenuation becomes zero, the frequency is called the **cutoff frequency** since the waves are no longer bound to the slab. Hence, at the cutoff frequency f_{co} for the odd TM modes, $\alpha = 0$ and $\beta = (2\pi f_{co})\sqrt{\mu_0\epsilon_0}$, $k_y = (2\pi f_{co})\sqrt{\mu_d\epsilon_d - \mu_0\epsilon_0}$. From (8.132), we have

$$\tan\Big(\frac{\omega_{co} d}{2}\sqrt{\mu_d\epsilon_d - \mu_0\epsilon_0}\Big) = 0,$$

or

$$f_{co} = \frac{(n-1)}{d\sqrt{\mu_d\epsilon_d - \mu_0\epsilon_0}}, \qquad n = 1, 2, 3, \cdots \tag{8.133}$$

The lowest odd TM mode, TM_1 mode ($n = 1$), has a zero cutoff frequency. Therefore, TM_1 mode can propagate along a dielectric-slab waveguide regardless of the thickness of the slab.

• **(ii) Even TM Modes:**

The other solution for equation (8.131) is that $E_o = 0$ but $E_e \neq 0$, and

$$\left(\frac{\epsilon_d}{k_y}\tan\frac{k_y d}{2} + \frac{\epsilon_0}{\alpha}\right) = 0,$$

or equivalently

$$[\omega^2(\mu_d\epsilon_d - \mu_0\epsilon_0) - k_y^2]^{1/2} = -\frac{\epsilon_0}{\epsilon_d}k_y \cot\frac{k_y d}{2}. \tag{8.134}$$

Since $E_o = 0$, the E_z component is an even function of y.

Similarly, the cutoff frequency is

$$f_{ce} = \frac{(n-\frac{1}{2})}{d\sqrt{\mu_d\epsilon_d - \mu_0\epsilon_0}}, \qquad n = 1, 2, 3, \cdots \tag{8.135}$$

Only the waves with frequencies higher than the cutoff frequency can propagate in the slab waveguide.

8.6.2 TE Waves Along A Dielectric Slab

For transverse electric waves, $E_z = 0$, and h_z satisfies the following equation

$$\frac{d^2 h_z(y)}{dy^2} + k_t^2 h_z(y) = 0, \tag{8.136}$$

where k_t^2 is given by (8.124). Similar to the TM waves, the solution for h_z can be written as

$$h_z(y) = \begin{cases} \left(H_o \sin\frac{k_y d}{2} + H_e\cos\frac{k_y d}{2}\right)e^{-\alpha(y-d/2)}, & y \geq d/2, \\ H_o \sin k_y y + H_e \cos k_y y, & |y| \leq d/2, \\ \left(-H_o\sin\frac{k_y d}{2} + H_e\cos\frac{k_y d}{2}\right)e^{\alpha(y+d/2)}, & y \leq -d/2, \end{cases} \tag{8.137}$$

where k_y and α are given by (8.126) and (8.128) respectively.

The other field components can be found from the h_z component using equations (8.2)–(8.5). The nonzero transverse components are given by

$$h_y(y) = \begin{cases} -\frac{j\beta}{\alpha}\left(H_o \sin\frac{k_y d}{2} + H_e\cos\frac{k_y d}{2}\right)e^{-\alpha(y-d/2)}, & y \geq d/2, \\ -\frac{j\beta}{k_y}(H_o \cos k_y y - H_e \sin k_y y), & |y| \leq d/2, \\ \frac{j\beta}{\alpha}\left(-H_o\sin\frac{k_y d}{2} + H_e\cos\frac{k_y d}{2}\right)e^{\alpha(y+d/2)}, & y \leq -d/2, \end{cases} \tag{8.138}$$

$$e_x(y) = \begin{cases} -\frac{j\omega\mu_0}{\alpha}\Big(H_o \sin\frac{k_y d}{2} + H_e \cos\frac{k_y d}{2}\Big)e^{-\alpha(y-d/2)}, & y \geq d/2, \\ -\frac{j\omega\mu_d}{k_y}(H_o \cos k_y y - H_e \sin k_y y), & |y| \leq d/2, \\ \frac{j\omega\mu_0}{\alpha}\Big(-H_o \sin\frac{k_y d}{2} + H_e \cos\frac{k_y d}{2}\Big)e^{\alpha(y+d/2)}, & y \leq -d/2. \end{cases} \tag{8.139}$$

Similar to TM waves, the continuity condition for the tangential component of electric field e_x at $y = d/2$ and at $y = -d/2$ requires

$$\begin{cases} \Big(\frac{\mu_d}{k_y} - \frac{\mu_0}{\alpha}\tan\frac{k_y d}{2}\Big)E_o - \Big(\frac{\mu_d}{k_y}\tan\frac{k_y d}{2} + \frac{\mu_0}{\alpha}\Big)E_e &= 0, \\ \Big(\frac{\mu_d}{k_y} - \frac{\mu_0}{\alpha}\tan\frac{k_y d}{2}\Big)E_o + \Big(\frac{\mu_d}{k_y}\tan\frac{k_y d}{2} + \frac{\mu_0}{\alpha}\Big)E_e &= 0. \end{cases} \tag{8.140}$$

Only under the following two conditions do this set of equations can be satisfied:

- **(i) Odd TE Modes:**

In this case, $H_e = 0$ but $H_o \neq 0$, and

$$\Big(\frac{\mu_d}{k_y} - \frac{\mu_0}{\alpha}\tan\frac{k_y d}{2}\Big) = 0,$$

or equivalently

$$[\omega^2(\mu_d\mu_d - \mu_0\mu_0) - k_y^2]^{1/2} = \frac{\mu_0}{\mu_d}k_y \tan\frac{k_y d}{2}. \tag{8.141}$$

The roots for this transcendental equation (8.141) give the discrete values of k_y. Since $H_o = 0$, the H_z component is an odd function of y.

Cutoff Frequency: The cutoff frequency for odd TE mode is the same as that for odd TM mode given by (8.133). The lowest odd TE mode, TE_1 mode ($n = 1$), has a zero cutoff frequency. Therefore, TE_1 mode can propagate along a dielectric-slab waveguide regardless of the thickness of the slab.

- **(ii) Even TE Modes:**

The other solution for equation (8.140) is that $H_o = 0$ but $H_e \neq 0$, and

$$\Big(\frac{\mu_d}{k_y}\tan\frac{k_y d}{2} + \frac{\mu_0}{\alpha}\Big) = 0,$$

or equivalently

$$[\omega^2(\mu_d\epsilon_d - \mu_0\epsilon_0) - k_y^2]^{1/2} = -\frac{\mu_0}{\mu_d}k_y \cot\frac{k_y d}{2}. \tag{8.142}$$

Since $H_o = 0$, the H_z component is an even function of y.

Similarly, the cutoff frequency for even TE modes is the same as for even TM modes given by (8.135).

8.7 Cavity Resonators

At UHF and higher frequencies, resonant circuits using R, L, and C elements become difficult because of the skin effect and radiation of the elements. In this section rectangular and circular cavity resonators are discussed.

8.7.1 Rectangular Cavity Resonators

If a rectangular resonator has dimensions a, b, and d, we can define a Cartesian coordinate system such that the cavity walls are located at $x=0$, $x=a$, $y=0$, $y=b$, $z=0$, and $z=d$. With respect to the z direction, we can have both TM_{mnp} and TE_{nmnp} modes, where mnp refer to the wave pattern in x, y, and z directions.

The difference between rectangular cavity and rectangular waveguides is the waves are no longer propagating in the z direction in cavities. They become standing waves. Therefore, instead of the $e^{\pm\gamma z}$ factor, the possible factor in z are $\cos\beta z$ and $\sin\beta z$, depending on the appropriate boundary conditions (zero tangential electric field components) at $z=0$ and $z=d$.

• **TM_{mnp} Modes:** For TM modes, $H_z=0$ but $E_z\neq 0$. The boundary conditions require that the form of TM_{mnp} modes to be

$$E_z(x,y,z)=E_0\sin\left(\frac{m\pi}{a}\right)\sin\left(\frac{n\pi}{b}\right)\cos\left(\frac{p\pi}{d}\right). \tag{8.143}$$

The other field components can be found from E_z as

$$E_x(x,y,z)=-\frac{1}{k_t^2}\left(\frac{m\pi}{a}\right)\left(\frac{p\pi}{d}\right)E_0\cos\left(\frac{m\pi}{a}\right)\sin\left(\frac{n\pi}{b}\right)\sin\left(\frac{p\pi}{d}\right), \tag{8.144}$$

$$E_y(x,y,z)=-\frac{1}{k_t^2}\left(\frac{n\pi}{b}\right)\left(\frac{p\pi}{d}\right)E_0\sin\left(\frac{m\pi}{a}\right)\cos\left(\frac{n\pi}{b}\right)\sin\left(\frac{p\pi}{d}\right), \tag{8.145}$$

$$H_x(x,y,z)=\frac{j\omega\epsilon}{k_t^2}\left(\frac{n\pi}{b}\right)E_0\sin\left(\frac{m\pi}{a}\right)\cos\left(\frac{n\pi}{b}\right)\cos\left(\frac{p\pi}{d}\right), \tag{8.146}$$

$$H_y(x,y,z)=-\frac{j\omega\epsilon}{k_t^2}\left(\frac{m\pi}{a}\right)E_0\cos\left(\frac{m\pi}{a}\right)\sin\left(\frac{n\pi}{b}\right)\cos\left(\frac{p\pi}{d}\right), \tag{8.147}$$

where

$$k_t^2=\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2. \tag{8.148}$$

In the above, $m,n=1,2,3,\cdots$ and $p=0,1,2,\cdots$.

From the definition $k_t^2=k^2-\beta^2$, we obtain the **resonant frequency** for TM_{mnp} mode of the cavity

$$f_{mnp}=\frac{1}{2\sqrt{\mu\epsilon}}\sqrt{\left(\frac{m}{a}\right)^2+\left(\frac{n}{b}\right)^2+\left(\frac{p}{d}\right)^2}. \tag{8.149}$$

• **TE_{mnp} Modes:** For TE modes, $E_z=0$ but $H_z\neq 0$. The boundary conditions require that the form of TE_{mnp} modes to be

$$H_z(x,y,z)=H_0\cos\left(\frac{m\pi}{a}\right)\cos\left(\frac{n\pi}{b}\right)\sin\left(\frac{p\pi}{d}\right). \tag{8.150}$$

The other field components can be found from H_z as

$$H_x(x,y,z)=-\frac{1}{k_t^2}\left(\frac{m\pi}{a}\right)\left(\frac{p\pi}{d}\right)H_0\sin\left(\frac{m\pi}{a}\right)\cos\left(\frac{n\pi}{b}\right)\cos\left(\frac{p\pi}{d}\right), \tag{8.151}$$

$$H_y(x,y,z) = -\frac{1}{k_t^2}\left(\frac{n\pi}{b}\right)\left(\frac{p\pi}{d}\right)H_0\cos\left(\frac{m\pi}{a}\right)\sin\left(\frac{n\pi}{b}\right)\cos\left(\frac{p\pi}{d}\right), \qquad (8.152)$$

$$E_x(x,y,z) = \frac{j\omega\mu}{k_t^2}\left(\frac{n\pi}{b}\right)H_0\cos\left(\frac{m\pi}{a}\right)\sin\left(\frac{n\pi}{b}\right)\sin\left(\frac{p\pi}{d}\right), \qquad (8.153)$$

$$E_y(x,y,z) = -\frac{j\omega\mu}{k_t^2}\left(\frac{m\pi}{a}\right)H_0\sin\left(\frac{m\pi}{a}\right)\cos\left(\frac{n\pi}{b}\right)\sin\left(\frac{p\pi}{d}\right), \qquad (8.154)$$

where k_t^2 is given by (8.148). In the above, $m, n = 0, 1, 2, 3, \cdots$ (but m and n cannot be zero simultaneously) and $p = 1, 2, 3, \cdots$.

The **resonant frequency** for TE_{mnp} mode is the same as that for TM_{mnp} mode as given by (8.149).

- **Dominant Mode:** The modes with lowest resonant frequency.
- **Degenerate Modes:** different modes having the same resonant frequency. If m, n, p are all nonzero, then TM_{mnp} and TE_{mnp} modes are always degenerate.

B. Quality Factor of Cavity Resonators

The finite conductivity in the conducting walls of cavities results in power loss which causes a decay in the stored energy inside the cavities. We define a **quality factor** Q as the ratio

$$Q = \frac{\omega W}{P_L}, \qquad (8.155)$$

where ω is the resonant frequency, W is the time-averaged energy stored at the resonant frequency, and P_L is the time-averaged power dissipated in the cavity.

For TE_{101} mode, we can find that

$$Q_{101} = \frac{\pi f_{101}\mu_0 abd(a^2+d^2)}{R_s[2b(a^3+d^3)+ad(a^2+d^2)]}, \qquad (8.156)$$

where $R_s = \sqrt{\pi f \mu_c/\sigma_c}$ is the intrinsic resistance of the conducting wall.

8.7.2 Circular Cavity Resonators

TM_{mnp} and TE_{mnp} modes for circular cavities can also be analyzed with the same procedures as for the rectangular cavities. The resonant frequency for TM_{mnp} mode is

$$f_{TM_{mnp}} = \frac{1}{2\pi\sqrt{\mu\epsilon}}\sqrt{\left(\frac{x_{mn}}{a}\right)^2+\left(\frac{p\pi}{d}\right)^2}, \qquad (8.157)$$

where d is the dimension of the cavity in z direction, and x_{mn} is the n-th zero of $J_m(x)$.

Similarly, the resonant frequency for TE_{mnp} mode is

$$f_{TE_{mnp}} = \frac{1}{2\pi\sqrt{\mu\epsilon}}\sqrt{\left(\frac{x'_{mn}}{a}\right)^2+\left(\frac{p\pi}{d}\right)^2}, \qquad (8.158)$$

where x'_{mn} is the n-th zero of $J'_m(x)$.

The quality factor for TM_{010} is given by

$$Q_{TM_{010}} = \left(\frac{\eta_0}{R_s}\right)\frac{2.405}{2(1+a/d)}. \quad (8.159)$$