Definitions and Facts about Zeros of MIMO Systems,

For SISO systems, a zero is a frequency for which the transfer function equals zero. For MIMO systems, the transfer function is a *matrix*. Intuitively, we would like a zero of a MIMO system to be a frequency for which the transfer function matrix has less than full rank. This intuition proves to be correct; however, there are a number of technical questions that must be addressed. These include

(i) How do we define the rank of a *matrix* whose elements are transfer functions?
(ii) What about *nonminimal* realizations?
(iii) What is a *pole-zero cancellation*?
(iv) When can a system have pole and a zero at the *same* frequency?

Historically, it took many years to work out a detailed understanding of the zeros of a MIMO system. Some additional references:

J.M.Maciejowski *Multivariable Feedback Design,* Addison Wesley 1989.
T. Kailath, *Linear Systems*, Prentice-Hall, 1980.
F.M.Callier and C.A.Desoer, *Multivariable Feedback Systems*, Springer Verlag, New York, 1982.
F.M.Callier and C.A.Desoer, *Linear System Theory*, Springer Verlag, New York, 1991.
K. Zhou, J.C. Doyle, and K. Glover, *Robust and Optimal Control*, Prentice-Hall, 1996.

This handout doesn't follow any of these identically; the reference that is most close is F.M.Callier and C.A.Desoer, *Linear System Theory*.

We shall consider the system

$$\dot{x} = Ax + Bu, \qquad x \in \mathbf{R}^n,\ u \in \mathbf{R}^p$$
$$y = Cx + Du, \qquad y \in \mathbf{R}^q$$

<u>Definition:</u> The *normal rank* of $P(s)$ is equal to r if $\operatorname{rank} P(s) = r$ for almost all values of s. ###

Hence, to find the normal rank of P(s), we just evaluate P(s) at a randomly chosen value of s, compute the rank of the resulting complex matrix, and this yields the value of the normal rank. (Just to be sure, you might want to check a couple values of s...)

Example 1: Consider the transfer function matrix $P(s)=\begin{bmatrix}\frac{1}{s+1} & \frac{1}{s}\\ \frac{1}{s} & \frac{1}{s+1}\end{bmatrix}$. It is easy to determine that $\text{normal rank}\, P(s)=2$. Note that

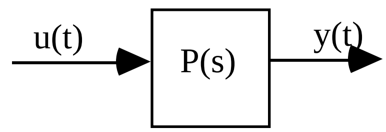
$$\det P(s)=\frac{-(2s+1)}{s^2(s^2+2s+1)}.$$

Hence, $\text{rank}\, P(\frac{-1}{2})=1$, even though $\text{normal rank}\, P(s)=2$. ###

Example 2: Consider the transfer function matrix $P(s)=\begin{bmatrix}\frac{1}{s+1} & \frac{s}{s+1}\\ \frac{1}{s} & 1\\ \frac{1}{s(s+2)} & \frac{1}{s+2}\end{bmatrix}$.

Note that the second column is equal to s times the first column. Hence, the columns of $P(s)$ are linearly dependent, and $\text{rank}\, P(s)=1$ at all values of s that are not poles of $P(s)$. It follows that $\text{normal rank}\, P(s)=1$. ###

Interpretation: Suppose that $P(s)$ is the transfer function of a linear system:



Note that $\text{normal rank}\, P(s)\le \min(p,q)$.

Suppose $p\le q$ (more outputs than inputs). Then if $\text{normal rank}\, P(s)<p$, it follows that the p columns of $P(s)$ are linearly dependent. This may be interpreted as the inputs to $P(s)$ being *redundant*.

Suppose $p\ge q$ (more inputs than outputs). Then if $\text{normal rank}\, P(s)<q$, it follows that the q rows of $P(s)$ are linearly dependent. This may be interpreted as the outputs of $P(s)$ being *redundant*.

In the event that $u(t)$ represents a vector of actuator inputs, and $y(t)$ represents a vector of sensor outputs, then the above conditions correspond to actuator and/or sensor redundancy.

System Inverses

If $\text{normal rank}\, P(s) = q$, then $P(s)$ has a *right inverse* $P^{-R}(s)$ satisfying

$$P(s)P^{-R}(s) = I_{\boldsymbol{q}}.$$

If $\text{normal rank}\, P(s) = p$, then $P(s)$ has a *left inverse* $P^{-L}(s)$ satisfying

$$P^{-L}(s)P(s) = I_p.$$

The left and/or right inverse is unique precisely when $p = q$, in which case we say that the system has an inverse $P^{-1}(s)$ satisfying both the above properties.

The inverse of a system is useful in theoretical derivations; however, it is usually improper, and thus does not have a state space realization. It is true that many design methodologies tend to construct a *rational approximation* to the inverse of a system. This is sometimes desirable, but can also lead to robustness difficulties, as we shall see later in the course.

Note: Suppose that $p = q$, and that D is invertible. Then a state space description of $P^{-1}(s)$ is given by

$$\dot{z} = (A - BD^{-1}C)z + BD^{-1}v$$
$$w = -D^{-1}Cz + D^{-1}v$$

Exercise: Verify that the above system is indeed an inverse. ###

Definition, Rosenbrock System Matrix: Consider the state space system (A, B, C, D). The associated *Rosenbrock System Matrix* (RSM) is given by

$$RSM(s) := \begin{bmatrix} sI - A & B \\ -C & D \end{bmatrix}. \qquad \text{###}$$

It turns out that the RSM will allow us to define zeros of a MIMO system. We now explore some of its properties. Our first result states that a system has an inverse precisely when its associated system matrix has maximum normal rank.

Lemma: $\text{normal rank}\, RSM(s) = n + \text{normal rank}\, P(s)$

Proof: It is easy to verify that the following identity holds:

$$\begin{bmatrix} I & 0 \\ C(sI-A)^{-1} & I \end{bmatrix}\begin{bmatrix} sI-A & B \\ -C & D \end{bmatrix} = \begin{bmatrix} sI-A & B \\ 0 & P(s) \end{bmatrix} . \tag{*}$$

By Sylvester's Inequality,

$$\begin{aligned} \text{normal rank}\begin{bmatrix} I & 0 \\ C(sI-A)^{-1} & I \end{bmatrix}\begin{bmatrix} sI-A & B \\ -C & D \end{bmatrix} &= \text{normal rank}\begin{bmatrix} sI-A & B \\ -C & D \end{bmatrix} \\ &= \text{normal rank}\begin{bmatrix} sI-A & B \\ 0 & P(s) \end{bmatrix} \\ &= n + \text{normal rank}\, P(s) \end{aligned}$$

###

Corollary:

$$\text{normal rank } P(s) = \min(p,q)$$

if and only if

$$\text{normal rank } RSM(s) = n + \min(p,q).$$ ###

Before we define zeros of a system, we invoke the following asumptions; the necessity for these assumptions will become clear in the sequel.

Assumptions:

(1) $\text{normal rank } P(s) = \min(p,q)$
(2) (A,B) is controllable
(3) (A,C) is observable

Definition (Transmission Zero): Suppose that Assumptions (1)-(3) are satisfied. Then z is a *transmission zero* of the system (A,B,C,D) if

$$\text{rank}\, RSM(z) < n + \min(p,q).$$ ###

This definition looks a little unmotivated. We now show that it does indeed make sense for scalar systems (i.e., those with one input and output).

<u>SISO Transmission Zeros:</u> Suppose that $p = q = 1$. Then

$$P(s) = \frac{N(s)}{D(s)} \qquad \text{where} \qquad N(s) = \det(RSM(s)) \text{ and } D(s) = \det(sI - A).$$

<u>Proof:</u> Since $p = q = 1$, the three matrices in (*) are all square; hence we may take their determinants. Using the facts that

$$\det(AB) = \det(A)\det(B)$$

and

$$\det\begin{bmatrix} I & 0 \\ C(sI-A)^{-1} & I \end{bmatrix} = 1$$

yields

$$\det\begin{bmatrix} sI-A & B \\ -C & D \end{bmatrix} = \det(sI-A)\det P(s)$$

$$= \det(sI-A)P(s)$$

###

It follows that the transmission zeros of a SISO system are equal to the zeros of the associated system matrix. Note that the assumptions of controllability and observability is crucial to this fact. The following example shows what happens when these assumptions are not satisfied.

<u>Example (Scalar Pole-Zero Cancellations):</u> Consider the SISO system

$$A = \begin{bmatrix} -2 & -1 \\ -1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad C = [1 \quad 1], \quad D = 0, \quad P(s) = \frac{s+1}{s^2+2s+1}$$

It is easy to verify that both eigenvalues of A are equal to -1, and that

$$\det RSM(s) = \det\begin{bmatrix} sI-A & B \\ -C & D \end{bmatrix}$$

$$= s+1$$

$$= N(s)$$

It follows that the Rosenbrock System matrix loses rank at $s = -1$; however, $s = -1$ is not a zero of the transfer function due to the cancellation with the eigenvalue of A. ###

For SISO systems, we have seen that the assumption of minimality implies that transmission zeros of a system correspond to zeros of its transfer function. For MIMO systems, the transfer function is a matrix; intuitively, the analogous property of a zero is that the transfer function matrix should *lose rank*. The only difficulty is that a matrix transfer function can have poles and zeros at the same value of s, as the following example shows:

$$P(s) = \begin{bmatrix} \frac{1}{s-1} & 0 \\ 0 & \frac{s-1}{s+1} \end{bmatrix}$$

Defining the rank of a matrix transfer function at one of its poles is problematic. (For example, $\det P(s) = \dfrac{1}{s+1}$, which has no finite zeros[1].) This is why the zeros are defined in terms of the rank of the system matrix instead of the rank of the transfer function. In cases for which the zero is not also a pole, it is true that the system matrix loses rank precisely when the transfer function matrix loses rank:

Theorem (MIMO Systems, z not a pole): Suppose that Assumptions (1)-(3) hold, that z is a transmission zero, and that z is *not* a pole of $P(s)$. Then

$$\operatorname{rank} P(z) < \min(p,q).$$

Proof: Follows from the identity

$$\begin{bmatrix} I & 0 \\ C(zI-A)^{-1} & I \end{bmatrix} \begin{bmatrix} zI-A & B \\ -C & D \end{bmatrix} = \begin{bmatrix} zI-A & B \\ 0 & P(z) \end{bmatrix}$$

together with the assumption that z is not an eigenvalue of A. ###

Transmission Blocking Property:

Transmission zeros are associated with modes of behavior wherein the input and states of a system are nonzero, yet the output equals zero. We now provide formulas for the inputs and initial states that produce zero output. We must handle separately the cases for which there are more outputs than inputs ($p \le q$) and for which there are more inputs than outputs ($p \ge q$).

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[1] We say that a strictly proper transfer function has one or more *zeros at infinity*.

Theorem (Transmission Blocking): Suppose that Assumptions (1)-(3) hold and that z is a transmission zero.

Case 1 ($p \le q$): There exist an input $u(t) = u_0 e^{zt}$, $u_0 \ne 0$, and an initial state x_0, such that $y(t) = 0, \forall t \ge 0$. Furthermore, if A has stable eigenvalues, then for this input, $y(t) \to 0, \forall x_0$.

Case 2 ($p \ge q$): Assume that z is not an eigenvalue of A. There exists a linear combination of outputs $k^T y(t) = k_1 y_1(t) + \ldots + k_q y_q(t)$ such that for any input $u(t) = u_0 e^{zt}$, u_0 arbitrary, there is an initial state x_0 such that $k^T y(t) = 0, \forall t \ge 0$. Furthermore, if A has stable eigenvalues, then $k^T y(t) \to 0, \forall x_0$.

Terminology (Zero Directions): We say that u_0 is the "input zero direction", x_0 is the "zero state direction", and that k is the "output zero direction". ###

Proof: I will do Case 1; my derivation is adapted from p. 399 of F.M.Callier and C.A.Desoer, *Linear System Theory*.

Suppose that $\text{rank}\, RSM(z) < n + p$. Then $RSM(z)$ has a nontrivial right nullspace. By the definition of nullspace, there exist vectors $x \in \mathbf{R}^n, u \in \mathbf{R}^p$ such that $x \ne 0$ and/or $u \ne 0$ and

$$\begin{bmatrix} zI - A & B \\ -C & D \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

We show that, in fact, both $x \ne 0$ *and* $u \ne 0$. To see this, suppose first that $x = 0$. Then

$$\begin{bmatrix} zI - A & B \\ -C & D \end{bmatrix} \begin{bmatrix} 0 \\ u \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} sI - A & B \\ -C & D \end{bmatrix} \begin{bmatrix} 0 \\ u \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \forall s$$

and it follows that Assumption 1 is violated. Next suppose that $u = 0$. Then

$$\begin{bmatrix} zI - A & B \\ -C & D \end{bmatrix} \begin{bmatrix} 0 \\ u \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} zI - A \\ C \end{bmatrix} x = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

and it follows that Assumption 2 is violated. (We will only use the fact that $u \neq 0$.)

Let the input be given by $u(t) = u_0 e^{zt}$, with $u_0 = -u$, and let the initial state be given by $x_0 = x$. Then

$$\begin{bmatrix} zI - A & B \\ -C & D \end{bmatrix} \begin{bmatrix} x_0 \\ -u_0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{cases} (zI - A)x_0 = Bu_0 \\ Cx_0 + Du_0 = 0 \end{cases}$$

Recall that

$$X(s) = (sI - A)^{-1} x_0 + (sI - A)^{-1} BU(s).$$

Substituting $U(s) = \dfrac{u_0}{s - z}$ yields

$$\begin{aligned} X(s) &= (sI - A)^{-1} x_0 + (sI - A)^{-1} B \frac{u_0}{s - z} \\ &= (sI - A)^{-1} \left\{ x_0 + \frac{Bu_0}{s - z} \right\} \\ &= (sI - A)^{-1} \left\{ x_0 + \frac{(zI - A)x_0}{s - z} \right\} \\ &= (sI - A)^{-1} \left\{ (sI - A) \frac{x_0}{s - z} \right\} \end{aligned}$$

It follows that $x(t) = e^{zt} x_0$, and thus $y(t) = Cx_0 e^{zt} + Du_0 e^{zt} = 0$.

Suppose next that all eigenvalues of A are stable, and assume (for simplicity of the derivation) that z is not an eigenvalue of A. Let x_0 be an arbitrary initial condition. Then the partial fraction expansion of the output has the form

$$\begin{aligned} Y(s) &= C(sI - A)^{-1} x_0 + \left\{ C(sI - A)^{-1} B + D \right\} \frac{u_0}{s - z} \\ &= \left\{ C(zI - A)^{-1} B + D \right\} \frac{u_0}{s - z} + \textit{terms due to eigenvalues of } A \end{aligned}$$

where $\left\{ C(zI - A)^{-1} B + D \right\} u_0$ is the residue at the pole $s = z$. It follows from the identity

$$\begin{bmatrix} I & 0 \\ C(zI-A)^{-1} & I \end{bmatrix}\begin{bmatrix} zI-A & B \\ -C & D \end{bmatrix} = \begin{bmatrix} zI-A & B \\ 0 & P(z) \end{bmatrix}$$

that this residue is equal to zero. Since all the eigenvalues of A are stable, $y(t) \to 0$ as $t \to \infty$. ###

<u>Zeros of the System Matrix:</u> Let z be a frequency for which

$$\operatorname{rank}\begin{bmatrix} zI-A & B \\ -C & D \end{bmatrix} < n + \min(p,q)$$

Such a value of z is called a "zero of the system matrix". If Assumptions (1)-(3) are satisfied, then we know that z is also a transmission zero. We now discuss what happens if any of these assumptions are violated.

If Assumption(1) is violated, then we know that $P(z)$ has less than full rank for every value of z. The system has redundant inputs and/or outputs.

Let Assumption (1) hold, assume that $p \geq q$, and suppose that Assumption (2) is violated. Let λ be an uncontrollable eigenvalue of (A,B). Then $\operatorname{rank} RSM(\lambda) < n+q$. Such zeros of $RSM(s)$ are termed *input decoupling zeros*, because they correspond to modes of the system that cannot be excited by the input.

Let Assumption (1) hold, assume that $p \leq q$, and suppose that Assumption (3) is violated. Let λ be an unobservable eigenvalue of (A,C). Then $\operatorname{rank} RSM(\lambda) < n+p$. Such zeros of $RSM(s)$ are termed *output decoupling zeros*, because they correspond to modes of the system that do not appear in the output.

It follows that the three Assumptions imply that all zeros of the system correspond to transmission zeros of the system.

<u>Invariance of System Zeros under State Feedback</u>

Consider the system

$$\dot{x} = Ax + Bu, \qquad x \in \mathbf{R}^n,\ u \in \mathbf{R}^p$$
$$y = Cx + Du, \qquad y \in \mathbf{R}^q$$

and assume that

$$\text{normal rank}\begin{bmatrix} sI-A & B \\ -C & D \end{bmatrix} = n + \min(p,q)$$

(We do not assume that (A,B) is controllable or that (A,C) is observable.)

Applying state feedback, $u = -Kx + r$, yields the system

$$\dot{x} = (A - BK)x + Br,$$
$$y = (C - DK)x + Dr$$

Lemma: $\text{rank}\begin{bmatrix} zI-A & B \\ -C & D \end{bmatrix} < n + \min(p,q) \Leftrightarrow \text{rank}\begin{bmatrix} zI-A+BK & B \\ -C+DK & D \end{bmatrix} < n + \min(p,q)$

Proof: Follows from Sylvester's Inequality applied to the factorization

$$\begin{bmatrix} zI-A+BK & B \\ -C+DK & D \end{bmatrix} = \begin{bmatrix} zI-A & B \\ -C & D \end{bmatrix}\begin{bmatrix} I & 0 \\ K & I \end{bmatrix}$$

###

Notes: (i) Zeros of the Rosenbrock System Matrix are *invariant under state feedback*.
(ii) If (A,B) is controllable and (A,C) is observable, then applying state feedback creates no new transmission zeros, and can "remove" transmission zeros only by creating a nonminimal realization; i.e., by shifting a pole under a transmission zero to obtain a pole-zero cancellation. (Several design procedures tend to do this; e.g., cheap control LQ regulators.)

Calculating zeros: The MATLAB command "tzero" finds zeros of the system matrix by solving a *generalized eigenvalue problem*. By the above comments, if Assumptions (1)-(3) are satisfied, then the solutions to this problem are also transmission zeros. Specifically:

($p \le q$) Suppose that $\text{rank}\, RSM(z) < n + p$. Then, by definition $\exists \begin{bmatrix} x \\ u \end{bmatrix} \ne \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ such that $\begin{bmatrix} zI-A & B \\ -C & D \end{bmatrix}\begin{bmatrix} x \\ u \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. Equivalently, $\begin{bmatrix} A & -B \\ C & -D \end{bmatrix}\begin{bmatrix} x \\ u \end{bmatrix} = z\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ u \end{bmatrix}$.

Finding values of x,u,z such that this equation is satisfied is a special case of the *generalized eigenvalue problem*:

Given compatibly dimensioned matrices M and N, find $v \neq 0$ and λ such that $Mv = \lambda N v$. In our case, the generalized eigenvalue problem is obtained by setting $M = \begin{bmatrix} A & -B \\ C & -D \end{bmatrix}$ and $N = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$.

(If $p > q$, then one may work with the transposes of the relevant matrices.)

<u>Note:</u> The "tzero" command will give you zeros of the system matrix; these will be transmission zeros only when Assumptions (1)-(3) are satisfied.

<u>Example:</u> Consider the transfer function

$$P(s) = \begin{bmatrix} \frac{2}{s^2+3s+2} & \frac{2s}{s^2+3s+2} \\ \frac{-2s}{s^2+3s+2} & \frac{-2}{s^2+3s+2} \end{bmatrix}$$

A minimal realization of $P(s)$ is given by

$$A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & -2 \\ -2 & 4 \\ -4 & 2 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Let's now use MATLAB to calculate transmission zeros:

```
»tzero(A,B,C,D)
ans =
    1.0000
```

Let's verify that $z = 1$ is a transmission zero by calculating the rank of the system matrix:

```
»RSM_1 = [1*eye(3)-A  B;-C  D] ;
»rank(RSM_1)
ans =
    4
```

let's now find the input zero and state zero directions by looking at the nullspace of $RSM(1)$:

```
»null(RSM_1)
ans =

    0.5345
   -0.5345
   -0.5345
   -0.2673
    0.2673
```

It follows that the zero state direction is $x_0 = \begin{bmatrix} 0.5345 \\ -0.5345 \\ -0.5345 \end{bmatrix}$, and that the input zero direction is $u_0 = \begin{bmatrix} 0.2673 \\ -0.2673 \end{bmatrix}$.

Let's now simulate the response of the system to initial condition x_0 and an input $u(t) = u_0 e^t$:

```
»x0 = [0.5345; -0.5345; -0.5345];
»u0 = [0.2673; -0.2673];
»t = linspace(0,5);
»u = exp(t);
»[y,x] = lsim(A,B*u0,C,D*u0,u,t,x0);
»plot(t,y)
»axis([0 5 -1 1])
»xlabel('time, seconds')
»stitle('response to x_{0} and u(t)=u_{0}e^{t}')
```