

Can Speculative Trading Explain the Volume–Volatility Relation?

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We derive a speculative trading model with endogenous informed trading that yields a conditionally heteroscedastic time series for trading volume and the squared price changes. We use half-hourly price-change and volume data for IBM during 1988 to test the model and estimate the structural parameters using the simulated method-of-moments estimation procedure. Although the model seems to do a reasonable job fitting the unconditional moments of the volume and the squared price change processes, it fares less well in fitting the relation between current trading volume and lags of trading volume and squared volume's (and its lag's) relation to squared price changes.

KEY WORDS: Market microstructure; Simulated method of moments estimation; Volume and volatility.

In recent years, there has been a renewed interest in the relation between trading volume and the volatility of share prices. Part of this interest has been fueled by recent episodes of high price volatility coupled with heavy trading volume in equity markets during the “market breaks” of 1987 and 1989. Another source of this interest has been the emergence of a theoretical literature that examines the interactions of market makers and speculative, informed traders. The goal of this literature is to understand better how information is incorporated into share prices, which in turn gives new insights into the allocational efficiency of markets. Many of these models have distinct implications for the relation between trading volume and price volatility. Examples of these works are those of Kyle (1985, 1986), Admati and Pfleiderer (1988, 1989), Chowdhry and Nanda (1991), Foster and Viswanathan (1990, 1993a), and many others. At the same time several authors have used transaction data to test the claims of this theoretical literature, including, but not restricted to, Glosten and Harris (1988), Hasbrouck (1989, 1991a,b), Stoll (1989), Madhavan and Smidt (1991), and Foster and Viswanathan (1993b). Typically, these tests have focused on the estimation of market depth (expressed as how aggressively the market maker adjusts prices after observing the current order flow) and the relation of market depth to price volatility and trading volume. For the purposes of our discussion we refer to this body of theoretical and empirical work as models of speculative trading because of their reliance on a better informed trader who uses his/her information to generate trading profits.

In this article we adopt a different approach to testing models of speculative trading by informed traders. We use a theoretical model of speculative trading to undertake a detailed examination of the statistical relation between trading volume and price volatility [in an independent paper, Anderson

(1995) presented a different approach]. This relation is the focus of the early work of Clark (1973) and Tauchen and Pitts (1983) and has continued to be of interest to researchers, as evidenced by the more recent work of Harris (1987), Lamoureux and Lastrapes (1991, 1994), Anderson (1995), Gallant, Rossi, and Tauchen (1992), LeBaron (1992), and Richardson and Smith (1993). Researchers in this area have documented several empirical regularities that appear to be at odds with the speculative trading literature. As an important example of this difference, consider the work of Gallant et al. (1992), who used semiparametric density estimation to compute the joint density of daily volume and price volatility. In motivating their work, they claim that existing theoretical models do not provide a rich enough characterization of price volatility and trading volume to “jointly account for major stylized facts—serially correlated volatility, contemporaneous volume–volatility correlation and excess kurtosis of price changes” (p. 202, emphasis ours).

In this article we present a model of speculative trading that predicts conditional heteroscedasticity in trading volume and the variance of price changes and positive autocorrelation in trading volume. The model further restricts the joint stochastic process of trading volume and the variance of price changes in a testable way. To accomplish this, we build on previous work of Foster and Viswanathan (1993a), who showed that these richer empirical implications can be achieved by assuming that the conditional joint distribution of the fundamental variables of the model is from the class of elliptical contoured distributions [models such as those of Kyle (1985) and Admati and Pfleiderer (1988) that rely on the normal distribution predict conditional homoscedasticity in price changes and trading volume].

The setup of our model is similar to that of Kyle (1985) and Admati and Pfleiderer (1988). We have a single market

maker, several traders who must decide whether to pay a fee to become informed, and many liquidity traders, all trading shares in a single asset whose liquidation value is changing each period. In this model the informed traders receive a signal about the liquidation value of the asset, and the market maker knows the sum of the orders received. Without knowing the identities of the traders that submitted the orders, the market maker sets a price at which traders are cleared. Hence the model contains the speculative trading features of many recent works in the microstructure literature. Additionally, with our representation of uncertainty, the model also yields positive autocorrelation in trading volume and a positive conditional correlation between volume and the variance of price changes that depends on the trading history.

The major focus of our article is to bring together the market microstructure literature and the statistical literature on volume and volatility by imbedding an explicit model of speculative trading in the stochastic volatility model. This focus also allows us to undertake structural estimation of speculative trading models as opposed to the usual reduced-form estimation that is conducted in the empirical market microstructure literature [see Easley, Kiefer, and O'Hara (1993) for independent structural estimations of other microstructure models]. Instead of simply estimating the market depth (an endogenous parameter in theoretical market microstructure models) as is typically done in empirical market microstructure literature, we allow for time variation in endogenous variables and estimate deeper structural parameters, like the cost of acquiring information.

We test the model using the simulated method of moments (SMM) advocated by Duffie and Singleton (1993), McFadden (1989), Pakes and Pollard (1989), and Ingram and Lee (1991) [see also Bansal, Gallant, Hussey, and Tauchen (1995) for a related approach that uses seminonparametric methods to choose moments]. We test our model using half-hour volume and price-change data for International Business Machines (IBM) during the year 1988 as reported by the Institute for the Study of Security Markets (ISSM). For IBM we find that most of the volume comes from intense trading by many informed traders, each of whom pays very little to acquire information that is not very precise. It appears that a substantial portion of the information about IBM comes from public announcements, and that relatively little reliable private news about the stock is created in 30-minute intervals.

Our work represents a first step in the integration of the market microstructure and statistical stochastic volatility literatures. An important limitation of our approach, however, is that we assume that private information is short-lived. Traders in our model make a one-shot decision in which information has no direct value in future periods (notwithstanding a dependence of current decisions on the trading history). Later work in the area should attempt to allow for a more general characterization with long-lived information.

Section 1 presents our basic model. Section 2 lists the empirical implications of our model and explains our implementation of the SMM. Section 3 describes our data, Section 4 contains our empirical results, and Section 5 concludes. The Appendix contains all of our proofs.

1. THE MODEL

In this section we introduce the market participants and the assumptions that we use to solve the model. We prove that there is a unique linear Nash equilibrium (Proposition 1), show that trading volume and the variance of price changes are conditionally heteroscedastic (Proposition 2), and show that volume is positively autocorrelated (Proposition 3).

We consider a multiperiod model with a market maker, informed traders, and liquidity traders who trade shares in a single asset that has a liquidation value, v_t , that changes each period. Informed traders are traders that have paid to see a signal that is related to the liquidation value of the asset in that period. All market participants are assumed to be risk-neutral. Before trading each period, all traders know the past trading history, including past liquidation values, past prices, past signals, and the number of shares of the asset that were traded. We summarize this past information in the set Ψ_{t-1} that is known at the beginning of time t . Often we refer to this information set as the market history. Based on Ψ_{t-1} , several traders choose to pay c to acquire a signal, s_t , that is related to the liquidation value of the asset at time t . We assume that all traders that choose to acquire information see the same signal and therefore make the same computation to decide whether to pay c (based on the market history) to see the signal, s_t . This means that the number of traders that are so informed at time t is a function of the past information, Ψ_{t-1} , and we label this function as $I(\Psi_{t-1})$.

Before orders are submitted at time t , the number of traders that choose to acquire information, $I(\Psi_{t-1})$, is made known to all traders and the market maker. After the signal, s_t , is observed by the informed traders, all traders submit their orders to the market maker, who observes the total order flow, y_t ; individual orders or the identity of the informed traders is not known by the market maker. This total order flow, which includes orders from liquidity traders, l_t , is used by the market maker to determine the new price, p_t , at which all orders are cleared. At the end of trading the total number of orders, y_t , the price, p_t , the liquidation value of the asset that period, v_t , and the private signal, s_t , are all made public and become part of the observable history of the market at the end of time t , Ψ_t . Thus, we follow Admati and Pfleiderer (1988) and Kyle (1985) in assuming that information has only value for one period.

We represent the uncertainty in the model with a joint distribution from the compound normal class of distributions (this is a subclass of the class of elliptically contoured distributions). Specifically, we assume that the underlying value process, the signal process, and the liquidity-trading process are determined by

$$\begin{aligned} v_t - v_{t-1} &= h_t x_t \\ l_t &= h_t z_t \\ s_t &= v_t - v_{t-1} + q_t = h_t x_t + h_t r_t \\ x_t &\sim N(0, \sigma_x^2) \\ z_t &\sim N(0, \sigma_z^2) \\ r_t &\sim N(0, \sigma_r^2), \end{aligned} \tag{1}$$

where x_t , z_t , and r_t are innovations that are assumed to be independently distributed through time. q_t is a representation of the noise in the signal about the true innovation in the liquidation value of the asset at time t , s_t . The specification in Expression (1) makes use of an unobservable latent variable, h_t , whose support is the open interval $(0, \infty)$ that is independent of the x_t , r_t , and z_t processes. Expression (1) states that, conditional on the realization of h_t , the value process, the signal process, and the liquidity trading are jointly normally distributed. Because no trader knows the realization of h_t , they can only compute a distribution for h_t conditioning on the observable history, Ψ_{t-1} . If we use the observable history, Ψ_{t-1} , to compute the conditional joint distribution of $v_t - v_{t-1}$, s_t , and l_t we find that it is not a multivariate normal distribution; rather it belongs to the compound normal class (these distributions are also referred to as mixtures of multivariate normals). In our setup, the latent variable, h_t , governs the conditional heteroscedasticity in volatility in the model. As we shall see, these assumptions about the distribution of the parameters of the model allow us to solve the model in the same fashion as other models of speculative trading but, at the same time, allow for a more general representation of the time series of the variance of price changes and trading volume.

To complete our characterization of the uncertainty in the model, we need to specify the stochastic process for the latent variable. At present we write the evolution of h_t in a general form, in terms of the recursive stationary stochastic transition function

$$h_{t+1} = G(h_t, h_{t-1}, \dots, h_{t-k}, \zeta_t). \quad (2)$$

When we test the model we assume a specific parametric form for $G(\cdot)$.

Given all of the preceding, we define the unconditional variance–covariance matrix of the innovation in the liquidation value of the asset, the liquidity trading, and the signal noise to be

$$\begin{pmatrix} \text{var}(v_t - v_{t-1}) & \text{cov}(v_t - v_{t-1}, l_t) & \text{cov}(v_t - v_{t-1}, q_t) \\ \text{cov}(l_t, v_t - v_{t-1}) & \text{var}(l_t) & \text{cov}(l_t, q_t) \\ \text{cov}(q_t, v_t - v_{t-1}) & \text{cov}(q_t, l_t) & \text{var}(q_t) \end{pmatrix} \\ = \begin{pmatrix} \Lambda & 0 & 0 \\ 0 & \sigma_l^2 & 0 \\ 0 & 0 & \sigma_q^2 \end{pmatrix} = \begin{pmatrix} E[h^2]\sigma_x^2 & 0 & 0 \\ 0 & E[h^2]\sigma_z^2 & 0 \\ 0 & 0 & E[h^2]\sigma_r^2 \end{pmatrix}. \quad (3)$$

Using the assumptions listed previously, we now describe the order-submission strategy of the informed traders. Using the information he/she acquires at time t and the market history, the i th informed trader submits an order, a_{it} . In submitting this order, the i th informed trader makes Nash equilibrium conjectures about the pricing schedule of the market maker and the orders of other informed traders. This means that the i th informed trader selects a_{it} to maximize the following expression:

$$E[(v_t - p_t(\Psi_{t-1}, y_t))a_{it} \mid \Psi_{t-1}, s_t]. \quad (4)$$

Expression (4) is simply the expected trading profits of the i th informed trader at time t , where $E[\cdot]$ is the expectations operator and $p(\Psi_{t-1}, y_t)$ is the price set by the market maker.

The market maker sets the price after he or she observes the total order flow, y_t , composed of the informed agents' orders and the liquidity trader's orders, while knowing the available public information, Ψ_{t-1} , and the number of traders who choose to acquire information, $I(\Psi_{t-1})$. This means that the price, p_t , is a function only of the total order flow, $y_t = I(\Psi_{t-1})a_{it} + l_t$, given the observable history. If we require that the price be an unbiased conditional expectation of the liquidation value of the asset at time t (the market-efficiency condition), we have the following:

$$p_t = p(\Psi_{t-1}, y_t) = E[v_t \mid \Psi_{t-1}, y_t]; \quad (5)$$

this price-setting behavior can be justified by Bertrand competition among market makers. Hence this model is an infinitely repeated version of the model of Kyle (1985) and Admati and Pfleiderer (1988) with different distributional assumptions.

Based on the assumptions and structure given previously, Proposition 1 characterizes the unique linear equilibrium of this model in each trading period. Notice that the equilibrium decision rules listed in Proposition 1 depend on the history Ψ_{t-1} through the number of traders that choose to acquire information.

Proposition 1. The unique linear Nash equilibrium in each trading period is

$$\begin{aligned} a_{it} &= \beta(\Psi_{t-1})s_t \\ p_t &= v_{t-1} + \lambda(\Psi_{t-1})y_t \\ &= v_{t-1} + \lambda(\Psi_{t-1})[I(\Psi_{t-1})\beta(\Psi_{t-1})s_t + l_t], \end{aligned}$$

where

$$\begin{aligned} \lambda(\Psi_{t-1}) &= \frac{\sqrt{I(\Psi_{t-1})\phi} \sigma_x}{1 + I(\Psi_{t-1}) \sigma_z} \\ \beta(\Psi_{t-1}) &= \sqrt{\frac{\phi}{I(\Psi_{t-1})} \frac{\sigma_z}{\sigma_x}}, \end{aligned}$$

where $\phi = \sigma_x^2 / (\sigma_x^2 + \sigma_r^2)$. The market maker's sensitivity to price change with respect to the order flow, $\lambda(\Psi_{t-1})$, and the intensity of trading by the informed traders, $\beta(\Psi_{t-1})$, depend on the history of past trades, liquidation values, private signals, and prices through the number of traders that decide to acquire information based on this information, $I(\Psi_{t-1})$.

If there are I informed traders ($I \geq 1$), each has expected profits of (when $I = 0$, the profit is defined to be 0)

$$\Pi(I, \Psi_{t-1}) = E[h_t^2 \mid \Psi_{t-1}] \frac{1}{1 + I} \sqrt{\frac{\phi}{I}} \sigma_z \sigma_x.$$

If c is the cost that traders must pay to become informed then, in equilibrium, the number of traders that acquire information is determined by the inequalities (when $I \geq 1$) $\Pi(I(\Psi_{t-1}) + 1, \Psi_{t-1}) < c \leq \Pi(I(\Psi_{t-1}), \Psi_{t-1})$ and when $I = 0$ by the inequality $\Pi(1, \Psi_{t-1}) < c$.

Proof. See Appendix.

Proposition 1 states that parameters such as the intensity of trade of the informed traders and the market maker's sensitivity to the order flow depend on the number of traders that

decide to acquire information. Proposition 1 also shows that the conditional second moment of the random variable h_t , given the history Ψ_{t-1} (which is a measure of the conditional variance of the value-change process), is a sufficient statistic for the number of traders who choose to acquire information at time t (see the expression for the profits to informed traders). The distributional assumptions imply that a high expected conditional variance for both the value-change process and the liquidity-trading process lead to higher payoffs to information acquisition and thus greater informed trading. With more traders seeking to become informed, there is more information released through trading; that is, prices are more informative and there is higher trading volume. In the remainder of this section, we make these ideas precise by computing the price change (and the squared price change) and trading volume and characterizing their time series properties in this model.

Given the price adjustment rule of the market maker, the intensity of trade of the informed traders, and the realized liquidity trading, the price change from $t - 1$ to t is

$$\begin{aligned}
 p_t - p_{t-1} &= v_{t-1} + \lambda(\Psi_{t-1})[I(\Psi_{t-1})\beta(\Psi_{t-1})s_t + l_t] \\
 &\quad - v_{t-2} - \lambda(\Psi_{t-2})[I(\Psi_{t-2})\beta(\Psi_{t-2})s_{t-1} + l_{t-1}] \\
 &= I(\Psi_{t-1})\lambda(\Psi_{t-1})\beta(\Psi_{t-1})s_t + \lambda(\Psi_{t-1})l_t + (v_{t-1} - v_{t-2}) \\
 &\quad - I(\Psi_{t-2})\lambda(\Psi_{t-2})\beta(\Psi_{t-2})s_{t-1} - \lambda(\Psi_{t-2})l_{t-1} \\
 &= \frac{I(\Psi_{t-1})\phi}{1 + I(\Psi_{t-1})}s_t + \lambda(\Psi_{t-1})l_t \\
 &\quad + \frac{1 + I(\Psi_{t-2})(1 - \phi)}{1 + I(\Psi_{t-2})}(v_{t-1} - v_{t-2}) \\
 &\quad - \frac{I(\Psi_{t-2})\phi}{1 + I(\Psi_{t-2})}q_{t-1} - \lambda(\Psi_{t-2})l_{t-1} \\
 &= \frac{I(\Psi_{t-1})\phi}{1 + I(\Psi_{t-1})}h_t(x_t + r_t) + \lambda(\Psi_{t-1})h_t z_t \\
 &\quad + \frac{1 + I(\Psi_{t-2})(1 - \phi)}{1 + I(\Psi_{t-2})}h_{t-1}x_{t-1} \\
 &\quad - \frac{I(\Psi_{t-2})\phi}{1 + I(\Psi_{t-2})}h_{t-1}r_{t-1} - \lambda(\Psi_{t-2})h_{t-1}z_{t-1} \\
 &= \frac{I(\Psi_{t-1})\phi}{1 + I(\Psi_{t-1})}h_t(x_t + r_t) + \frac{\sqrt{I(\Psi_{t-1})\phi}}{1 + I(\Psi_{t-1})}\frac{\sigma_x}{\sigma_z}h_t z_t \\
 &\quad + \frac{1 + I(\Psi_{t-2})(1 - \phi)}{1 + I(\Psi_{t-2})}h_{t-1}x_{t-1} \\
 &\quad - \frac{I(\Psi_{t-2})\phi}{1 + I(\Psi_{t-2})}h_{t-1}r_{t-1} - \frac{\sqrt{I(\Psi_{t-2})\phi}}{1 + I(\Psi_{t-2})}\frac{\sigma_x}{\sigma_z}h_{t-1}z_{t-1}. \quad (6)
 \end{aligned}$$

From Expression (6), notice that the price change is orthogonal to the information known at time $t - 1$; this is a result of our assumptions of risk neutrality and perfect competition among market makers. The conditional variance and other higher moments of the price-change process, however, are not independent of this history. Rather, they depend on information at time $t - 1$ (i.e., before the updated price is announced) about h_{t-1} and h_{t-2} . The last equation in Expression (6) has some similarities to the conditional variance structure assumed by Hsieh (1991) and Nelson (1991) (they made these assumptions for returns, not price changes). Although our

assumptions about the exogenous value-change process is like that of Nelson (1991), the resulting price process is more complicated because it depends on both h_{t-1} and h_{t-2} , as well as endogenous parameters such as the number of traders that acquire information in both periods, $I(\Psi_{t-1})$ and $I(\Psi_{t-2})$.

We define trading volume in the same way as Admati and Pfleiderer (1988) and Foster and Viswanathan (1990). Specifically, trading volume each period, V_t , is computed as half of the orders from the informed traders, plus half of the orders from liquidity traders, plus half of the orders traded with the market maker (noncrossed orders):

$$\begin{aligned}
 V_t &= \frac{1}{2}|I(\Psi_{t-1})\beta(\Psi_{t-1})s_t| + \frac{1}{2}|l_t| \\
 &\quad + \frac{1}{2}|I(\Psi_{t-1})\beta(\Psi_{t-1})s_t + l_t| \\
 &= \frac{1}{2}|I(\Psi_{t-1})\beta(\Psi_{t-1})h_t(x_t + r_t)| + \frac{1}{2}|h_t z_t| \\
 &\quad + \frac{1}{2}|I(\Psi_{t-1})\beta(\Psi_{t-1})h_t(x_t + r_t) + h_t z_t| \\
 &= \frac{h_t}{2}[|I(\Psi_{t-1})\beta(\Psi_{t-1})(x_t + r_t)| \\
 &\quad + |z_t| + |I(\Psi_{t-1})\beta(\Psi_{t-1})(x_t + r_t) + z_t|]. \quad (7)
 \end{aligned}$$

The temporal correlation, and hence the conditional heteroscedasticity, in trading volume comes from the term h_t and the dependence of the number of traders who choose to acquire information on the market history.

Using the expressions for the trading volume and the price change, Equations (6) and (7), respectively, we turn to an examination of the relation between the conditional squared price changes and volume (Proposition 2) and the autocorrelation in volume (Proposition 3). In the statement of each proposition we refer to other authors who provide some empirical evidence in support of its claim.

Proposition 2. Trading volume and the variance of price changes exhibit the following properties:

1. Trading volume and the variance of price changes are conditionally positively correlated (see Gallant et al. 1992):

$$\text{cov}[(p_t - p_{t-1})^2, V_t | \Psi_{t-1}] > 0.$$

2. The variance of price changes is conditionally heteroscedastic (see Gallant et al. 1992; Lamoureux and Lastrapes 1991; Nelson 1991):

$$\begin{aligned}
 \text{var}[p_t - p_{t-1} | \Psi_{t-1}, y_t] &= m(\Psi_{t-1}, y_t^2) \\
 \text{var}[p_t - p_{t-1} | \Psi_{t-1}] &= m(\Psi_{t-1}).
 \end{aligned}$$

3. Expected trading volume is conditional heteroscedastic (see Gallant et al. 1992):

$$\begin{aligned}
 E[V_t | \Psi_{t-1}, y_t] &= n(\Psi_{t-1}, y_t^2) \\
 E[V_t | \Psi_{t-1}] &= n(\Psi_{t-1}).
 \end{aligned}$$

Proof. See Appendix.

Proposition 2 can be used to derive an empirically testable set of restrictions on the stochastic processes of trading volume and the squared price change. In addition to these properties, we are interested in the intertemporal correlation of

trading volume and the squared price change. To focus on this intertemporal correlation, we need to make an additional assumption about the latent process, h_t . This assumption is composed of two statements that imply a positive dependence through time in any stochastic process.

Affiliation Assumption. The latent variable process satisfies the following affiliation restrictions. Denote the vector (h_t, \dots, h_{t-k}) by m_t and let $f(\cdot)$ denote a joint density function. Then

$$f(h_{t+1} \wedge h'_{t+1} \mid m_t \wedge m'_t) f(h_{t+1} \vee h'_{t+1} \mid m_t \vee m'_t) > f(h'_{t+1} \mid m'_t) f(h_{t+1} \mid m_t)$$

and

$$f(m_t \wedge m'_t) f(m_t \vee m'_t) > f(m'_t) f(m_t),$$

where \wedge and \vee are the component-wise minimum and maximum operators and (h_{t+1}, m_t) , (h'_{t+1}, m'_t) are arbitrary values.

These restrictions on the stationary finite-dimensional distributions of the h_t process are similar to the affiliation assumptions used by Milgrom and Weber (1982) in their analysis of auctions. This affiliation assumption implies that if there is a large value for h_{t-1} , then the realizations of h_t and h_{t+1} are more likely to be large than to be small for arbitrary t . With this additional assumption on the latent-variable process, we obtain positive intertemporal correlation in trading volume and squared price changes and positive unconditional correlation between trading volume and price changes.

Proposition 3. Trading volume and squared pricing changes are positively autocorrelated and the unconditional cross-correlation between squared price changes and trading volume is positive (see Gallant et al. 1992; Karpoff 1987; Tauchen and Pitts 1983); that is,

$$\begin{aligned} \text{cov}[V_t, V_{t-1}] &> 0 \\ \text{cov}[(p_t - p_{t-1})^2, (p_{t-1} - p_{t-2})^2] &> 0 \\ \text{cov}[V_t, (p_t - p_{t-1})^2] &> 0. \end{aligned}$$

Proof. See Appendix.

Another interesting issue is the usefulness of current price and volume variables in predicting future prices and volumes. We consider two cases. In the first, all past information is made public. In the second, only the history of prices and volumes is publicly available. We state the results for these two cases in Proposition 4.

Proposition 4. When the complete history of price, volume, and other information previously assumed to be made public is available to all traders, then trading volume is a redundant variable for predicting functions of future prices and volumes. The history of prices is needed to predict future prices and trading volume, however. When only the history of prices and trading volume is available, both prices and trading volume are useful in predicting functions of future prices and volumes.

Proof. See Appendix.

When prices, volumes, and other information that we previously assumed to be publicly available is available to all traders, then volume information is redundant and is not required. This is because the other variables can be used to compute the trading volume. If what we have assumed to be public information is omitted, then price changes contain the information about this news and any private information. In this case, a history of trading volume may be useful to predict future prices and trading volume [see Blume, Easley, and O'Hara (1994) for a model in which trading volume serves this role]. In our model, only the variances and higher moments of future price changes are predictable. By construction, prices follow a martingale, and thus price changes cannot be forecasted.

Taken together, these four propositions are consistent with several documented empirical regularities. To test this model, however, we need to do more than ensure that it is consistent with past research. There are at least two reasons for a specific test of this model. First, there are alternative models that can explain some of the existing empirical regularities, and we would like to be able to differentiate among these competing theories (examples are Blume et al. 1994; Harris and Raviv 1993; Wang 1993). Second, the statement that a model is consistent with past empirical regularities is much weaker than the statement that the model is consistent with the data. That is, we need to use the model to give us appropriate test procedures; only then can we see if the model's relation to the previous literature is coincidental or substantial. Because speculative trading models have received so much attention in the theoretical literature, we need a well-specified test of whether they can explain the observed stochastic process of price changes and trading volume, using test procedures that are meaningful to this class of models. Principally, we need to estimate the deep parameters of the model (which are, or are used to determine, liquidity trading, the noise in the informed trader's signal, the cost of acquiring information, the parameters of the latent-variable process, and the innovations in the liquidation-value process) with specific restrictions on the moments (and cross-moments) of trading volume and price changes to see whether they make economic sense. This is what we do in the remainder of the article.

2. TEST OF THE MODEL

In this section, we outline how we implement our test of the model presented in Section 1. In doing so we describe our assumed specification of the latent-variable process, h_t ; introduce the SMM estimation technique; list our solution technique for the simulation of the model; and motivate our choice of the price-change and trading-volume moment restrictions that we use in our tests.

2.1 Specification of the Latent-Variable Process

The model of Section 1 and the related propositions are designed to provide a set of testable statements about the stochastic processes for the price change and trading volume of an asset. In particular, using the model we can compute any number of moment and cross-moment restrictions on

these data series, be they cross-sectional or intertemporal. In contrast the early tests employed by Tauchen and Pitts (1983) and Harris (1987) are based on unconditional cross-sectional restrictions and on different models of speculative trading. Anderson (1995) built a market microstructure model that is more closely tied to the original Tauchen and Pitts (1983) approach.

The deep parameters of the model are σ_x^2 , σ_z^2 , σ_r^2 , c , and the parameters that determine the h_t process, and it is these parameters that we need to estimate. The other parameters—like the unconditional variances of the value-change process, the signal noise, and liquidity trading—are derived from these parameters. To estimate the deep parameters, we need a reasonable specification of the stochastic process for the latent variable, h_t . Hence our test is a joint test of both the model and our assumed specification of the latent-variable process.

Although the specification of the h_t process is crucial in determining the exact temporal structure of the joint stochastic process of trading volume and price changes, the model does not pose any restrictions on this process, and it allows for arbitrary temporal dependence in h_t . Our assumed structure for h_t is motivated by the work of Gallant, Hsieh, and Tauchen (1991), Hsieh (1991), and Nelson (1991) and is given by

$$\ln(h_{t+1}) = \alpha + \gamma \ln(h_t) + \zeta_t \quad (8)$$

or

$$h_{t+1} = e^\alpha h_t^\gamma e^{\zeta_t}, \quad (9)$$

where ζ_t is a normally distributed, mean zero, and variance σ_ζ^2 process that is intertemporally independent. This specification satisfies the affiliation assumption that we introduced earlier and is generally referred to as a stochastic volatility model because the volatility of price changes is assumed to be driven by the latent variable [see Anderson's (1994) survey article for further discussion of the stochastic volatility model]. In our model both the latent variable and its lag affect the volatility of price changes [see Expression (6)]. Notice that Expressions (8) and (9) imply that the unconditional stationary distribution of h_t is lognormal with mean $\alpha/(1-\gamma)$ and variance parameter $\sigma_\zeta^2/(1-\gamma^2)$. Hence the additional deep parameters that come from the h_t process are α , γ , and σ_ζ^2 .

Our setup does not match that of Nelson (1991). He assumed that the variance of price changes or returns is linear in the logarithms in past observations (this is a standard assumption in the autoregressive conditional heteroscedasticity literature). Our approach is to use a latent variable that is linear in the logarithms of its past values and some unobservable shock. Hence, in our model, traders forecast the conditional second moment of h_t with the observable history, which results in its conditional second moment being highly nonlinear with respect to the observable variables.

This specification of the stochastic process of the latent variable, h_t , means that it is linear in logarithms and is unconditionally lognormal. As a consequence, the price change, Δp_t , is distributed as an unconditional lognormal mixture of normal distributions. Although this approach is theoretically attractive, it is also related to the empirical work of Tauchen

and Pitts (1983), Harris (1987), and Richardson and Smith (1993).

2.2 Estimation Method

As we state in Section 1 and in the introduction, we want our tests to use moment restrictions based on the price-change and trading-volume series predicted by the model. Taking these restrictions to observed data allows us to accept or reject the model and, perhaps more importantly, if the model is rejected have a better understanding of why it was rejected. What makes this test difficult is that many of the moments that we are interested in using cannot be computed in a closed form. For example, with endogenous informed trading, the decision to acquire information depends, in a nonlinear fashion, on the conditional second moment of the latent variable h_t , given the trading history (which includes variables that are unobservable to an econometrician). Rather than ignore some moments or not test the model, the natural method to test the model is to use the SMM procedure (see Duffie and Singleton 1993; Ingram and Lee 1991; McFadden 1989; Pakes and Pollard 1989). Essentially, this approach allows the use of moments computed from a simulation of the model rather than analytic moments, which makes tests feasible. Our description and implementation of the SMM procedure follows closely the discussion of Duffie and Singleton (1993). We set up our model in the SMM framework, discuss how the observed data are used, and describe how the SMM procedure estimates the parameters. We note at this point that we use analytic moments for those moments that the model predicts are expected to have a zero mean (see Sec. 2.4). These particular moments are 0 for all parameter values and cannot be used for identifying any of the parameters. These moment conditions represent a specification test, and hence we use for them the analytic moment of 0 in our estimation.

Duffie and Singleton's (1993) approach requires continuous parameter spaces for all relevant variables. Although McFadden (1989) and Pakes and Pollard (1989) considered discrete choice variables, they required the observed variables to be independent and identically distributed. In our model the number of traders that choose to acquire information each period is from the natural numbers. To ensure continuous parameter spaces, we amend our model as follows. Let the number of traders who acquire information, given the history Ψ_{t-1} , be a real, nonnegative number determined by

$$\Pi(I(\Psi_{t-1}), \Psi_{t-1}) = c, \quad (10)$$

where we assume that $I \geq 1$. These changes do not affect the economic intuition about what induces a trader to pay c to acquire the signal s_t . The restriction that $I \geq 1$ is needed to avoid the discontinuity that arises at $I = 0$ (in what follows, we will ignore the restriction that $I \geq 1$ because it is not likely to be binding for the very liquid security that we consider).

Using this additional assumption, we state our model in the framework of Duffie and Singleton (1993). To begin, the state vector in our model is

$$Y_t' = (h_t \quad x_t \quad r_t \quad z_t \quad I_t), \quad (11)$$

which evolves according to $Y_t = H(Y_{t-1}, \epsilon_t, \theta^*)$, where

$$Y_t = \begin{pmatrix} \ln h_t \\ x_t \\ r_t \\ z_t \\ I_t \end{pmatrix} = \begin{pmatrix} \alpha + \gamma \ln h_{t-1} + \sigma_\zeta \epsilon_t^1 \\ \sigma_x \epsilon_t^2 \\ \sigma_r \epsilon_t^3 \\ \sigma_z \epsilon_t^4 \\ G(\cdot) \end{pmatrix} \quad (12)$$

is the process that generates the observed data; here $G(\cdot)$ is the function linking the number of informed traders to the history (see Proposition 1). Here $\epsilon_t = (\epsilon_t^1, \epsilon_t^2, \epsilon_t^3, \epsilon_t^4, \epsilon_t^5)'$ is a multivariate unit normal vector that is independent through time. By scaling the elements of ϵ_t by the appropriate standard deviations (for the h_t process we must also adjust for the effects from h_{t-1}), we generate the appropriate innovations to the elements of Y_t . $\theta = (\alpha, \gamma, \sigma_\zeta^2, \sigma_x^2, \sigma_r^2, \sigma_z^2, c)'$ is a vector containing the parameters we need to estimate. We define θ^* to be the true parameters that are consistent with the observed data, and we use SMM to estimate θ^* from all possible θ that lie in a compact subset of \mathcal{R}^7 (there are seven parameters). In addition, the true value θ^* is assumed to lie in the interior of this compact subset. The parameter c is included and is used to determine the number of traders that choose to acquire information, I_t , which is an element of the state vector, Y_t .

In SMM we match moments from the data and moments from a simulation of the model that are computed with observable stochastic processes that are functions of the state vector. In our case, the observable stochastic processes are the price change, Δp_t , and the trading volume, V_t . In Section 2.4 we discuss the moments that we use for the SMM estimation. In this section we note that we use moments of up to five lags of the price change and volume data. Because one needs a prior period's realization of the price to compute a price change, we need six lags of the state variables to compute these moments. Let $Z_t = (Y_t, Y_{t-1}, \dots, Y_{t-6})'$. Then the contribution, in period t , to the sum used to compute these moments of the price change and volume process for the observed data is $f_t(Z_t, \theta^*) = f_t^{\theta^*}$. If there are T observations of data available to compute the moments, then the contributions to the sum used to compute the moments of the observed data in each period are $\{f_t^{\theta^*} = f_t(Z_t, \theta^*)\}_{t=1}^T$.

In SMM one computes moments from simulated data in which the simulation has $\tau(T)$ observations and $\tau(T) \rightarrow \infty$ as $T \rightarrow \infty$. Given arbitrary parameters, θ , and some initial value of the state process, Y_0 , with a pseudorandom independent and identically distributed $\widehat{\epsilon}_t$ series (this is a unit normal vector in our implementation), one can compute a simulated state process, \widehat{Y}_t , and the contribution to the sum used to compute moments from the simulated data, f_t^θ , according to

$$\begin{aligned} \widehat{Y}_t &= H(\widehat{Y}_{t-1}, \widehat{\epsilon}_t, \theta) \\ f_t^\theta &= f_t^\theta(\widehat{Z}_t, \theta). \end{aligned} \quad (13)$$

For any feasible vector of parameter values, θ , we define the difference between the observed and simulated moments as

$$g_T(\theta) = \frac{1}{T} \sum_{i=1}^T f_i^{\theta^*} - \frac{1}{\tau(T)} \sum_{j=1}^{\tau(T)} f_j^\theta, \quad (14)$$

where the expectation is that $g_T(\theta) \rightarrow 0$ as $T \rightarrow \infty$. The SMM estimate $\widehat{\theta}$ is the θ that minimizes the quadratic form

$$g_T(\theta)' W g_T(\theta), \quad (15)$$

where W is a weighting matrix that incorporates the Newey and West (1987) correction for serial correlation. The form of the weighting matrix is chosen so that the estimates are consistent and are asymptotically normally distributed. Specifically, the asymptotic distribution of the SMM estimator is given by (see Duffie and Singleton 1993; Ingram and Lee 1991; McFadden 1989; Pakes and Pollard 1989)

$$\sqrt{T}(\widehat{\theta} - \theta^*) \sim N(0, (1 + \tau)(D'WD)^{-1}), \quad (16)$$

where $\tau = \lim_T(T/\tau(T))$ and $D = \text{plim}_T \partial g_T(\widehat{\theta})/\partial \theta$. If the simulation size is large relative to the sample size, the SMM estimator will have little efficiency loss relative to the generalized method of moments (GMM) estimator [for a more detailed analysis of the relation between the simulation size, $\tau(T)$, the sample size, T , and the efficiency of SMM relative to GMM, see Foster, Richardson, and Smith 1993].

Although the SMM approach is clearly a natural approach to test our model given the learning about an underlying latent variable that occurs in our model, we need to point out that the small-sample properties of our approach are not known for models as complex as ours. In particular, Jacquier, Polson, and Rossi (1994) showed in the context of the simple stochastic volatility model that GMM estimation and approximate Kalman-filter estimation have poor small properties relative to a Bayesian Monte Carlo approach, especially when the true autocorrelation coefficients are close to 1.

2.3 Simulation Implementation

In this section we describe how we compute the simulated values of our model that are used in the SMM procedure. In our simulations, we draw 10,000 simulated values. We first describe our technique for choosing the initial value of the model's parameters, and detail our computation of the conditional second moment of the latent-variable process.

To choose our initial value of the h_t process we use its unconditional stationary distribution given the parameter vector, θ . This means that we draw h_1 from a lognormal distribution with mean $\alpha/(1 - \gamma)$ and variance $\sigma_\zeta^2/(1 - \gamma^2)$. We note that (in principle at least) drawing from the unconditional stationary distribution of the parameter h_t implies that the simulated series should be stationary because we have eliminated the dependence on the arbitrarily chosen initial value (see Duffie and Singleton 1993 for further discussion).

In our simulation we need to compute the conditional moment of h_t given the history Ψ_{t-1} , which is used in determining the number of traders that choose to acquire information. Although the unconditional distribution of h_t is a lognormal distribution, the conditional distribution need not be lognormal. The approach we take is to approximate the two first moments of the conditional distribution by the lognormal distribution. We have also compared our approximation to a second method that uses the inverse of the gamma distribution as the conditional distribution and found virtually no

difference. In the Bayesian time series literature, such approximations are standard (e.g., see West and Harrison 1989, pp. 368–369).

Our choice of these approximate distributions is motivated by our need to find a simple distribution to use in the updating process. In particular we are not interested in the updated conditional distribution of the h_t process; rather we are only concerned with its second noncentral moment. Moreover, we can always take the view that these updating approximations are used by traders—the traders are not expected to use updating rules that are very cumbersome to implement. We note, however, that recent advances in applying Bayesian Monte Carlo methods could enable more accurate approximations of the kind of conditional distribution that we have in our model. In particular, Jacquier et al. (1994) showed how to use the inverse of gamma distribution to accurately approximate conditional distributions in the simpler stochastic volatility model. Our approximations are also related to the usual generalized Kalman filter. Instead of normal distribution approximations, however, we are approximating by distributions on the positive real line.

We estimate the conditional second moment through time in the following manner [see also Lamoureux and Lastrapes (1994) for an approach to estimating moments of unknown latent variable]. Suppose that we start with a conditional prior on h_t (given the history Ψ_{t-1}) that is approximated by a lognormal distribution with mean μ_t and variance σ_t . Then the posterior distribution of h_t , given the observations of $\Delta v_t = v_t - v_{t-1} = h_t x_t$, $q_t = h_t r_t$, and $l_t = h_t z_t$ (which is equivalent to observing $\Delta p_t, s_t, v_t - v_{t-1}$) is

$$\begin{aligned} \eta(h_t | \Delta v_t, q_t, l_t) &= \frac{\eta(h_t, \Delta v_t, q_t, l_t)}{\int_0^\infty \eta(h_t, \Delta v_t, q_t, l_t) dh_t} \\ &= \frac{v(\Delta v_t, q_t, l_t | h_t) \eta(h_t)}{\int_0^\infty v(\Delta v_t, q_t, l_t | h_t) \eta(h_t) dh_t} \\ &= \frac{v(\Delta v_t | h_t) v(q_t | h_t) v(l_t | h_t) \eta(h_t)}{\int_0^\infty v(\Delta v_t | h_t) v(q_t | h_t) v(l_t | h_t) \eta(h_t) dh_t}. \end{aligned} \quad (17)$$

The numerator of Equation (17) can be rewritten as

$$\begin{aligned} v(\Delta v_t | h_t) v(q_t | h_t) v(l_t | h_t) \eta(h_t) &= \frac{1}{h_t^3} \frac{1}{\sqrt{2\pi}\sigma_x} \frac{1}{\sqrt{2\pi}\sigma_r} \frac{1}{\sqrt{2\pi}\sigma_z} \\ &\times \exp \left[-\frac{(\Delta v_t)^2}{2h_t^2\sigma_x^2} - \frac{q_t^2}{2h_t^2\sigma_r^2} - \frac{l_t^2}{2h_t^2\sigma_z^2} \right] \\ &\times \frac{1}{h_t \sqrt{2\pi}\sigma_t} \exp \left[-\frac{(\log_e h_t - \mu_t)^2}{2\sigma_t^2} \right]. \end{aligned} \quad (18)$$

Integrating over the density function defined in Expressions (17) and (18), we calculate the two moments $E[h_t^\gamma | \Delta v_t, q_t, l_t]$ and $E[h_t^{2\gamma} | \Delta v_t, q_t, l_t]$. But

$$\begin{aligned} E[h_{t+1} | \Psi_t] &= e^\alpha E[h_t^\gamma | \Psi_t] E[e_t^\zeta] \\ E[h_{t+1}^2 | \Psi_t] &= e^{2\alpha} E[h_t^{2\gamma} | \Psi_t] E[e_t^{2\zeta}], \end{aligned} \quad (19)$$

where we have used Equation (9). Thus, given our initial approximate prior, we obtain the conditional first and second

moment for h_{t+1} . The conditional second moment immediately determines the number of informed traders. If we approximate the posterior by a lognormal distribution, we obtain the new parameters μ_{t+1} and σ_{t+1}^2 by

$$\begin{aligned} E[h_{t+1} | \Psi_t] &= \exp \left[\mu_{t+1} + \frac{1}{2} \sigma_{t+1}^2 \right] \\ E[h_{t+1}^2 | \Psi_t] &= \exp [2\mu_{t+1} + 2\sigma_{t+1}^2]. \end{aligned} \quad (20)$$

We rewrite Expression (20) in terms of μ_{t+1} and σ_{t+1}^2 ,

$$\begin{aligned} \mu_{t+1} &= 2 \log_e \{ E[h_{t+1} | \Psi_t] \} - \frac{1}{2} \log_e \{ E[h_{t+1}^2 | \Psi_t] \} \\ \sigma_t^2 &= \log_e \{ E[h_{t+1}^2 | \Psi_t] \} - 2 \log_e \{ E[h_{t+1} | \Psi_t] \}, \end{aligned} \quad (21)$$

which we use to compute a new approximate posterior for the next time period.

2.4 Choice of Moments

Our choice of moments in part reflects those of earlier studies, such as Tauchen and Pitts (1983), Harris (1987), Gallant et al. (1992), and Richardson and Smith (1993). With the exception of that of Gallant et al. (1992), these works focus on unconditional moments. Here, we use both unconditional moments and various time series moments.

Table 1 contains a complete list of the moments we use. Essentially, we are interested in the mean, variance, skewness, and kurtosis of both the price change and the trading volume. In addition we include the cross-correlation between the squared price changes and volume (the well-known volume–volatility relation), the cross-correlation between absolute price changes and trading volume, and the cross-correlation between the kurtosis of the price changes and the trading volume. Each of the moments listed is computed with various lags (up to five) of the price-change and volume process. In total, 52 moments are used to estimate the seven parameters given by θ , which means there are 45 overidentifying restrictions. After this base case is computed, we also consider the same moment conditions using up to two lags for comparison purposes. An alternative strategy, which we do not pursue here, is to take the approach pioneered by Gallant and

Table 1. Moment Conditions Used in SMM Estimation

$E[\Delta p_t]$
$E[\Delta p_t^2]$
$E[\Delta p_t^3]$
$E[\Delta p_t^4]$
$E[\Delta p_t \Delta p_{t-i}], i = 1, 2, 3, 4, 5$
$E[\Delta p_t^2 \Delta p_{t-i}^2], i = 1, 2, 3, 4, 5$
$E[V_t]$
$E[V_t^2]$
$E[V_t V_{t-i}], i = 1, 2, 3, 4, 5$
$E[\Delta p_t V_{t-i}], i = 0, 1, 2, 3, 4, 5$
$E[\Delta p_{t-i} V_t], i = 0, 1, 2, 3, 4, 5$
$E[\Delta p_{t-i}^2 V_t], i = 1, 2, 3, 4, 5$
$E[\Delta p_t^2 V_{t-i}], i = 0, 1, 2, 3, 4, 5$
$E[\Delta p_t^2 V_{t-i}^2], i = 0, 1$
$E[\Delta p_t^4 V_{t-i}], i = 1, 2, 3, 4, 5$
$E[\Delta p_t^4 V_{t-i}^2]$

NOTE: This table lists the moment conditions used in the SMM estimation of the model. The model is estimated with half-hourly transactions data for IBM in 1988, as reported by ISSM.

Tauchen (1994) and use the score of the density of an auxiliary model (that approximates the distribution of the observed data) to generate moments.

3. DATA

To test our model, we use the time series of half-hourly trading volume and quote midpoints changes for IBM in 1988. With $6\frac{1}{2}$ hours of trading a day and 253 trading days in 1988, we have 3,289 observations of price and volume data. The data are taken from the ISSM tapes. For transactions in each half-hour interval, we compute the trading volume and quote midpoint. We then compute the sum, on a half-hourly basis, of the absolute trading volume [consistent with Expression (7)] and changes in the quote midpoints. Following the procedure recommended by Lee and Ready (1991), we use quotes that are at least five seconds older than the transaction to determine which bid–ask quote was available for each transaction.

We use data from an individual stock rather than an index (see Gallant et al. 1992) for two reasons. The first reason is that we want to be consistent with our theory, which is based on trading in shares of a single asset. Second, in using half-hour data, there is likely to be a severe nonsynchronous trading problem with the index (see Lo and MacKinlay 1988; Miller, Muthuswamy, and Whaley 1994), which may induce spurious positive autocorrelations in the data. We chose IBM because it is a well-known bellwether stock. Because of the computational intensity of our estimation procedure, we estimated the model for only one stock.

We use the quote midpoint rather than the transaction price to avoid any first-order negative serial correlation in price-change process that is the result of the transaction price moving from the bid price to the ask price [see Hasbrouck (1989, 1991a,b) for a similar rationale on using quote midpoints rather than transaction prices]. If there is a fixed-cost component to the bid–ask spread, then using transaction prices includes this value and contaminates our estimates because we did not incorporate a fixed-cost component in the market maker's price-adjustment rule in our model.

Our choice of half-hour data represents the smallest interval that we believe we can reasonably use to test our model. A smaller time interval is preferred because there is an aggregation problem in using volume data in larger intervals: The time series properties of a time-aggregated random variable will be different from the time series properties of the disaggregated random variable. We cannot use transactions-level data because our model assumes a discrete time interval in which several traders pool their orders and then trade occurs. With transactions data we are often using orders that emanate from a single trader rather than a group of traders. Moreover, our model does not give an endogenous transaction time; trades are assumed to occur on a regular basis, and the half-hour intervals are consistent with this structure.

One final concern is that the model and tests outlined in Sections 2 and 3 do not allow for nontrading periods such as overnight, weekends, and holidays. The cessation of trading caused by the market schedule is known to induce

seasonalities in the price-change and trading-volume processes (e.g., see Foster and Viswanathan 1993b). Because the model as described previously requires a stationary time series, we have to make one of at least two potential adjustments. The first is to alter the model to include an explicit adjust for the nontrading periods. One possibility is to add dummy values to the variance terms in the value-change process and liquidity-trading process for opening time periods after an overnight cessation of trading, closing time periods, and opening time periods after a cessation of trading longer than overnight. This adds six more parameters to be estimated (two variance terms each requiring three dummy variables) and quintuples the number of moment conditions we have to work with (the original set given in Sec. 2.4 and replications for the a standard opening period, the closing period, and an opening period after a longer than overnight closing). This approach complicates an already cumbersome estimation problem.

Another approach, which we use, is to whiten the observed price-change and trading-volume series for the effects of the market schedule before the parameters of the model are estimated. This procedure was used by Gallant et al. (1992). Essentially, we standardize the price-change and trading-volume series for the mean and variance effects that can be attributed to the opening and closing time period. Anderson and Bollerslev (1994) suggested that such dummy-variable adjustments may be insufficient to eliminate all seasonalities. In what follows we describe the procedure used to whiten the price-change series; an identical procedure is used for the trading-volume time series. We use ordinary least squares to estimate the following:

$$\begin{aligned}\Delta p_t &= \delta + \delta^o 1_t^o + \delta^k 1_t^k + \delta^b 1_t^b + \iota_t \\ \iota_t^2 &= \omega + \omega^o 1_t^o + \omega^k 1_t^k + \omega^b 1_t^b + \kappa_t,\end{aligned}\quad (22)$$

where o is used to denote time periods that include the open after the market has been closed overnight, k is used to denote time periods that include the close, and b is used to denote time periods that include the open when the market has been closed longer than overnight. The first equation of Expression (22) adjusts the price change for mean shifts at the open and close. The second equation of Expression (22) computes the variance shifts in the price-change process at the open and the close. We use the estimated coefficients to scale the price changes as follows. If a price change is from an opening period after the market was closed overnight, then the whitened price change, $\widehat{\Delta p}_t$, is

$$\widehat{\Delta p}_t = \sqrt{\frac{\omega}{\omega + \omega^o}} [\Delta p_t - \delta^o]. \quad (23)$$

We compute standardized values for both types of opening-period price changes and for the closing-period price changes and use these values in our SMM estimation. A similar procedure gives the standardized logarithm of trading volume for each period. We note that augmented Dickey–Fuller tests rejected the unit-root hypothesis for the standardized price change and standardized logarithm of trading-volume data at the 1% level.

4. RESULTS

In this section we outline the results of our test of the model. We first list our parameter estimates and examine the goodness of fit of the model. Next we show how the fitted density of the simulated data compares to the fitted density of the observed data. Finally, using the simulated data we present descriptive statistics for some of the parameters of interest in the model, such as the number of informed traders, the intensity of trading by the informed traders, the market maker's sensitivity to the order flow, and the composition of the trading volume (what portion comes from informed traders and what portion comes from liquidity traders).

Table 2 contains our estimates of the model's parameters and of the minimized criterion value. In our minimizations we used a modified Levenberg–Marquardt algorithm with a finite-difference Jacobian. Given the optimized criterion value and the optimal parameters, we searched whether some other values of the parameters achieved a lower criterion function and found none. Given the high computational burden of our estimation procedure, these checks are not exhaustive.

Because $g_T(\theta)$ is asymptotically normally distributed, we know that, under the null hypothesis that $g_T = 0$, the expression $Tg_T'Wg$, is distributed as a χ^2 with 45 df for our base case (52 moment equations less the 7 parameters to be estimated). From Table 2 we see that the reported χ^2 value of 346.1805 clearly rejects the model. We also estimate the model for our two-lag case, with 21 df (28 moment equations less the 7 parameters to be estimated). Notice that the model is still rejected (the χ^2 statistic is 307.936) but the coefficients are little changed from the five-lag case. We minimized the SMM criterion function with a local gradient procedure. Because the model was rejected, any economic inferences based on these parameters must be made with care.

The variance parameters for the model suggest that the liquidity-trading variance is very large (but, as we shall see, even with this large variance, liquidity trading is a small portion of the total trading volume). Additionally, the conditional variance of the noise in the signal, σ_z^2 , is substantially larger than the conditional variance of the value process, σ_x^2 (by conditional we mean given the value of the latent variable, h_t). This suggests that the information that the informed traders receive each half-hour is relatively noisy and

not especially valuable relative to the innovation on the value process. This is consistent with the findings of Hasbrouck (1991b), who found that 31.8% of the total change in value is revealed by the actions of speculative traders (his high capitalization sample). For his model, this means that approximately 60% of the traders are informed, hence a substantial proportion of trading volume comes from informed traders. For our estimation, 20.42% of the total change in value is revealed by the actions of informed traders, which is lower than that found by Hasbrouck (1991b). Finally, the cost of becoming informed each half-hour is small (c is estimated to be .28 cents), which is consistent with the relative lack of precision of the signal received. For example, with 250 trading days a year and 13 half-hour intervals a day, it would cost a trader \$9.10 to follow one stock for a year. This number is not unlike the prorated cost of subscribing to a newsletter for a year. Although there is little new information about IBM that is produced each half-hour, these estimates suggest that new, precise information about IBM may come from public sources rather than private sources that are revealed through trading. As a consequence, the cost of private information is consistent with its precision.

In Table 3 we present the observed and simulated moments for each of the 52 moment conditions that we use. For this large number of moment conditions we find that many have observed and simulated moments that are quite close. The moments for which there is the largest absolute difference between the observed and simulated values are trading volume and its relation to various lags of trading volume and higher moments of the price changes and various lags of the squared trading volume. Hence, at least for the raw moments (those computed in the first stage of the SMM minimization), we find that it is the moments having to do with volume and lagged volume, and squared price changes and lagged volume squared that fit the worst. To further investigate which moment conditions are resulting in the rejection of the model, we use the standard errors in the variance–covariance matrix from the second stage of the SMM estimation. By scaling the means of the moment conditions by their estimated standard errors, we can compute a t statistic for each moment condition. These values are reported in Table 3. It appears that there is no single moment condition that is driving the rejection.

Because of the unreliability of standard errors calculated via numerical derivatives in the simulated method of moments, we calculate a quadratic approximation to the criterion function around its minimum and use this to compute confidence intervals. This approximation is calculated by choosing two values of a specific parameter around its optimum, fixing its value at each of these two points, and then reoptimizing over the remaining parameters. This yields a value of the criterion function subject to a constraint on one of the parameters. With three estimates of the criterion function (one the unconstrained optimized estimate and the others constrained estimates), we can plot a quadratic approximation to the criterion function over a relevant range. The curvature of the criterion function indicates the precision with which we can estimate various parameters—a flat approximate function

Table 2. Parameter Estimates

Parameter	Estimate (5 lags)	Estimate (2 lags)
σ_x^2	.5938	.5828
σ_z^2	3,127,818.4168	3,136,568.6096
σ_f^2	2.3139	2.2714
σ_c^2	.1240	.1155
γ	.7480	.7203
α	-.1418	-.1525
c	.0028	.0020
χ^2	346.1805	307.9360

NOTE: This table lists the parameter estimates for the model using half-hourly quote mid-point changes and the logarithm of trading volume for IBM (as reported by the ISSM) in 1988. Observed price and volume data have been adjusted for time-of-day effects. We report the estimates based on two and five lags in the choice of moment conditions (see Table 1).

Table 3. Estimated Moments for Both Simulated and Observed Data

Moment	Simulated value	Observed value	t statistic
$E[\Delta p_t^2]$.6848	.0866	.0670
$E[\Delta p_t^4]$.0335	.0519	.0334
$E[V_t]$	10.1501	10.1282	-.0111
$E[V_t^2]$	104.6121	107.1861	.0723
$E[\Delta p_t^2 \Delta p_{t-1}^2]$.0097	.0125	.0215
$E[\Delta p_t^2 \Delta p_{t-2}^2]$.0071	.0112	.0380
$E[\Delta p_t^2 \Delta p_{t-3}^2]$.0068	.0125	.0387
$E[\Delta p_t^2 \Delta p_{t-4}^2]$.0069	.0086	.0271
$E[\Delta p_t^2 \Delta p_{t-5}^2]$.0054	.0103	.0583
$E[\Delta p_t^2 V_t]$.8085	.9198	.0377
$E[\Delta p_t^2 V_{t-1}]$.7318	.8682	.0498
$E[\Delta p_t^2 V_{t-2}]$.7204	.8664	.0531
$E[\Delta p_t^2 V_{t-3}]$.7126	.8891	.0602
$E[\Delta p_t^2 V_{t-4}]$.7099	.8985	.0643
$E[\Delta p_t^2 V_{t-5}]$.7052	.9048	.0675
$E[\Delta p_t V_t]$	2.0375	2.2026	.0550
$E[\Delta p_{t-1} V_t]$	1.9119	2.1605	.0837
$E[\Delta p_{t-2} V_t]$	1.8957	2.1238	.0771
$E[\Delta p_{t-3} V_t]$	1.8835	2.1560	.0892
$E[\Delta p_{t-4} V_t]$	1.8730	2.1796	.0988
$E[\Delta p_{t-5} V_t]$	1.8657	2.1956	.1059
$E[\Delta p_{t-1}^2 V_t]$.7385	.8908	.0532
$E[\Delta p_{t-2}^2 V_t]$.7260	.8712	.0501
$E[\Delta p_{t-3}^2 V_t]$.7195	.9020	.0600
$E[\Delta p_{t-4}^2 V_t]$.7113	.9138	.0666
$E[\Delta p_{t-5}^2 V_t]$.7062	.9217	.0723
$E[\Delta p_t^4 V_{t-1}]$.3706	.4968	.0251
$E[\Delta p_t^4 V_{t-2}]$.3614	.5080	.0278
$E[\Delta p_t^4 V_{t-3}]$.3560	.5496	.0320
$E[\Delta p_t^4 V_{t-4}]$.3586	.5577	.0331
$E[\Delta p_t^4 V_{t-5}]$.3483	.5593	.0348
$E[V_t V_{t-1}]$	103.4813	104.1534	.0182
$E[V_t V_{t-2}]$	103.3621	101.5458	-.0480
$E[V_t V_{t-3}]$	103.2667	101.8281	-.0425
$E[V_t V_{t-4}]$	103.1975	102.0897	-.0371
$E[V_t V_{t-5}]$	103.1431	102.2801	-.0326
$E[\Delta p_t^2 V_t^2]$	9.5899	10.2056	.0180
$E[\Delta p_t^2 V_{t-1}^2]$	7.9236	9.2497	.0433
$E[\Delta p_t^4 V_{t-1}^2]$	4.1389	5.2258	.0200
$E[\Delta p_t]$.0000	-.0016	-.0050
$E[\Delta p_t^3]$.0000	-.0098	-.0308
$E[\Delta p_t \Delta p_{t-1}]$.0000	.0068	.0546
$E[\Delta p_t \Delta p_{t-2}]$.0000	.0054	.0531
$E[\Delta p_t \Delta p_{t-3}]$.0000	.0012	.0115
$E[\Delta p_t \Delta p_{t-4}]$.0000	.0003	.0031
$E[\Delta p_t \Delta p_{t-5}]$.0000	-.0020	-.0187
$E[\Delta p_t V_t]$.0000	-.0096	-.0027
$E[\Delta p_t V_{t-1}]$.0000	-.0200	-.0058
$E[\Delta p_t V_{t-2}]$.0000	-.0101	-.0030
$E[\Delta p_t V_{t-3}]$.0000	-.0059	-.0018
$E[\Delta p_t V_{t-4}]$.0000	.0035	.0010
$E[\Delta p_t V_{t-5}]$.0000	-.0087	-.0026

NOTE: This table lists the 52 moments for both the simulated and observed data for the moment conditions given in Table 1. The data are half-hourly quote midpoint changes and the logarithm of trading-volume data for IBM (as reported by the ISSM) in 1988. Observed price and volume data have been adjusted for time-of-day effects. The simulation results are based on the model in the text, using the estimated parameters in Table 2. Simulated moments listed as .00 are those where we know that the analytic moment from the model is .00.

indicates little effect from changes in the parameter estimate. In addition, a confidence interval for the parameter can be computed using the criterion difference test (we discuss this in more detail in what follows).

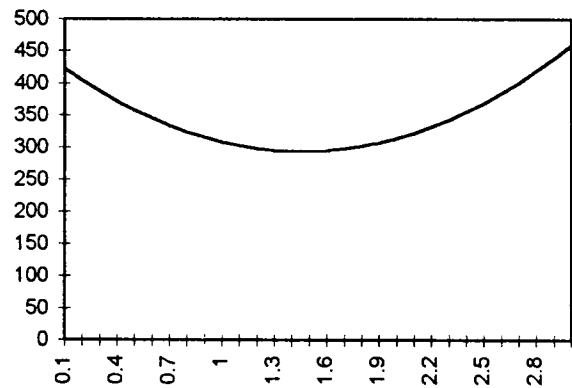


Figure 1. Quadratic Approximation to the Chi-Squared Value for Variance of x .

Although in principle the quadratic approximation could be computed for each parameter, this is not feasible due to the computation time involved. As a consequence, we choose only one parameter to fit the quadratic approximation, σ_x^2 , the parameter that determines the volatility of the true-value process. The optimized value of σ_x^2 at the optimal parameters is .5938 (the associated criterion function value is 346.1805). The two other values we pick are .587479 and 2.34996 (one smaller and one larger). When we reoptimize over the other parameters to find the constrained value of the criterion function, we obtain values of 346.9492 and 349.0158, respectively. These estimates suggest that the criterion function does not exhibit much curvature with respect to the parameter σ_x^2 . In fact, an examination of Figure 1 shows that the approximate function has considerably more curvature than is found in the data (it falls below 300 around the value of $\sigma_x^2 = 1.3$). Hence the local sensitivity of the criterion function (or lack thereof) seems to indicate that the method of simulated moments is not able to estimate very reliably this parameter. This should be borne in mind while considering some of the descriptive statistics we present. The flatness of the criterion function around the individual estimates does not mean that the model is not rejected. The criterion function generates χ^2 statistics that are so large for a large region around the optimal estimate that we can be quite confident in our rejection of the model.

Given the quadratic approximation to the criterion function, we computed the confidence intervals on the parameter σ_x^2 using the criterion difference test. We followed the approach of Press, Flannery, Teukolsky, and Vetterling (1988, pp. 532–537) (use of constant χ^2 boundaries as confidence limits). The confidence interval we obtain at the 90% significance level is [.4952, .6912]. Although this confidence interval seems reasonable, we believe (as we have stated in the previous paragraph) that the criterion function is flatter than the quadratic approximation would indicate and hence that the confidence interval is probably wider (at least on one side).

In Table 4 we present descriptive statistics for the deep parameters of the model. First, there are many traders who choose to acquire information (mean of 2,404 and range of 530 to 13,170) and their trading accounts for a large portion

Table 4. Descriptive Statistics for Simulated Model Parameters

Variable	Mean	Std. dev.	Minimum	Maximum	ρ_1	ρ_2	ρ_3	ρ_4	ρ_5
l	2,404.07	1,108.28	530.36	13,170.29	.739	.539	.307	.295	.218
β	22.65	4.74	9.04	45.04	.773	.584	.432	.316	.233
λ	4.29	.90	1.72	8.53	.773	.584	.432	.316	.234
h	.65	.38	.06	4.33	.721	.521	.384	.281	.209
μ	-.57	.33	-1.66	.81	.752	.564	.420	.309	.229
σ^2	.18	.02	.15	.25	.194	.049	.013	-.004	-.001
ΔP	-.004	.26	-1.66	2.00	-.009	.008	-.007	.005	.007
V	10.15	1.26	.12	13.58	.287	.214	.153	.110	.081
V^i	23,860.49	29,210.94	.08	396,140.00	.360	.228	.177	.126	.084
V^l	470.48	484.05	.11	5,995.38	.237	.158	.119	.095	.072
V^m	23,859.47	29,205.17	.00	394,887.90	.358	.227	.177	.126	.084

NOTE: This table contains descriptive statistics for the simulated data at the estimated parameters. Total trading volume has been transformed by the natural logarithm, and both total volume and price change have been adjusted for opening and closing seasonalities. The λ values, other than the autocorrelation-function estimates, have been multiplied by 10^6 . The parameters are as defined in the text, and V^i , V^l , and V^m are the trading volume (in number of shares) from the informed traders, the liquidity traders, and the market maker, respectively.

of the trading volume (on average, 23,860 shares, where 470 shares are from liquidity traders and 23,859 shares are met by the market maker). This is consistent with our interpretation that it costs little to become informed and that means that the traders will receive relatively imprecise information; however, intense competition between the informed traders ensures that the price reflects all of the imprecise information that they acquire. These numbers are, however, not entirely plausible given our existing knowledge of the markets. This underscores the fact that the model was rejected and that the parameters should be interpreted with care.

The intensity of trade of the informed traders and the sensitivity of the market maker show considerable variation (β has an average value of 22.65 and ranges from 9.04 to 45.04, whereas λ times 10^6 ranges between 1.72 and 8.53, with an average value of 4.29). The average estimate of λ is consistent with those reported in other works (e.g., see Foster and Viswanathan 1993b; Glosten and Harris 1988) and is not constant through time, as is typically assumed in the literature ($\rho_1 = .773$). Finally, note that the persistence in the estimate of the variance of the latent process (ρ_1 of σ^2 is .194) means that other model parameters will also be persistent (for example, ρ_1 of the liquidity-trading volume, V^l , is .237 and is less persistent than the informed trading volume, which depends on the number of informed traders). The ability of our model to create these intertemporal dependencies is what makes it attractive for our purposes and allows it to better describe observed data.

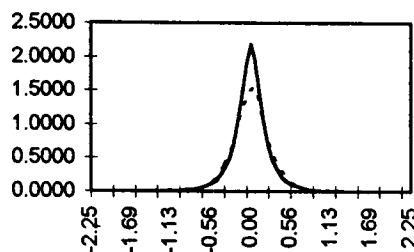


Figure 2. Estimated Densities From Simulated and Actual Price-Change Data: —, Simulated; - - -, Actual.

We also present some univariate kernel estimates of the density functions for the scaled unconditional price change and unconditional scaled logarithm of trading volume. We compute the densities for both the simulated and observed data. For all time series we compute the density with a Gaussian kernel and choose the window width according to Silverman (1986, p. 45), who suggested a window width for a sample of size T with a standard deviation of σ to be $1.06\sigma T^{1/5}$. Figure 2 is a plot of the unconditional price-change density, and Figure 3 is a plot of the trading-volume densities. In both cases the simulated and observed densities are similar. In particular, the simulated price changes appear to fit the observed scaled price changes quite well.

5. CONCLUSIONS

We have presented a model of speculative trading that is partially consistent with the volume–volatility relation documented by several researchers. We use a speculative trading model in which a lognormal latent variable is used to mix conditionally normal parameters, thereby generating persistence in trading volume and squared price changes. Using moment conditions from the model, we estimate its deep parameters with an SMM procedure for IBM in 1988. Although we reject the model, we learn several things. It appears that many informed traders pay little to receive relatively imprecise information and that the bulk of trading comes due to intense

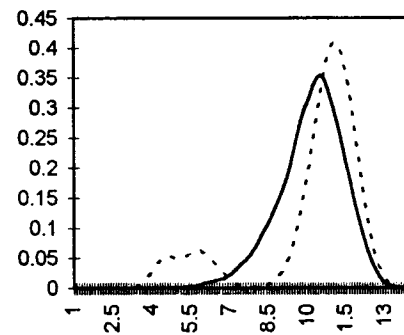


Figure 3. Estimated Densities From Simulated and Actual Volume Data: —, Simulated; - - -, Actual.

competition between these informed traders. Hence it may be the case that material information about IBM is revealed through public disclosure and there is much less private information for IBM that is revealed through trading (at least in relation to the changes in the liquidation-value process). Moreover, it appears that the model is unable to explain the relation between current trading volume and lags of trading volume and squared volume's (and its lag's) relation to squared price changes. After scaling these values by their standard errors it is less clear that these moment conditions are responsible for the model's demise.

The answer to the question posed by the title of the article is "No, our model of speculative trading cannot explain the volume–volatility relation for IBM in 1988." Although our article is clearly an early attempt at using optimizing models of trading to explain the volume–volatility relation, we believe that this general approach has promise. We hope (perhaps optimistically) that further work along these lines will lead to more fruitful answers and expect that other models that use long-lived private information will shed even more light on this complex subject.

ACKNOWLEDGMENT

We owe a special debt to Ravi Bansal, Tom Smith, and especially George Tauchen for valuable discussions on this article. Comments from Anat Admati, John Coleman, Darrell Duffie, Gerard Gennotte, Charlie Jacklin, Allan Kleidon, Narayana Kocherlakota, Anant Madhavan, Paul Pfleiderer, Mark Rubinstein, Matt Spiegel, Ingrid Werner, two anonymous referees, seminar participants at the 1993 American Finance Association meetings in Boston, and the Duke New York Stock Exchange Symposium in honor of Kalman Cohen, University of California at Berkeley, University of Iowa, and Stanford University are greatly appreciated. Viswanathan thanks the Business Associates Fund, the Isle Maligne Fund, and the Futures and Options Research Center (FORCE) of the Fuqua School of Business for financial support. Part of this research was done while Foster was at Duke University.

APPENDIX: PROOFS OF PROPOSITIONS

Proof of Proposition 1

If the market maker uses a linear pricing rule of the form $p(y, s) = v_{t-1} + \alpha(\Psi_{t-1}) + \lambda(\Psi_{t-1})y_t$, the i th informed trader maximizes

$$\begin{aligned} & E[(v_t - v_{t-1} - \alpha(\Psi_{t-1}) - \lambda(\Psi_{t-1})y_t)a_{it} \mid \Psi_{t-1}, s_t] \\ &= E \left[(\Delta v_t - \alpha(\Psi_{t-1}) - \lambda(\Psi_{t-1}) \sum_{j \neq i} a_{jt} \right. \\ &\quad \left. - \lambda(\Psi_{t-1})a_{it} - \lambda(\Psi_{t-1})l_t)a_{it} \mid \Psi_{t-1}, s_t \right] \\ &= \left[\phi s_t - \alpha(\Psi_{t-1}) - \lambda(\Psi_{t-1}) \sum_{j \neq i} a_{jt} \right] a_{it} - \lambda(\Psi_{t-1})[a_{it}]^2, \end{aligned} \quad (\text{A.1})$$

where we have substituted $y = \sum_{j \neq i} a_{jt} + l_t$ and simplified. The first-order condition from this maximization is

$$\phi s_t - \alpha(\Psi_{t-1}) - \lambda(\Psi_{t-1}) \sum_{j \neq i} a_{jt} - 2\lambda(\Psi_{t-1})a_{it} = 0, \quad (\text{A.2})$$

and the second-order condition is $-2\lambda(\Psi_{t-1}) < 0$, or $\lambda(\Psi_{t-1}) > 0$. The first-order condition can be rearranged as

$$a_{it} = \frac{1}{\lambda(\Psi_{t-1})} \left[\phi s_t - \alpha(\Psi_{t-1}) - \lambda(\Psi_{t-1}) \sum_{j \neq i} a_{jt} \right]. \quad (\text{A.3})$$

Thus, a_{it} is independent of i and

$$a_{it} = \frac{\phi}{[(1 + I(\Psi_{t-1}))\lambda(\Psi_{t-1})]} \left[s_t - \frac{\alpha(\Psi_{t-1})}{\phi} \right]. \quad (\text{A.4})$$

Hence $\beta(\Psi_{t-1}) = \phi / [(I(\Psi_{t-1}) + 1)\lambda(\Psi_{t-1})]$.

The market maker takes the linear strategy of the informed traders as given and sets a price $p_t = p(\Psi_{t-1}, y_t) = E[v \mid \Psi_{t-1}, y_t]$. Because of the structure we have assumed that, on the liquidation value, signal, and liquidity-trading processes, the information in the observable history up to time $t - 1$ is informative only about the latent variable, h_t . Consequently, the conditional distribution of $(\Delta v_t, s_t, l_t)$ is multivariate compound normal and the conditional expectation of Δv_t given y_t is linear, where

$$\begin{aligned} & E[\Delta v_t \mid \Psi_{t-1}, y_t] \\ &= \frac{I(\Psi_{t-1})\beta(\Psi_{t-1})E[h_t^2 \mid \Psi_{t-1}]\sigma_x^2}{I(\Psi_{t-1})^2\beta(\Psi_{t-1})^2E[h_t^2 \mid \Psi_{t-1}](\sigma_x^2 + \sigma_r^2) + E[h_t^2 \mid \Psi_{t-1}]\sigma_z^2} \\ &\times \left[y_t - \frac{I(\Psi_{t-1})}{(1 + I(\Psi_{t-1}))\lambda(\Psi_{t-1})} \left(E[s_t \mid \Psi_{t-1}] - \frac{\alpha(\Psi_{t-1})}{\phi} \right) \right]. \end{aligned} \quad (\text{A.5})$$

Thus

$$\lambda(\Psi_{t-1}) = \frac{I(\Psi_{t-1})\beta(\Psi_{t-1})\sigma_x^2}{I(\Psi_{t-1})^2\beta(\Psi_{t-1})^2(\sigma_x^2 + \sigma_r^2) + \sigma_z^2} \quad (\text{A.6})$$

and (using $E[s_t \mid \Psi_{t-1}] = 0$)

$$\alpha(\Psi_{t-1}) = \frac{I(\Psi_{t-1})}{1 + I(\Psi_{t-1})} \frac{\alpha(\Psi_{t-1})}{\phi}. \quad (\text{A.7})$$

This yields $\alpha(\Psi_{t-1}) = 0$. Equation (A.6) can be rearranged to give

$$\begin{aligned} \lambda(\Psi_{t-1})[I(\Psi_{t-1})^2\beta(\Psi_{t-1})^2(\sigma_x^2 + \sigma_r^2) + \sigma_z^2] \\ = I(\Psi_{t-1})\beta(\Psi_{t-1})\sigma_x^2. \end{aligned} \quad (\text{A.8})$$

Simplifying, and substituting $\beta(\Psi_{t-1}) = \phi / [(1 + I(\Psi_{t-1}))\lambda(\Psi_{t-1})]$ yields

$$\begin{aligned} \lambda(\Psi_{t-1}) \left[\frac{\phi^2 I(\Psi_{t-1})^2}{(1 + I(\Psi_{t-1}))^2 \lambda(\Psi_{t-1})^2} (\sigma_x^2 + \sigma_r^2) + \sigma_z^2 \right] \\ = \frac{\phi I(\Psi_{t-1})}{(1 + I(\Psi_{t-1}))\lambda(\Psi_{t-1})} \sigma_x^2. \end{aligned} \quad (\text{A.9})$$

This yields

$$\begin{aligned} \lambda(\Psi_{t-1}) &= \frac{\sqrt{I(\Psi_{t-1})\phi} \sigma_x}{1 + I(\Psi_{t-1}) \sigma_z} \\ \beta(\Psi_{t-1}) &= \sqrt{\frac{\phi}{I(\Psi_{t-1})}} \frac{\sigma_z}{\sigma_x}. \end{aligned} \quad (\text{A.10})$$

The expected profits when $I(\Psi_{t-1})$ traders choose to acquire information is

$$\begin{aligned} \Pi(I(\Psi_{t-1}), \Psi_{t-1}) &= E[E[(v_t - p_t)a_t | s_t, \Psi_{t-1}] | \Psi_{t-1}] \\ &= E[(\phi - I(\Psi_{t-1})\lambda(\Psi_{t-1})\beta(\Psi_{t-1}))\beta(\Psi_{t-1})s_t^2 | \Psi_{t-1}] \\ &= (\phi - I(\Psi_{t-1})\lambda(\Psi_{t-1})\beta(\Psi_{t-1}))\beta(\Psi_{t-1})E[s_t^2 | \Psi_{t-1}] \\ &= \phi \frac{1}{1 + I(\Psi_{t-1})} \sqrt{\frac{\phi}{I(\Psi_{t-1})}} \frac{\sigma_z}{\sigma_x} (\sigma_x^2 + \sigma_r^2) E[h_t^2 | \Psi_{t-1}] \\ &= E[h_t^2 | \Psi_{t-1}] \frac{1}{1 + I(\Psi_{t-1})} \sqrt{\frac{\phi}{I(\Psi_{t-1})}} \sigma_z \sigma_x, \end{aligned} \quad (\text{A.11})$$

which immediately leads to the condition determining the equilibrium number of traders that acquire information as stated in the proposition.

Proof of Proposition 2

Part 1. First note that

$$\begin{aligned} (p_t - p_{t-1})^2 &= \lambda(\Psi_{t-1})^2 [I(\Psi_{t-1})\beta(\Psi_{t-1})s_t + l_t]^2 \\ &+ (v_{t-1} - p_{t-1})^2 + 2\lambda(\Psi_{t-1})[I(\Psi_{t-1})\beta(\Psi_{t-1})s_t + l_t] \\ &\quad \times (v_{t-1} - p_{t-1}) \end{aligned} \quad (\text{A.12})$$

and

$$V_t = \frac{1}{2} [|I(\Psi_{t-1})\beta(\Psi_{t-1})s_t| + |l_t| + |I(\Psi_{t-1})\beta(\Psi_{t-1})s_t + l_t|]. \quad (\text{A.13})$$

Conditional on Ψ_{t-1} , we can simplify $E[(p_t - p_{t-1})^2 V_t | \Psi_{t-1}]$ to obtain

$$\begin{aligned} E[(p_t - p_{t-1})^2 V_t | \Psi_t] &= (v_{t-1} - p_{t-1})^2 E[V_t | \Psi_{t-1}] \\ &+ \frac{\lambda(\Psi_{t-1})^2}{2} [E[|I(\Psi_{t-1})\beta(\Psi_{t-1})s_t + l_t|^3 | \Psi_{t-1}] \\ &+ E[|I(\Psi_{t-1})\beta(\Psi_{t-1})s_t|^3 | \Psi_{t-1}] \\ &+ E[|l_t|^3 | \Psi_t] + E[(I(\Psi_{t-1})\beta(\Psi_{t-1})s_t)^2 |l_t| | \Psi_{t-1}] \\ &+ E[|I(\Psi_{t-1})\beta(\Psi_{t-1})s_t||l_t|^2 | \Psi_{t-1}]]. \end{aligned} \quad (\text{A.14})$$

In addition,

$$\begin{aligned} E[(p_t - p_{t-1})^2 | \Psi_{t-1}] &= \lambda(\Psi_{t-1})^2 E[(I(\Psi_{t-1})\beta(\Psi_{t-1})s_t + l_t)^2 | \Psi_{t-1}] \\ &+ (v_{t-1} - p_{t-1})^2. \end{aligned} \quad (\text{A.15})$$

This leads to the following expression for the conditional covariance between the squared price changes and volume,

given Ψ_t :

$$\begin{aligned} \text{cov} [(p_t - p_{t-1})^2, V_t | \Psi_{t-1}] &= \frac{\lambda(\Psi_{t-1})^2}{2} [E[|I(\Psi_{t-1})\beta(\Psi_{t-1})s_t + l_t|^3 | \Psi_{t-1}] \\ &- (E[(I(\Psi_{t-1})\beta(\Psi_{t-1})s_t + l_t)^2 | \Psi_{t-1}] \\ &\times E[|I(\Psi_{t-1})\beta(\Psi_{t-1})s_t + l_t| | \Psi_{t-1}]) \\ &+ E[|I(\Psi_{t-1})\beta(\Psi_{t-1})s_t|^3 | \Psi_{t-1}] \\ &- (E[(I(\Psi_{t-1})\beta(\Psi_{t-1})s_t)^2 | \Psi_{t-1}] \\ &\times E[|I(\Psi_{t-1})\beta(\Psi_{t-1})s_t + l_t| | \Psi_{t-1}]) \\ &+ E[|l_t|^3 | \Psi_{t-1}] - (E[l_t^2 | \Psi_{t-1}] E[|l_t| | \Psi_{t-1}]) \\ &+ E[|I(\Psi_{t-1})\beta(\Psi_{t-1})s_t|^2 |l_t| | \Psi_{t-1}] \\ &- (E[|I(\Psi_{t-1})\beta(\Psi_{t-1})s_t|^2 | \Psi_{t-1}] E[|l_t| | \Psi_{t-1}]) \\ &+ E[|I(\Psi_{t-1})\beta(\Psi_{t-1})s_t||l_t|^2 | \Psi_{t-1}] \\ &- (E[|I(\Psi_{t-1})\beta(\Psi_{t-1})s_t| | \Psi_{t-1}] E[|l_t|^2 | \Psi_{t-1}])]. \end{aligned} \quad (\text{A.16})$$

The last two terms that involve cross-moments are 0 because x_t and r_t are independent of z_t . The remaining terms are positive if $E[|x|^3] > E[x^2]E[|x|]$; that is, the random variables $|x|$ and x^2 are positively correlated. From Jensen's inequality, with the convex functions x^2 and x^3 (for positive x), it follows that

$$\begin{aligned} E[|x|^3] &> (E[x^2])^{(3/2)} \\ E[|x|^3] &> (E[|x|])^3. \end{aligned} \quad (\text{A.17})$$

Putting these two inequalities together yields the desired result.

Proof of Parts 2 and 3. The conditional heteroscedasticity is obvious from the dependence of price changes and volume on h_t and on the information set via the number of informed traders. Affiliation ensures that the information in the past history is relevant in predicting h_t , and the number of informed traders. The fact that the square of the history is only relevant follows from symmetry.

Proof of Proposition 3

The volume today, V_t , depends on the infinite history of observations up to that point $(i_{t-1}, i_{t-2}, \dots)$, where $i_t = (\Delta v_t, q_t, l_t)$. Because we assume that the process (x_t, r_t, z_t, h_t) is stationary and ergodic, so is the process i_t [because i_t is a measurable function of the original process; see Breiman (1968, proposition 6.6, p. 105, and proposition 6.31, p. 119)]. By the martingale convergence theorem (Billingsley 1986, p. 492, theorem 35.5), it follows for integrable $f(w)$ that

$$E[f(w) | C_n] \rightarrow E[f(w) | C], \quad (\text{A.18})$$

where $C_n \subset C_{n+1}$ and $C = \cup C_n$.

In our application, C_n is the Borel field generated by $(i_{t-1}, i_{t-2}, \dots, i_{t-n})$ and C is that generated by the infinite history $\Psi_{t-1} = (i_{t-1}, i_{t-2}, \dots)$. Moreover, $f(w)$ is just the conditional second moment given the history $E[h_t^2 | \Psi_{t-1}]$.

Thus

$$E[h_t^2 | i_{t-1}i_{t-2}, \dots, i_{t-n}] \rightarrow E[h_t^2 | i_{t-1}i_{t-2}, \dots]. \quad (\text{A.19})$$

Consider now the history up to n periods in the past (where n is arbitrary), $(i_{t-1}, \dots, i_{t-n})$. Given the stationary distribution (which exists from our assumptions), we first show that $i_t, i_{t-1}, \dots, i_{t-n}$ is affiliated.

First, it is easy to show, using the affiliation assumption, that $h_t, h_{t-1}, \dots, h_{t-n}$ are affiliated for arbitrary n . Then, $h_{t+1}, h_t, \dots, h_{t-n}, |i_t|, \dots, |i_{t-n}|$ is affiliated (where n is arbitrary). The proof is as follows:

$$\begin{aligned} & f(h_{t+1}, |i_t|, \dots, |i_{t-n}|, h_t, \dots, h_{t-n}) \\ &= f(h_{t+1}, |i_t|, \dots, |i_{t-n}| | h_t, \dots, h_{t-n}) f(h_t, \dots, h_{t-n}) \\ &= f(h_{t+1} | h_t, \dots, h_{t-k}) \left(\prod_{r=1}^n f(|i_{t-r}| | h_{t-r}) \right) \\ & \quad \times f(h_t, \dots, h_{t-n}). \quad (\text{A.20}) \end{aligned}$$

The first term of Expression (A.20) satisfies the first part of the affiliation assumption, and the last term of expression (A.20) satisfies the second part of the affiliation assumption. Thus we need only show that the second set of terms satisfies the monotone likelihood ratio property (MLRP) property. But the second set of terms $f(|i_t| | h_t) = f(|\Delta v_t| | h_t) f(|q_t| | h_t) f(|l_t| | h_t)$. Thus we can consider each term and use m_t to signify any of the three elements of i_t [i.e., one of $(\Delta v_t, q_t, l_t)$]. The MLRP follows because

$$\begin{aligned} & \frac{f(|m'_t| | h'_t)}{f(|m_t| | h_t)} > \frac{f(|m'_t| | h'_t)}{f(|m_t| | h_t)} \\ & \Leftrightarrow \frac{\frac{1}{\sqrt{2\pi}h'_t\sigma} \exp[-(m'_t/h'_t\sigma)^2]}{\frac{1}{\sqrt{2\pi}h_t\sigma} \exp[-(m_t/h_t\sigma)^2]} \\ & > \frac{\frac{1}{\sqrt{2\pi}h'_t\sigma} \exp[-(m_t/h'_t\sigma)^2]}{\frac{1}{\sqrt{2\pi}h_t\sigma} \exp[-(m_t/h_t\sigma)^2]} \\ & \Leftrightarrow (h'_t)^2 [(m'_t)^2 - (m_t)^2] > (h_t)^2 [(m'_t)^2 - (m_t)^2]. \quad (\text{A.21}) \end{aligned}$$

Because the last statement is true, the affiliation property follows. Thus the variables $h_{t+1}, |i_t|, \dots, |i_{t-n}|, h_t, \dots, h_{t-n}$ are affiliated. Because subsets of affiliated variables are affiliated, the desired result follows.

Consider trading volume when we only consider updating the second moment of h_{t+1} and h_t given the history i_1, \dots, i_t , and start with the stationary distribution for the conditional second moment. First we note that h_t^2 is a monotone function of h_t and, from Milgrom and Weber (1982, theorem 5), $E[h_t^2 | |i_t|, \dots, |i_{t-n}|]$ is monotone in the conditioning variables. Because the number of traders $I(E[h_t^2 | |i_t|, \dots, |i_{t-n}|])$ is a monotone increasing function of the conditional second moment given the restricted history, volume at time $t+1$ and time t are nondecreasing functions of $h_{t+1}, h_t, |i_t|, \dots, |i_{t-n}|$. Then, by the affiliation property, $E_n[V_{t+1}V_t] - E_n[V_{t+1}]E_n[V_t] > 0$ for all n .

Now as we condition further on the history, we know that $E[h_{t+1}^2 | |i_t|, \dots, |i_{t-n}|] \rightarrow E[h_{t+1}^2 | \Psi_{t-1}]$. If the conditional expectation given the history, Ψ_{t-1} , is absolutely continuous with respect to the Lebesgue measure, then

the points of possible discontinuity for the convergence of $I(E[h_t^2 | |i_t|, \dots, |i_{t-n}|])$ to $I(E[h_t^2 | |i_t|, \dots])$ are countable and hence a set of measure 0.

Hence we get that $E[V_{t+1}V_t] - E[V_{t+1}]E[V_t] > 0$ using the dominated convergence theorem. The proof of the convergence of $E_n[V_t]$ to $E(v_t)$ is as follows. By the triangle inequality, the market maker's volume is strictly less than that of the liquidity traders and informed traders, and thus we need only show that the dominated convergence theorem applies to the informed trader's volume. By the Cauchy–Schwarz inequality,

$$\begin{aligned} & E\left[\frac{h_t}{2} I(\Psi_{t-1}) \beta(\Psi_{t-1}) | (x_t + r_t) \right] \\ & < E\left[\frac{h_t^2}{4} (x_t + r_t)^2\right] E\left[(I(\Psi_{t-1}) \beta(\Psi_{t-1}))^2\right] \\ & = E\left[\frac{h_t^2}{4} (x_t + r_t)^2\right] E\left[I(\Psi_{t-1}) \phi \frac{\sigma_z^2}{\sigma_x^2}\right] \\ & < E\left[\frac{h_t^2}{4} (x_t + r_t)^2\right] E\left[E[h_t^2 | \Psi_{t-1}] (\phi)^{3/2} \frac{\sigma_z^3}{\sigma_x}\right] \\ & < E\left[\frac{h_t^2}{4} (x_t + r_t)^2\right] E\left[h_t^2 (\phi)^{(3/2)} \frac{\sigma_z^3}{c\sigma_x}\right], \quad (\text{A.22}) \end{aligned}$$

where the fact that $I < (1+I)\sqrt{I}$ for $I \geq 1$ and the equation determining $I(\Psi_{t-1})$ (see Proposition 1) has been used. Because the unconditional expectations exist, domination follows, and we can use the dominated convergence theorem. A similar proof applies for the other terms.

To prove the proposition for squared price changes, note that

$$\begin{aligned} p_t - p_{t-1} &= \left[a_t \phi h_t (x_t + r_t) + b_t \sqrt{\phi} \frac{\sigma_x}{\sigma_z} h_t z_t \right] + \Delta v_{t-1} \\ & \quad - \left[a_{t-1} \phi h_{t-1} (x_{t-1} + r_{t-1}) + b_{t-1} \sqrt{\phi} \frac{\sigma_x}{\sigma_z} h_{t-1} z_{t-1} \right], \quad (\text{A.23}) \end{aligned}$$

where $a_t = I(\Psi_{t-1})/(1+I(\Psi_{t-1}))$ and $b_t = \sqrt{I(\Psi_{t-1})}/(1+I(\Psi_{t-1}))$.

Define

$$\begin{aligned} \varphi_t &= (a_t \phi h_t (x_t + r_t))^2 + \left(b_t \sqrt{\phi} \frac{\sigma_x}{\sigma_z} h_t z_t \right)^2 \\ \zeta_t &= (h_t x_t)^2. \quad (\text{A.24}) \end{aligned}$$

Then we obtain (after simplification) that

$$\begin{aligned} & E[(p_t - p_{t-1})^2 (p_{t-1} - p_{t-2})^2] \\ & = E[\{\varphi_t + \zeta_{t-1} + \varphi_{t-1}\} \{\varphi_{t-1} + \zeta_{t-2} + \varphi_{t-2}\}] \quad (\text{A.25}) \end{aligned}$$

and

$$\begin{aligned} & E[(p_t - p_{t-1})^2] = E[\varphi_t + \zeta_{t-1} + \varphi_{t-1}] \\ & E[(p_{t-1} - p_{t-2})^2] = E[\varphi_{t-1} + \zeta_{t-2} + \varphi_{t-2}]. \quad (\text{A.26}) \end{aligned}$$

Thus covariance is positive provided terms of the following equality holds:

$$E[\varphi_t \varphi_{t-1}] > E[\varphi_t] \times E[\varphi_{t-1}]. \quad (\text{A.27})$$

Expanding, we obtain that

$$\begin{aligned}
E[\varphi_i \varphi_{i-1}] &= E \left[\left[(a_i \phi h_i (x_i + r_i))^2 + \left(b_i \sqrt{\phi} \frac{\sigma_x}{\sigma_z} h_i z_i \right)^2 \right] \right. \\
&\quad \times \left. \left[(a_{i-1} \phi h_{i-1} (x_{i-1} + r_{i-1}))^2 + \left(b_{i-1} \sqrt{\phi} \frac{\sigma_x}{\sigma_z} h_{i-1} z_{i-1} \right)^2 \right] \right] \\
&= E \left[\left[(a_i \phi h_i)^2 (\sigma_x^2 + \sigma_r^2) + (b_i h_i)^2 \phi \sigma_x^2 \right] \right. \\
&\quad \times \left. \left[(a_{i-1} \phi h_{i-1})^2 (\sigma_x^2 + \sigma_r^2) + (b_{i-1} h_{i-1})^2 \phi \sigma_x^2 \right] \right] \\
&= E \left[\left(h_i^2 \phi \sigma_x^2 \frac{I(\Psi_{i-1})}{1 + I(\Psi_{i-1})} \right) \left(h_{i-1}^2 \phi \sigma_x^2 \frac{I(\Psi_{i-2})}{1 + I(\Psi_{i-2})} \right) \right], \quad (\text{A.28})
\end{aligned}$$

where we have used the fact that x_t , r_t , and z_t are independent processes (cross-sectionally and across time) and that

$$a_i^2 \phi^2 (\sigma_x^2 + \sigma_r^2) + b_i^2 \phi \sigma_x^2 = \phi \sigma_x^2 \frac{I(\Psi_{i-1})}{1 + I(\Psi_{i-1})}. \quad (\text{A.29})$$

Similarly,

$$\begin{aligned}
E[\varphi_i] &= h_i^2 \phi \sigma_x^2 \frac{I(\Psi_{i-1})}{1 + I(\Psi_{i-1})} \\
E[\varphi_{i-1}] &= h_{i-1}^2 \phi \sigma_x^2 \frac{I(\Psi_{i-2})}{1 + I(\Psi_{i-2})}. \quad (\text{A.30})
\end{aligned}$$

The affiliation assumption and the monotonicity of $I/(1+I)$ in I and that of $I(\Psi_{i-1})$ in the conditional second moment of h_i given the history, $E[h_i^2 | \Psi_{i-1}]$, yield the covariance inequality $E[\varphi_i \varphi_{i-1}] > E[\varphi_i] E[\varphi_{i-1}]$. The proof for the other terms is similar except for the following terms:

$$\begin{aligned}
E[\varphi_{i-1} \varphi_{i-1}] &> E[\varphi_{i-1}] E[\varphi_{i-1}] \\
E[\varphi_{i-1} \zeta_{i-1}] &> E[\varphi_{i-1}] E[\zeta_{i-1}]. \quad (\text{A.31})
\end{aligned}$$

The first equation in Equation (A.31) is true because variances are positive. That the second equation in Equation (A.31) is true is shown next. First,

$$\begin{aligned}
E[\varphi_{i-1} \zeta_{i-1}] &= E \left[\left((a_i \phi h_i (x_i + r_i))^2 + \left(b_i \sqrt{\phi} \frac{\sigma_x}{\sigma_z} h_i z_i \right)^2 \right) (h_{i-1} x_{i-1})^2 \right] \\
&= E \left[(a_i \phi h_i (x_i + r_i))^2 (h_{i-1} x_{i-1})^2 \right. \\
&\quad \left. + \left(b_i \sqrt{\phi} \frac{\sigma_x}{\sigma_z} h_i z_i \right)^2 (h_{i-1} x_{i-1})^2 \right] \\
&= E \left[(h_{i-1} \phi a_{i-1})^2 (\sigma_x^2 + \sigma_r^2) h_{i-1}^3 \sigma_x^2 \right. \\
&\quad \left. + (b_{i-1} h_{i-1})^2 \phi \sigma_x^2 h_{i-1}^2 \sigma_x^2 \right] \\
&= E \left[(h_{i-1} \phi a_{i-1})^2 (\sigma_x^2 + \sigma_r^2) h_{i-1} \sigma_x^2 \right. \\
&\quad \left. + (b_{i-1} h_{i-1})^2 \phi \sigma_x^2 h_{i-1}^2 \sigma_x^2 \right] + 2 \sigma_x^2 (\sigma_x^2 + \sigma_r^2) \phi E[h_{i-1}^4 a_{i-1}^2] \\
&= E \left[\left(\phi \sigma_x^2 \frac{I(\Psi_{i-2})}{1 + I(\Psi_{i-2})} h_{i-1}^2 \right) h_{i-1}^2 \sigma_x^2 \right] \\
&\quad + 2 \sigma_x^2 (\sigma_x^2 + \sigma_r^2) \phi \times E \left[h_{i-1}^4 \left(\frac{I(\Psi_{i-2})}{1 + I(\Psi_{i-2})} \right)^2 \right]. \quad (\text{A.32})
\end{aligned}$$

The last term is clearly positive. The affiliation argument discussed previously ensures that

$$\begin{aligned}
E \left[\left(\phi \sigma_x^2 \frac{I(\Psi_{i-2})}{1 + I(\Psi_{i-2})} h_{i-1}^2 \right) h_{i-1}^2 \sigma_x^2 \right] \\
> E \left[\left(\phi \sigma_x^2 \frac{I(\Psi_{i-2})}{1 + I(\Psi_{i-2})} h_{i-1}^2 \right) \right] E[h_{i-1}^2 \sigma_x^2] \\
= E[\varphi_{i-1}] E[\zeta_{i-1}]. \quad (\text{A.33})
\end{aligned}$$

This completes our proof. Thus squared price changes are positively autocorrelated.

The last part of the proposition deals with the unconditional covariance between squared price changes and volume. We prove this as follows. First,

$$\begin{aligned}
\text{cov}[(p_t - p_{t-1})^2 V_t] &= \text{cov}[(\varphi_t + \zeta_{t-1} + \varphi_{t-1}) V_t] \\
&= \text{cov}[\varphi_t V_t] + \text{cov}[\zeta_{t-1} V_t] + \text{cov}[\varphi_{t-1} V_t]. \quad (\text{A.34})
\end{aligned}$$

The fact that $\text{cov}[\zeta_{t-1} V_t] > 0$ and that $\text{cov}[\varphi_{t-1} V_t] > 0$ can be proved using methods identical to that used in the proof for squared price changes. Thus we consider the term $\text{cov}[\varphi_t V_t]$. The proof proceeds as follows. First,

$$\begin{aligned}
\text{cov}[\varphi_t V_t] &= \text{cov} \left[\left\{ (a_t \phi h_t (x_t + r_t))^2 + \left(b_t \sqrt{\phi} \frac{\sigma_x}{\sigma_z} h_t z_t \right)^2 \right\} \right. \\
&\quad \times \left. \left\{ \frac{h_t}{2} I(\Psi_{t-1}) \beta(\Psi_{t-1}) |x_t + r_t| + \frac{h_t}{2} |z_t| \right. \right. \\
&\quad \left. \left. + \frac{h_t}{2} |I(\Psi_{t-1}) \beta(\Psi_{t-1}) (x_t + r_t) + z_t| \right\} \right]. \quad (\text{A.35})
\end{aligned}$$

Consider the first covariance term. Expanding,

$$\begin{aligned}
E \left[\left\{ (a_t \phi h_t (x_t + r_t))^2 + \left(b_t \sqrt{\phi} \frac{\sigma_x}{\sigma_z} h_t z_t \right)^2 \right\} \right. \\
\times \left. \left\{ \frac{h_t}{2} I(\Psi_{t-1}) \beta(\Psi_{t-1}) |x_t + r_t| \right\} \right] \\
> E \left[\left\{ (a_t \phi h_t)^2 (\sigma_x^2 + \sigma_r^2) + (b_t h_t \phi)^2 \right\} \right] \\
\times E \left[\frac{h_t}{2} I(\Psi_{t-1}) \beta(\Psi_{t-1}) \frac{2}{\sqrt{2\pi}} \sqrt{\sigma_x^2 + \sigma_r^2} \right] \\
= E \left[\left\{ (a_t \phi h_t (x_t + r_t))^2 + \left(b_t \sqrt{\phi} \frac{\sigma_x}{\sigma_z} h_t z_t \right)^2 \right\} \right. \\
\times \left. E \left[\frac{h_t}{2} I(\Psi_{t-1}) \beta(\Psi_{t-1}) |x_t + r_t| \right] \right], \quad (\text{A.36})
\end{aligned}$$

where we have used the fact that $E[|x|^3] > E[x^2] E[|x|]$ [see the discussion preceding Equation (A.17) for a proof of this fact]. The proof for the second covariance term involving the liquidity volume is similar.

We now consider the last covariance term in Equation (A.35) that concerns the market maker's volume.

First,

$$\begin{aligned} \text{cov} \left[\left\{ (a_t \phi_t h_t (x_t + r_t))^2 + \left(b_t \sqrt{\phi} \frac{\sigma_x}{\sigma_x} h_t z_t \right)^2 \right\} \right. \\ \left. \times \left\{ \frac{h_t}{2} |I(\Psi_{t-1}) \beta(\Psi_{t-1})(x_t + r_t) + z_t| \right\} \right] \\ = \text{cov}[\lambda(\Psi_{t-1})^2 |y_t|^3]. \quad (\text{A.37}) \end{aligned}$$

Second, we know that, if x is normally distributed with mean 0 and variance σ ,

$$E[|x|^3] = 2\sqrt{2}\pi\sigma^3. \quad (\text{A.38})$$

Using this fact,

$$\begin{aligned} E[\lambda(\Psi_{t-1})^2 |y_t|^3] \\ = E[\lambda(\Psi_{t-1})^2 E[|y_t|^3 | \Psi_{t-1}]] \\ = E[\lambda(\Psi_{t-1})^2 2\sqrt{2}\pi (E[h_t^2 | \Psi_{t-1}] \\ \times \{(I(\Psi_{t-1})\beta(\Psi_{t-1}))^2(\sigma_x^2 + \sigma_r^2) + \sigma_z^2\})^{(3/2)}] \\ = 2E \left[E[h_t^2 | \Psi_{t-1}] \phi \sigma_x^2 \frac{I(\Psi_{t-1})}{1 + I(\Psi_{t-1})} \sqrt{2}\pi (E[h_t^2 | \Psi_{t-1}] \right. \\ \left. \times \{(I(\Psi_{t-1})\beta(\Psi_{t-1}))^2(\sigma_x^2 + \sigma_r^2) + \sigma_z^2\}) \right] \\ > E \left[E[h_t^2 | \Psi_{t-1}] \phi \sigma_x^2 \frac{I(\Psi_{t-1})}{1 + I(\Psi_{t-1})} \right] \\ \times E[\sqrt{2}\pi (E[h_t^2 | \Psi_{t-1}] \\ \times \{(I(\Psi_{t-1})\beta(\Psi_{t-1}))^2(\sigma_x^2 + \sigma_r^2) + \sigma_z^2\})], \quad (\text{A.39}) \end{aligned}$$

where we have used affiliation in the last stage in a manner similar to prior proofs. This completes the proof that the unconditional covariance between price changes and volume is positive.

Proof of Proposition 4

Given the history of $(\Delta p_t, s_t, \Delta v_t)$, one can infer l_t . Given these variables and knowing the nature of the model, $I(\Psi_{t-1})$ can be calculated and thus V_t is known. Hence V_t is a deterministic function of the history of the variables $(\Delta v_t, q_t, l_t)$. Hence, given these variables, it is redundant in forming conditional expectations of functions of future prices and volumes. The price-change history is not irrelevant. Knowing $(s_t, \Delta v_t, V_t)$ does not completely reveal l_t . Even if I , the number of traders, is given, the knowledge of I , β , s_t , and V_t leads to two solutions for l_t , one positive and one negative. When I itself cannot be computed (because of the inability to derive lagged values of l_t), the situation is further compounded. Thus the information in prices is not redundant.

When only the history of prices and volumes is known, neither the price change nor the volume process is redundant. The volume process will be useful in forecasting the conditional second moment of the h_t process.

[Received May 1993. Revised February 1995.]

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