Basics of Logic Design:
Boolean Algebra, Logic Gates, and the ALU
(Combinational Logic)

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Duke University

Slides are derived from work by
Daniel J. Sorin (Duke), Alvy Lebeck (Duke), and Drew Hilton (Duke)
Reading

- Appendix B (parts 1, 2, 3, 5, 6, 7, 8, 9, 10)
- This material is covered in MUCH greater depth in ECE/CS 350 – please take ECE/CS 350 if you want to learn enough digital design to build your own processor
What We’ve Done, Where We’re Going

Top Down

(Application) 
Compiler | Firmware

Operating System

(Circuit Design)

CPU | Memory | I/O system

Digital Design

Software

Interface Between HW and SW

Instruction Set Architecture, Memory, I/O

Hardware

(Almost) Bottom UP to CPU
Computer = Machine That Manipulates Bits

- Everything is in binary (bunches of 0s and 1s)
  - Instructions, numbers, memory locations, etc.
- Computer is a machine that operates on bits
  - Executing instructions → operating on bits

- Computers physically made of transistors
  - Electrically controlled switches
- We can use transistors to build logic
  - E.g., if this bit is a 0 and that bit is a 1, then set some other bit to be a 1
  - E.g., if the first 5 bits of the instruction are 10010 then set this other bit to 1 (to tell the adder to subtract instead of add)
How Many Transistors Are We Talking About?

Pentium III
• Processor Core 9.5 Million Transistors
• Total: 28 Million Transistors

Pentium 4
• Total: 42 Million Transistors

Core2 Duo (two processor cores)
• Total: 290 Million Transistors

Core2 Duo Extreme (4 processor cores, 8MB cache)
• Total: 590 Million Transistors

Core i7 with 6-cores
• Total: 2.27 Billion Transistors

How do they design such a thing? Carefully!
Abstraction!

• Use of **abstraction** (key to design of any large system)
  • Put a few (2-8) transistors into a **logic gate** (or, and, xor, ...)
  • Combine gates into logical functions (add, select, ....)
  • Combine adders, shifters, etc., together into modules
    Units with well-defined interfaces for large tasks: e.g., decode
  • Combine a dozen of those into a core...
  • Stick 4 cores on a chip...
Boolean Algebra

• First step to logic: Boolean Algebra
  • Manipulation of True / False (1/0)
  • After all: everything is just 1s and 0s

• Given inputs (variables): A, B, C, P, Q...
  • Compute outputs using logical operators, such as:

  • NOT: !A (= \sim A = \overline{A})
  • AND: A&B (= A \cdot B = A*B = AB = A\land B) = A&&B in C/C++
  • OR: A | B (= A+B = A \lor B) = A || B in C/C++
  • XOR: A ^ B (= A \oplus B)
  • NAND, NOR, XNOR, Etc.
### Truth Tables

- Can represent as **truth table**: shows outputs for all inputs

<table>
<thead>
<tr>
<th>a</th>
<th>NOT (a)</th>
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<tbody>
<tr>
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Any Inputs, Any Outputs

- Can have any # of inputs, any # of outputs
- Can have arbitrary functions:

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<tr>
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Let’s Write a Truth Table for a Function…

- Example:
  
  \[(A \& B) \mid !C\]

Start with Empty TT
  
  Column Per Input
  
  Column Per Output

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- Example:
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Start with Empty TT

- Column Per Input
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Fill in Inputs

- Counting in Binary

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Let’s write a Truth Table for a function…

- Example:
  
  \[(A \& B) \mid \neg C\]

Start with Empty TT
  
  Column Per Input
  
  Column Per Output

Fill in Inputs
  
  Counting in Binary

Compute Output
  
  \[(0 \& 0) \mid \neg 0 = 0 \mid 1 = 1\]

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- Example:
  \((A \& B) \mid !C\)

Start with Empty TT
  - Column Per Input
  - Column Per Output

Fill in Inputs
  - Counting in Binary

Compute Output
  \((0 \& 0) \mid !1 = 0 \mid 0 = 0\)

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Let’s write a Truth Table for a function…

• Example:
  
  \[(A \& B) \mid !C\]

Start with Empty TT
  Column Per Input
  Column Per Output

Fill in Inputs
  Counting in Binary

Compute Output
  \[(0 \& 1) \mid !0 = 0 \mid 1 = 1\]
Let’s write a Truth Table for a function…

- Example:
  
  \((A \& B) \mid \neg C\)

### Start with Empty TT
- Column Per Input
- Column Per Output

### Fill in Inputs
- Counting in Binary

### Compute Output

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Logisim example: `basic_logic.circ: example1`
Suppose I turn it around...

- Given a Truth Table, find the formula?

Hmmm..

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Suppose I turn it around…

- Given a Truth Table, find the formula?

Hmmm …

Could write down every “true” case

Then OR together:

\[

\begin{align*}
\neg A \land \neg B \land \neg C \land \\
\neg A \land \neg B \land C \land \\
\neg A \land B \land \neg C \land \\
A \land B \land \neg C \land \\
A \land B \land C
\end{align*}
\]

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Suppose I turn it around...

- Given a Truth Table, find the formula?

Hmmm..

Could write down every “true” case
Then OR together:

\[
\begin{align*}
(!A \& \!B \& \!C) \; & | \\
(!A \& \!B \& C) \; & | \\
(!A \& B \& \!C) \; & | \\
(A \& B \& \!C) \; & | \\
(A \& B \& C) \; & |
\end{align*}
\]

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Suppose I turn it around...

- Given a Truth Table, find the formula?

Hmmm..
Could write down every “true” case
Then OR together:

\[ \neg A \land \neg B \land \neg C \lor \neg A \land \neg B \land C \lor \neg A \land B \land \neg C \lor A \land B \land \neg C \lor A \land B \land C \]
Suppose I turn it around…

- This approach: “sum of products”
  - Works every time
  - Result is right...
  - But really ugly

\[(\neg A \land \neg B \land \neg C) \lor \neg A \land B \land \neg C \lor \neg A \land B \land C \lor A \land B \land \neg C \lor A \land B \land C\]

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Suppose I turn it around...

- This approach: “sum of products”
  - Works every time
  - Result is right...
  - But really ugly

\[(\neg A \land \neg B \land \neg C) \lor (\neg A \land \neg B \land C) \lor (\neg A \land B \land \neg C) \lor (A \land B \land \neg C) \lor (A \land B \land C)\]

Could just be \((A \land B)\) here?
Suppose I turn it around…

• This approach: “sum of products”
  • Works every time
  • Result is right...
  • But really ugly

(\neg A \land \neg B \land \neg C) \lor
(\neg A \land \neg B \land C) \lor
(\neg A \land B \land \neg C) \lor
(A \land B)

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Suppose I turn it around…

- This approach: “sum of products”
  - Works every time
  - Result is right...
  - But really ugly

\[(\neg A \land \neg B \land \neg C) \lor (\neg A \land \neg B \land C) \lor (\neg A \land B \land \neg C) \lor (A \land B)\]

Could just be \((\neg A \land \neg B)\) here
Suppose I turn it around…

- This approach: “sum of products”
  - Works every time
  - Result is right...
  - But really ugly

\[ (!A \& !B) \mid (!A \& B \& !C) \mid (A\&B) \]

Could just be (!A \& !B) here
Suppose I turn it around…

- This approach: “sum of products”
  - Works every time
  - Result is right...
  - But really ugly

\[
(!A \& !B) \mid (!A \& B \& !C) \mid (A \& B)
\]

Looks nicer…
Can we do better?
Just did some of these by intuition. but

- Somewhat intuitive approach to simplifying
- This is **math**, so there are formal rules
  - Just like “regular” algebra
Boolean Function Simplification

- Boolean expressions can be simplified by using the following rules (bitwise logical):
  
  - $A \& A = A$
  - $A \& 0 = 0$
  - $A \& 1 = A$
  - $A \& \neg A = 0$
  - $\neg \neg A = A$
  - $A \& (B \| C) = (A \& B) \| (A \& C)$
  - $A \| (A \& B) = A$

- & and | are both commutative and associative

$A$ and $\neg A$ are subsumed.
DeMorgan’s Laws

• Two (less obvious) Laws of Boolean Algebra:
  • Let’s push negations inside, flipping & and |

\[
\neg (A \& B) = \neg A \mid \neg B
\]

\[
\neg (A \mid B) = \neg A \& \neg B
\]

• You should try this at home – build truth tables for both the left and right sides and see that they’re the same
## Summary of all Boolean axioms

<table>
<thead>
<tr>
<th>Name</th>
<th><strong>AND form</strong></th>
<th><strong>OR form</strong></th>
</tr>
</thead>
<tbody>
<tr>
<td>Identity law</td>
<td>1 &amp; A = A</td>
<td>0</td>
</tr>
<tr>
<td>Null law</td>
<td>0 &amp; A = 0</td>
<td>1</td>
</tr>
<tr>
<td>Idempotent law</td>
<td>A &amp; A = A</td>
<td>A</td>
</tr>
<tr>
<td>Inverse law</td>
<td>A &amp; !A = 0</td>
<td>A</td>
</tr>
<tr>
<td>Commutative law</td>
<td>A &amp; B = B &amp; A</td>
<td>A</td>
</tr>
<tr>
<td>Associative law</td>
<td>(A&amp;B) &amp; C = A &amp; (B&amp;C)</td>
<td>(A</td>
</tr>
<tr>
<td>Distributive law</td>
<td>A</td>
<td>(B&amp;C) = (A</td>
</tr>
<tr>
<td>Absorption law</td>
<td>A &amp; (A</td>
<td>B) = A</td>
</tr>
<tr>
<td>De Morgan’s law</td>
<td>!(A&amp;B) = !A</td>
<td>!B</td>
</tr>
<tr>
<td>Double negation law</td>
<td>!!A = A</td>
<td></td>
</tr>
</tbody>
</table>

Adapted from [http://studytronics.weebly.com/boolean-algebra.html](http://studytronics.weebly.com/boolean-algebra.html)
Simplification Example:

\[
\neg (\neg A \lor \neg (A \land (B \lor C)))
\]

DeMorgan’s

\[
\neg \neg A \land \neg \neg (A \land (B \lor C))
\]

Double Negation Elimination

\[
A \land (A \land (B \lor C))
\]

Associativity of \&

\[
(A \land A) \land (B \lor C)
\]

\[
A \land A = A
\]

\[
A \land (B \lor C)
\]
You try this:

Come up with a formula for this Truth Table
Simplify as much as possible

Sum of Products:

\((\neg A \land \neg B \land \neg C) \lor (\neg A \land B \land \neg C) \lor (A \land \neg B \land C) \lor (A \land B \land C)\)

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</table>
You try this:

Simplify:

\[ (\neg A \land \neg B \land \neg C) \lor (\neg A \land B \land \neg C) \]

Regroup (associative/commutative):

\[ ((\neg A \land \neg C) \land \neg B) \lor ((\neg A \land \neg C) \land B) \]

Un-distribute (factor):

\[ (\neg A \land \neg C) \land (\neg B \lor B) \]

OR identities:

\[ (\neg A \land \neg C) \land \text{true} = (\neg A \land \neg C) \]
You try this:

Come up with a formula for this Truth Table
Simplify as much as possible

Sum of Products:

\[ \overline{A} \land \overline{C} \lor (A \land \overline{B} \land C) \lor (A \land B \land C) \]

Result of simplifying

You can simplify this part in the same way…

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</table>
You try this:

Come up with a formula for this Truth Table

Simplify as much as possible

Sum of Products:

$$(!A \land !C) \lor (A \land C)$$
Applying the Theory

- Lots of good theory
- Can reason about complex Boolean expressions
- But why is this useful?
**Boolean Gates**

- **Gates** are electronic devices that implement simple Boolean functions (building blocks of hardware)

- **AND** (\(a, b\))
  - Symbol: \(\cdot\)
- **OR** (\(a, b\))
  - Symbol: \(\lor\)
- **NAND** (\(a, b\))
  - Symbol: \(\neg\)
- **NOR** (\(a, b\))
  - Symbol: \(\lor\neg\)
- **XOR** (\(a, b\))
  - Symbol: \(\oplus\)
- **XNOR** (\(a, b\))
  - Symbol: \(\oplus\neg\)
- **NOT** (\(a\))
  - Symbol: \(\neg\)
Guide to Remembering your Gates

- This one looks like it just points its input where to go
  - It just produces its input as its output
  - Called a buffer

- A circle always means negate (invert)
Guide to Remembering your Gates

**AND** \((a, b)\)

- Straight like an A

**OR** \((a, b)\)

- Curved, like an O

**XOR** \((a, b)\)

- XOR looks like OR (curved line), but has two lines (like an X does)

**NAND** \((a, b)\)

- Circle means NOT

**NOR** \((a, b)\)

**XNOR** \((a, b)\)

- (XNOR is 1-bit “equals” by the way)

**NOT** \((a)\)
Brief Interlude: Building An Inverter

$\text{NOT}(a)$

$V_{dd} = \text{power} = 1$

P-type: switch is “on” if input is 0

N-type: switch is “on” if input is 1

ground = 0
Boolean Functions, Gates and Circuits

- **Circuits** are made from a network of gates.

\[(\neg A \land \neg C) \lor (A \land C)\]
A few more words about gates

- Gates have inputs and outputs
  - If you try to hook up two outputs, bad things happen (your processor catches fire)

```
 a
 b
```

```
 c
 d
```

- If you don’t hook up an input, it behaves kind of randomly (also not good, but not set-your-chip-on-fire bad)
Introducing the Multiplexer ("mux")
Introducing the Multiplexer ("mux")

Selector (S)

Input A

1

Input B

0

Output

1

"A"
Introducing the Multiplexer ("mux")

Selector (S)

Input A

Input B

Output

“B”

”mux”

0

1
Introducing the Multiplexer ("mux")

Selector ($S$)

Input A

Input B

Output

"B"
Let’s Make a Useful Circuit

• Pick between 2 inputs (called 2-to-1 MUX)
  • Short for multiplexor

• What might we do first?
  • Make a truth table?
    • S is selector:
      • S=0, pick A
      • S=1, pick B

• Next: sum-of-products
  \[ (!A \land B \land S) \lor \]
  \[ (A \land \neg B \land \neg S) \lor \]
  \[ (A \land B \land \neg S) \lor \]
  \[ (A \land B \land S) \]

• Simplify
  \[ (A \land \neg S) \lor (B \land S) \]
Circuit Example: 2x1 MUX

Draw it in gates:

\[
MUX(A, B, S) = (A \land \neg S) \lor (B \land S)
\]

So common, we give it its own symbol:

Logisim example
basic_logic.circ : mux 2x1
Example 4x1 MUX

The / 2 on the wire means “2 bits”
Arithmetic and Logical Operations in ISA

- What operations are there?
- How do we implement them?
  - Consider a 1-bit Adder
Designing a 1-bit adder

• What boolean function describes the low bit?
  • XOR

• What boolean function describes the high bit?
  • AND

\[
\begin{array}{c c c}
0 + 0 &= 00 \\
0 + 1 &= 01 \\
1 + 0 &= 01 \\
1 + 1 &= 10 \\
\end{array}
\]
Designing a 1-bit adder

- Remember how we did binary addition:
  - Add the **two bits**
  - Do we have a **carry-in** for this bit?
  - Do we have to **carry-out** to the next bit?

```
01101100
+ 00101100
-----
10011001
```
Designing a 1-bit adder

- So we’ll need to add three bits (including carry-in)
- Two-bit output is the *carry-out* and the *sum*

<table>
<thead>
<tr>
<th>a</th>
<th>b</th>
<th>C_in</th>
<th>Result</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>00</td>
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<td>0</td>
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<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>11</td>
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</tbody>
</table>

Turn into expression, simplify, circuit-ify, yadda yadda yadda…
A 1-bit Full Adder

\[
\begin{array}{ccc}
\text{a} & \text{b} & \text{Cin} \\
0 & 0 & 0 \\
0 & 0 & 1 \\
0 & 1 & 0 \\
0 & 1 & 1 \\
1 & 0 & 0 \\
1 & 0 & 1 \\
1 & 1 & 0 \\
1 & 1 & 1 \\
\end{array}
\]

<table>
<thead>
<tr>
<th>a</th>
<th>b</th>
<th>C_{in}</th>
<th>Sum</th>
<th>C_{out}</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
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Logisim example
basic_logic.circ - full-adder
Example: 4-bit adder

```
Example: 4-bit adder

4-bit adder

Full Adder

Full Adder

Full Adder

Full Adder

C_{out}

S3

S2

S1

S0

a3 b3

a2 b2

a1 b1

a0 b0

Logisim example
basic_logic.circ : 4bit-adder
```
Subtraction

• How do we perform integer subtraction?
• What is the hardware?
  • Recall: hardware was why 2’s complement was good idea

• Remember: Subtraction is just addition
  \[ X - Y = \]
  \[ X + (-Y) = \]
  \[ X + (\sim Y + 1) \]
Example: Adder/Subtractor

![Diagram of a 4-bit adder/subtractor using full adders]

- Inputs: $a_3, b_3, a_2, b_2, a_1, b_1, a_0, b_0$
- Outputs: $C_{out}$
- Intermediate signals: $S_3, S_2, S_1, S_0$

Logisim example:
basic_logic.circ - 4bit-addsub
Overflow

- We can detect **unsigned overflow** by looking at CO.
- How would we detect **signed overflow**?
  - If adding positive numbers and result “is” negative.
  - If adding negative numbers and result “is” positive.
  - At most significant bit of adder, check if CI != CO.
  - Can check with XOR gate.
Add/Subtract With Overflow Detection

Signed Overflow

Unsigned Overflow

Add/Sub

Logisim example
basic_logic.circ : 4bit-addsub2
ALU Slice

Add/sub

Cin

a

b

Add/sub

Cout

Add/sub

F

Q

<table>
<thead>
<tr>
<th>A</th>
<th>F</th>
<th>Q</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>a + b</td>
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<tr>
<td>1</td>
<td>0</td>
<td>a - b</td>
</tr>
<tr>
<td>-</td>
<td>1</td>
<td>NOT b</td>
</tr>
<tr>
<td>-</td>
<td>2</td>
<td>a OR b</td>
</tr>
<tr>
<td>-</td>
<td>3</td>
<td>a AND b</td>
</tr>
</tbody>
</table>

Logisim example
basic_logic.circ : alu-slice
The ALU

Overflow

Is non-zero?

ALU Slice

\[ Q_{n-1} \]

\[ Q_{n-2} \]

\[ b_{n-1} \quad a_{n-1} \]

\[ b_{n-2} \quad a_{n-2} \]

ALU Slice

\[ Q_1 \]

\[ Q_0 \]

\[ b_1 \quad a_1 \]

\[ b_0 \quad a_0 \]

ALU control

Logisim example

basic_logic.circ - alu
Alternate ALU design

- Previous design did ALU stuff for each bit, then chained them.

- Can also do each word-size operation and mux the resulting words.
Abstraction: The ALU

- General structure
- Two operand inputs
- Control inputs

- We can build circuits for
  - Multiplication
  - Division
  - They are more complex

\[ \log_2(\text{num\_of\_operations\_supported}) \]
Another Operations We Might Want: Shift

- Remember the << and >> operations?
  - Shift left/shift right?
  - How would we implement these?
- Suppose you have an 8-bit number $b_7b_6b_5b_4b_3b_2b_1b_0$
- And you can shift it left by a 3-bit number $s_2s_1s_0$

- Option 1: Truth Table?
  - $2^{11} = 2048$ rows? Yuck.

...but you can do it. Truth table gives this expression for output bit 0:
Building a bit shifter

- Simpler problem: A shift-by-one circuit, all controlled by the same 1 bit input ($s_0$)
Building a bit shifter

- Simpler problem: A shift-by-two circuit, all controlled by the same 1 bit input ($s_1$)
Building a bit shifter

- Simpler problem: A shift-by-four circuit, all controlled by the same 1 bit input ($s_2$)

```
    b_7  |  out_7
    s_2  |
    b_6  |  out_6
    s_2  |
    b_5  |  out_5
    s_2  |
    b_4  |  out_4
    s_2  |
    b_3  |  out_3
    s_2  |
    b_2  |  out_2
    s_2  |
    b_1  |  out_1
    s_2  |
    b_0  |  out_0
    s_2  |
```

Literal

0
Now shifted by 3-bit number

- Full problem: stick them all together, controlled by 3-bit value $s_{2:0}$
Now shifted by 3-bit number

- Example: shift by 000
Now shifted by 3-bit number

- Example: shift by 011
Summary

- Boolean Algebra & functions
- Logic gates (AND, OR, NOT, etc)
- Multiplexors
- Adder
- Arithmetic Logic Unit (ALU)
- Bit shifting