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A CONSTRAINT ON THE EXISTENCE OF SIMPLE TORSION-FREE LIE MODULES

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ABSTRACT. For any simple Lie algebra L with Cartan subalgebra H the classification of all simple H-diagonalizable L-modules having a finite-dimensional weight space is known to depend on determining the simple torsion-free L-modules of finite degree. It is further known that the only simple Lie algebras which admit simple torsion-free modules of finite degree are those of types A_n and C_n. For the case of A_n we show that there are no simple torsion-free A_n-modules of degree k for $n \geq 4$ and $2 \leq k \leq n-2$. We conclude with some examples showing that there exist simple torsion-free A_n-modules of degrees $1, n-1$, and n.

Let L be a finite-dimensional simple Lie algebra over the complex numbers $\mathbb{C}$, and H be a fixed Cartan subalgebra of L. An H-diagonalizable L-module M is said to be: of *degree* k provided each weight space of M is of dimension k; *torsion-free* provided each root vector of L acts on each weight space of M in a one-to-one manner; and *pointed* provided it is simple and has a 1-dimensional weight space.

Theorem 4.18 of [F] describes simple H-diagonalizable L-modules having a finite-dimensional weight space as the amalgam of highest weight modules and torsion-free modules, thereby reducing the classification of such modules to determining the simple torsion-free modules of degree $k < \infty$. The complete classification of pointed modules was presented in [BL1]. In studying modules of minimal weight space dimension greater than 1, we must study simple torsion-free modules of degree greater than 1. In [F], Fernando showed that only A_n, $n \geq 1$, and C_n, $n \geq 2$, possess torsion-free modules. It is easy to see that no simple torsion-free modules of minimal weight space dimension greater than 1 exist for A_1. The existence of simple torsion-free A_n-modules of arbitrary degree for $n = 2, 3$ has been established in [BBL].

In this note, we further restrict the existence of torsion-free A_n-modules by proving

Main Theorem. *There are no simple torsion-free A_n-modules of degree k for $n \geq 4$ and $2 \leq k \leq n-2$.*

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We consider A_n to be the $(n+1)\times(n+1)$ traceless matrices over $\mathbb{C}$. Let $\{E_{ij}|1\le i, j\le n+1\}$ be a standard set of matrix units and $\{\varepsilon_i|1\le i\le n+1\}$ be a standard basis of $\mathbb{C}^{n+1}$. Naturally, we think of $\Delta=\{\alpha_i=\varepsilon_i-\varepsilon_{i+1}|1\le i\le n\}$ as a base of the root system $\Phi=\{\pm(\varepsilon_i-\varepsilon_j)|1\le i<j\le n\}$ of A_n. If $i\le j$ and $\alpha=\alpha_i+\cdots+\alpha_j=\varepsilon_i-\varepsilon_{j+1}$, then the root vectors X_α, $X_{-\alpha}$ are $E_{i,j+1}$, $E_{j+1,i}$, respectively, and $h_\alpha=E_{ii}-E_{j+1,j+1}$. $H=\operatorname{span}_{\mathbb{C}}\{h_\alpha|\alpha\in\Delta\}$ is our fixed Cartan subalgebra. In subsequent sections we will have occasion to use other simple bases for the root system Φ of A_n. In all cases we order the simple basis elements to preserve the connectedness of adjacent roots.

Let $U=U(A_n)$ be the universal enveloping algebra of A_n and $U_0=U_0(A_n)=\{u\in U|[u,h]=0, \forall h\in H\}$. Using Theorem 2.1 of [BL2], the elements $u\in U_0$ can be characterized as linear combinations of products of h_α, $\alpha\in\Delta$, and elements of the form

$$u=E_{i_1,i_2}\cdots E_{i_{l-1},i_l}E_{i_l,i_1}, \tag{1}$$

where $i_1,\dots,i_l$ are distinct in $\{1,\dots,n+1\}$.

In the torsion-free setting that we are working in it is easy to see that M is a simple U-module if and only if for any weight Θ M_Θ is a simple U_0-module with $M=UM_\Theta$. This means that we can restrict our attention to an arbitrary weight space M_Θ and so, at our convenience, we can choose Θ such that it satisfies the General Assumption below.

The Main Theorem is proved by a sequence of lemmas under the following

General Assumption. *M is a torsion-free A_n-module of degree $k<\infty$ and $n\ge 2$. Using the torsion-free assumption we fix a weight Θ of M such that $\Theta(h_\alpha)\ne 0$ for all $\alpha\in\Phi$. Denote the restriction of $X_{-\alpha}X_\alpha$ to the weight space M_Θ by $\mathscr{A}_\alpha$.*

Let i, j, m be any three distinct values in $\{1,\dots,n+1\}$. Set

$$c_1(i,j,m)=E_{ji}E_{ij},\quad c_2(i,j,m)=E_{mj}E_{jm},\quad c_3(i,j,m)=E_{mi}E_{im},$$
$$c_4(i,j,m)=E_{mi}E_{ij}E_{jm},\quad c_5(i,j,m)=E_{mj}E_{ji}E_{im},$$
$$h_1(i,j,m)=E_{ii}-E_{jj},\quad\text{and}\quad h_2(i,j,m)=E_{jj}-E_{mm}.$$

Drop (i,j,m) to simplify notation. One can show that the following identities hold in U:

$$[c_1,c_2]=[c_2,c_3]=-[c_1,c_3]=c_5-c_4, \tag{2}$$
$$[c_1,c_4]=-c_2c_1+c_3c_1-c_4h_1+c_3h_1-c_5+c_4, \tag{3}$$
$$[c_2,c_4]=c_2c_1-c_3c_2-c_4h_2+c_5-c_4, \tag{4}$$
$$[c_1,[c_1,c_2]]=2c_2c_1-2c_3c_1+(c_5+c_4)h_1-2c_3h_1+2[c_1,c_2]. \tag{5}$$

Let $\tilde{c}_i$ denote the action of c_i on M_Θ and $\tilde{h}_i=\Theta(h_i)$. We use $W\le M_\Theta$ to indicate that W is a subspace of M_Θ.

Lemma 6. *If $W\le M_\Theta$ is invariant under $\mathscr{A}_\alpha$ for all $\alpha\in\Phi$, then W is a U_0-submodule of M_Θ.*

Proof. It suffices to show that W is invariant under all u of form (1). By hypothesis, this is true if $l=2$. Assume $l=3$ so that $u=c_4(i,j,m)=E_{mi}E_{ij}E_{jm}$. Since $\Theta(h_1)\ne 0$, we can use (2) and (5) to solve for $\tilde{c}_4$ and $\tilde{c}_5$

and see that $\tilde{c}_4$ and $\tilde{c}_5$ leave W invariant. Assume that W is invariant under all elements of form (1) with $l < m$, and observe that in U_0

$$(E_{i_1,i_2}\cdots E_{i_{m-1}i_m}E_{i_m,i_1})(E_{i_{m-1},i_1}E_{i_1,i_{m-1}}) = (E_{i_1,i_2}\cdots E_{i_{m-2},i_{m-1}}E_{i_{m-1},i_1})(E_{i_{m-1},i_m}E_{i_m,i_1}E_{i_1,i_{m-1}}). \tag{7}$$

Since the two expressions of form (1) on the right-hand side of (10) leave W invariant as does the invertible operator $E_{i_{m-1},i_1}E_{i_1,i_{m-1}}$, the result follows by induction. □

Notation. For each $1 \le m \le n$, let Φ^m be the root subsystem generated by $\Delta^m = \{\alpha_m, \dots, \alpha_n\} \subseteq \Delta$. If $\alpha, \beta, \alpha+\beta \in \Phi$, then $A(\alpha, \beta)$ denotes the A_2 subalgebra of A_n generated by $X_{\pm\alpha}$ and $X_{\pm\beta}$.

If $\mathscr{A}_\alpha$ acts like a complex multiple of the identity map on $W \le M_\Theta$, then we say that $\mathscr{A}_\alpha$ is a scalar on W.

Lemma 8. *Let $\alpha, \beta, \alpha+\beta \in \Phi$ and $W \le M_\Theta$. Then*

(i) *if $\mathscr{A}_\alpha$ is a scalar on W and W is invariant under $\mathscr{A}_\beta$, then W is a $U_0(A(\alpha, \beta))$-module,*

(ii) *if $\mathscr{A}_\alpha$ and $\mathscr{A}_\beta$ are both scalars on W, then all elements in $U_0(A(\alpha, \beta))$ are scalars on W, and*

(iii) *if $\mathscr{A}_\alpha$ is a scalar on W and $\mathscr{A}_\beta$ is a scalar on W for all $\beta \in \Phi$ such that $\alpha+\beta \in \Phi$, then $\mathscr{A}_\gamma$ is a scalar on W for all $\gamma \in \Phi$.*

Proof. Let $w \in W$ and $\mathscr{A}_\alpha$ be a scalar on W. Without loss of generality, we may assume that $X_\alpha = E_{ij}$ and $X_\beta = E_{jm}$. Then with $\tilde{c}_1 = \mathscr{A}_\alpha$ and $\tilde{c}_2 = \mathscr{A}_\beta$ we have that $0 = [\tilde{c}_1, \tilde{c}_2]w = (\tilde{c}_5 - \tilde{c}_4)w$ and so, by (2), $[\tilde{c}_r, \tilde{c}_s]w = 0$ for $1 \le r, s \le 3$. By (2) and (5), $\tilde{c}_4$ and $\tilde{c}_5$ are expressible in terms of $\tilde{c}_1, \tilde{c}_2, \tilde{c}_3$. Therefore,

$$0 = [\tilde{c}_1, \tilde{c}_4]w = (-\tilde{c}_2\tilde{c}_1 + \tilde{c}_3\tilde{c}_1 - \tilde{c}_4\tilde{h}_1 + \tilde{c}_3\tilde{h}_1)w, \tag{9}$$

$$0 = [\tilde{c}_2, \tilde{c}_4]w = (\tilde{c}_2\tilde{c}_1 - \tilde{c}_3\tilde{c}_2 - \tilde{c}_4\tilde{h}_2)w, \tag{10}$$

and we have

$$\tilde{c}_2\tilde{c}_1(\tilde{h}_1 + \tilde{h}_2)w = \tilde{c}_3(\tilde{c}_1\tilde{h}_2 + \tilde{c}_2\tilde{h}_1 + \tilde{h}_1\tilde{h}_2)w. \tag{11}$$

Since, if $w \ne 0$, the left-hand side of equality (11) is a nonzero element in W, $(\tilde{c}_1\tilde{h}_2 + \tilde{c}_2\tilde{h}_1 + \tilde{h}_1\tilde{h}_2)$ is an invertible operator on W and we see that $\tilde{c}_3$ leaves W invariant, (i) follows from this and Lemma 6. Also, if $\mathscr{A}_\beta$ is a scalar on W, then (11) implies $\tilde{c}_3$ is a scalar on W and (10) implies that $\tilde{c}_4$ is and (ii) follows.

Now for (iii), without loss of generality, take $\alpha_1 = \alpha$ in Δ. Clearly, $\mathscr{A}_{\alpha_2}$ and $\mathscr{A}_{-(\alpha_1+\alpha_2+\alpha_3)}$ are scalars on W. Part (ii) implies $\mathscr{A}_{\alpha_1+\alpha_2}$ is a scalar on W. $\mathscr{A}_{\alpha_3}$ is a scalar on W since it denotes the action of an element of

$$U_0(A(\alpha_1+\alpha_2, -(\alpha_1+\alpha_2+\alpha_3))).$$

Continuing in this fashion one gets that $\mathscr{A}_{\alpha_i}$ is a scalar on W for all $\alpha_i \in \Delta$, and the result follows from part (ii). □

Lemma 12. *Fix an eigenvalue* λ_i *of* $\mathscr{A}_{\alpha_i}$ *for each* $\alpha_i \in \Delta$, *and let* W_i *be the corresponding eigenspace. For each* $1 \le j \le n$, *define* $W^{(j-1)} = W_1 \cap \cdots \cap W_{j-1}$. *If* $\mathscr{A}_{\alpha_j}$ *is a scalar on* $W^{(j-1)} \neq 0$ *and* $k > 1$, *then* M *is not simple.* (*In particular, if for some* $\beta \in \Phi$ $\mathscr{A}_\beta$ *is a scalar on* M_Θ, *then* M *is not simple.*)

Proof. Assume that M is simple of degree $k \ge 2$. $W^{(j-1)} \neq 0$ is invariant under $\mathscr{A}_\alpha$ for each $\alpha \in \Phi^{j+1}$ because $\mathscr{A}_\alpha \mathscr{A}_{\alpha_i} = \mathscr{A}_{\alpha_i}\mathscr{A}_\alpha$ for $1 \le i \le j-1$. Lemma 8(i) implies $W^{(j-1)}$ is invariant under $\mathscr{A}_\alpha$ for $\alpha \in \Phi^j$ and in turn for $\alpha \in \Phi^{j-1}$, $\alpha \in \Phi^{j-2}, \ldots, \alpha \in \Phi^1 = \Phi$. If $W^{(j-1)} \neq M_\Theta$, then by Lemma 6, $W^{(j-1)}$ is a proper U_0-submodule of M_Θ, which implies that M is not simple.

If $W^{(j-1)} = M_\Theta$, then let l be the smallest subscript such that $\mathscr{A}_{\alpha_l}$ is not a scalar on M_Θ. By Lemma 6 and Lemma 8(ii), such an l must exist. Note that $\Delta' = \{\alpha_1' = \alpha_l, \alpha_2' = \alpha_{l-1}, \ldots, \alpha_l' = \alpha_1, \alpha_{l+1}' = -(\alpha_{l+1} + \alpha_l + \cdots + \alpha_1), \alpha_{l+2}' = -\alpha_{l+2}, \ldots, \alpha_n = -\alpha_n\}$ is a base such that if we let W_1' be an eigenspace corresponding to $\mathscr{A}_{\alpha_1'}$, then $\mathscr{A}_{\alpha_2'}$ is a scalar on $M_\Theta \neq W^{(1)'} = W_1' \neq 0$. We are now back to the case handled above. □

Lemma 13. *Let* $n \ge 4$. *If the degree* k *of* M *satisfies* $2 \le k \le n-2$ *and* v *is an eigenvector of* $\mathscr{A}_\alpha$ *for each* $\alpha \in \Delta^3$, *then* M *is not simple.*

Proof. Suppose that M is simple, and for $3 \le i \le n$ let λ_i be the eigenvalue of $\mathscr{A}_{\alpha_i}$ belonging to v. Let W_i be the eigenspace of $\mathscr{A}_{\alpha_i}$ belonging to λ_i. By Lemma 12, $\dim W_i < k \le n-2$. Since we can complete $\{\alpha_1' = \alpha_3, \ldots, \alpha_{n-2}' = \alpha_n\}$ to a base Δ' for Φ and apply Lemma 12 to get in turn $\dim(W_3 \cap W_4) < n-3$, $\dim(W_3 \cap W_4 \cap W_5) < n-4, \ldots, \dim(W_3 \cap \cdots \cap W_n) < n-(n-1) = 1$. But by assumption, $v \in W_3 \cap \cdots \cap W_n$. This contradiction gives the result. □

Corollary 14. *If* $n \ge 4$ *and* M *is simple, then for* $\alpha \in \Phi$ *the eigenspace of* $\mathscr{A}_\alpha$ *are all of dimension greater than* 1.

Proof. If $W = \mathbb{C}v$ is a 1-dimensional eigenspace of $\mathscr{A}_\alpha$, then take $\alpha_1 = \alpha$ in base Δ and note that $\mathscr{A}_{\alpha_3}, \ldots, \mathscr{A}_{\alpha_n}$ all have v as an eigenvector. Lemma 13 implies that M is not simple. □

Proof of Main Theorem. Assume that M is simple torsion-free of degree $k \le n-2$ and $n \ge 4$. If $k = 2$, then we have the contradiction concerning the dimension of an eigenspace of $\mathscr{A}_\alpha$ given to us by Lemma 12 and Corollary 14. Hence, $k \ge 3$.

This proves that the result is true for $n = 4$ and begins our inductive proof. Now, assume that the result is true for all m, $4 \le m < n$, and $n \ge 5$.

As our first step, we establish the claim that, when $n \ge 5$, $k = n-2$ and each operator $\mathscr{A}_\alpha$ has a unique eigenvalue whose eigenspace has dimension $d = n-3$. For $n = 5$, Lemma 12 and Corollary 14 imply that the claim is true. Continuing, we assume that $n \ge 6$. Let V be either an eigenspace or a generalized eigenspace of $\mathscr{A}_\alpha$. By Corollary 14, $\dim V \ge 2$. Suppose that $2 \le \dim V \le n-4$. Take a base $\Delta = \{\alpha = \alpha_1, \ldots, \alpha_n\}$. Let A_m be the subalgebra generated by the root vectors in Φ^3. $U(A_m)V$ is a torsion-free A_m-module, and since $m = n-2 \ge 4$, the induction assumption says that there is a 1-dimensional $U_0(A_m)$-submodule of V, contrary to Lemma 13. If $\mathscr{A}_\alpha$ has two distinct eigenvalues with corresponding dimensions d_1 and d_2, then $k \ge d_1 + d_2 > 2n-8$ and so we have arrived at the contradiction that $2n-8 < k \le n-2$ or $n < 6$. Hence, the claim is true.

Our second claim is that if $n \geq 5$ and W_i is the unique eigenspace of $\mathscr{A}_{\alpha_i}$ of dimension $n-3$ for $\alpha_i \in \Delta$, then $\dim(W_{i_1} \cap \cdots \cap W_{i_j}) \geq n-2-j$. This is known for $j = 1$. Inductively assume that $\dim(W_{i_1} \cap \cdots \cap W_{i_{j-1}}) \geq n-2-(j-1) = n-1-j$. Then

$$\begin{aligned} n-2 &\geq \dim(W_{i_1} \cap \cdots \cap W_{i_{j-1}} + W_{i_j}) \\ &= \dim(W_{i_1} \cap \cdots \cap W_{i_{j-1}}) + \dim(W_{i_j}) - \dim(W_{i_1} \cap \cdots \cap W_{i_j}) \\ &\geq n-1-j+n-3-\dim(W_{i_1} \cap \cdots \cap W_{i_j}). \end{aligned}$$

Thus $\dim(W_{i_1} \cap \cdots \cap W_{i_j}) \geq n-2-j$.

When $j = n-3$, $\dim(W_{i_1} \cap \cdots \cap W_{i_j}) \geq 1$. If $W_1 \cap \cdots \cap W_{j-1} = W_1 \cap \cdots \cap W_j$, then $\mathscr{A}_{\alpha_j}$ is a scalar on $W = W_1 \cap \cdots \cap W_{j-1}$ and M is not simple by Lemma 13. Therefore, $W_1 \supset W_1 \cap W_2 \supset \cdots \supset W_1 \cap \cdots \cap W_{n-3} \neq \{0\}$ and so $\dim(W_1 \cap \cdots \cap W_{n-4}) = 2$ and $\dim(W_1 \cap \cdots \cap W_{n-3}) = 1$. If, for some base Δ, $\dim(W_1 \cap \cdots \cap W_{n-4} \cap W_n) = 1$, then since $\dim(W_1 \cap \cdots \cap W_{n-3}) = 1$ and A_{α_n} leaves $W_1 \cap \cdots \cap W_{n-3}$ invariant, we have $W_1 \cap \cdots \cap W_{n-3} \subset W_n$ and so $W_1 \cap \cdots \cap W_{n-3} = W_1 \cap \cdots \cap W_{n-4} \cap W_n$. This implies $W_1 \cap \cdots \cap W_{n-3} \subset W_{n-2}$ and $\mathscr{A}_{\alpha_{n-2}}$ is a scalar on $W_1 \cap \cdots \cap W_{n-3}$, contrary to Lemma 12. Hence, we assume that for all bases $W_1 \cap \cdots \cap W_{n-4} \subset W_n$. Consider the bases

$$\Delta' = \{\alpha_1 \ldots, \alpha_{n-3}, \alpha_{n-2}+\alpha_{n-1}, -\alpha_{n-1}, \alpha_{n-1}+\alpha_n\}$$

and

$$\Delta'' = \{\alpha_1 \ldots, \alpha_{n-4}, \alpha_{n-3}+\alpha_{n-2}+\alpha_{n-1}, -\alpha_{n-1}, -\alpha_{n-2}, \alpha_{n-2}+\alpha_{n-1}+\alpha_n\}.$$

Since $\mathscr{A}_{\alpha_{n-2}+\alpha_{n-1}+\alpha_n}$ and $\mathscr{A}_{-(\alpha_{n-1}+\alpha_n)}$ are scalars on $W_1 \cap \cdots \cap W_{n-4}$, Lemma 8(ii) says that $\mathscr{A}_{\alpha_{n-2}}$ is a scalar on $W_1 \cap \cdots \cap W_{n-3} \subset W_1 \cap \cdots \cap W_{n-4}$, and we again have a contradiction to Lemma 12. □

Example. We provide examples of simple torsion-free A_n-modules of degrees $1, n-1$, and n. To this end we first observe that the algebra $\mathrm{gl}(n+1, \mathbb{C})$ can be identified with an operator algebra as follows. Let $\{x_1, \ldots, x_{n+1}\}$ denote a set of commuting variables, and set X_i (respectively, ∂_i) to be multiplication (respectively, partial differentiation) by the variable x_i. Then the map which identifies the standard basis matrices E_{ij} with the operator $X_i\partial_j$ is a Lie algebra isomorphism. The restriction of this map to the subalgebra A_n of $n+1 \times n+1$ traceless matrices provides an operator realization of A_n and a realization of the natural representation space $V(\omega_1) = \mathrm{span}_{\mathbb{C}}\{x_1, \ldots, x_{n+1}\}$.

Fix $\vec{a} = (a_1, \ldots, a_{n+1}) \in \mathbb{C}^{n+1}$ where none of the components are integers, and consider the space

$$N(\vec{a}) = \mathrm{span}_{\mathbb{C}}\left\{x^{\vec{b}} = x_1^{b_1} \cdots x_{n+1}^{b_{n+1}} | b_i - a_i \in \mathbb{Z}, \text{ for all } i \text{ and } \sum_{i=1}^{n+1}(b_i - a_i) = 0\right\}.$$

Using the realization of A_n in terms of operators we easily see that $N(\vec{a})$ is a simple torsion-free A_n-module of degree 1. In fact in [BL1], it is shown that every simple torsion-free A_n-module of degree 1 is equivalent to such a module.

We now consider the tensor product module M given by $M = N(\vec{a} - \varepsilon_1) \otimes V(\omega_1)$, where $\vec{a} - \varepsilon_1 = (a_1 - 1, \alpha_2, \ldots, a_{n+1})$. In [BBL], it is shown that the tensor product of a torsion-free module of finite degree and a finite-dimensional module is torsion-free. Hence, M is torsion-free.

For each $i = 1, \dots, n+1$, we define

$$v_i = x_1^{a_1} \cdots x_i^{a_i - 1} \cdots x_{n+1}^{a_{n+1}} \otimes x_i .$$

It is easily verified that the elements v_i span the Θ weight space of M where $\Theta(E_{ii} - E_{i+1 i+1}) = a_i - a_{i+1}$. By direct computation we observe that subspaces

$$\mathscr{W}_1 = \operatorname{span}_{\mathbb{C}} \left\{ \sum_{i=1}^{n+1} a_i v_i \right\}, \qquad \mathscr{W}_0 = \operatorname{span}_{\mathbb{C}} \left\{ \sum_{i=1}^{n+1} c_i v_i \,\middle|\, \sum_{i=1}^{n+1} c_i = 0 \right\}$$

of M_Θ are invariant under $\mathscr{A}_\alpha$ for all $\alpha \in \Phi$ and hence, by Lemma 6, they are U_0 submodules. This means that $M_0 = U\mathscr{W}_0$ and $M_1 = U\mathscr{W}_1$ are submodules of M with $(U\mathscr{W}_i)_\Theta = W_i$. M_1 is isomorphic to $N(\vec{a})$.

Suppose that M_0 contains the proper nonzero simple submodule V of degree $k \leq n-1$, and hence by our Main Theorem $k = 1$ or $k = n-1$. Suppose that $k = n-1$. Then no vector of the form $v_i - v_j$ with $i \neq j$ is in V because such vectors generate $\mathscr{W}_0$ and hence generate M_0 as we see from

$$X_i \partial_k X_k \partial_i (v_i - v_j) = a_i(a_k + 1)(v_i - v_j) - a_k(v_i - v_k) \quad \text{for all } k \neq i, j.$$

By a dimension argument, we know that $(v_1 - v_2) - b(v_1 - v_3) \in V$ for some $b \in \mathbb{C}$ and, since $-v_2 + v_3 \notin V$, we know that $b \neq 1$. Computing we have

$$\begin{aligned} &X_1 \partial_4 X_4 \partial_1 ((v_1 - v_2) - b(v_1 - v_3)) \\ &\quad = [a_1(a_4 + 1)(v_1 - v_2) - a_4(v_1 - v_4)] - b[a_1(a_4 + 1)(v_1 - v_3) - a_4(v_1 - v_4)], \end{aligned}$$

from which it follows that $v_1 - v_4 \in V$. Hence, $k \neq n-1$ and so $k = 1$. This means that there is some $v = \sum_{i=1}^{n+1} d_i v_i \in V \cap M_\Theta$ with not all $d_i = 0$ for $i \geq 3$ which is an eigenvector for every $X_i \partial_k X_k \partial_i$.

$$\begin{aligned} X_1 \partial_2 X_2 \partial_1 v = {} & (d_1(a_1 - 1)(a_2 + 1) + d_1 + d_2 a_1) v_1 + (d_1 a_2 + d_2 a_1 a_2) v_2 \\ & + d_3 a_1 (a_2 + 1) v_3 + \cdots + d_{n+1} a_1 (a_2 + 1) v_{n+1} \end{aligned}$$

tells us that the eigenvalue for this operator is $a_1(a_2 + 1)$, and it forces $d_1 = s a_1$ and $d_2 = s a_2$ for some $s \in \mathbb{C}$. Similar calculations show that $d_i = s a_i$. Therefore, M_0 is a simple torsion-free A_n-module of degree n and $M = M_1 \oplus M_0$ unless $\sum_{i=1}^{n+1} a_i = 0$, in which case we have a composition series $\{0\} \subset M_1 \subset M_0 \subset M$. When $M_1 \subset M_0$, the quotient M_0/M_1 is a simple torsion-free A_n-module of degree $n-1$.

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