DUKE THE FUQUA SCHOOL OF BUSINESS FUQUA School of Business

BA 513/STA 234: Ph.D. Seminar on Choice Theory
Professor Robert Nau
Spring Semester 2008

Readings for class #11: Auction theory (updated April 2, 2008)

Primary readings:

1. "Auctions and Bidding" by R. Preston McAfee and John McMillan, *J. Economic Literature* (1987)
2. "Combinatorial Auction Design" by Aleksandar Pekec and Michael Rothkopf, available at this link: http://faculty.fuqua.duke.edu/%7Epekec/bio/PekecRothkopfCombAuctDesign.pdf

Supplementary readings:

3. "Auction Theory: A Guide to the Literature" by Paul Klemperer, *J. Economic Surveys* (1999)
4. "Auctions and Bidding: A Primer" by Paul Milgrom, *J. Economic Perspectives* (1989)
5. "A Theory of Auctions and Competitive Bidding" by Paul Milgrom and Robert Weber, *Econometrica* (1982)
6. "The Simple Economics of Optimal Auctions" by Jeremy Bulow and John Roberts, *J. Political Economy* (1989)

Notes on auction theory

An auction is a market mechanism in which a price for an item is determined from bids submitted by competing buyers. As an alternative to posted prices, auctions are useful in situations where prices are inherently unstable and current market conditions are better known to buyers than sellers (e.g., T-bills or catches of fresh fish) and/or where goods are not standardized (e.g., artwork, oil leases, assets of bankrupt firms). As an alternative to negotiated prices, auctions have the advantage that they can be delegated to unsupervised—even robotic—agents and they offer fewer opportunities for kickbacks and behind-the-scenes deals than negotiated agreements. In recent years there has been an upsurge of interest in both the theory and applications of auctions. The internet has made it possible for anyone to easily buy or sell goods in worldwide auctions, while the opening of new markets and deregulation of old ones has created demands for novel auction forms to determine prices for non-standard commodities such as electric power and segments of the radio spectrum. Auction models are widely regarded as the most important and successful application of the theory of games of incomplete information as well as a test bed for studying the process of price formation among dispersed agents with heterogeneous information.

The striking empirical facts about auctions are that (a) only four major types are widely used in practice: English (open ascending-bid), Dutch (open descending bid), first-price sealed-bid, and second-price sealed-bid, and (b) they are not interchangeable (each has its own characteristic domains of application, and English auctions predominate). The striking theoretical facts are that (a) under strong simplifying assumptions about risk neutrality, symmetry, regularity, and independence of private values among bidders, the four major auction forms are quasi-optimal and produce identical revenues, (b) under somewhat more realistic assumptions of dependent values, English auctions yield the highest revenues among the four major types, and (c) under more general conditions of asymmetry, dependence, and/or risk aversion, the auction mechanisms that are "optimal" bear very little resemblance to those used in practice.

The standard game-theoretic approach to modeling an auction is to frame it as a game of incomplete information in which each player's *type* consists of either her own private valuation of the item (the so-called "private value" setting), or else a signal that represents private information about the item's value to all agents (the so-called "common value" setting), or else a signal that represents private information in a more general model of dependent private values. As is usual in games of incomplete information, the common prior assumption is invoked: the joint distribution of types is assumed to be common knowledge. Usually one of two simplifying assumptions is imposed on the common prior: in the private-value setting the distributions of the types of individual agents are typically assumed to be *independent* (and perhaps also symmetric), and in the dependent-value setting agents' signals are assumed to be *affiliated* (a generalized form of positive correlation). A strategy profile for such a game consists of a *bid function* for each agent, which specifies how the agent should bid as a function of her type. Thus, it is as if the agents commit themselves to strategies for bidding before they know their own types. If the types and possible bids are continuously distributed, this means that each agent's strategy is an infinite-dimensional vector. The set of possible strategies is therefore huge, and the set of *equilibrium* strategies can be exceedingly complex, so that additional strong simplifying assumptions are usually imposed. For example, attention is often restricted to simple parametric bidding functions that are symmetric between agents (e.g., bids that are linear functions of types), or else the revelation principle is invoked so that it suffices for each player to truthfully reveal her type to a central authority who then determines the outcome of the auction according to some rule that is incentive compatible and individually rational.

Ordinary economic intuition suggests that, when a single item is sold at auction, its price should fall somewhere between the second-highest valuation and the highest valuation among the participating bidders, because (only) such a price will equalize supply and demand. (The supply is one unit, and the demand will equal one unit when the price is set so that exactly one bidder has a valuation greater than or equal to the price.) One important question, then, is *where* in this interval the price will fall. The situation is usually analyzed from the perspective of the seller, who is interested in maximizing revenue. One of the most basic results of auction theory is the *revenue equivalence theorem*. Here is the version of the theorem given by Milgrom and Weber (1982):

> "Theorem 0: Assume that a particular auction mechanism is given, that the IPV [independent private value] model applies, and that the bidders adopt strategies which constitute a noncooperative equilibrium. Suppose that at equilibrium the bidder who

values the object most highly is certain to receive it, and that any bidder who values the object at its lowest possible level has an expected payment of zero. Then the expected revenue generated for the seller by the mechanism is precisely the expected value of the object to the second-highest evaluator.

At the symmetric equilibria of the English, Dutch, first-price, and second-price auctions, the conditions of the theorem are satisfied. Consequently, the expected selling price is the same for all four mechanisms; this is the so-called 'revenue-equivalence' result. It should be noted that Theorem 0 has an attractive economic interpretation. No matter what competitive mechanism is used to establish the selling price of the object, on average the sale will be at the lowest price at which supply (a single unit) equals demand."

Note the additional conditions (besides the symmetric IPV assumptions) that are imposed on the solution in order to obtain this result: the mechanism must award the item to the bidder with the highest valuation, and any bidder with the lowest possible valuation must expect to pay zero. A stronger version of the theorem, due to Myerson, states that revenue equivalence holds if the mechanism has the properties that the seller's expected revenue only depends on the probabilities with which different bidders win the object, as a function of everyone's values, and on the expected utilities that the bidders receive in the situations where their values are at their minima. So any auctions that agree in these respects must yield the same revenues. However, these are not necessarily *optimal* revenues for the seller unless seller also sets a reserve price, as we'll see later. We'll also see later that in situations where values are *asymmetric*, mechanisms which violate these conditions (by occasionally awarding the item to a lower-valuation bidder and/or imposing penalties on low-valuation bidders) may yield higher revenues.

It is intuitively obvious that the Dutch (descending bid) auction is strategically equivalent to a first-price sealed-bid auction, since in both cases the highest bidder wins and pays her bid amount without observing anyone else's bids. (In practice, though, the real-time aspect of a Dutch auction makes it behaviorally different from a first-price sealed-bid auction: a bidder who is impatient may be tempted to jump the gun.) It is also obvious that an English (ascending bid) auction is strategically *almost* equivalent to a second-price sealed-bid auction, since in both cases the highest-valuation bidder wins and pays an amount equal to the second-highest valuation. (Actually, the latter equivalence is clearer for the "Japanese" variant of the ascending bid auction, in which the price is raised continuously and bidders drop out one by one rather than affirming that they wish to raise the current bid.) The equivalence is not perfect because in an ascending-bid auction the observed sequence of bids or drop-outs may be informative to higher-valuation bidders, although that effect is explicitly ruled out by the IPV assumptions.

The not-so-obvious part of the revenue equivalence theorem is the fact that Dutch auctions yield the same expected revenue as English auctions, and first-price sealed-bid auctions yield the same expected revenue as second-price sealed bid auctions (under the strong assumptions of the IPV model). Of course, bidders should not use the same *strategies* in first-price and second-price auctions. In a second-price auction it is a weakly dominant strategy to simply *bid your own valuation*, expecting to pay the amount of the second-highest valuation in the event that you win. In contrast, in the symmetric equilibrium of a first-price auction, it is optimal to bid an amount equal to the expected value of the second-highest valuation, conditional on your own valuation turning out to be the highest. Hence, in a first-price auction, you should "shade" your bid below your own valuation by an amount equal to the expected distance to the second-highest valuation, assuming yours is the highest. In the simple special case of the symmetric IPV model where the value distributions are *uniform* on the same interval [0, *M*], it is optimal for player *i* to bid $v_i(n\text{-}1)/n$, where v_i is her own valuation and *n* is the total number of bidders. Thus, if there are only two bidders, each bids exactly one-half her own valuation; if there are three bidders, each bids two-thirds of her own valuation, and so on. As number of bidders increases, the expected value of the winning bid increases, and in the limit the seller extracts all the surplus.

Because the optimal strategy in the English auction is weakly dominant, it is robust against non-equilibrium behavior on the part of competing bidders, whereas the optimal strategy in the Dutch/first-price auction is not. This is no doubt one reason—but not the only reason—why English auctions are the form most commonly observed in practice. The optimal strategy in the second-price auction is similarly transparent and robust, although second-price auctions are less common than English auctions for other reasons. (Because the bids are sealed, they are more vulnerable to cheating by both buyers and sellers, and they yield lower expected revenues when values are dependent, as discussed below, because less information is revealed during bidding.)

The general model of the "benchmark" case of symmetric, risk-neutral, IPV bidders is as follows. Let $f(v)$ and $F(v)$ denote the probability density and cumulative distribution functions of the common value distribution. For the English and second-price auctions, in which each bidder bids her own valuation, the expected "rent" of the winner is the expected value of the difference between the 1^{st} and 2^{nd} order statistics of the joint (i.e., product) distribution, which are denoted $v_{(1)}$ and $v_{(2).}$ It can be shown that this expected difference is the expected value of the quantity:

$$[1 - F(v)]/f(v)$$

…where the expectation is taken with respect to the distribution of $v_{(1)}$. (Note that the cumulative distribution function of $v_{(1)}$ is just $F(v)^n$ if there are *n* bidders.) The expected payoff to the seller is the winning bidder's value minus her rent, which is the expected value of the quantity $J(v_{(1)})$, where:

$$J(v) \;=\; v - [1 - F(v)]/f(v)$$

$J(v)$ is the so-called **marginal revenue function** that turns out to play a critical role elsewhere as well. The reasons for this name are discussed in the paper by Bulow and Roberts and in Appendix B of the paper by Klemperer,[1] but here is a handwaving explanation. Suppose that the seller faces a single buyer whose value is drawn from the distribution whose cdf is *F*. If the single item were offered for sale at price *v*, the (expected) quantity sold would be $q = 1{-}F(v)$, which is the probability that the bidder's value is greater than or equal to *v*, and so the (expected) revenue would be $r = v(1{-}F(v))$. The marginal (expected) revenue is then $dr/dq = (dr/dv)/(dq/dv) = J(v)$. The following figure illustrates the situation for a normal value distribution with a mean of 20 and a standard deviation of 5:

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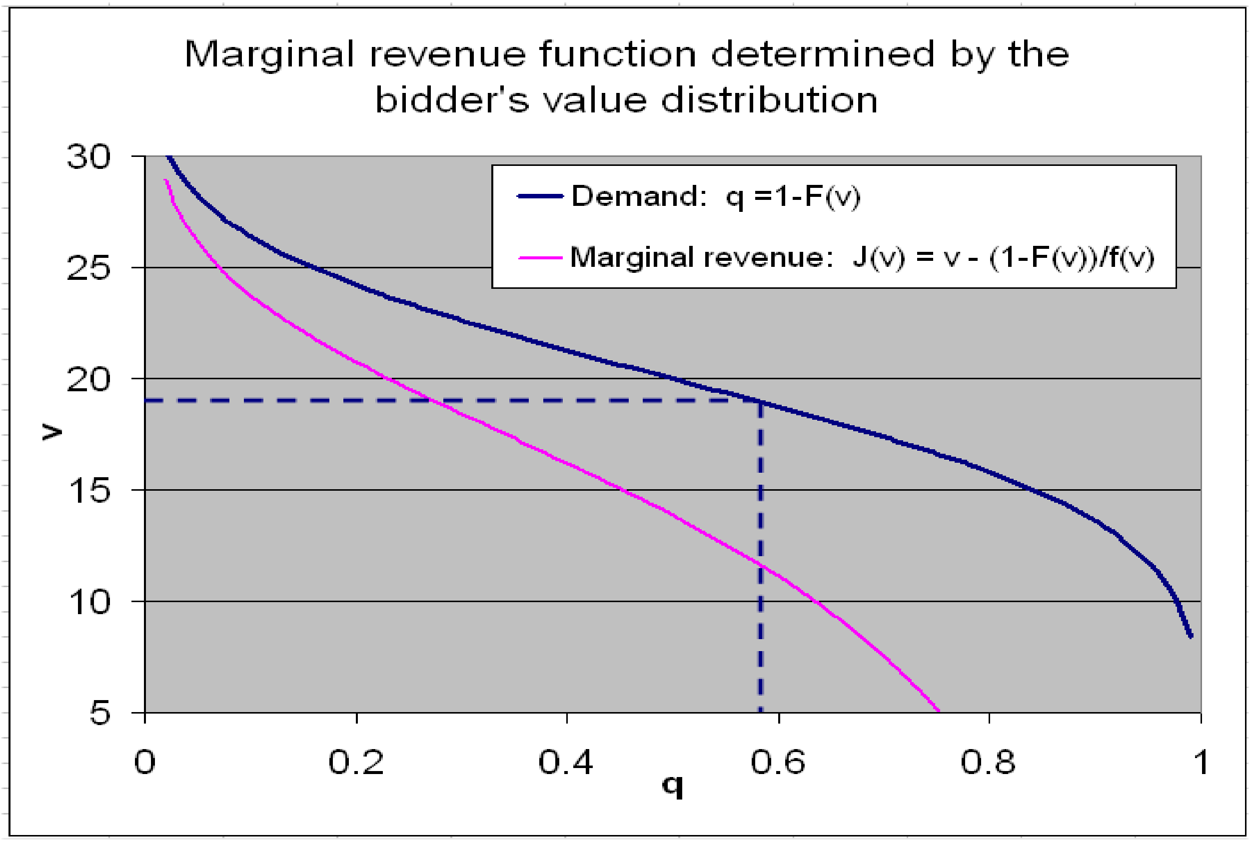
[1] Klemperer's explanation is as follows: "Imagine a firm whose demand curve is constructed from an arbitrarily large number of bidders whose values are independently drawn from a bidder's value distribution. When bidders have private values, a bidder's 'marginal revenue' is defined as the marginal revenue of this firm at the price that equals the bidder's actual value." Under this interpretation, the recipe for the optimal auction looks very much like "third-degree price discrimination."



The area enclosed by the rectangle is the (expected) revenue at a given price (*v*=19 in this case), and its derivative with respect to *q*, which is implicitly a function of *v*, is *J*(*v*).

In the Dutch or 1st-price auction, the bidder's problem does not have a dominant-strategy solution. Instead, the usual approach is to apply the Bayesian Nash equilibrium concept to a game in which each player is assumed to use the same bid function for mapping values to bids. Note that this involves some huge assumptions, namely that there is really common knowledge, and everyone really does try to solve the game by calculating the symmetric Nash equilibrium, and they are all able to do it (not easy!) The formula for the symmetric equilibrium bid function for the Dutch/first-price auction is:

$$b(v_i) = v_i - \frac{\int_0^{v_i} F(x)^{n-1} dx}{F(v_i)^{n-1}}$$

Evaluating the term on the right, which is the optimal amount by which to shade one's bid, is a non-trivial computational problem. However, it can be shown that the seller's expected revenue in this equilibrium solution is the expected value of $J(v_{(1)})$, the same as in the dominant-strategy solution of the English/2nd-price auction.

Although the expected revenue of both types of auction is the same, the *probability distribution* of revenue is not the same: the revenue in the Dutch/first-price auction has a *smaller variance* than the revenue in the English/second-price auction, because the ultimate sale price is based on the winner's forecast of the second-highest valuation rather than the actual second-highest valuation. (This is a regression-to-the mean effect: forecasts of uncertain quantities typically have a lower variance than the quantities themselves.) Hence, if the seller is risk averse while the buyers are risk neutral, the seller should prefer the Dutch/first-price auction form. Also, if the buyers are risk averse, that will work to the advantage of the seller in a Dutch/first-price auction: the buyers will shade their bids by less than they would have if they were risk neutral, because they will prefer to increase their probability of winning at the expense of a lower gain in the event that they win. (By comparison, in an English/second-price auction, the bidder's optimal strategy is unaffected by her degree of risk aversion.) If risk averse bidders are uncertain about the number of other bidders, this also tends to raise the average price in a first-price auction.

A fundamental tool for the modeling of more general auctions is the so-called **Revelation Principle**. Define an auction *mechanism* as a process that takes bids as inputs and produces as outputs the determination of the winner and how much each of the bidders (not necessarily only the winner!) needs to pay. A *direct* mechanism is one in which each bidder is simply asked to report his value for the item. It is *incentive-compatible* if it is in each bidder's interest to report truthfully (assuming others do too!) The Revelation Principle states that *for any mechanism there is a direct-incentive-compatible mechanism that produces the same outcome.* In particular, this is true for an *optimal* (expected revenue maximizing) mechanism, so the study of the properties of optimal mechanisms can be greatly simplified by invoking this principle. McAfee and McMillan describe its significance as follows (p. 713):

> "The significance of the Revelation Principle is that it shows that the modeler can limit his search for the optimal mechanism to the class of direct, incentive-compatible mechanisms. The number of possible selling procedures is huge; hence it is useful to be able to restrict attention to one relatively simple class of mechanisms. The Revelation Principle is purely a theoretical technique; few, if any, resource allocation procedures in practical use are direct, incentive-compatible mechanisms. But using the Relevation Principle does facilitate solving for that resource-allocation mechanism that is optimal subject to the constraints imposed by the asymmetry of information. The optimal direct mechanism is found as the solution to a mathematical programming problem involving two kinds of constraints: first, incentive-compatibility or self-selection constraints, which state that the bidders cannot gain by misrepresenting their valuations; and second, individual-rationality or free-exit constraints, which state that the bidders would not be better off if they refused to participate."

The Revelation Principle can be used to show that the four standard auction mechanisms are optimal auctions for the benchmark symmetric IPV model if they include a **reserve price,** i.e., a minimum bid that the seller requires in order to part with the item. Not surprisingly, the optimal value of the reserve price turns out to depend on the value that the seller attaches to the item in the event that it goes unsold. If the marginal revenue functions are strictly increasing for all agents (as they usually are), then an optimal auction necessarily awards the item to the bidder with the highest value, provided that it is greater than the reserve price. Interestingly, *the optimal reserve price depends only on the common value distribution and not on the number of bidders*. The optimal reserve price r satisfies $J(r) = v_0$, where v_0 is the seller's own valuation of the item. Equivalently, the optimal reserve price is determined by $r = J^{-1}(v_0)$. This is easiest to see in the case of 1 buyer: the buyer will pay exactly r if his value exceeds r, which occurs with probability $1-F(r)$. The seller's expected value is then: $r[1- F(r)] + v_0F(r)$. Setting the derivative w.r.t. r equal to zero (first-order condition for an optimum) yields $v_0 = J(r)$. The optimal auction comes with the following caveat: the presence of a reserve price means that the outcome is *not necessarily ex post efficient*: the item may go unsold even though there is a bidder who values the item more than does the seller. (Alas, another impossibility theorem: you can't have both the desiderata of maximal expected revenue and ex post efficiency.)

In the more general *asymmetric* IPV model, each bidder is permitted to have a different value distribution, although the value distributions are still assumed to be independent and commonly known. The solution for the optimal (expected-revenue maximizing) auction under these conditions, which can be obtained by invoking the Revelation Principle, turns out to be a natural generalization of the solution for the symmetric case, merely with appropriate heterogeneity in the marginal revenue functions. (This was shown by Myerson in 1981.) Let the cumulative distribution function of bidder i's distribution be denoted by F_i, with corresponding density function f_i, and define the personal marginal revenue function of agent i in the natural way:

$$J_i(v) = v - (1-F_i(v))/f_i(v).$$

The quantity $J_i(v)$ is called *virtual utility* by Myerson, because it acts like a surrogate utility function for the i^{th} bidder. The recipe for the optimal auction then goes like this: ask each player to truthfully reveal her type (i.e., her valuation v_i) to a central authority. The central authority then computes $J_i(v_i)$ for every agent (based on the known value distributions) and awards the item to the agent for whom this quantity is largest, if it is positive. If the largest such quantity is negative, the item is not sold. This effectively means that t*he seller's reserve price varies across bidders, depending on their value distributions*. The reserve price for a given bidder is the valuation at which her marginal revenue is zero. The price that the winner is charged is equal to the lowest valuation that she could have reported and still have won, and the expected revenue turns out to equal the expected marginal revenue of the winning bidder. Notice that the optimal auction mechanism entails a significant problem of measurement in the general asymmetric case, since it requires the seller and buyers to agree on the value distribution to be attributed to each buyer.

Recall that in the symmetric IPV model an optimal auction need not be ex post efficient. In the more general asymmetric IPV model, the result of an optimal auction can be ex post inefficient in an even stranger way: the auction may be won by a bidder who does not have the highest valuation and who may even be logically incapable of having the highest valuation. Bulow and Roberts give the following example: suppose there are two bidders, one whose valuation is uniformly distributed on [0, 10] and the other whose valuation is uniformly distributed on [10, 30]. The seller obviously would not want to hold an English auction and sell the unit at the second-highest valuation, whose expected value is only 5. It might seem as though the presence of the first bidder is simply irrelevant and that the best mechanism for the seller would be to just set a reserve price to force the second bidder to bid somewhere above her minimum valuation of 10. Under that mechanism, the optimal reserve price would be 15, yielding an expected revenue of 11.25. (With probability ¾, the second bidder would bid the reserve price, and with probably ¼, she would decline to bid and the item would go unsold.) Under the *optimal* mechanism, if

$v_1 \geq 5$ and $v_1 > v_2 - 10$, bidder 1 wins and pays the larger of 5 (her reserve price) and $v_2 - 10$; whereas if $v_2 \geq 15$ and $v_2 > v_1 + 10$, bidder 2 wins and pays the larger of 15 (*her* reserve price) and $v_1 + 10$; while if both $v_1 < 5$ and $v_2 < 15$, the item goes unsold. Under the latter mechanism, bidder 1 wins with probability 0.1875 (with an average winning bid of 5.56), while bidder 2 wins with probability 0.6875 (with an average winning bid of 16.06), and the expected revenue is 12.1. At first glance this result seems very counterintuitive, but it can be explained roughly as follows. The first bidder is effectively subsidized, i.e., "handicapped," by adding 10 to her bid, after which both bidders engage in an ordinary second-price auction with a common reserve price of 15. This trick makes it optimal for both players to bid their true values. Thus, by effectively subsidizing the disadvantaged bidder, the auction is made more competitive and the second bidder is forced to bid more aggressively than she would in an ordinary auction against a helpless opponent (or against a reserve price alone). Of course, the threat of awarding the item to a low-valuation bidder is credible only if the bidders are somehow prevented from re-selling the object among themselves, otherwise arbitrage would sabotage the forced competition. Also note that if the seller knows the value distributions of both agents, as the mechanism requires, she can earn almost as much revenue from an optimal posted price (11.25) as from the optimal auction (12.1). More precisely, the different in expected revenue between the optimal auction and the optimal posted price (12.1 – 11.25 = 0.85) is much less than the revenue difference between the optimal auction and a suboptimal English auction without a reserve price (12.1 – 5 = 7.1). Hence, at least in this example, most of the "value added" of the optimal auction is due to the exploitation of detailed information about private value distributions which is not assumed to be available in standard auction models.

As the previous example shows, an optimal auction mechanism can yield an expected revenue to the seller that is greater than the expected value of the second-highest valuation. The seller's position is potentially even better in the case where the bidders' valuations are **dependent**: it may be possible to construct a mechanism in which the seller captures all the surplus—i.e., the expected revenue equals the expected value of the highest valuation. The sort of mechanism needed to achieve such a result is strange indeed: typically it awards the item to the highest-valuation bidder at that bidder's valuation while requiring all bidders to accept a side bet which has zero expectation if they bid truthfully and negative expectation if they lie to try to obtain the item at a lower price. Thus, bidders other than the winner may experience gains and losses from participating in the auction. An example is given by Myerson (1981). Suppose there are two bidders, each of whom can have one of two possible values, 10 or 100, with:

$$\Pr(10, 10) = \Pr(100, 100) = 1/3$$

$$\Pr(10, 100) = \Pr(100, 10) = 1/6$$

In the optimal mechanism, if both bidders have value 100, the object is sold to one of them (chosen at random) for a price of 100. If one has a value of 100 and the other a value of 10, it is sold to the high-value bidder for 100 while the low-value bidder must pay 30 but receives nothing. If both bidders have values of 10, each is paid 15 units of money and the object is then sold to one of them (chosen at random) for a price of 10. The rationale for this mechanism as follows: if a bidder reveals his value to be 10, he is forced to take a side bet in which he loses 30 if the other's value is 100 and wins 15 if the other's value is 10, which has zero expected value if he is truthful. If he is lying, it has a negative expected value which exactly cancels the expected gain from obtaining the item cheaply. This renders truth-telling (weakly) optimal. When bidders have independent values but are risk averse, optimal auctions may also involve transfer payments that penalize low bidders and subsidize high bidders who do not win.

When values are positively dependent between players, the **"winner's curse"** phenomenon arises in the standard auction mechanisms: the knowledge that one's own bid is the highest conveys information that the other bids were lower than expected, implying that the value of the item is also less than expected. Milgrom and Weber (1982) analyze this case using a general model in which every agent's value has a similar functional dependence on a set of informational variables, one of which is private information and some of which may be observable by the seller. The dependence of every agent's value on the other agents' private information variables is assumed to be symmetric, and all the informational variables are assumed to be *affiliated*, which is a generalized form of positive correlation in which a higher value of any one variable implies higher values for the others. The agents are also assumed to be risk neutral. Under these conditions, the revenue equivalence among the four major auction forms breaks down. The rankings of the auction forms in terms of expected revenue are: English > second-price sealed bid > Dutch = first-price sealed bid. Milgrom and Weber point out that these results provide an explanation for Casady's observation that " an estimated 75 percent, or even more, of all auctions in the world are conducted on an ascending-bid basis."

The intuition for the inferiority of the Dutch/first-price auctions in the presence of dependent values is that the winner's curse is most potent in a first-price auction: in a second-price auction, the curse is mitigated by the fact that knowing that other bids are lower than expected also implies that the second-lowest bid (which the winner will pay) is lower than expected. The intuition for the superiority of the English auction over the second-price auction is that in the former the bidders are able to glean information from the sequence of bids (i.e., the prices at which low-valuation bidders drop out), which lessens the winner's curse even further. An ascending-bid auction can be imagined to unfold in two phases. In phase 1, the first n–2 bidders drop out at observable price levels and thereby reveal their private information, after which the remaining two bidders engage in an ordinary second-price auction in the presence of more information than they started with. The availability of affiliated public information can be shown to raise prices on average rather than lower them. For the same reason, if the seller is able to obtain information affiliated with the bidders' valuations, then it is to her advantage to commit herself in advance to revealing it completely and truthfully. Meanwhile, if a bidder has private information, she should *not* reveal it: if one bidder reveals her private information to another, it reduces her expected surplus to zero! ("It is more important to a bidder that his information be private than that it be precise.") Also, when values are dependent, optimal reserve prices may vary with the type of auction and with the number of bidders.

When risk aversion is introduced in the dependent-values framework, the results are mixed, because the revenue effects of risk aversion are opposite to those of affiliation. (Recall that Dutch/first-price auctions raise more revenue from risk averse bidders.)

Combinatorial auctions have emerged as an important frontier of auction theory and practice in the last 10 or 15 years, through their use in high-stakes applications such as allocating segments of the radio spectrum for cellular phone networks, and many well-known auction theorists have become highly-paid consultants. Combinatorial auctions are used to simultaneously sell multiple items in situations where the valuation of any one item to a bidder depends on which other items are also obtained, and they typically allow all-or-nothing bids (or more complicated logical bids) on combinations of these items. Combinatorial auctions pose a number of new issues. First, the bidder's problem is much more complex: in principle, bidders may have to assess values and submit bids for all possible subsets of items. Second, the bidder's problem also has cooperative angles: a bidder on one set of items benefits from a competitor's high bid on a complementary set of items. Third, the determination of the winners is also more complex: finding the allocation that maximizes revenue (or some other objective) is equivalent to the "set packing" problem on hypergraphs, a prototypical NP-complete problem. For all these reasons, combinatorial auctions present huge analytic challenges to both buyers and sellers, and their implementation has turned out to require a great deal of experimentation and tinkering outside the bounds of the theory, as well as heavy lobbying by the potential bidders. Not surprisingly, the winners and losers in novel high-stakes auctions are often determined by how the rules are written, not how the game is eventually played. The paper by Pekec and Rothkopf delves into the complexity issues associated with the design of combinatorial auctions.

Here is a simple example: suppose the highest bids on combinations of three items (a, b, c) are as follows:

{a} - $1
{b} - $3
{c} - $2
{a, b} - $5
{a, c} - $5
{b, c} - $4
{a, b, c} - $6

The revenue-maximizing allocation is to sell {a, c} for $5 and {b} for $3, yielding $8. This problem can be solved by inspection, but in situations with large numbers of items and bidders, the solution of both buyer's and seller's problem can be very difficult, both computationally and strategically.

One type of mechanism that is potentially applicable to combinatorial auctions is the **Vickrey-Clarke-Groves mechanism**, which is a generalization of the second-price auction.

> "The VCG mechanism proceeds as follows. First solve the winner determination problem to maximize the sum of the agents' utilities before payments… Call this sum of utilities *a*. Then, to determine winning bidder *i*'s payment, remove that bidder's bid, and see what the maximum sum of the utilities before payments would have been with only the remaining bids. Call this sum of utilities b_i. Winning bidder *i* must pay $b_i - a + v_i$ where v_i is the value of winning bidder *i*'s bid. Effectively, it is the externality the bidder imposed on the other bidders (before payments). We observe that this payment is negative if the bidder's presence makes the other bidders better off (before payments)…" [from Conitzer & Sandholm 2006: "Failures of the VCG Mechanism in Combinatorial Auctions and Exchanges"]

Here's an example from Pekec and Rothkopf's 2003 paper**:** two items are for sale (a and b) and there are two bidders (#1 and #2). #1 bids $10 for {a}, $5 for {b}, and $15 for {a,b}. #2 bids

$1 for {a}, $6 for {b}, and $12 for {a,b}. The VCG solution is to sell {a} to #1 for $10 and {b} to #2 for $6, for a gross revenue of $16. #1 then receives a refund of $4 (the difference between 16 and 12), and #2 receives a refund of $1 (the difference between $16 and $15). Hence the actual payments are $6 and $5 for a net revenue of $11. One problem with the VCG mechanism is that it may be "revenue deficient" because of the refunds. It's the subject of much theoretical research but generally regarded as impractical except in special cases.

Various other mechanisms are used in practice for auctioning multiple items. When the items are identical, the **uniform price mechanism** can be used: items are awarded to highest bidders; and the bid on last item awarded is the price for all items sold. This is another generalization of the second-price auction in which the optimal strategy for the bidders is the simple and obvious one of bidding their own true values. **Iterative combinatorial auctions** are commonly used in e-business and by the FCC. These allow bidders to learn about rivals' values over multiple rounds but pose complex communication problems and create opportunities for strategizing and collusion.

Comments

The results sketched above, which include the best-known theorems and stylized facts about auctions, just begin to scratch the surface of auction theory. Some of them jibe with intuition and experience (e.g., ascending-bid and second-price auctions have more transparent bidding strategies and raise the most revenue in the plausible case of dependent values) while others are somewhat paradoxical (e.g., optimal auctions have peculiar rules and may be ex post inefficient). There are abundant grounds for questioning whether these results capture the most fundamental issues in auction design or represent realistic models of price formation among agents with heterogeneous information. Milgrom (1989) candidly points out:

> "The emphasis of much of recent bidding theory has been on ranking auctions on the basis of the expected receipts they generate. Sometimes this approach is taken to the extreme of determining institutions that maximize expected receipts, on the grounds that such institutions will be the ones chosen by the auctioneer/seller. The results of these maximization problems are, for all but the simplest environments, auctions of outlandishly complicated forms involving payments by the seller to losers, required side bets among the bidders, and so on. That such forms are not observed in practice indicates that the 'optimal auctions' theory in which the auctioneer can tailor a specific institution to each environment may be a poor way to explain actual institutions. The common auction institutions are all simple and robust, working well in a variety of environments, used by desperate sellers as well as by those with market power bordering on monopoly, and usually leading to a tolerably efficient allocation of the items being sold. Comparisons of robustness, efficiency, transaction costs, and immunity to cheating offer an important alternative to the revenue-based approaches for explaining the popularity of specific auction institutions."

The possibility of cheating is an important but often unmodeled issue in auction design. Sellers can cheat by employing "shills" (false bidders) in ascending-bid auctions, or by fabricating bids in second-price sealed-bid auctions, unless the true identities and bids of all the bidders are ultimately observable. (If you have ever bought anything at an internet auction site, you may well have wondered if some of the competing bidders were friends of the seller, or perhaps even robots.) Hence the robustness of these auction forms against irrational bidding by competitors is somewhat offset by their greater vulnerability to cheating on the part of sellers. On the other hand, buyers can also cheat by bidding collusively. In an ascending-bid or second-price auction, a group of high-valuation bidders can form a cartel, with one bidder bidding on behalf of the cartel and winning the item by paying the highest valuation outside the cartel, which is presumably less than the second-highest valuation among the cartel members. Later a secondary auction is held within the cartel, with the winner paying the second-highest valuation within the cartel, and the profit is shared among the cartel members. Bidding cartels can be countered to some extent by setting higher reserve prices. First-price sealed-bid auctions are also vulnerable to collusive behavior by buyers if the bids are not eventually publicly revealed.

A more profound issue is whether the potential buyers and sellers have anything approaching common knowledge of each others' value distributions, or even very precise estimates of their own valuations prior to the announcement of the auction. (Indeed, one of the attractions of internet auctions is that they offer opportunities for well-informed sellers to take advantage of naive bidders, and vice versa.) The beauty of auctions is that they work with or without common knowledge. A seller who puts a non-standard item up for sale at auction can leave it up to better-informed (and hopefully non-colluding) buyers to determine its value, and the buyers can, in turn, look to each other for clues to aid in the articulation of their own preferences. This observation suggests another possible explanation for the popularity of English auctions, namely that, more than the other auction forms, they facilitate the social construction of preferences through the feedback offered by an affirmative sequence of ascending bids. The announcement that an item will be sold at auction may also set in motion a process whereby potential buyers gather information about the item and perhaps discover value in it that may have been previously unquantified or even unsuspected. The seller and potential buyers need not have a well-defined and agreed-upon model of reciprocal beliefs about the value of an item in order for the auction to yield benefits to all. This property of auctions (and decentralized mechanisms in general) exemplifies the philosophy of the Austrian school of economics, whose spokesman Friedrich Hayek is quoted by McAfee and McMillan:

> "The economic problem of society is… not merely a problem of how to allocate 'given' resources—if 'given' is taken to mean given to a single mind which deliberately solves the problem set by these 'data.' It is rather a problem of how to secure the best use of resources known to any of the members of society, for ends whose relative importance only these individuals know. Or, to put it briefly, it is a problem of the utilization of knowledge which is not given to anyone in its totality."

Perhaps models of auctions with "uncommon priors" could shed more light on the comparative advantages and disadvantages of different auction processes in the latter regard. Insofar as models of optimal auctions depend sensitively on common knowledge of the joint value distribution attributed to the bidders, it might be interesting to consider whether some kind of augmented bidding mechanism, in which buyers and sellers place bets on the outcome of the bidding process, might be able to flesh out the value distribution and let arbitrage arguments play a role in the characterization of a rational outcome.

Guide to the readings:

1. “Auctions and Bidding” by R. Preston McAfee and John McMillan, *J. Economic Literature* (1987)

   IMHO, this last paper is the most thorough and readable of the many surveys of auction theory. Interestingly, it opens with a quotation from Hayek (an Austrian economist) who states that “The economic problem of society is … not merely a problem of how to allocate ‘given’ resources—if ‘given’ is taken to mean given to a single mind which deliberately solves the problem set by these ‘data.’ It is rather a problem of how to secure the best use of resources known to any of the members of society, for ends whose relative importance only these individuals know. Or, to put it briefly, it is a problem of the utilization of knowledge which is not given to anyone in its totality.” McAfee and McMillan argue that auctions help solve the economic problem described by Hayek—which is certainly true—but standard theoretical models of auctions nevertheless start from the assumption that very precise data on value distributions are ‘given’ and that it is possible to ‘deliberately solve the problem’ of auction design.

2. “Combinatorial Auction Design” by Aleksandar Pekec and Michael Rothkopf.

   This paper describes the computational and other practical problems raised by combinatorial auctions.

3. “Auction Theory: A Guide to the Literature” by Paul Klemperer, *J. Economic Surveys* (1999)

   This broad survey also appears as the introduction to the 2-volume set of readings *The Economic Theory of Auctions* recently published by Elgar, in which most of the classic papers on auction theory are reproduced.

4. “Auctions and Bidding: A Primer” by Paul Milgrom, *J. Economic Perspectives* (1989)

   A candid overview of auction theory, emphasizing intuition over technical detail.

5. “A Theory of Auctions and Competitive Bidding” by Paul Milgrom and Robert Weber, *Econometrica* (1982)

   One of the classic papers in the field, focusing on the general dependent-value model. Very technical, but with a good deal of readable commentary.

6. “The Simple Economics of Optimal Auctions” by Jeremy Bulow and John Roberts, *J. Political Economy* (1989)

   Provides an explanation of the marginal-revenue interpretation of optimal auctions which can be easily understood by anyone who instinctively feels that *Q* should be plotted on the horizontal axis and *P* should be plotted on the vertical axis.